Variational source conditions and stability estimates for inverse electromagnetic medium scattering problems

Frederic Weidling
joint work with Thorsten Hohage

Institute for Numerical and Applied Mathematics
Georg-August University of Göttingen

Inverse Problems for PDEs
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Outline

Tikhonov regularization

General Setup

- Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces
- $F\colon \operatorname{dom}(F) \subset \mathbb{X} \to \mathbb{Y}$ be a (possibly nonlinear) forward operator
- $f^\dagger \in \operatorname{dom}(F)$ the true solution
- $g^\delta \in \mathbb{Y}$ observed data with $\|g^\delta - F(f^\dagger)\|_{\mathbb{Y}} \leq \delta$

Tikhonov regularization: Find an approximate solution

$$f_\alpha^\delta \in \operatorname*{arg\,min}_{f \in \operatorname{dom}(F)} \left[\frac{1}{\alpha} \left\| F(f) - g^\delta \right\|_{\mathbb{Y}}^2 + \Omega(f) \right],$$

where Ω is an appropriate penalty term.

Distance to true solution

Spectral source conditions:

$$f^\dagger = \varphi\left(F'[f^\dagger]^* F'[f^\dagger]\right)\omega$$

with an index function φ, $\rightsquigarrow$ additional restrictive requirements for nonlinear operators (tangential cone condition)

H. Engl, M. Hanke and A. Neubauer. *Regularization of inverse problems*, Kluwer, 1996.

Variational source conditions (VSC):

$$\forall f \in \mathrm{dom}(F): \quad \beta\Delta_\Omega(f, f^\dagger) \leq \Omega(f) - \Omega(f^\dagger) + \psi\left(\left\|F(f) - F(f^\dagger)\right\|_{\mathbb{Y}}^2\right)$$

for a concave index function ψ and a $\beta \in (0, 1]$.
First used (with $\psi(t) = c\sqrt{t}$) in

B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. *A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators*. **Inverse Problems** 23:987–1010, 2007.

Advantages of VCSs

- simplify proofs, e.g. one can easily show that a VSC implies the convergence rate
$$\beta\Delta_\Omega(f_\alpha^\delta, f^\dagger) \leq 4\psi(\delta^2)$$
for the optimal choice of the regularization parameter α
  - M. Grasmair. *Generalized Bregman distances and convergence rates for non-convex regularization methods.* **Inverse Problems** 26:115014, 2010.
  - F. Werner and T. Hohage. *Convergence rates in expectation for Tikhonov-type regularization of inverse problems with Poisson data.* **Inverse Problems** 28:104004, 2012.
- for linear operators between Hilbert space even necessary conditions for certain rates of convergence
  - J. Flemming, B. Hofmann, and P. Mathé. *Sharp converse results for the regularization error using distance functions.* **Inverse Problems**, 27:025006, 2011.
- no differentiability assumption $\rightsquigarrow$ no restrictive assumption connecting F and F' needed (tangential cone condition)
- allow extension to Banach spaces and general data misfit/penalty terms

...but

Verification of VSCs:

- Reformulations of VSC with $\psi(x) = \sqrt{x}$ for a phase retrieval and an option pricing problem.

B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. *A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators*. **Inverse Problems** 23:987–1010, 2007.

- Spectral source conditions imply VSCs.
- For linear operators F with ℓ^q penalty term via the range of F^*

M. Burger, J. Flemming, and B. Hofmann. *Convergence rates in ℓ^1-regularization if the sparsity assumption fails.* **Inverse Problems**, 29:025013, 2013.

S. W. Anzengruber, B. Hofmann, and R. Ramlau. *On the interplay of basis smoothness and specific range conditions occurring in sparsity regularization.* **Inverse Problems**, 29:125002, 2013.

J. Flemming and M. Hegland. *Convergence rates in ℓ^1-regularization when the basis is not smooth enough.* **Applicable Analysis**, 94:464–476, 2015.

- Acoustic scattering

T. Hohage and F. Weidling. *Verification of a variational source condition for acoustic inverse medium scattering problems.* **Inverse Problems**, 31:075006, 2015.

$\leadsto$ few verifications so far

VSC vs. stability estimates

Let $K \subset \mathrm{dom}(F)$ be some smoothness class (e.g. a Sobolev ball), and Δ a symmetric error measure.

