# Solving a Free Boundary Value Problem via an Inverse Problem Algorithm

Rainer Kress, University of Göttingen

joint work with

Houssem Haddar, INRIA & Ecole Polytechnique Palaiseau

Inverse Problems for PDEs, Bremen, March 2016

# A New Algorithm for the Bernoulli Free Boundary Problem

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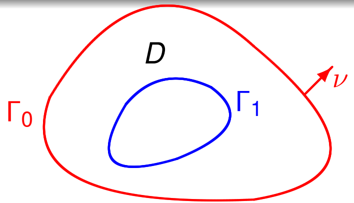
# The Bernoulli free boundary value problem




Determine $\Gamma_0$ such that the unique harmonic function $w \in H^1(D)$ with boundary values

$$w = 0 \quad \text{on } \Gamma_0, \qquad w = 1 \quad \text{on } \Gamma_1$$

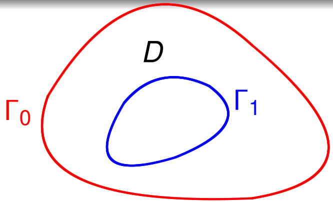
satisfies

$$-\frac{\partial w}{\partial \nu} = \lambda \quad \text{on } \Gamma_0$$

where $\Gamma_1$ is known and $\lambda$ is a given positive constant.

Main idea: Employ conformal mapping

# Free b.v.p versus inverse b.v.p.




$\Gamma_0$ is unknown $\qquad$ $\Gamma_1$ is known

Free boundary value problem: Cauchy data given on the unknown boundary $\Gamma_0$

Inverse boundary value problem: Cauchy data given on the known boundary $\Gamma_1$

# Outline

1. The Bernoulli free boundary value problem
2. The conformal mapping algorithm for an inverse boundary value problem
3. The iterative algorithm for the Bernoulli problem
4. Numerical examples
5. Conclusion and outlook

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Only in two dimensions because of conformal mapping involved.

# Free boundary problems for potential flows

The velocity field $V$ of an incompressible irrotational flow satisfies

$$\operatorname{div} V = 0 \quad \text{and} \quad \operatorname{curl} V = 0.$$

There exists a harmonic function $v$, called velocity potential, such that

$$V = \operatorname{grad} v.$$

In two dimensions, in addition a stream function $w$ can be introduced as a conjugate harmonic of $v$ (such that $v + iw$ is holomorphic.)

The velocity $V = \operatorname{grad} v$ is orthogonal to the equipotential lines $v = \text{const}$, and the lines $v = \text{const}$ and $w = \text{const}$ of the two conjugate harmonic functions are orthogonal.

Therefore the lines $w = \text{const}$ represent the streamlines of the potential flow.

# Free boundary problems for potential flows

In free boundary value problems for such potential flows both the free boundaries and the known rigid boundaries must be streamlines, i.e., the stream function $w$ is constant on both.

When the free boundary is the interface between the fluid and a surrounding gas the free streamline is in equilibrium with the gas, that is, the pressure $p$ of the fluid at the free streamline is constant. By Bernoulli's law

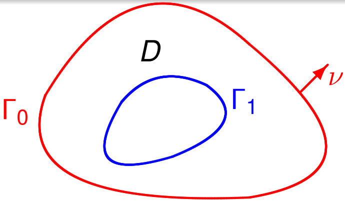
$$\frac{1}{2}|V|^2 + p = \mathrm{const}$$

this implies

$$|\operatorname{grad} w| = \mathrm{const}.$$

Therefore the normal derivative of the stream function $w$ must be constant on the free boundary.

# The Bernoulli free boundary value problem




$$w = 0 \quad \text{on } \Gamma_0 \quad \text{and} \quad w = 1 \quad \text{on } \Gamma_1$$

and

$$-\frac{\partial w}{\partial \nu} = \lambda \quad \text{on } \Gamma_0$$

where $\lambda$ is a given positive constant (Hopf's lemma!)

