Geophysics Wavelet Library: Applications of the Continuous Wavelet Transform to the Polarization and Dispersion Analysis of Signals

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Abstract

In the present paper, we propose a software package developed by the authors and based on the continuous wavelet transform that allows to perform the direct and inverse continuous wavelet transform, 2C and 3C polarization analysis and filtering, modeling the dispersed and attenuated wave propagation in the time-frequency domain and optimization in signal and wavelet domains with the aim to extract velocities and attenuation parameters from a seismogram. The novelty of this package is that we incorporate the continuous wavelet transform into the library where the kernel is the time-frequency polarization and dispersion analysis. This library has a wide range of potential applications and can be particularly suitable in geophysical problems that we illustrate with the analysis of synthetic, geomagnetic or real seismic data.

Key words: Continuous wavelet transform, signal processing, dispersion, polarization, MATLAB

1 Introduction

Frequency-dependent measurements, or time-frequency analysis (TFR) offer additional insight and performance in any applications where Fourier techniques have been used. This analysis consists of examining the variation of

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the frequency content of a signal with time and is particularly suitable in geo-
physical applications. We can emphasize three directions to use TFR for the
analysis of geophysical data. As the first very helpful tool, time-frequency rep-
resentations can be incorporated in polarization analysis (Soma et al. (2002); Schimmel and Gallart (2005); Kulesh et al. (2007a,c); Diallo et al. (2005b, 2006b); Pinnegar (2006); Pacor et al. (2007)). It is also possible to model
dispersive and dissipative wave propagation in the time-frequency domain
(Kulesh et al. (2005a,b)). Finally, the time-frequency analysis is suitable for
an estimate of the phase velocity (the group velocity) and the attenuation
coefficient (Levshin et al. (1972); Prosser et al. (1999); Pedersen et al. (2003);
Holschneider et al. (2005); Kulesh et al. (2007b)).

The continuous wavelet transform (CWT) gives a suitable general framework
for solving these types of problems; this approach is powerful and elegant,
but is not the only available for the practical applications. Other TFR meth-
ods such as the Gabor transform, the S-transform (Schimmel and Gallart
(2005)) or bilinear transforms like the Wigner-Ville (Pedersen et al. (2003))
or smoothed Wigner-Ville transform can be used as well. The relative perfor-
manence of time-frequency analysis from different TFR approaches is primarily
controlled by the frequency resolution capability that motivated the use of
CWT in the present work.

This article summarizes our previous works aimed to polarization and dis-
persion analysis of signals in the wavelet domain and offers the Geophysics
Wavelet Library (GWL) — a new free software package based on CWT and
having the following key features:

(1) object-based implementation of main data types like a vector, an axis, a
matrix, a multi-channel signal, a multi-channel wavelet spectrum;
(2) object-based implementation of main mathematical objects like Morlet
and Cauchy wavelets, some function approximations, 2C and 3C polar-
ization parameters and dispersion parameters;
(3) command line and MATLAB interface for transformations like the Fourier
transform, the direct and inverse CWT, 2C and 3C polarization trans-
forms (Diallo et al. (2005b,a, 2006b,a)), linear and nonlinear deformations of a wavelet spectrum introduced by Xie et al. (2003); Kulesh et al.
(2005a);
(4) command line interface for the optimization in signal and wavelet do-
mains using the algorithm of Levenberg-Marquardt optimization with the
aim to extract velocities and attenuation parameters from a seismogram
(Holschneider2005GJI);
(5) data import from tabulated and plain ASCII files and data export into
ASCII files.

One can find and download in internet a great number of other free or com-
mmercial wavelet-based software, for example:

(1) **ImageLib** is a C++ class library providing image processing and related facilities. The main set of classes provides a variety of image and vector types, with additional modules supporting scalar and vector quantization, the discrete wavelet transforms, and simple histogram operations;

(2) **LIFTPACK** is a software package written in C for fast calculation of 1D and 2D Haar and biorthogonal wavelet transforms using the lifting scheme;

(3) **The Rice Wavelet Toolbox** is a collection of MATLAB M-files and C MEX-files for 1D and 2D wavelet and filter bank design, analysis, and processing. The toolbox provides tools for denoising and interfaces directly with MATLAB code for hidden Markov models in the wavelet domain and wavelet regularized deconvolution;

(4) **MathWorks’ Wavelet Toolbox** for MATLAB implements standard wavelet families, including Daubechies wavelet filters, complex Morlet and Gaussian, real reverse biorthogonal, and discrete Meyer. It has interactive tools for continuous and discrete wavelet analysis and methods for adding wavelet families;

(5) **MR/1** is a large package written in C++ and IDL (Interactive Data Language, Research Systems Inc.) for filtering, deconvolution, object detection and analysis, vision modeling, compression, registering, etc. A wide range of wavelet and other multi-scale transforms is supported;

(6) **Wavelet Explorer** - a package for Wavelets in Mathematica. It includes common filters such as Daubechies’ extremal phase and least asymmetric filters, spline filters, and contains transforms to wavelet bases, wavelet packet bases, or local trigonometric bases in one and two dimensions as well as data compression and denoising.

However, all these software packages do not have any feature tailored for geophysical problems. With GWL, we try to address these limitations and incorporate CWT into the library where the kernel is the time-frequency polarization and dispersion analysis. The main purpose of this article is to show not only mathematical aspects of this problem, but also some peculiarities of implementation.

This paper is organized as follows. After a short overview of GWL structure and implementation technology, we briefly introduce the direct and inverse wavelet transforms and some their properties that we will need afterwards. Then, we describe three different wavelet based polarization analysis methods for two components, three components or more. Next, we introduce wavelet deformation algebra and describe three approximative methods for the wave propagation modeling in the time-frequency domain. In the last section, we demonstrate the applications of these propagation models in two inversion algorithms for the characterization of dispersion and attenuation properties of surface waves. In each section, we give base equations and short mathe-
mathematical description of every method, show how the method is implemented in GWL, illustrate it with using synthetic, geomagnetic or real seismic data, and compare with other methods.

2 GWL structure and implementation technology

GWL includes three logical levels: the library level, the level of command line tools and the interface level as shown in Fig. 1.

The main part of the library level is a C++ hierarchical object library called PPP. This library uses the C++ Standard Template Library, an ANSI C command line parser (argtable2 version 2.6) and a C subroutine library for computing the discrete Fourier transform (FFTW version 3.0.1, Frigo and Johnson (1998)).

GWL command line level is a set of independent C++ modules which are based on PPP library and provide a command line interface for all methods implemented in this library. After the compilation, we obtain a set of executable modules placed in GWL/bin directory. We tested the compilation of these modules using Linux based GCC compiler (version 3.3.x) and Windows based Borland C++ compiler (Borland C++ Builder, version 5.0).

To perform a calculation using GWL, we run certain modules from GWL/bin directory in the appointed order. The calculation parameters have to be given by command line. The data exchange between different modules is implemented by the data files with binary stream formats. After the calculation process is finished, we obtain the collection of ASCII or binary files with calculation results.
In general, we can plot these calculation results saved in ASCII format using any plotting software. To reduce the required space on hard drive and to improve the plotting performance, we can also store the results as binary files. Toward this end, we developed an especial MATLAB package placed on interface level of GWL in the directory GWL/mshell. This package allows us to read all binary formats supported in GWL and plot GWL objects like a multi-channel signal or a multi-channel wavelet spectrum using height-level subroutines based on the standard MATLAB plotting commands. This package is developed as a M-file library and do not need to run any installation before using the program.

The second possibility to use the GWL/mshell library is an integration of the calculation process with plotting subroutines using a M-file program. In this way we added some procedures into GWL/mshell for the directly execution of the modules from the GWL command line level.

To demonstrate the calculation technique using GWL, we stored many examples into GWL/solutions directory. This folder together with GWL MATLAB tools constitutes a part of GWL interface level. Diallo et al. (2005b,a, 2006b,a); Kulesh et al. (2005a,b); Holschneider et al. (2005) preciously used all these examples to show the application possibilities of the CWT for dispersion and polarization analysis of synthetic, geomagnetic and real seismic data.