Variational source condition: $\forall f^\dagger \in K, f \in \mathrm{dom}(F)$:

$$\beta\Delta(f, f^\dagger) \leq \Omega(f) - \Omega(f^\dagger) + \psi\left(\|F(f) - F(f^\dagger)\|_{\mathbb{Y}}^2\right)$$

Conditional stability estimate: $\forall f_1, f_2 \in K$:

$$\beta\Delta(f_1, f_2) \leq \qquad\qquad \psi\left(\|F(f_1) - F(f_2)\|_{\mathbb{Y}}^2\right)$$

VSC $\Longrightarrow$ Stability:

- W.l.o.g. $\Omega(f_1) \geq \Omega(f_2)$, choose $f_1 = f^\dagger$, $f_2 = f$

Stability $\overset{???}{\Longrightarrow}$ VSC:

- sign of $\Omega(f) - \Omega(f^\dagger)$ unknown
- VSC must hold on the larger set $\mathrm{dom}(F)$

Time-harmonic Maxwell equations

Time-harmonic Maxwell equations:

$$\nabla \times E - \mathrm{i}\kappa H = 0$$
$$\nabla \times H + \mathrm{i}\kappa n E = 0$$

with wave number κ and refractive index n:

$$\kappa := \omega\sqrt{\mu_0 \epsilon_0} \qquad n(x) := \frac{1}{\epsilon_0}\left(\epsilon(x) + \mathrm{i}\frac{\sigma(x)}{\omega}\right)$$

- μ_0 magnetic permeability, assumed to be constant
- $\epsilon(x) > 0$ electric permittivity, with $\epsilon - \epsilon_0$ compactly supported
- $\sigma(x) \geq 0$ conductivity, compactly supported

Direct problem

Given incident field $(E^{\mathrm{i}}, H^{\mathrm{i}})$ fulfilling

$$\nabla \times E - \mathrm{i}\kappa H = 0 \qquad \nabla \times H + \mathrm{i}\kappa n E = 0$$

for $n = 1$ find the scattered field $(E^{\mathrm{s}}, H^{\mathrm{s}})$ such that Silver-Müller radiation condition is fulfilled

$$\lim_{|x|\to\infty} \left(H^{\mathrm{s}}(x) \times x - |x|\, E^{\mathrm{s}}(x)\right) = 0$$

and $(E, H) = (E^{\mathrm{i}}, H^{\mathrm{i}}) + (E^{\mathrm{s}}, H^{\mathrm{s}})$ solves the pdes.

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Classical solution theory

Let the refractive index satisfy:

$$n \in \mathfrak{D} := \left\{ n \in C^{1,\alpha}(\mathbb{R}^3) \colon \operatorname{supp}(1-n) \subset B(\pi), \Re(n) > 0, \Im(n) \geq 0 \right\}$$

Then there exists a unique solution $(E, H) \in C^1$.

Inverse near field problem

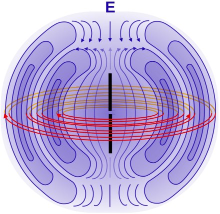


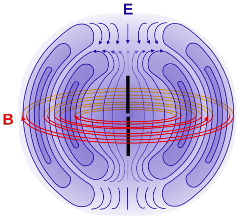
Image from wikipedia

Incident fields are generated by dipoles

$$E^{\mathrm{i}}_{y,a}(x) = -\frac{1}{\mathrm{i}\kappa}\nabla \times \nabla \times a\Phi(x,y)$$
$$H^{\mathrm{i}}_{y,a}(x) = \nabla \times a\Phi(x,y)$$

for all $y \in \partial B(R)$ with $R > \pi$, $a \in \mathbb{R}^3$ and $\Phi(x,y) = \mathrm{e}^{\mathrm{i}\kappa|x-y|}/(4\pi\,|x-y|)$

Inverse near field problem



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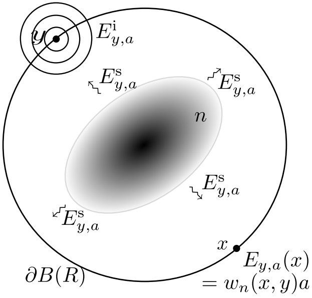
Measurements are the matrices

$$E_{y,a}(x) = w_n(x,y)a$$

on $\partial B(R) \times \partial B(R)$.

