Enhancing residual-based techniques with shape reconstruction features in EIT

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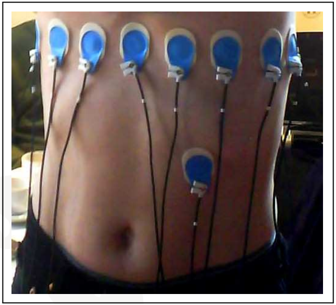
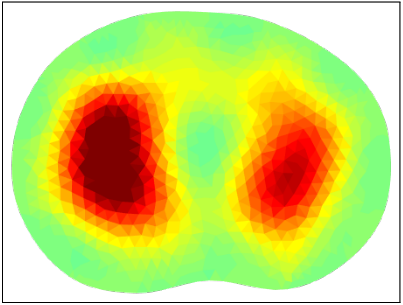
Electrical impedance tomography (EIT)

Source: (left) Bastian Harrach, (right) Kyoung Hun Lee

- Apply electric currents on subject's boundary
- Measure necessary voltages

$\rightsquigarrow$ Reconstruct conductivity inside subject.

Mathematical Model

$\Omega \subset \mathbb{R}^n$:	imaged body, $n \geq 2$
$\sigma(x)$:	conductivity
$u(x)$:	electric potential
g:	current density

Inside Ω, $u(x)$ solves:

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0$$

Idealistic model for boundary measurements (continuum model):

$$\sigma \nabla u \cdot \nu = g, \quad \text{on } \partial\Omega$$

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \tag{1}$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves (1)}\}\,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma) : L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Calderón problem

- σ is uniquely defined by $\Lambda(\sigma)$: Kohn and Vogelius (1984), Sylvester and Uhlmann (1987)
  Nachman (1996), Astala and Päivärinta (2006)

- Recovering of σ from $\Lambda(\sigma)$: D-bar method, Factorization method, Enclosure method, NOSER algorithm, GREIT algorithm ...

Minimization problem of the linearized residual

Recovering of σ from Λ_{meas}

- Find σ such that $\Lambda(\sigma) = \Lambda_{\text{meas}}$?
- Find σ such that $\|\Lambda_{\text{meas}} - \Lambda(\sigma)\|^2 \to \text{min!}$
- Linearized Tikhonov:
  $\|\Lambda_{\text{meas}} - \Lambda(\sigma_0) - \Lambda'(\sigma_0)(\sigma - \sigma_0)\|^2 + \alpha\|\sigma - \sigma_0\|^2 \to \text{min!}$

Advantage:

- Flexible and fast
- Good reconstruction images

Drawback:

- Images usually contain ringing artifacts
- Convergence unclear
  - Convergence against true sol for exact data $\Lambda(\sigma)$?
  - Convergence against true sol for noisy data Λ_{meas}?

Linearization and shape reconstruction

Theorem (Harrach/Seo, SIMA 2010)

Let κ, σ, σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Longrightarrow \quad \mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

$\mathrm{supp}_{\partial\Omega}$: outer support ($=$ supp $+$ parts unreachable from $\partial\Omega$)

⇝ Linearized EIT equation contains correct shape information

Monotonicity-based imaging

- Inclusion detection: For $\sigma = 1 + \chi_D$ with unknown D, use $\tau = 1 + \chi_B$, with small ball B.

$$B \subseteq D \quad \Longrightarrow \quad \tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

- Algorithm: Mark all balls B with $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$
- Result: upper bound of D.

Only an upper bound? Converse monotonicity relation?

Theorem (Harrach/Ullrich, SIMA 2013)

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

Proof: Monotonicity & localized potentials

Shape can be reconstructed by linearized monotonicity tests

$\leadsto$ fast, rigorous, allows globally convergent implementation

$\leadsto$ very sensitive to noise

Improving residuum-based methods

Let $\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \gamma\chi_D, \gamma \in L^\infty_+(\Omega)$.

- Pixel partition $\Omega = \bigcup_{k=1}^m P_k$
- Monotonicity tests

$$\beta_k \in [0,\infty] \text{ max. values s.t. } \quad \beta_k \Lambda'(1)\chi_{P_k} \geq \Lambda(\sigma) - \Lambda(1)$$

- Let $a := 1 - \frac{1}{1+\inf_D \gamma}$, then $\beta_k \geq a$ if $P_k \subseteq D$.