3 The continuous wavelet transform

In this section, we introduce two modules gwlCwt and gwlIwt from the command line level. These procedures implement the direct and inverse CWT.

3.1 The direct wavelet transform

The wavelet transform of a real or complex signal \( S(t) \in L^2(\mathbb{R}) \) with respect to a real or complex mother wavelet \( g(t) \) is the set of \( L^2 \)-scalar products of all dilated and translated wavelets with an arbitrary signal to be analyzed Holschneider (1995):

\[
W_g S(t, a) = \langle g_{t,a}, S \rangle = \int_{-\infty}^{+\infty} \frac{1}{a} g^* \left( \frac{\tau - t}{a} \right) S(\tau) d\tau, \quad a \in \mathbb{R}, t \in \mathbb{R}, 
\]

where \( g_{t,a} = \frac{1}{a} g((\tau - t)/a) \) is generated from \( g(t) \) through dilation \( a \) and translation \( t \). The symbol \((\cdot)^*\) denotes the complex conjugate. The inverse of
the scale \( a \) may be associated with a frequency measured in units of the wavelet central frequency. If the central frequency of the wavelet is assumed to be \( f_0 \), each scale \( a \) can be related to the physical frequency \( f \) by \( a = f_0 / f \). Therefore, if we select a wavelet with a unit central frequency, it is possible to obtain the physical frequency directly by taking the inverse of the scale. This approach is implemented in the module \( \text{gwlCwt} \) with the parameter \(-\text{wttype}=0\), which calculates the wavelet spectrum related to the physical frequency \( f \) and time \( t \), or in the time-frequency domain.

The procedure (1) is very slow and therefore cannot be used for long signals. We can construct a more effective algorithm if we note the fact that the wavelet transform can be expressed in terms of the Fourier transform \( \hat{S}(\zeta) \) of \( S(t) \) as

\[
W_g S(t, f) = \int_{-\infty}^{+\infty} \hat{g}^*(\zeta / f) \exp(2\pi it\zeta) \hat{S}(\zeta) \, d\zeta.
\]

This fast approach is implemented in the module \( \text{gwlCwt} \) as well, but with the parameter \(-\text{wttype}=1\), where FFT procedure (Press et al. (1992)) or FFTW3 procedure is used for the convolution calculation. In comparison with the approach (1), this algorithm requires a signal with \( 2^n \) points’ length.

### 3.2 Wavelets

The wavelet \( g(t) \) is assumed to be a function which is well localized in the time and frequency and obeys the oscillation condition \( \int_{-\infty}^{+\infty} g(t) \, dt = 0 \). The choice of the wavelet and the initialization of the wavelet parameter are represented in the module \( \text{gwlCwt} \) using the parameters \(-\text{wavelet} \) and \(-\text{wavepar} \).

![Representation of Morlet wavelet in (a) time and (b) frequency domain, \( \sigma = 0.7 \).](image)

(1) \(-\text{wavelet=morlet} \) corresponds to the progressive complex Morlet wavelet (Holschneider (1995)). This wavelet is shown in Fig. 2 and can be written with its Fourier transform \( \hat{g}(\omega) \) as
\( g(t) = \exp(2\pi it) \exp(-t^2/(2\sigma^2)), \)
\( \hat{g}(\omega) = \sigma \exp(-\sigma^2(\omega - 2\pi)^2/2), \)

where \( \omega \) is the circular frequency and parameter \( \sigma \) describes the variance of the wavelet.

(2) The second wavelet implemented in GWL is the complex Cauchy wavelet (-wavelet=cauchy):
\[
g(t) = \left(1 - \frac{2\pi it}{p-1}\right)^{-p},
\]
\[
\hat{g}(\omega) = \frac{(p-1)^p}{\sqrt{2\pi(p-1)!}} \left(\frac{\omega}{2\pi}\right)^{p-1} \exp\left(-\frac{\omega(p-1)}{2\pi}\right).
\]

Both Morlet and Cauchy wavelets are progressive, i.e. their Fourier coefficients for negative frequencies are zero. This feature allows us to separate the wavelet spectrum into progressive and regressive components (gwlCwt --ftype=2),

\[
W_gS(t, f) = W_g^+S(t, f) + W_g^-S(t, f),
\]

where
\[
W_g^+S(t, f) = \begin{cases} W_gS(t, f), f \geq 0 \\ 0, f < 0 \end{cases}, \quad W_g^-S(t, f) = \begin{cases} 0, f \geq 0 \\ W_gS(t, f), f < 0 \end{cases}
\]

An example of this separation is shown in Fig. 3, where we consider a two-component synthetic seismogram related to the Rayleigh wave. From two signals \( S_x(t) \) and \( S_z(t) \) corresponding to the orthogonal components of the record, we construct a complex signal
\[
Z(t) = S_x(t) + iS_z(t).
\]

Next, we perform CWT (2) for this complex signal using Cauchy wavelet \( (p = 5) \), and plot only the absolute values of the complex wavelet coefficients as gray-scaled images separately for the progressive and the regressive components (5) of the wavelet spectrum.

3.3 The inverse wavelet transform

We implemented the inverse CWT in the module gwlIwt:
Fig. 3. Wavelet transform of Rayleigh wave arrival. (a) Radial $S_r(t)$ and vertical $S_z(t)$ components of the synthetic 2-C seismograms. (b) Its progressive and regressive wavelet transform. (c) Hodogram showing the particle motion over the entire time window. Solid lines in the panel (b) indicate the maximum of modulus of the wavelet spectrum.

$$S(t) = \mathcal{M}_h \mathcal{W}_g S(t, f) = \frac{1}{C_{g,h}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(f(t - \tau)) \mathcal{W}_g S(\tau, f) \, d\tau \, df,$$  

where $h(t)$ is the wavelet used for the inverse wavelet transform $\mathcal{M}_h$. In the general case, this wavelet can be different from the analyzing wavelet $g(t)$.

As in the case of the direct integration (1), the procedure (7) is very slow. However, one can choose for the inverse CWT the $\delta$-function as the wavelet $h(t)$ (\texttt{gwt} \texttt{-wavelet=delta}), which gives us a rather simple and fast reconstruction formula

$$S(t) = \mathcal{M}_h \mathcal{W}_g S(t, f) = \frac{1}{C_{g,\delta}} \int_{-\infty}^{+\infty} \mathcal{W}_g S(t, f) \, df.$$  

$C_{g,h}$ in eq. (7) and $C_{g,\delta}$ in eq. (8) are the normalization coefficients related to the direct and inverse mother wavelets:

$$C_{g,h} = \int_{0}^{+\infty} \left( \hat{g}^*(\omega) \hat{h}(\omega) + \hat{g}^*(-\omega) \hat{h}(-\omega) \right) \frac{d\omega}{\omega}.$$ 

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We need to calculate this coefficient not for all applications; therefore we implemented the parameter - - ampl in the module gwlIwt, which defines the normalization mode of the inverse signal.

4 Polarization properties for two-component data and polarization filtering

GWL contains several modules for the polarization analysis and polarization filtering of two-component and three-component signals. Polarization analysis for two-component data can be carried out using the module gwlET2D while the polarization filtering is performed using the other module gwlET2DFilter.

Given a signal from three-component record, with \( S_x(t) \), \( S_y(t) \), and \( S_z(t) \) representing the seismic traces recorded in three orthogonal directions, any combination of two orthogonal components can be selected for the polarization analysis. There are several methods to perform such analysis; some of them we implemented in the module gwlET2D using the parameter - - type:

1. - - type=rene performs the complex trace analysis (CTA) proposed by Rene et al. (1986),
2. - - type=morozov corresponds to the method proposed by Morozov and Smithson (1996) and adapted to the two-component case,
3. - - type=scovar allows us to calculate the polarization properties by the eigenanalysis of the cross-energy matrix of a two-component record (Flinn (1965); Kanasewich (1981); Jurkevics (1988)).