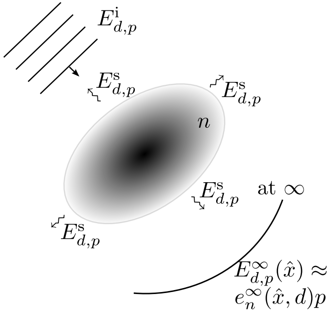


Inverse far field problem

Incident fields are plane waves

$$E^{\mathrm{i}}_{d,p}(x) = d \times (p \times d)\mathrm{e}^{i\kappa d\cdot x}$$
$$H^{\mathrm{i}}_{d,p}(x) = \frac{1}{\mathrm{i}\kappa}\nabla \times E^{\mathrm{i}}_{d,p}(x)$$

for all directions $d \in \partial B(1)$ and polarization $p \in \mathbb{R}^3$.



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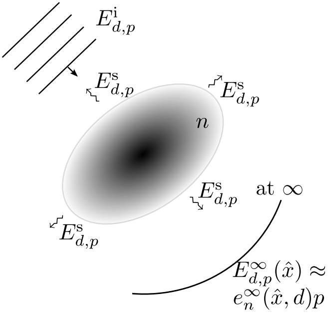

Far field expansion

$$E^{\mathrm{s}}_{p,d}(x) = \frac{\mathrm{e}^{\mathrm{i}\kappa r}}{r}\left(E^{\infty}_{d,p}(\hat{x}) + o(1)\right)$$

as $r = |x| \to \infty$ for measurement direction $\hat{x} = x/r \in \partial B(1)$.

Inverse far field problem

Incident fields are plane waves

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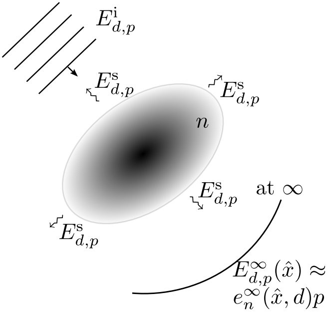

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as $r = |x| \to \infty$ for measurement direction $\hat{x} = x/r \in \partial B(1)$.

Measurements are the matrices defined by

$$E^{\infty}_{d,p}(\hat{x}) = e^{\infty}_{n}(\hat{x}, d)p$$

Functional setup

Define

$$\mathfrak{D}_b := \left\{ n \in C^{1,\alpha}(\mathbb{R}^3) \colon \operatorname{supp}(1-n) \subset B(\pi), \Re(n) \geq b, \Im(n) \geq 0 \right\}$$

and let $m > 7/2$ and set $\mathbb{X} = H^m([-\pi,\pi]^3)$ with norm

$$\|f\|_{H^m}^2 = \sum_{\gamma \in \mathbb{Z}^3} \langle \gamma \rangle^m \left| \widehat{f}(\gamma) \right|^2$$

Functional setup

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Near field operator

$$F_{\mathrm{n}} \colon H^m([-\pi,\pi]^3) \cap \mathfrak{D}_b \to (L^2(\partial B(R) \times \partial B(R)))^{3\times 3}, \quad n \mapsto w_n.$$

Far field operator

$$F_{\mathrm{f}} \colon H^m([-\pi,\pi]^3) \cap \mathfrak{D}_b \to (L^2(\partial B(1) \times \partial B(1)))^{3\times 3}, \quad n \mapsto e_n^\infty.$$

Main theorem

Theorem

Assume that $7/2 < m < s$, $s \neq 2m + 3/2$ *and* $n^\dagger \in \mathfrak{D}_b$ *satisfies* $\|n^\dagger\|_{H^s} \leq C_s$ *for some* $C_s \geq 0$. *Then a VSC holds true for the operator* F_{n} *with* $\beta = 1/2$, *and* ψ *given by*

$$\psi_{\mathrm{n}}(t) := A\left(\ln(3 + t^{-1})\right)^{-2\nu}, \qquad \nu := \min\left\{\frac{s-m}{m+5/2}, \frac{s-m}{s-m+1}\right\},$$

where the constant $A > 0$ *depends on* m, s, b, C_s, κ *and* R.

Remark: Instability results show optimality up to value of exponent

N. Mandache. *Exponential instability in an inverse problem for the Schrödinger equation.* **Inverse Problems**, 17:1435–1444, 2001.

Corollaries

Corollary (convergence rate)

Under the assumptions of the previous theorem the error bound

$$\left\| n_\alpha^\delta - n^\dagger \right\|_{H^m} \leq 2\sqrt{2A} \left(\ln(3 + \delta^{-2}) \right)^{-\nu}$$

holds true for Tikhonov regularization for optimal α.