Lemma 1 (Harrach/**M.**, submitted, 2015) $P_k \subseteq D$ if and only if $\beta_k > 0$

- $R(\kappa) \in \mathbb{R}^{s\times s}$: Discretization of lin. residual $\Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\kappa$

  (e.g. Galerkin proj. to fin.-dim. space)
- Lemma 1 $\rightsquigarrow$ upper bound for κ: β_k
- β_k may be $+\infty$ $\rightsquigarrow$ new upper bound for κ: $\min\{a, \beta_k\}$

Theorem 2 (Harrach/**M.**, submitted, 2015)
The monotonicity-constrained residuum minimization problem

$$\|R(\kappa)\|_{\mathrm{F}} \to \text{min!} \quad \text{s.t.} \quad \kappa|_{P_k} = \text{const.},\ 0 \leq \kappa|_{P_k} \leq \min\{a, \beta_k\}$$

possesses a unique solution $\hat{\kappa}$, and $P_k \subseteq \operatorname{supp} \hat{\kappa}$ iff $P_k \subseteq \operatorname{supp}(\sigma - 1)$.
Moreover, $\hat{\kappa}|_{P_k} = \min\{a, \beta_k\}$.

Proof of Theorem 2:

- Existence of minimizer:
  - $\kappa \mapsto \|\mathbf{R}(\kappa)\|_F^2$ continuous.
  - The admissible set is compact.

- **Step 1:** If $\kappa|_{P_k} = \text{const.},\ 0 \leq \kappa|_{P_k} \leq \min\{a, \beta_k\}$, then

$$\Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\kappa \leq 0.$$

- **Step 2:** If $\hat{\kappa} = \sum \hat{\alpha}_k \chi_k$ is a minimizer, then $\operatorname{supp} \hat{\kappa} \subseteq D$.
  - Step 1 $\rightsquigarrow \Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\hat{\alpha}_k\chi_k \leq \Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\hat{\kappa} \leq 0.$
  - Lemma 1 + definition of $\beta_k \rightsquigarrow$ if $\hat{\alpha}_k > 0$, then $P_k \subseteq D$.

Proof of Theorem 2:

- **Step 3:** If $\hat{\kappa}$ is a minimizer, then $\hat{\kappa}|_{P_k} = \min\{a, \beta_k\}$
  - If there exists a pixel P_k such that $\hat{\kappa}(x) < \min\{a, \beta_k\}$ in P_k, we can choose $h > 0$ such that $\hat{\kappa} + h\chi_k = \min(a, \beta_k)$ in P_k.
  - $\mathbf{R}(\hat{\kappa})$ and $\mathbf{R}(\hat{\kappa} + h\chi_k)$ symmetric $\rightsquigarrow$ real eigenvalues.
  - $\mathbf{R}(\hat{\kappa} + h\chi_k) = \mathbf{R}(\hat{\kappa}) + h\mathbf{S}_k$.
  - Poincaré's inequality and the unique continuation principle $\rightsquigarrow$ $\mathbf{S}_k$ is positive definite matrix $\rightsquigarrow$ $\lambda_i(\mathbf{S}_k) > 0$.
  - Weyl's Inequalities

$$\lambda_i(\hat{\kappa} + h\chi_k) \geq \lambda_i(\hat{\kappa}) + h\lambda_N(\mathbf{S}_k) > \lambda_i(\hat{\kappa}) \quad \text{for all} \quad i \in \{1, \ldots, N\}.$$

  - Step 1 $\rightsquigarrow$ $\lambda_i(\hat{\kappa}) \leq 0, \lambda_i(\hat{\kappa} + h\chi_k) \leq 0$.
  - Thus,
    $\|\mathbf{R}(\hat{\kappa} + h\chi_k)\|_F^2 - \|\mathbf{R}(\hat{\kappa})\|_F^2 = \sum_{i=1}^N |\lambda_i(\hat{\kappa} + h\chi_k)|^2 - \sum_{i=1}^N |\lambda_i(\hat{\kappa})|^2 < 0$,
    which contradicts the minimality of $\hat{\kappa}$.