4.1 Complex trace analysis in the wavelet domain

The above-listed methods operate only in the time domain and the estimated attributes represent an average over all frequencies and therefore do not provide information about their frequency dependency. To cope with these limitations inherent to time-frequency resolution, we previously proposed a method based on the CWT (Diallo et al. (2006b)). In its full generality, an elliptically polarized rotating signal (6) is described by following geometric parameters (Fig. 4):

1. \( R \): the semi-major axis \( R \geq 0 \),
2. \( r \): the semi-minor axis \( R \geq r \geq 0 \),
3. \( \rho = r/R \): the ellipticity ratio, \( \rho \in [0, 1] \),
4. \( \theta \): the tilt angle, which is the angle of the semi-major axis with the horizontal axis, \( \theta \in (-\pi/2, \pi/2] \),
(5) $\Delta \phi$: the phase difference between $S_x(t)$ and $S_z(t)$ components.

The main idea to obtain instantaneous polarization attributes is the approximation of the complex signal $Z(t)$ at the time point $t$ by a turning ellipse $C(\tau)$ on the complex plane (Fig. 4.a). The most general turning ellipse is parameterized by two complex numbers, $A^+$ and $A^-$ and two real numbers $\Omega^+$ and $\Omega^-$: $C(\tau) = A^+ \exp (i\Omega^+ \tau) + A^- \exp (-i\Omega^- \tau)$.

When a complex progressive wavelet is used, the property (5) of CWT allows us to represent the wavelet spectrum as a superposition of its progressive and regressive components. Let us consider the instantaneous angular frequency defined as the derivative of the complex spectrum’s phase: $\Omega^\pm(t, f) = \pm \partial \arg W^\pm_g Z(t, f) / \partial t$. Then, near time instant $t$, each component can be represented as follows:

$$W_g Z(t + \tau, f) \simeq W^+_g Z(t, f) \exp (i\Omega^+(t, f) \tau) + W^-_g Z(t, f) \exp (-i\Omega^-(t, f) \tau),$$

which yields the time-frequency spectrum for each of the parameters:

$$R(t, f) = |W^+_g Z(t, f)| + |W^-_g Z(t, f)| / 2,$$
$$r(t, f) = ||W^+_g Z(t, f)| - |W^-_g Z(t, f)|| / 2,$$
$$\theta(t, f) = \arg [W^+_g Z(t, f) W^-_g Z(t, f)] / 2,$$
$$\Delta \phi(t, f) = \arg \left( \frac{W^+_g Z(t, f) + W^-_g Z(t, f)^*}{W^+_g Z(t, f) - W^-_g Z(t, f)^*} \right) \mod \pi.$$  

The module gw1ET2D with the input parameter --type=complex performs the calculation of polarization attributes in the time domain, if the input object is a signal, and in the time-frequency domain using eq. (9) in the case when a spectrum object is given as input.

This method can be applied not only for seismic data. We also used this polarization analysis for an examination of characteristics of geomagnetic Pi2
pulsations where we analyzed the time dependence of the phase difference (Kulesh et al. (2007c)). An important application of such information is the determination of oscillation properties of Pi2 pulsation during the event, since it provides information to reveal the excitation mechanism of Pi2 pulsations.

To demonstrate the time-frequency polarization analysis, we examine some records from a magnetic ground station (HER station, 124.425° co-latitude, 19.225° longitude) around a Pi2 pulsation for three events occurred on 7 December 1996, 4 January 1997, and 14 January 1997. The north and the east components of the records as well as computed ratio and the tilt angle in the time-frequency domain are shown in the Fig. 5. It is found that the ellipticity ratio is small, which indicates oscillations are dominant in the direction defined by the tilt angle. Tilt angles close to 0° indicate that the major polarization axis is directed almost along the north-south direction.

Fig. 5. Polarization properties of three geomagnetic records.

4.2 Filtering using polarization attributes

If we analyze the seismic data, an advantage of the method (9) is the possibility to perform the complete wave-mode separation/filtering process in the wavelet domain and the ability to provide the frequency dependence of ellipticity, which contains important information on the subsurface structure. Using 2-C synthetic and real seismic shot gathers, Diallo et al. (2006b) showed how to use the method to separate different wave types and identify zones of interfering wave modes.
With the extension of the polarization analysis to the wavelet domain, we can construct filtering algorithms to separate different wave types based on the instantaneous attributes by a combination of constraints posed on the range of the reciprocal ellipticity \( \rho(t, f) \) and the tilt angle \( \theta(t, f) \). Formally the algorithm can be represented as

\[
Z^f(t) = M_h \mathcal{E}_{\rho, \theta} W_g Z(t, f),
\]

\[
\mathcal{E}_{\rho, \theta}(t, f) = \begin{cases} 
W_g Z(t, f) & \text{for } \rho(t, f) \in P_\rho \text{ and } \theta(t, f) \in P_\theta, \\
0 & \text{otherwise,} \end{cases}
\]

(10)

where \( \mathcal{E}_{\rho, \theta} \) is the filter operator of the wavelet-spectrum. The sets \( P_\rho \) and \( P_\theta \) define the range of \( \rho \) and \( \theta \), which are kept in the filtered signal. In the Table 1 we summarize filter settings that can be used to detect signals with specific polarization.

<table>
<thead>
<tr>
<th>Filter name</th>
<th>Notation</th>
<th>( P_\rho )</th>
<th>( P_\theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear and horizontal</td>
<td>( \mathcal{E}_{LH} )</td>
<td>([0, \rho_f] \cup [1 - \rho_f, 1])</td>
<td>([-\theta_f, \theta_f])</td>
</tr>
<tr>
<td>Linear and vertical</td>
<td>( \mathcal{E}_{LV} )</td>
<td>([0, \rho_f] \cup [1 - \rho_f, 1])</td>
<td>([-\pi/2, -\theta_f] \cup [\theta_f, \pi/2])</td>
</tr>
<tr>
<td>Elliptical and horizontal</td>
<td>( \mathcal{E}_{EH} )</td>
<td>([\rho_f, 1 - \rho_f])</td>
<td>([-\theta_f, \theta_f])</td>
</tr>
<tr>
<td>Elliptical and vertical</td>
<td>( \mathcal{E}_{EV} )</td>
<td>([\rho_f, 1 - \rho_f])</td>
<td>([-\pi/2, -\theta_f] \cup [\theta_f, \pi/2])</td>
</tr>
</tbody>
</table>

Table 1
Classification of wave types using the polarization attributes. \( \rho_f \) and \( \theta_f \) are two filter parameters.

The filtering approach (10) accordingly to the filter settings defined in the Table 1 is implemented in the module \texttt{gwlET2DFilter}. However, the object structure of PPP library allows us to extend the set of the filter types.

In the Fig. 6, we show an example of polarization filter calculated using the module \texttt{gwlET2DFilter}. We consider here three different events showed above in the Fig. 5, but now we analyze the whole signal within the day containing a Pi2 pulsation. With the aim to extract the Pi2 pulsation from all these signals, we use Fourier band-pass filter and the so-called ”total horizontal filter” \( \mathcal{E}_{LH} + \mathcal{E}_{EH} \), which is defined in the same frequency band as Fourier filter, but with additional restriction on the tilt angle \( |\theta| < 0.6 \text{rad.} \) We can see that the polarization filter picks out the Pi2 pulsation clearer than Fourier band-pass filter. The next example is a division of previous horizontal polarized signals into the linear \( \mathcal{E}_{LH} \) and elliptical \( \mathcal{E}_{EH} \) parts, where \( \rho_f = 0.3 \). As expected, all of Pi2 pulsations are placed only on the left panels demonstrated linearly polarized signals, because Pi2 pulsations on the nightside are predominantly linearly polarized.
Fig. 6. An example of polarization filtering as applied to geomagnetic records.

5 Polarization analysis of three-component data and polarization filtering

In this section, we briefly introduce two wavelet-based polarization methods appropriated for the analysis of three-component records. This analysis and corresponding polarization filtering are implemented in the modules `gwlET3D` and `gwlET3DFilter`.