Applications of Bernoulli's free boundary problem also occur in electro- and magnetostatics, for example in optimal design of insulation layers for coaxial cables.

# Some history

**Lewy** 1952 Analyticity of the free boundary

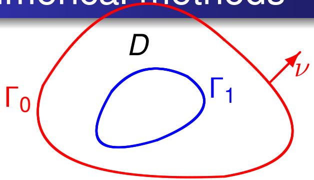
**Beuerling** 1957 Existence of a solution
Contemporary existence proofs are based on variational methods in the sense of shape optimization.

**Tepper** 1974 Uniqueness of the solution, provided $\Gamma_1$ is convex.

**Tepper** 1975 If $\Gamma_1$ is convex or starlike, then so is $\Gamma_0$.

**Flucher, Rumpf** 1997 Counter examples showing that convexity for $\Gamma_1$ is necessary for uniqueness.

# Numerical methods

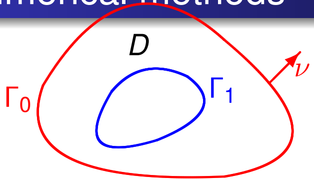



$$w = 1 \text{ on } \Gamma_1$$
$$w = 0 \text{ on } \Gamma_0$$
$$-\partial_\nu w = \lambda \text{ on } \Gamma_0$$

So-called trial methods from shape optimization.

1. Make an initial guess for the free boundary
2. Solve the boundary value problem with one of the conditions on the free boundary omitted.
3. If the remaining boundary condition is satisfied up to some specified tolerance accept the solution.
   Otherwise use the remaining boundary condition to update the free boundary and go back to step 2.

# Numerical methods

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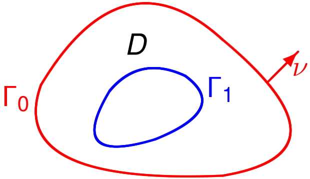
**Kuster, Gremaud, Touzani** 2007
**Ben Abda, Bouchon, Peichl, Sayeh, Touzani** 2013
**Harbrecht, Mitrou** 2014

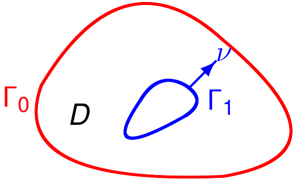
# Our method




$$w = 1 \text{ on } \Gamma_1$$
$$w = 0 \text{ on } \Gamma_0$$
$$-\partial_\nu w = \lambda \text{ on } \Gamma_0$$

1. Make an initial guess for the unknown normal derivative $g := \partial_\nu w$ on $\Gamma_1$.
2. Solve the inverse problem to obtain $\Gamma_0$ from $w = 0$ on $\Gamma_0$ and the Cauchy data on $\Gamma_1$ by the conformal mapping method.
3. If $-\partial_\nu w = \lambda$ on $\Gamma_0$ is satified up to some specified tolerance accept the solution.
   Otherwise update $g$ by the solution to $w = 1$ on $\Gamma_1$ and $-\partial_\nu w = \lambda$ on $\Gamma_0$

# A related inverse problem




$$u \in H^1(D)$$

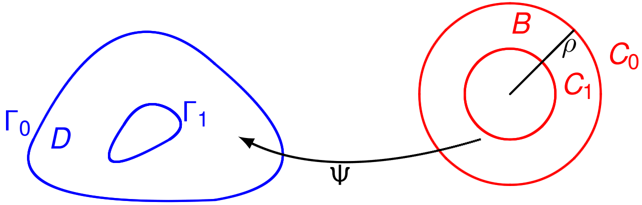
$$\Delta u = 0 \quad \text{in } D$$

$$u = 0 \quad \text{on } \Gamma_0$$

**Given** $f = u|_{\Gamma_1}$ and $g = \left.\dfrac{\partial u}{\partial \nu}\right|_{\Gamma_1}$, **find** boundary $\Gamma_0$

# Conformal mapping method

Riemann mapping theorem for doubly connected domains: There exists a uniquely determined radius $\rho$ and a bijective holomorphic function $\Psi : B \to D$ such that $\Psi(C_1) = \Gamma_1$ and $\Psi(C_0) = \Gamma_0$. ($\Psi$ is unique up to rotations of $B$.)