Corollary (stability estimate)

Suppose $\frac{7}{2} < m < s$, $s \neq 2m + 3/2$ and n_1 and n_2 satisfy $n_j \in \mathfrak{D}_b$ with $\|n_j\|_{H^s} \leq C_s$ for $j = 1, 2$ and some $C_s > 0$. Then

$$\|n_1 - n_2\|_{H^m} \leq \sqrt{2A} \left[\ln \left(3 + \|F_{\mathrm{n}}(n_1) - F_{\mathrm{n}}(n_2)\|^{-2}_{(L^2(\partial B(R) \times \partial B(R)))^{3\times 3}} \right) \right]^{-\nu}.$$

VSC for far field data

Theorem

Under the assumptions of the previous theorem the operator F_{f} fulfills for all $0 < \theta < 1$ a VSC with $\beta = 1/2$ and ψ given by

$$\psi_{\mathrm{f}}(t) := B\left(\ln(3 + t^{-1})\right)^{-2\nu\theta}$$

with a constant $B > 0$ depending only on m, s, C_s, κ, b and θ.

Remark: One obtains similar corollaries as in the near field case.

Stability result comparison

	new	**Hähner**	**Caro**	**Lai**
data	near and far field	far field	Cauchy	Cauchy
validity	global	local anywhere	global	local around 0
stability of	σ, ϵ	σ, ϵ	σ, ϵ, μ	σ
norm	H^m	L^∞	H^1	H^{-s}
exponent	< 1	$1/15$	unknown, $< 1/3$	≤ 1
special		strong norm image space		Hölder-logarithmic

P. Hähner. *Stability of the inverse electromagnetic inhomogeneous medium problem.* **Inverse Problems**, 16:155–174, 2000.

P.Caro. *Stable determination of the electromagnetic coefficients by boundary measurements.* **Inverse Problems**, 26:105014, 2010.

R.-Y. Lai, V. Isakov and J.-N. Wang. *Increasing stability for the conductivity and attenuation coefficient.* arXiv:1505.00108, 2015.

Proof in a nutshell

- Reformulate VSC to

$$\left\langle n^{\dagger}, n^{\dagger} - n \right\rangle_{H^m} \leq \frac{1}{4} \left\| n^{\dagger} - n \right\|_{H^m}^{2} + \psi(\delta)$$

- Show VSC holds independently of δ outside of a ball
- Split in high and low frequencies
  - Use Alessandrini type identity and CGOs to bound low frequencies
  - Use higher smoothness of true solution for high frequencies
- Choose occurring parameter in dependence of δ such that right hand side is approximately minimal

Alessandrini type estimate

Connection between potentials and data

Lemma

Let $2R > R' > R > \pi$, $m > 7/2$ and assume that n_1 and n_2 are two refractive indices satisfying $n_j \in \mathfrak{D}_b \cap H^m$ such that $\|n_j\|_{H^m} \leq C_m$ for some $C_m \geq 0$. Let $E_j, H_j \in C^1(B(2R)) \cap L^2(B(2R))$ be solutions to the pdes in $B(2R)$ for $n = n_j$ for $j = 1, 2$. Then the estimate

$$\left| \int_{B(\pi)} (n_1 - n_2) E_1 E_2 \, \mathrm{d}x \right|$$
$$\leq C \|w_1 - w_2\|_{(L^2((\partial B(R))^2))^{3\times 3}} \|E_1\|_{L^2(B(R'))} \|E_2\|_{L^2(B(R'))}$$

holds true, where w_j is the near field scattering data for $n = n_j$ for $j = 1, 2$ and C depends on κ, R, b and C_m.

P. Hähner. *Stability of the inverse electromagnetic inhomogeneous medium problem.* **Inverse Problems**, 16:155–174, 2000.

General idea

Desired solutions E_j are of the form

$$E_j = \eta_j \mathrm{e}^{\mathrm{i}\zeta_j \cdot x} \qquad \zeta_j \in \mathbb{C}^3 \qquad \zeta_1 = \overline{\zeta_2} \qquad \zeta_j \cdot \zeta_j = \kappa^2$$

to obtain bounds on Fourier coefficient $\zeta_1 + \overline{\zeta_2}$.