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5.1 Polarization ellipse

Morozov and Smithson (1996) proposed a method based on a variational principle that allows generalization to any number of components, and they briefly addressed the possibility of using the instantaneous polarization attributes for wavefield separation and shear-wave splitting identification. Using GWL, we can execute this algorithm, if we run the module `gwlET3D --type=morozov` for an input object having a signal format.

Diallo et al. (2005b) extended the method of Morozov and Smithson (1996) to the wavelet domain in order to use the instantaneous attributes for filtering and wavefield separation for any number of components. To introduce this method, we define $W_g S(t, f)$ as the component-wise calculated progressive wavelet spectra ($f > 0$, wavelet is progressive) of the multicomponent signal $S(t) = [S_x(t), S_y(t), S_z(t)]$. The resulting semi-major axis and the semi-minor axis in this approach are

$$\begin{align*}
R(t, f) &= \Re \{W_g S(t, f) \exp (-i\psi_0(t, f))\}, \\
r(t, f) &= \Re \{W_g S(t, f) \exp (-i(\psi_0(t, f) + \pi/2))\}, \\
\psi_0(t, f) &= \frac{1}{2} \arg [A(t, f) + \varepsilon B(t, f)] + \pi n, \quad n \in \mathbb{N}, \quad \varepsilon \simeq 0, \\
A(t, f) &= \frac{1}{2} \sum_k W_g S_k(t, f)^2, \\
B(t, f) &= \frac{1}{2} (\sum_k W_g S_k(t, f))^2.
\end{align*} \tag{11}$$

Fig. 7. Three-component real seismograms showing radial, transversal and vertical components respectively.

Using the modules `gwlET3D --type=morozov` and `gwlET3DFilter` with an input object in the spectrum format, we show here an example of polarization filtering based on the approach (11). The real seismograms (Fig. 7) used in this example are three-component recordings from an explosive-source experiment aimed at imaging the Dead Sea Transform in the Middle East (DESERT-Group (2000)). The strong surface wave arrivals can be observed between 3s.
and 5s.

Fig. 8. Example of wave-type separation applied to the three-component seismograms.

We perform two different polarization filters (See, e.g. Diallo et al. (2005b) for more information):

1. `gwlET3DFilter --type=morozov --filter=ellixy` extracts a seismic arrival with a polarization in the horizontal $(x−y)$ plane. It selects the region in the time-frequency plane where $θ_z(t, f)$ is relatively small and then sets the corresponding $R(t, f)$ and $r(t, f)$ to zero followed by an inverse CWT, where

$$θ_k(t, f) = \arccos \left( \frac{|v_k(t, f)|}{∥v(t, f)∥} \right) \in \left[ 0, \frac{π}{2} \right], \quad k = x, y, z$$

are the different angles between the planarity vector $v(t, f) = R(t, f) \times r(t, f)$ and the orthogonal axes $x$, $y$ and $z$ at each point $(t, f)$. Fig. 8a corresponds to this filter. The strong arrival observed on the filtered seismogram would probably correspond to P-to-S conversion or to P-wave energy leaking into the horizontal component due to the offset. It is important to note that the energy arrival from the filtered vertical component is very weak. It suggests that our filter is capable of detecting the desired signal, expected to consist mainly of the $x$- and $y$-components.

2. `gwlET3DFilter --type=morozov --filter=norm` performs another application of polarization analysis, which is important in geologically complex
areas. This is the removal of out-of-plane energy. This is achieved by nor-
malizing the different cosine directions $\theta_k(t, f)$ so that the range of each
direction is between 0 and 1. Then, each normalized $\theta_k(t, f)$ is multiplied
by the wavelet transform of the signal component along the corresponding
$k$-axis ($k = x, y, z$), followed by an inverse CWT. Fig. 8b shows the result
of this procedure. This filtering has the effect of reducing the influence
of energy arrivals that come from directions not parallel to the direction
where each component has its highest sensitivity to particle motion.

As an additional example, Pacor et al. (2007) used the method (11) for spec-
tral analysis and multicomponent polarisation analyses on the Gubbio Piana
(central Italy) recordings to identify the frequency content of the different
phases composing the recorded wavefield and to highlight the importance of
basin-induced surface waves in modifying the main strong ground-motion pa-
rameters.

5.2 Polarization ellipsoid

Unfortunately, there is no mathematically exact a priori definition for the in-
stantaneous polarization attributes of a multicomponent signal. Therefore any
attempts to produce one are usually arbitrary. The above described method is
by design restricted to the characterization of an ellipse. In more general terms,
particle motions captured with three-component recordings can be character-
ized by a polarization ellipsoid. Several methods are proposed in the liter-
ature to introduce such an approximation. They are based on the analysis
of the covariance matrix of multicomponent recordings and principal compo-
nents analysis using singular value decomposition (Kanasewich (1981); Vidale
(1986); Park et al. (1987); Jurkevics (1988); Jackson et al. (1991)). The mod-
ule gw1ET3D --type=scovar fulfils a covariance analysis in the case if an input
object has a signal format.

Kulesh et al. (2007a) and Diallo et al. (2005a) extended the covariance method
to the time-frequency domain. Following this way, we use an approximate
analytical formula to compute the elements of the covariance matrix for a
time window which is derived from an averaged instantaneous frequency of
the multicomponent record. For this, we approximate each voice of the wavelet
transform with frequency $f$ around time $t$ using the modulus and phase of the
wavelet coefficients as

$$
W_gS_k(t + \tau, f) \simeq |W_gS_k(t, f)| \cos [\Omega_k(t, f)\tau + \arg W_gS_k(t, f)],
$$

where $\Omega_k(t, f) = \partial \arg W_gS_k(t, f)/\partial t$. 

Using this approximation, the entries of the cross-energy matrix \( M(t, f) = [M_{km}(t, f)] \) can be calculated as

\[
M_{km}(t, f) = |\mathcal{W}_g S_k(t, f)||\mathcal{W}_g S_m(t, f)| \cdot \\
\left\{ \text{sinc} \left[ \frac{\Omega_k(t, f) - \Omega_m(t, f)}{2} \Delta t_{km}(t, f) \right] \cos (A^-_{km}(t, f)) + \right. \\
\left. \text{sinc} \left[ \frac{\Omega_k(t, f) + \Omega_m(t, f)}{2} \Delta t_{km}(t, f) \right] \cos (A^+_{km}(t, f)) \right\} - \mu_{km} \mu_{mk},
\]

\[ A_{km}^\pm(t, f) = \text{arg} \mathcal{W}_g S_k(t, f) \pm \text{arg} \mathcal{W}_g S_m(t, f), \quad k, m = x, y, z, \]  

where \( \text{sinc}(x) \) indicates the sine cardinal function and the mean values \( \mu_{km} \) are defined as

\[
\mu_{km} = \Re[\mathcal{W}_g S_k(t, f)] \text{sinc}[\Delta t_{km}(t, f) \Omega_k(t, f)/2].
\]

For example, we can define \( \Delta t_{km}(t, f) \) in a specific manner for each entry of the cross-matrix energy \( M(t, f) \) as

\[
\Delta t_{km}(t, f) = \frac{4\pi n}{\Omega_k(t, f) + \Omega_m(t, f)}, \quad n \in \mathbb{N}.
\]

The eigenanalysis performed on \( M(t, f) \) yields the principal component decomposition of the energy. Such a decomposition produces three eigenvalues \( \lambda_1(t, f) \geq \lambda_2(t, f) \geq \lambda_3(t, f) \) and three corresponding eigenvectors \( v_k(t, f) \) that fully characterize the magnitudes and directions of the principal components of the ellipsoid that approximates the particle motion in the considered time window \( \Delta t_{km}(t, f) \):

- the major half-axis \( \mathbf{R}(t, f) = \sqrt{\lambda_1(t, f)} v_1(t, f)/||v_1(t, f)|| \);
- the minor half-axis \( \mathbf{r}(t, f) = \sqrt{\lambda_3(t, f)} v_3(t, f)/||v_3(t, f)|| \);
- the second minor half-axis \( \mathbf{r}_s(t, f) = \sqrt{\lambda_2(t, f)} v_2(t, f)/||v_2(t, f)|| \);
- the reciprocal ellipticity \( \rho(t, f) = ||\mathbf{r}_s(t, f)||/||\mathbf{R}(t, f)|| \);
- the minor reciprocal ellipticity \( \rho_1(t, f) = ||\mathbf{r}(t, f)||/||\mathbf{r}_s(t, f)|| \);
- the dip angle \( \delta(t, f) = \arctan(\sqrt{v_{1,x}(t, f)^2 + v_{1,y}(t, f)^2/v_{1,z}(t, f)}) \);
- the azimuth \( \alpha(t, f) = \arctan(v_{1,y}(t, f)/v_{1,x}(t, f)) \).