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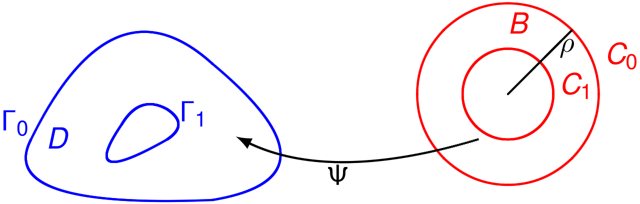


**1.** Solve a nonlocal and nonlinear ordinary differential equation for the boundary values $\Psi|_{C_1}$ and a nonlinear equation for the radius $\rho$.

# Conformal mapping method

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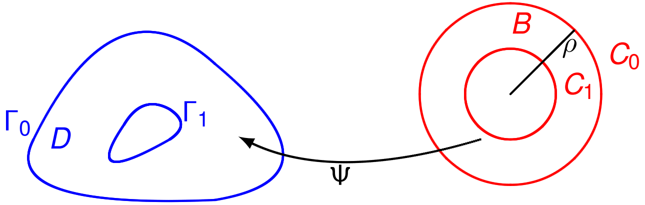
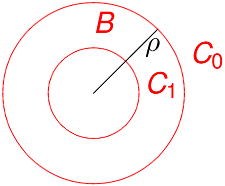



**2.** Knowing $\rho$ and $\Psi|_{C_1}$, solve Cauchy problem for the holomorphic function $\Psi$ in annulus $B$ via a regularized Laurent expansion and obtain $\Gamma_0 = \Psi(C_0)$

# The Cauchy problem

Assume we know the radius $\rho$ and $\Psi$ on $C_1$, i.e.,

$$\Psi(e^{it}) = \sum_{k=-\infty}^{\infty} b_k e^{ikt}, \quad 0 \le t \le 2\pi.$$



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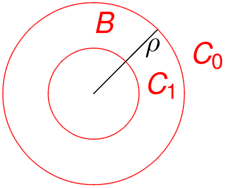


Obtain $\Psi$ in $B$ via **Laurent expansion**

$$\Psi(z) = \sum_{k=-\infty}^{\infty} b_k z^k, \quad \rho \le |z| \le 1,$$

and the unknown boundary by

$$\Gamma_0 = \Psi(C_0) = \left\{ \sum_{k=-\infty}^{\infty} \rho^k b_k e^{ikt}, \quad 0 \le t \le 2\pi. \right\}$$

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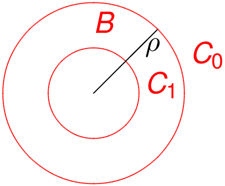


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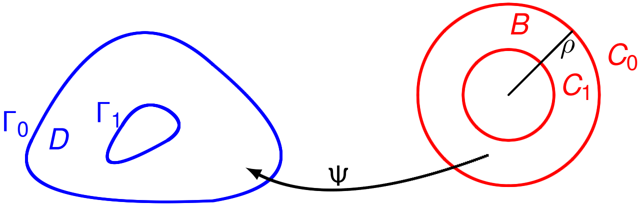
and the unknown boundary by

$$\Gamma_0 \approx \left\{ \sum_{k=-N}^{N} \rho^k b_k e^{ikt}, \quad 0 \le t \le 2\pi. \right\}$$