Inhomogeneous case

- Not possible
- Aim: construct solutions close to these
- Strategy: transformation to Helmholtz like equation

Helmholtz type equation

Construct Helmholtz type equation for $(E', H') = (n^{1/2}E, H)$ with $E^{\mathrm{i}} = \eta \mathrm{e}^{\mathrm{i}\zeta\cdot x}$

$$(\Delta + \kappa^2)\begin{pmatrix} E' \\ H' \end{pmatrix} = \mathcal{Q}\begin{pmatrix} E' \\ H' \end{pmatrix}$$

$$\mathcal{Q}(x) := \kappa^2(1-n)1_6 + \mathrm{i}\kappa n^{-1/2}\begin{pmatrix} 0_3 & -\nabla n\times \\ \nabla n\times & 0_3 \end{pmatrix} - \begin{pmatrix} D\left(\frac{\nabla n}{n}\right) & 0_3 \\ 0_3 & 0_3 \end{pmatrix} - \frac{1}{4}n^{-2}\begin{pmatrix} 1_3(\nabla n\cdot\nabla n) & 0_3 \\ 0_3 & 0_3 \end{pmatrix} + \frac{1}{2}n^{-1}\begin{pmatrix} 1_3(\Delta n) & 0_3 \\ 0_3 & 0_3 \end{pmatrix}$$

In order to apply the usual CGO theory one needs to bound $\|\mathcal{Q}\|_2$ for $n \in \mathfrak{D}_b \cap H^m$, $\|n\|_{H^m} \le C_m$.

D. Colton and L. Päivärinta. *The uniqueness of a solution to an inverse scattering problem for electromagnetic waves.* **Archive for rational mechanics and analysis**, 119:59–70, 1992.

Form and estimate of CGOs

For $\Im(\zeta) \geq C \|\mathcal{Q}\|_2$ one obtains existence of solution to the time-harmonic Maxwell equations of the form

$$E(x,\zeta,\eta) = \mathrm{e}^{\mathrm{i}\zeta\cdot x}\left[\eta + f(x,\zeta,\eta)\zeta + V(x,\zeta,\eta)\right], \quad x \in B(R')$$

fulfilling the estimate

$$\|f(\cdot,\zeta,\eta)\|_{L^2(B(R'))} + \|V(\cdot,\zeta,\eta)\|_{L^2(B(R'))} \leq C\frac{|\eta|}{|\Im(\zeta)|}.$$

Inserting in Alessandrini estimate

Lemma

Let $R > \pi$, $m > 7/2$ and n_1 and n_2 be two refractive indices such that $n_j \in \mathfrak{D}_b \cap H^m$ such that $\|n_j\|_{H^m} \leq C_m$ with $C_m \geq 0$ and corresponding near field data w_j for $j = 1, 2$. Let

$$t \geq C(1+\kappa^2)C_m^2 b^{-2}$$

and $1 \leq \varrho \leq 2\sqrt{\kappa^2 + t^2}$. Then there exists a constant C depending only on R, κ, b and C_m such that

$$|(\widehat{n_1} - \widehat{n_2})(\gamma)| \leq C\left(\|w_1 - w_2\|_{(L^2((\partial B(R))^2))^{3\times 3}} \mathrm{e}^{3Rt} + \|n_1 - n_2\|_{H^m} \frac{\varrho}{t}\right)$$

holds true for all $\gamma \in \mathbb{Z}^3$ with $|\gamma| \leq \varrho$.

Remark: The better estimate on $\|\mathcal{Q}\|_2$ improves previous results which had an additional factor of ϱ^3.

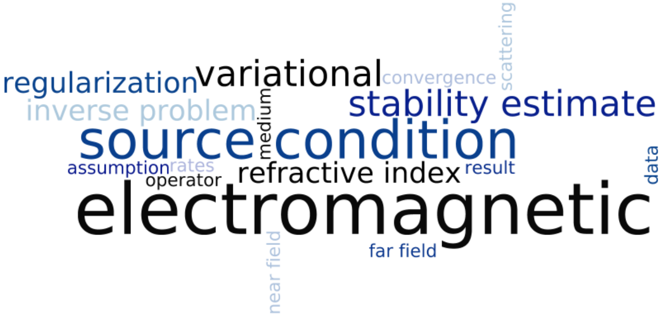
Summary

- First proof of (logarithmic) convergence rates for Tikhonov applied to electromagnetic medium scattering under Sobolev smoothness
- A compatible stability estimate for electromagnetic medium scattering
- Proof
  - improves parameter dependence of previous results
  - shows that ideas of the proof of the VSC for acoustic medium scattering can be applied to other problems

but ...

- We cannot guarantee that a global minimum of the Tikhonov functional can be computed. This still requires F' and conditions such as the tangential cone condition.

Thank you for your attention.



F. Weidling and T. Hohage. *Variational source conditions and stability estimates for inverse electromagnetic medium scattering problems*, **arxiv**, 1512.06586, 2015.