We implemented the method (12)-(13) in the module \texttt{gw1ET3DFilter --type=acovar} both for a signal and for a wavelet spectrum depending on the input object format. When the instantaneous frequencies are the same for all components, this method produces the same results as those by Morozov and Smithson (1996) in terms of polarization parameters.
Let us demonstrate the adaptive covariance method as applied to earthquake data analysis. Fig. 9a shows three-component seismograms of the $M_w = 7.8$ earthquake at the Chilean-Bolivian border of June 13, 2005 recorded at the GRA1 station of the GRF array in northern Bavaria (Germany). The hypocentral depth was estimated as 114 km.

![Seismogram](image)

**Fig. 9.** (a) The real 3-C seismic record. (b)-(d) The wavelet transforms of these seismograms performed componentwise.

In Fig. 10 we compare the major and minor axes of polarization computed from the adaptive covariance method and the standard covariance method (See, e.g. Diallo et al. (2006a) for more information). As one can see by comparing the black and gray curves, the standard method represents a smoothed version of the instantaneous attributes from the adaptive method. Furthermore, since the time window is fixed for the standard method, it is not possible to characterize polarization attributes of a seismic event with a period lower than that of the time window used for the analysis. We circumvent this problem with the adaptive covariance method through the adaptive selection of the time window (13).

An interesting feature in the seismograms (Fig. 9a) is the presence SKS arrivals. This observation indicates a kind of anisotropy of the upper mantle. The degree of anisotropy is generally quantified by the estimate of the amount and the direction of this shear wave splitting. Here we are not trying to perform a state-of-art analysis of the observed SKS phase for continental shear wave splitting, our aim is to show how the wavelet based polarization method can be used to improve shear wave splitting analysis in general. Therefore, we will focus our attention on the time window for the SKS arrival showed in Fig.
Fig. 10. Comparison of the polarization attributes obtained from the adaptive covariance method (ACM) with those computed using the standard covariance method (SCM). (a) Major polarization axis, (b) second major polarization axis, (c) minor polarization axis.

11a.

Fig. 11. Polarization analysis restricted to the time window corresponding to the SKS phase arrival: (a) three-component seismogram for the considered time window, (b) the wavelet transform of the radial component, (c) the reciprocal ellipticity $\rho(t, f)$, (d) the minor reciprocal ellipticity $\rho_1(t, f)$.

Fig. 11b corresponds to the wavelet transform of the radial component which entails most of the energy from the SKS arrivals. The image of $\rho(t, f)$ for this phase reveals some degree of ellipticity which varies between 0.2 to a
maximum of 0.4 as indicated by the color code. This is a weak ellipticity, but it is consistent with the amount of SKS splitting for typical teleseismic observations. It is also important to note that the average reciprocal ellipticity ratio for the SKS is about three times higher than its counterpart for the Pdiff, which we used as reference for linear polarization. For instance, in an illustration such as Fig. 11d, any noticeable frequency dependencies of the splitting parameters with frequency or a comparatively large $\rho_1(t, f)$ with respect to $\rho(t, f)$ should be a hint towards a shear wave splitting occurring at different scales and different directions.

6 Modeling of wave propagation using a diffeomorphism in wavelet space

In order to analyze the dynamical behavior of multivariate signals using the continuous wavelet transforms one can be interesting to investigate a diffeomorphic deformation of the wavelet space. These deformations establish algebra of wavelet pseudodifferential operators acting on signals. We included into GWL two modules related to these operators — `gwlDiffeoLin` and `gwlDiffeoDisp`.

In the most general case, a wavelet deformation operator can be defined as

$$O[\mathcal{D}] : S(t) \mapsto \mathcal{M}_h \mathcal{D} \mathcal{W}_g S(t, f), \quad \mathcal{D} : \mathbb{H} \to \mathbb{H},$$

$$\mathbb{H} := \{(t, f) : t \in \mathbb{R}, f > 0\}.$$

6.1 Linear diffeomorphism

The simplest deformation is a set of all linear maps; we implemented it in the module `gwlDiffeoLin` as follows:

$$\mathcal{D}_L : (t, f) \mapsto (\alpha t + \beta / f, \delta f).$$

In the time domain this action is given by the following expression (Xie et al. (2003)):

$$O[\mathcal{D}_L] : S(t) \mapsto \left( C^+ C^- + \frac{\text{sign}(\alpha)(C^+ - C^-)}{2} i\mathcal{H} \right) S(\alpha t),$$

$$C^\pm = 2\pi \int_0^\infty \hat{g}^\pm(\pm \delta \omega) \hat{h}(\pm \alpha \omega) \exp(\pm i \beta \omega) \frac{d\omega}{\omega},$$

$$\mathcal{M}_h \mathcal{D} \mathcal{W}_g S(t, f) = \mathcal{M}_h \mathcal{D} \mathcal{W}_g S(t, f),$$

$$\mathcal{D} : \mathbb{H} \to \mathbb{H},$$

$$\mathbb{H} := \{(t, f) : t \in \mathbb{R}, f > 0\}.$$
where $\mathcal{H}$ denotes the Hilbert transform.

In the particular case, if the analyzing wavelet $g(t)$ and a source signal $S(t)$ are both progressive, we have $O[D_{\mathcal{C}}] : S(t) \mapsto C^+ S(\alpha t)$. If we select $\alpha = 1$, the propagated signal can be simply obtained by the multiplication of the source signal with the constant $C^+$. This interesting mathematical result is demonstrated in the Fig. 12, where we consider a complex Morlet wavelet as a source signal $S_1(t)$. We apply the CWT procedure using Cauchy wavelet and the linear propagator $D_{\mathcal{C}} : (t, f) \mapsto (t + 3.3/f, f)$ and, finally, we calculate IWT following by the multiplication by the constant $C^+$. After this procedure we obtain the same signal $S_2(t) = S_1(t)$, but its wavelet spectrum is different from the source wavelet spectrum.

![Fig. 12. An example of linear propagator applied to the progressive Morlet wavelet as the source signal.](image)

### 6.2 Describing wave dispersion and attenuation with diffeomorphism

The linear propagator (14) is not relevant for practical geophysical applications, but it demonstrates the idea how to model a signal propagation using some diffeomorphic operators in the time-frequency domain. In this context we have shown previously in Kulesh et al. (2005a) how the wavelet transform of the source and the propagated signals are related through a transformation operator that explicitly incorporates the phase and group velocities as well as the attenuation factor of the medium.