Need to regularize because of **exponential ill-posedness**

# ODE for boundary correspondence map



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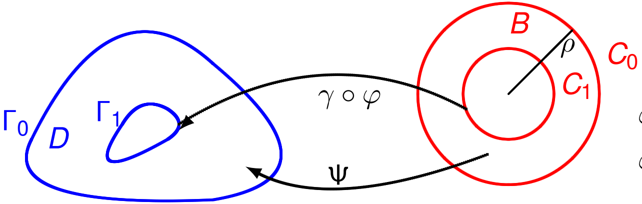


$$\varphi(0) = 0$$
$$\varphi(2\pi) = 2\pi$$

$$\Gamma_1 = \{\gamma(\tau) : \tau \in [0, 2\pi]\}$$

$$\boxed{\varphi(t) := \gamma^{-1}(\Psi(e^{it})),\ t \in [0, 2\pi]}$$

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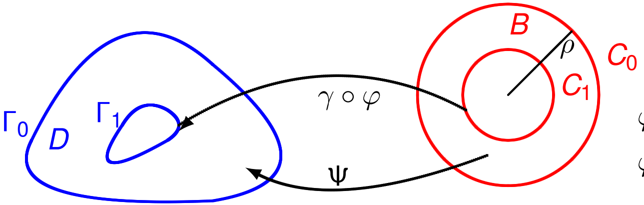


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$u, \tilde{u}$ in $D$ $v = u \circ \Psi, \tilde{v} = \tilde{u} \circ \Psi$ in $B$

$$\frac{\partial \tilde{v}}{\partial t} = \frac{\partial \tilde{u}}{\partial s} \gamma' \frac{d\varphi}{dt} \quad \Rightarrow \quad \frac{\partial v}{\partial \nu} = \frac{\partial u}{\partial \nu} \gamma' \frac{d\varphi}{dt}$$

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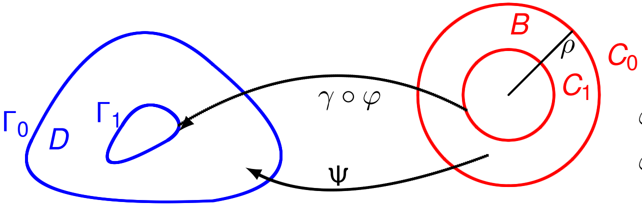


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$$\boxed{\frac{d\varphi}{dt} = \frac{A_\rho(f \circ \gamma \circ \varphi)}{\gamma' \circ \varphi \; g \circ \gamma \circ \varphi}}$$

$A_\rho$ = Dirichlet-to-Neumann map for $B$ with $v = 0$ on $C_0$

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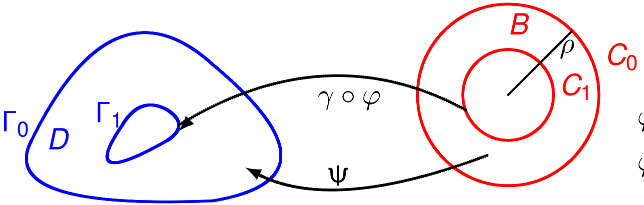


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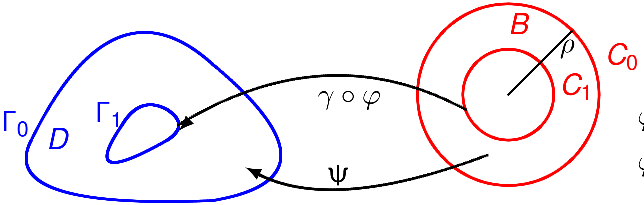


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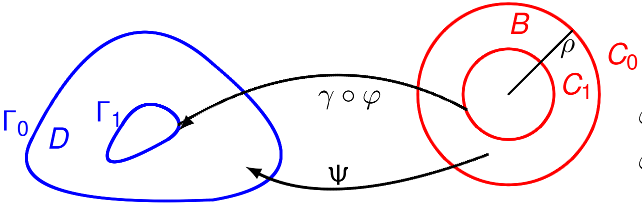
$A_\rho =$ Dirichlet-to-Neumann map for $B$ with $v = 0$ on $C_0$

$$\boxed{\rho = \exp\left(\frac{\int_0^{2\pi} f \circ \gamma \circ \varphi\, dt}{\int_{\Gamma_1} g\, ds}\right)}$$