Assume that $S_k(t)$ and $S_m(t)$ represent two signals observed at two stations, a distance $D_{mk} = D_m - D_k$ apart. If the dispersive and dissipative characteristics of the medium are represented by the frequency-dependent wavenumber $k(f)$
and attenuation coefficient $\alpha(f)$, the relation between the Fourier transforms of these signals reads

\[
\hat{S}_m(f) = D_F \hat{S}_k(f) = \exp \left( -i \mathbb{K}(f) D_{mk} - 2\pi i n \right) \hat{S}_k(f), \quad \text{or}
\]

\[
\hat{T}_{mk}(f) = D_C \hat{T}_{rr}(f) = \hat{S}_m(f) \cdot \hat{S}_k(f) = \exp \left( -\alpha(f) \Delta D_{mk} \right) \exp \left( -2\pi ik(f) D_{mk} - 2\pi i n \right) \hat{T}_{rr}(f),
\]

where $n \in \mathbb{N}$ is any integer number and $\mathbb{K}(f)$ is the complex wavenumber, which can be defined by real functions $k(f)$ and $\alpha(f)$ as $\mathbb{K}(f) = 2\pi k(f) - i\alpha(f)$. $\hat{T}_{mk}(f)$ is the cross-correlation in the Fourier domain, when we take a trace $S_r(t)$ as reference and two other traces $S_m(t)$ at a distance $D_m$ and $S_k(t)$ at a distance $D_k$ (distance with respect to the position of the reference). $\hat{T}_{rr}(f) = |\hat{S}_r(f)|^2$, $\Delta D_{mk} = D_m + D_k$.

Let us assume that frequency-depended wavenumber and attenuation are slowly varying with regard to the frequency range of the mother wavelet. For moderate dispersion, the complex wavenumber can be approximated by the first two terms of its Taylor series around the frequency $f$. Next, we assume that the attenuation shows nearly linear frequency dependence. In such a case, $\alpha'(f) \sim 0$ and the asymptotic propagator in the wavelet space has the form Kulesh et al. (2005a):

\[
W_g S_m(t, f) = D_W W_g S_k(t, f) = \\
= \exp \left( -\alpha(f) D_{mk} \right) \exp \left( -i \psi_1(f) \right) W_g S_k(t - k'(f) D_{mk}, f), \quad \text{or}
\]

\[
W_g T_{mk}(t, f) = D_C W_g T_{rr}(t, f) = \\
= \exp \left( -\alpha(f) \Delta D_{mk} \right) \exp \left( -i \psi_1(f) \right) W_g T_{rr}(t - k'(f) D_{mk}, f),
\]

where $\psi_1(f) = 2\pi [k(f) - fk'(f)] D_{mk} + 2\pi n$.

In the special case, with the assumption that the analyzing wavelet has a linear phase (with time-derivative approximately equal to $2\pi$, as it is the case for the Morlet wavelet (3)), the approximation (16.1) can be written in terms of the phase $C_p(f)$ and group $C_g(f)$ velocities as:

\[
W_g S_m(t, f) = \exp \left( -\alpha(f) D_{mk} \right) \left| W_g S_k \left( t - \frac{D_{mk}}{C_p(f)}, f \right) \right| \cdot \\
\exp \left[ i \text{arg} W_g S_k \left( t - \frac{D_{mk}}{C_p(f)} - \frac{n}{f C_p(f)}, f \right) \right],
\]

where $C_p(f) = \frac{f}{k(f)}$, $C_g(f) = \frac{1}{k'(f)} = \frac{C_p^2(f)}{C_p(f) - fC_p^2(f)}$.

22
The relationship (17) has the following interpretation. The group velocity is a function that "deforms" the image of the absolute value of the source signal’s wavelet spectrum, the phase velocity "deforms" the image of the wavelet spectrum phase, and the attenuation function determines the frequency-dependent real coefficient by which the spectrum is multiplied.

This behavior is demonstrated in Fig. 13, where we consider a synthetic signal \( S_1(t) \). In this example, we use a propagation phase and group velocities which are not based on a physical model. These frequency-dependent velocities are shown in Figs. 13c,d. We perform the propagation of signal \( S_1(t) \) using the equation (15.1) and obtain a propagated counterpart \( S_2(t) \). The gray-scaled images in Fig. 13 show the absolute values and phases of wavelet spectra \( \mathcal{W}_g S_1(t, f) \) and \( \mathcal{W}_g S_2(t, f) \). We see that the deformations of images labeled as \( |\mathcal{W}_g S_2(t, f)| \) and \( \arg \mathcal{W}_g S_2(t, f) \) agree in general with velocities curves \( C_g(f) \) and \( C_p(f) \) accordingly, but have small distinctions that demonstrate the asymptotic properties of the equation (17).

The procedure (16) is implemented in the module \texttt{gwlDiffeoDisp} with the parameter --prop=1, while the procedure (17) can be executed with the parameter --prop=2. This module has an additional parameter --acorr which can be used in the case if the source signal is given by a cross-correlation \( T_{rr} \), and we obtain after the propagation the cross-correlation as well. Second feature implemented in the module \texttt{gwlDiffeoDisp} is the possibility to use as input a two-component signal in the complex form (6). Detailed description of this calculation potentiality is given by Kulesh et al. (2005b) as applied to the dispersion analysis of Rayleigh waves.

Fig. 13. Propagated synthetic signal and its wavelet transform: (a),(c) are the power (absolute value squared) of the wavelet coefficients and (b),(d) are the corresponding phase images. The lines in (c) and (d) show frequency-dependent group and phase velocities used in propagation model.
6.3 The case with non-linear frequency-dependent attenuation

Note that in the propagation models (16)-(17), the frequency-dependent wavenumber and attenuation are independent and therefore do not satisfy the causality constraint. In order to satisfy it, we approximate the complex wavenumber as in previous section by the first two terms of its Taylor series around the frequency \( f \). However, instead of the assumption \( \alpha'(f) \sim 0 \), a special wavelet like Cauchy wavelet (4) can be used, which allows to derive a relationship between the wavelet transforms of signals observed at two different stations in terms of complex wavenumber:

\[
W_g S_m(t, f) = D_{CC} W_g S_k(t, f) = \exp \left( -i\frac{\alpha(f)}{f_\alpha(f)} - \frac{D_{mk}}{2\pi} \Re K'(f), \frac{f}{f_\alpha(f)} \right),
\]

where \( f_\alpha(f) = 1 - \frac{D_{mk}}{p-1} \Im K'(f) \).

This propagator is implemented in the module \texttt{gwlDiffeoDisp} with the parameter \texttt{-prop=3}.

6.4 Propagation in the space of polarization parameters

A special feature of the propagator (16) is the possibility to join it with the polarization properties (9). If a two-component input signal is given in the complex form (6) and the frequency-depended wavenumber and attenuation are independent, we can obtain the dispersive propagator in the space of polarization properties as follows:

\[
R_m(t, f) = \exp \left( -\alpha(f)D_{mk} \right) R_k(t - k'(f)D_{mk}, f),
\]

\[
r_m(t, f) = \exp \left( -\alpha(f)D_{mk} \right) r_k(t - k'(f)D_{mk}, f),
\]

\[
\theta_m(t, f) = \theta_k(t - k'(f)D_{mk}, f), \quad \Phi_m(t, f) = \Phi_k(t - k'(f)D_{mk}, f) - \psi_1(f),
\]

where we use the signed minor half-axis \( r(t, f) = |W_g^+ Z(t, f)| - |W_g^- Z(t, f)| \) and the additional phase parameter \( \Phi(t, f) = (\arg W_g^+ Z(t, f) - \arg W_g^- Z(t, f))/2 \).

In the special case of Morlet wavelet, this propagator can be rewritten in terms of the phase \( C_p(f) \) and group \( C_g(f) \) velocities as:
\[ R_m(t, f) = \exp (-\alpha(f)D_{mk}) R_k(t - D_{mk}/C_g(f), f), \]
\[ r_m(t, f) = \exp (-\alpha(f)D_{mk}) r_k(t - D_{mk}/C_g(f), f), \]
\[ \theta_m(t, f) = \theta_k(t - D_{mk}/C_p(f), f), \quad \Phi_m(t, f) = \Phi_k(t - D_{mk}/C_p(f), f). \] (20)

These relationships between the polarization parameters observed at two stations \( m \) and \( k \) allow us to construct the following propagation procedure (polarization propagator):

\[ W_g^\pm Z_m(t, f) = (R_m(t, f) \pm r_m(t, f)) \exp (i(\theta_m(t, f) \pm \Phi_m(t, f))), \]
\[ Z_m(t) = M_h \left( W_g^+ Z_m(t, f) + W_g^- Z_m(t, f) \right). \] (21)

We implemented this propagator in the module `gwlDiffeoDisp` with the parameter `-prop=4` or `-prop=5` depending on which relationship (19) or (20) used for the propagation procedure (21).