Recall $f = u|_{\Gamma_1}$ and $g = \partial_\nu u|_{\Gamma_1}$,

# ODE for boundary correspondence map

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# Picard iteration

$$\frac{d\varphi}{dt} = \frac{\int_{\Gamma_1} g\, ds}{2\pi\, \gamma' \circ \varphi\; g \circ \gamma \circ \varphi}$$

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$$\Rightarrow \qquad 1 = \frac{\int_{\Gamma_1} g\,ds}{4\pi^2} \int_0^{2\pi} \frac{1}{|\gamma' \circ \varphi_n|\, g \circ \gamma \circ \varphi_n}\, dt$$

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$$\frac{d\varphi}{dt} = \frac{\int_{\Gamma_1} g\,ds}{2\pi\,\gamma'\circ\varphi\; g\circ\gamma\circ\varphi} - \frac{\int_{\Gamma_1} g\,ds}{4\pi^2}\int_0^{2\pi} \frac{1}{|\gamma'\circ\varphi|\; g\circ\gamma\circ\varphi}\,dt + 1$$

$$\varphi(0) = 0$$

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$$\frac{d\varphi_{n+1}}{dt} = \frac{\int_{\Gamma_1} g\, ds}{2\pi\ \gamma' \circ \varphi_n\ g \circ \gamma \circ \varphi_n} - \frac{\int_{\Gamma_1} g\, ds}{4\pi^2} \int_0^{2\pi} \frac{1}{|\gamma' \circ \varphi_n|\ g \circ \gamma \circ \varphi_n}\, dt + 1$$

$$\varphi_{n+1}(0) = 0$$

# Update the normal derivative on $\Gamma_1$

Green's integral formula for $w = 1$ on $\Gamma_1$ and $-\partial_\nu w = \lambda$ on $\Gamma_0$

$$w(x) = \int_{\Gamma_1} \Phi(x,\cdot) g \, ds - \int_{\Gamma_0} \left\{ \lambda \Phi(x,\cdot) + \frac{\partial \Phi(x,\cdot)}{\partial \nu} \, \psi \right\} ds, \quad x \in D,$$

with $g = \partial_\nu w|_{\Gamma_1}$ and $\psi = w|_{\Gamma_0}$.

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Two integral equations

$$\frac{\psi}{2} + \int_{\Gamma_0} \frac{\partial \Phi(x,\cdot)}{\partial \nu} \, \psi \, ds - \int_{\Gamma_1} \Phi(x,\cdot) g \, ds = -\lambda \int_{\Gamma_0} \Phi(x,\cdot) \, ds, \quad x \in \Gamma_0,$$

and

$$\int_{\Gamma_0} \frac{\partial \Phi(x,\cdot)}{\partial \nu} \, \psi \, ds - \int_{\Gamma_1} \Phi(x,\cdot) g \, ds = -\lambda \int_{\Gamma_0} \Phi(x,\cdot) \, ds - 1, \quad x \in \Gamma_1,$$

for the two unknowns $\psi$ and $g$.

## The Algorithm

1. Choose an initial guess $g$.
2. Apply the conformal mapping method to obtain an approximation for $\Gamma_0$.
   a. Solve ODE for boundary correspondence map $\varphi$
   b. From $\varphi$ obtain $\Gamma_0$ by solving Cauchy problem via truncated Laurent series.

## The Algorithm

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   - a. Solve ODE for boundary correspondence map $\varphi$
   - b. From $\varphi$ obtain $\Gamma_0$ by solving Cauchy problem via truncated Laurent series.
3. Solve mixed Dirichlet–Neumann problem

$$w = 1 \quad \text{on } \Gamma_1, \quad -\partial_\nu w = \lambda \quad \text{on } \Gamma_0$$

by boundary integral equation for the unknowns

$$\partial_\nu w|_{\Gamma_1} \quad \text{and} \quad w|_{\Gamma_0}$$

If $\|g - \partial_\nu w|_{\Gamma_1}\|_\infty < \delta$ for a given tolerance $\delta$ terminate the iteration. Otherwise update $g := \partial_\nu w|_{\Gamma_1}$ and go back to Step 2.