### 6.5 Parametrization of dispersion and attenuation

To perform each of above-mentioned propagators in the module `gwlDiffeoDisp`, we need to define first a dispersion model. To prepare a dispersion model, the module `gwlDispModel` can be used. We redefine some models in this module:

1. `gwlDispModel -wn=gauss -atn=gauss` corresponds to the three-parameter exponential approximation for the wavenumber \( k(f) \) and the attenuation function \( \alpha(f) \):
   \[ F(f) = p_1 f + p_2 f \exp \left(-f^2/(2p_2^2)\right). \]

2. In this module we can also parameterize \( \alpha(f) \) and \( k(f) \) using polynomials functions of power \( N \) (`--wn=polin --atn=polin`):
   \[ F(f) = \sum_{k=1}^{N} p_k \frac{1}{k} f^k. \]

3. For the parameterization with B-spline functions of order four (`--wn=bspline --atn=bspline`) we define \( \alpha(f) \) and \( k(f) \) as
   \[ F(f) = \sum_{k=0}^{N-1} p_k B \left( \frac{f - f_1 - k\Delta}{\Delta} + 3 \right), \quad \Delta = \frac{f_2 - f_1}{N - 3}. \]
where \([f_1, f_2]\) is the frequency interval in which we want to define this function.

(4) We can also use an isotropic linear viscoelastic model (Cole-Cole model introduced by Lu and Hanyga (2004)), where the frequency-dependent wavenumber and attenuation are related (-\(\text{wn=colecole} - \text{atn=colecole}\)):

\[
\begin{align*}
  k(f) &= \Re \mathcal{K}(2\pi f)/(2\pi), \quad \alpha(f) = -\Im \mathcal{K}(2\pi f), \\
  \mathcal{K}(\omega) &= \frac{\omega}{\sqrt{\sigma(\omega)}}, \quad \sigma(\omega) = M_r \frac{1+(i\omega\tau_\varepsilon)\gamma}{1+(i\omega\tau_\sigma)\gamma},
\end{align*}
\]

Fig. 14 shows the dispersion and attenuation behavior of the medium within the framework of this model.

![Fig. 14](image)

Fig. 14. (a) Phase and group velocities and (b) attenuation curve for Cole-Cole model with parameters: \(M_r = 7.87 \cdot 10^6, \gamma = 0.4, \tau_\varepsilon = 4.73 \cdot 10^{-4}, \tau_\sigma = 1.71 \cdot 10^{-4}\).

Note that in the case of independent wavenumber and attenuation like for exponential, polynomial and B-spline approximation, we can combine different types of parametrization for \(k(f)\) and \(\alpha(f)\).

7  An estimate of the phase and group velocity and the attenuation

Equation (17) and Fig. 13 allows us to formulate the idea how the frequency-dependent phase velocity can be obtained using the wavelet spectra’ phases of source and propagated signals. On the one hand, we can calculate the correlations between two wavelet spectra to obtain approximately the phase velocities of each mode in a signal (Kulesh et al. (2007b)). On the other hand, one can formulate an optimization problem and solve it through the minimization of an appropriately defined cost function to precisely define the parameters of one mode (Holschneider et al. (2005)). Both of these possibilities are implemented in modules `gwlTransFK`, `gwlOptiSI` and `gwlOptiSP`. 
7.1 Wavelet based frequency-velocity analysis

Using the correlations between two spectra, we can perform "frequency-velocity" analysis on the analogy of frequency-wavenumber method (Capon (1969)) for a seismogram \( S_k(t), k = 1, N \). The main part of this analysis consists of the calculation of correlation spectrum \( \mathbf{M}(f, c) \) as follows (Kulesh et al. (2007b)):

\[
\mathbf{M}(f, c) = \int_{t_{\text{min}}}^{t_{\text{max}}} \left| \sum_{k,m} A_k(\tau, f) A_m^*(\tau - \frac{D_{mk}}{c}, f) \right| d\tau = \int_{t_{\text{min}}}^{t_{\text{max}}} \left| \sum_{k,m} \exp(iB_k(\tau, f)) \exp\left(-iB_m\left(\tau - \frac{D_{mk}}{c}, f\right)\right) \right| d\tau, \tag{22}
\]

where \([t_{\text{min}}, t_{\text{max}}]\) indicates the total time range for which the wavelet spectrum was calculated, \( c \in [C_{\text{p}}^{\text{min}}, C_{\text{p}}^{\text{max}}] \) is an unbound variable corresponding to the phase velocity, \( A_k \) is a complex-valued wavelet phase and \( B_k \) is a real-valued wavelet phase.

Correlation spectrum \( \mathbf{M}(f, c) \) is calculated using only the correlations between phases of wavelet spectra; these phases do not contain any amplitude information. Using this fact, we can use an alternative definition of a multi-trace correlation expression:

\[
\mathbf{M}(f, c) = \int_{t_{\text{min}}}^{t_{\text{max}}} \prod_{k=1}^{N} \left| B_k\left(\tau - \frac{D_{1k}}{c}, f\right) \right| d\tau. \tag{23}
\]

To demonstrate this concept, we analyze a synthetic seismogram having two propagated modes with different wavenumbers, but with the same frequency content (Fig. 15a). The phase velocities \( C_1(f) \) and \( C_2(f) \) used for this seismogram generation are plotted in Figs. 15b,c as solid curves; fundamentally this situation describes first symmetric and asymmetric modes of Lamb wave. We consider the situation without attenuation: \( \alpha(f) = 0 \).

\[
\hat{S}_m(f) = \exp\left(-2\pi iD_{mk}f/C_1(f)\right)\hat{S}_k(f) + \exp\left(-2\pi iD_{mk}f/C_2(f)\right)\hat{S}_k(f). \tag{24}
\]

We perform the "frequency-velocity" analysis of this synthetic seismogram using both equations (22) and (23). They are implemented in the module `gwlTransFK` with the parameter `-corr=cphase` and `-corr=arg` respectively. The gray-scaled background image in Fig. 15b shows the power function of normalized correlation coefficients for real-valued wavelet phases using equation (23), and Fig. 15c shows the correlation result where complex-valued
phases and equation (22) are used. The agreement of the correlation coefficients’ maximum lines with theoretical phase velocities is very good in both methods.

Four extra "pseudo-modes" presented in Fig. 15b,c are justified on the basis of $2\pi$-cycle skips between the stations introduced in equation (15) and remaining in the wavelet propagator (17) as the $n/f$ term. These "pseudo-modes" can be filtered by analyzing the group velocities.

7.2 Inversion for the dispersion and attenuation characteristics of the medium

Given the recorded signals at two or more stations we now intend to develop an approach to simultaneously estimate $k(f)$ (and hence $k'(f)$) and the attenuation coefficient $\alpha(f)$.

For given parametrization of dispersion and attenuation functions, finding an acceptable set of parameters can be thought of as an optimization problem that seeks to minimize a cost function $\chi^2$ and can be formulated as follows,

$$\chi^2(\alpha(f, p), k(f, q)) \rightarrow \text{min}, \quad p \in \mathbb{R}^P, \quad q \in \mathbb{R}^Q,$$

where $P$ is the number of parameters used to model the attenuation $\alpha(f)$ and $Q$ is the number of parameters used to model the wavenumber $k(f)$. $p$ and $q$ represent the vectors of parameters describing $\alpha(f)$ and $k(f)$ respectively. This cost function involves a propagator described in the previous section depending on the nature of the signal to be analyzed. In the following, we...
intend to discuss the different steps involved in our inversion algorithm as depicted in the chart of Fig. 16 (Holschneider et al. (2005)).

Fig. 16. Flowchart showing the sequence of optimizations to be performed in order to extract the dispersive and dissipative characteristics of individual modes (that can be identified with the CWT) from multimode surface wave records.