# A convergence result

Combine the three parts of one iteration step into regularized iteration operator $A_N$. (Recall $N$ as truncation index for Cauchy problem.)

**Theorem (Haddar, K. 2015)**

*Assume that $\Gamma_1$ is sufficiently close to a circle and $\lambda > e^{-1}$. Then the iterations $g_{n+1} = A_N g_n$, $n = 0, 1, 2, \dots,$ converge provided the initial guess $g_0$ is sufficiently close to the exact normal derivative $g$ for the circle.*

**Idea of proof** Show that if $\Gamma_1$ is a circle then the Fréchet derivative of $A_N$ has spectral radius less than one. Apply a perturbation argument and the mean value theorem to show that $A_N$ is a contraction.

# Numerical examples

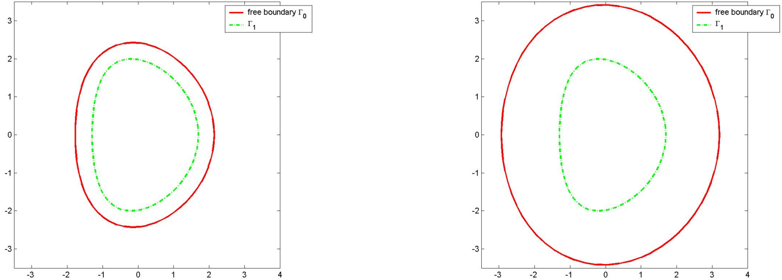


Figure: Free boundary for $\lambda = 2$ (left) and $\lambda = 0.5$ (right)

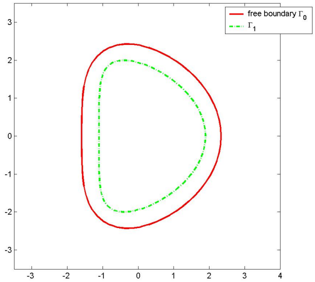


Figure: Free boundary for $\lambda = 2$ (left) and $\lambda = 0.5$ (right)

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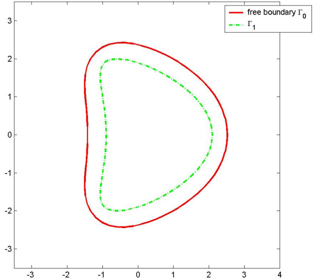


Figure: Free boundary for $\lambda = 2$ (left) and $\lambda = 0.5$ (right)

# Conclusion

The examples indicate satisfactory performance of our algorithm although we violated the principle not to destroy stability of a problem by incorporating an ill-posed component in its numerical solution.

# Trefftz' integral equation for Bernoulli problem

Recall integral equation for the update of unknown normal derivative on $\Gamma_1$.

Green's integral formula for $w = 1$ on $\Gamma_1$ and $-\partial_\nu w = \lambda$ on $\Gamma_0$ yields the two integral equations

$$-\frac{\psi}{2} - \int_{\Gamma_0} \frac{\partial \Phi(x,\cdot)}{\partial \nu}\,\psi\,ds + \int_{\Gamma_1} \Phi(x,\cdot) g\,ds = \lambda \int_{\Gamma_0} \Phi(x,\cdot)\,ds, \quad x \in \Gamma_0,$$

and

$$-\int_{\Gamma_0} \frac{\partial \Phi(x,\cdot)}{\partial \nu}\,\psi\,ds + \int_{\Gamma_1} \Phi(x,\cdot) g\,ds = \lambda \int_{\Gamma_0} \Phi(x,\cdot)\,ds + 1, \quad x \in \Gamma_1,$$

for the two unknowns $g = \partial_\nu w|_{\Gamma_1}$ and $\psi = w|_{\Gamma_0}$

# Trefftz' integral equation for Bernoulli problem

Green's integral formula for $w = 1$ on $\Gamma_1$, $w = 0$ on $\Gamma_0$ and $-\partial_\nu w = \lambda$ on $\Gamma_0$ yields the two integral equations

$$\int_{\Gamma_1} \Phi(x,\cdot) g \, ds = \lambda \int_{\Gamma_0} \Phi(x,\cdot) \, ds, \quad x \in \Gamma_0,$$

and

$$\int_{\Gamma_1} \Phi(x,\cdot) g \, ds = \lambda \int_{\Gamma_0} \Phi(x,\cdot) \, ds + 1, \quad x \in \Gamma_1,$$

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**Trefftz** 1916

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**To do:** Analyze convergence of Newton iterations simultaneously for $g$ and $\Gamma_0$.
Compare performance with trial methods.

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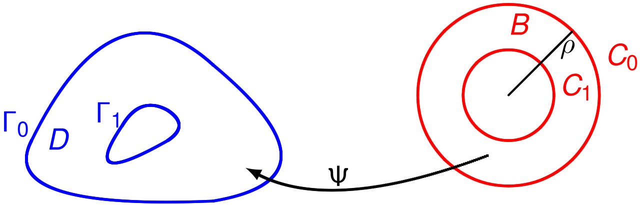
**To do:** Analyze convergence of Newton iterations simultaneously for $g$ and $\Gamma_0$.
Compare performance with trial methods.
**Done!**

# Another conformal mapping approach



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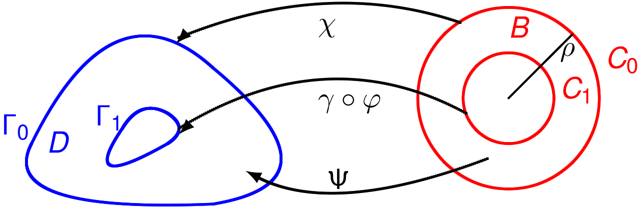


$$\varphi(t) := \gamma^{-1}(\Psi(e^{it})), \quad \chi(t) := \Psi(\rho e^{it}), \quad t \in [0, 2\pi]$$

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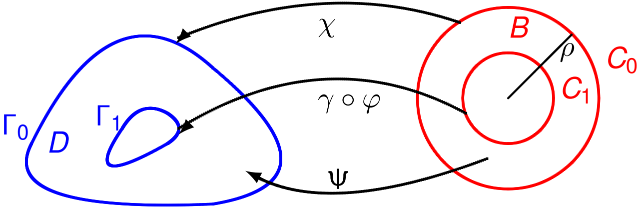
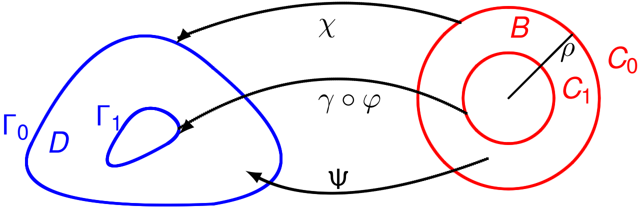



$$\varphi(t) := \gamma^{-1}(\Psi(e^{it})), \quad \chi(t) := \Psi(\rho e^{it}), \quad t \in [0, 2\pi]$$

$$v = w \circ \Psi, \quad \Delta v = 0, \quad v|_{C_1} = 1, \quad v|_{C_0} = 0$$

$$v(x) = 1 - \frac{\ln |x|}{\ln \rho}$$

$$-\frac{\partial w}{\partial \nu} = \lambda \quad \text{on } \Gamma_0 \quad \Leftrightarrow \quad |\chi'(t)| = \frac{1}{\lambda \ln \rho}, \quad t \in [0, 2\pi]$$