(1) The first step will consist of seeking a good initial condition by performing an image matching using the modulus of the wavelet transforms of a pair of traces (the module gw10ptiSP - cmpl=3). In this case the cost function $\chi^2$ is defined as

$$\chi^2(p, q) = \sum_{m,k} \int \int |W_{g}S_k(t, f)| - |D_{W}(p, q)W_{g}S_m(t, f)|^2 \, dt \, df. \quad (25)$$

The optimization is carried out over the whole frequency range of the signal. At the end we get the frequency-dependent derivative of the wavenumber $k'(f)$ and attenuation $\alpha(f)$. At this stage we need to distinguish between the case where the analyzed signal consists only of one coherent arrival from the case where it consists of several coherent arrivals. In the former case, the derived functions are meaningful and characterize those analyzed event. However in the latter, these functions cannot be easily interpreted since on one hand they were obtained from an optimization that only takes the modulus of the transform into account, which means that the phase information is lost, and on the other hand, the signals involved consist of many overlapping arrivals.

(2) If only one single phase is observed in all the traces $S_k(t)$, it will be enough to minimize a cost function that involves some selected seismic traces in order to estimate the attenuation and phase velocity using the
module gwlOptiSI. A possible definition of this cost function may be

$$\chi^2(p, q) = \sum_{m,k} \int \left| S_k(t) - \mathcal{F}^{-1} \mathcal{D}_F(p, q) \hat{S}_m(f) \right|^2 dt,$$  \hspace{1cm} (26)

where $\mathcal{F}^{-1}$ indicates the inverse Fourier transform, and where the relation between $S_m$ and $S_k$ is described by (15.1). In order to reduce the effect of uncorrelated noise in our estimates, it is preferable to use a propagator based on the cross-correlations (15.2),

$$\chi^2(p, q) = \sum_{m,k} \int \left| T_{mk}(t) - \mathcal{F}^{-1} \mathcal{D}_C(p, q) \hat{T}_{rr}(f) \right|^2 dt.$$  \hspace{1cm} (27)

Minimizing the cost function based on the cross-correlations is more advantageous for two reasons. On one hand, the effect of random noise is canceled, on the other, with geophones laid out symmetrically around the source in a seismic survey, cross-correlations of traces from seismic waves propagating in opposite directions can be combined in the optimization. For single phase arrival in all traces, the output at the end of step 2 yields the desired result.

(3) In the case where the observed signals consist of a mixture of different wave types and modes, a cascade of optimizations in the Fourier domain and the wavelet domain will be necessary in order to fully determine the dispersion and attenuation characteristics specific to each coherent arrival. Firstly, we perform the optimization on the modulus of the transforms in which case the attenuation for the specified event is derived (gwlOptiSP --cmpl=3):

$$\chi^2(p, q) = \sum_{m,k} \int \left| |W_g T_{mk}(t, f)| - |\mathcal{D}_{CW}(p, q) W_g T_{rr}(t, f)| \right|^2 dt df. \hspace{1cm} (28)$$

Next, we perform an optimization involving the argument of the wavelet transforms which will finally provide the phase and group velocity curves of the analyzed coherent arrival (gwlOptiSP --cmpl=4):

$$\chi^2(p, q) = \sum_{m,k} \int \int \left| \arg W_g T_{mk}(t, f) - \arg \mathcal{D}_{CVW}(p, q) W_g T_{rr}(t, f) \right|^2 dt df. \hspace{1cm} (29)$$

This optimization can be repeated to characterize each coherent arrival separately.

Since the dependence of the cost functions (25)-(29) on the parameters $p$ and $q$ is highly non-linear, each function may have several local minima. To obtain the global minimum that corresponds to the true parameters, a non-linear least-squares minimization method that proceeds iteratively from a reasonable set of initial parameters is required. In the present contribution, we use the Levenberg-Marquardt algorithm (Press et al. (1992)).
Fig. 17. Observed seismograms obtained from a shallow seismic experiment using a sledgehammer as a source.

As an example, we previously analyzed field data in Holschneider et al. (2005) where we successfully estimated the phase and group velocities as well as attenuation using this minimization procedure. The experimental data showed in Fig. 17 consist of a shallow seismic survey (stations along a line) at Kerpen, a particular site in the Lower Rhine embayment where the buried scarp of a historically active fault is presumed. Several profiles of 48 channels with 2 m inter-receiver spacing were collected using hammer blows as seismic source. We selected a seismogram profile with prominent low frequency, high amplitude arrivals that correspond to the surface wave arrivals we intend to characterize.

To check the quality of such an inversion, we evaluated the slowness using two alternative methods as described by Holschneider et al. (2005): CAPON high resolution method Capon (1969) and MUSIC high resolution method Schmidt (1986).

We selected the complete waveform windows of sub-arrays corresponding to subsections A and B along the shot profile. The horizontal wavenumbers $k$ is sampled equidistantly in one dimension and a set of 200 discrete frequencies is spaced equidistantly on a logarithmic frequency scale from 10 to 50 Hz. In Fig. 18 we show the results of this analysis for the two sub-arrays. Each sub-figure pictures the results of the MUSIC approach as gray-scaled background image. For better visibility we normalized the maxima to one for each individual analysis frequency. The superimposed contour lines give the result from Capon’s analysis method. In order to compare these results with the phase velocity estimates from the CWT-based method, we finally superimposed the corresponding estimates as solid curves.
Fig. 18. Comparison of the slowness estimates obtained for subsection A (a) and subsection B (b) from the CWT method (solid curve), Capon’s high resolution f-k method (contour lines) and the MUSIC algorithm (gray-scaled background image).

For subsection A, all three methods show a consistent estimate of the phase velocities from the seismic records. For subsection B, we recognize less consistency between the different estimates of the phase velocities. It is interesting to note that the contour plot of Capon’s analysis results do not reach the frequency range where the surface wave is highly attenuated. Also the gray-scale background depicting the result from the MUSIC approach shows a decreased resolution of the velocity in this frequency range, indicated by the peak broadening along the vertical scale. The region where the phase velocity estimates from all methods are similar corresponds to those, where the energy of the surface arrival is significantly high (around 25-35 Hz).

8 Conclusion

We propose a software package which implements some methods of polarizations and dispersion analysis in wavelet-domain using the continuous wavelet transform:

(1) a method for computing instantaneous attributes of 2-C signals in the time-frequency domain. The advantage of this method over previous techniques (Rene et al. (1986)) is that both the time and frequency dependence of the attributes can be obtained and used for wave mode separation and filtering;
(2) an extension of the polarization analysis technique for multicomponent data initially proposed by Morozov and Smithson (1996) into the time-frequency domain;

(3) a method for the estimation of instantaneous polarization attributes based on an approximation to the covariance matrix and an extension of the adaptive covariance method to the time-frequency domain. The advantage of the proposed method over the standard method is that the length of the window size for the covariance computation is adaptively adjusted with the help of the instantaneous frequencies from the different components;

(4) some methods to establish a link between the continuous wavelet transforms of a signal and its propagated counterpart in a dispersive and attenuating medium. The advantage of using the proposed propagator over traditional methods such as the Wigner-Ville or time frequency reassignment for dispersion curves estimates is that the full dispersion and dissipation characteristics are explicitly expressed and therefore can be easily extracted;

(5) an approach to use this wavelet propagator in the method of ”frequency-velocity” analysis in analogy to the classical frequency-wavenumber (f-k) analysis methods. Using this method, the determination of several mode branches is feasible;

(6) a method of simultaneous computations of both phase and group velocity in the wavelet domain. The method owes its robustness to the fact that the minimization process involves not only the modulus but also the phase of the wavelet transform thus making it possible, in principle, to reconstruct the dispersed signal from the manipulated wavelet coefficients.

The mathematical aspects of all these methods have been separately published in (Kulesh et al. (2005a,b); Holschneider et al. (2005); Diallo et al. (2005b, 2006b,a, 2005a)). In this paper, we showed that all these methods can be logically combined into one library. All examples presented in this paper are generated using this library and are included into the installation of our software.

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