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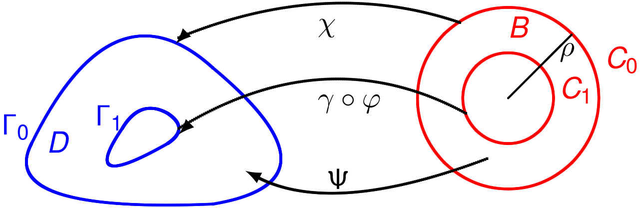
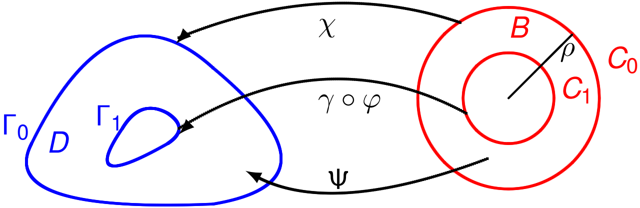



$$|\chi'(t)| = \frac{1}{\lambda \ln \rho}, \quad t \in [0, 2\pi]$$

Cauchy integral formula

$$\Psi(z) = \frac{\rho}{2\pi} \int_0^{2\pi} \frac{\chi(\tau)\, e^{i\tau}}{\rho\, e^{i\tau} - z}\, d\tau - \frac{1}{2\pi} \int_0^{2\pi} \frac{\gamma(\varphi(\tau))\, e^{i\tau}}{e^{i\tau} - z}\, d\tau, \quad z \in B.$$

# Another conformal mapping approach




$$|\chi'(t)| = \frac{1}{\lambda \ln \rho}\,, \qquad t \in [0, 2\pi]$$

$$\chi(t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\chi(\tau)\, e^{i\tau}}{e^{i\tau} - e^{it}}\, d\tau - \frac{1}{2\pi}\int_0^{2\pi} \frac{\gamma(\varphi(\tau))\, e^{i\tau}}{e^{i\tau} - \rho\, e^{it}}\, d\tau, \;\; t \in [0, 2\pi]$$

$$\gamma(\varphi(t)) = \frac{\rho}{2\pi}\int_0^{2\pi} \frac{\chi(\tau)\, e^{i\tau}}{\rho\, e^{i\tau} - e^{it}},\, d\tau - \frac{1}{2\pi}\int_0^{2\pi} \frac{\gamma(\varphi(\tau))\, e^{i\tau}}{e^{i\tau} - e^{it}}\, d\tau, \;\; t \in [0, 2\pi]$$

**To do:** See whether solving these three nonlinear equations for $\rho, \chi$ and $\varphi$ works.

# References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Thank you

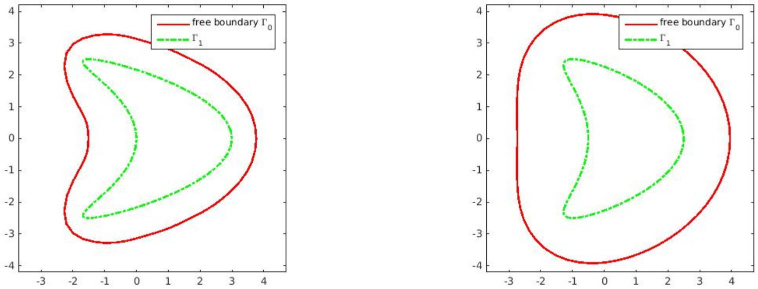


Figure: Free boundary for $\lambda = 1$ (left) and $\lambda = 0.5$ (right)

# Numerical examples

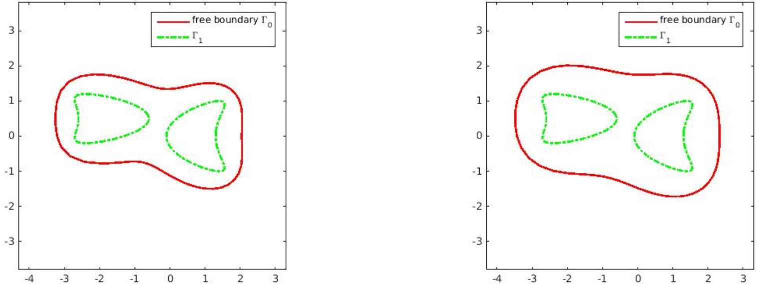


Figure: Free boundary for $\lambda = 1$ (left) and $\lambda = 0.5$ (right)