- **Step 4:** If $P_k \subseteq D$, then $P_k \subseteq \operatorname{supp}\hat{\kappa}$.
  - $\hat{\kappa}$ is a minimizer + Step 3 $\rightsquigarrow$ $\hat{\kappa} = \sum_{k=1}^{P} \min(a, \beta_k)\chi_k$.
  - $P_k \subseteq D$ + Lemma 1 $\rightsquigarrow$ $\min(a, \beta_k) > 0 \rightsquigarrow P_k \subseteq \operatorname{supp}\hat{\kappa}$.

- Uniqueness of minimizer:
  - Step 3 $\rightsquigarrow$ a unique minimizer $\hat{\kappa} = \sum_{k=1}^{P} \min(a, \beta_k)\chi_k$.
  - This minimizer fulfills

$$\begin{aligned} \hat{\kappa} &= a \quad \text{in } P_k, \text{ if } P_k \subseteq D, \\ \hat{\kappa} &= 0 \quad \text{in } P_k, \text{ if } P_k \nsubseteq D. \end{aligned}$$

Improving residuum-based methods

Convergence for noisy data: $\|\Lambda^{\delta}(\sigma)-\Lambda(\sigma)\|_{\mathrm{op}} \leq \delta$

- $R^{\delta}(\kappa) \in \mathbb{R}^{s\times s}$: Discretization of lin. residual for noisy data $\Lambda^{\delta}(\sigma)$

$$\Lambda^{\delta}(\sigma)-\Lambda(1)-\Lambda'(1)\kappa$$

- $\beta_k^{\delta} \in [0,\infty]$ max. values s.t. $\beta_k^{\delta}\Lambda'(1)\chi_{P_k} \geq \Lambda^{\delta}(\sigma)-\Lambda(1)$

Then $\beta_k^{\delta} \geq \beta_k$

$\rightsquigarrow$ $P_k \subseteq D \Rightarrow \beta_k^{\delta} \geq \mathbf{a}$

$\rightsquigarrow$ $\beta_k^{\delta} = 0 \Rightarrow P_k \nsubseteq D$

Theorem 3 (Harrach/**M.**, submitted, 2015)

The monotonicity-constrained residuum minimization problem

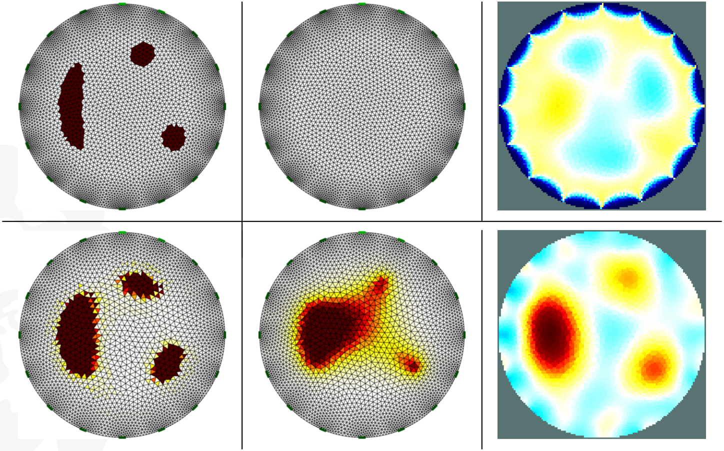
$$\|R^{\delta}(\kappa)\|_{\mathrm{F}} \to \text{min!} \quad \text{s.t.} \quad \kappa|_{P_k} = \text{const.},\ 0 \leq \kappa|_{P_k} \leq \min\{\mathbf{a}, \beta_k^{\delta}\}$$

possesses a solution $\hat{\kappa}^{\delta}$, and $\hat{\kappa}^{\delta} \to \hat{\kappa}$ pointwise.

Proof of Theorem 3: unique minimality of $\hat{\kappa}$

Numerical experiment: Simulated data 10% noise

Reference body: ball diameter 2. Inclusions: one half-elipse + two small balls diameter 0.2

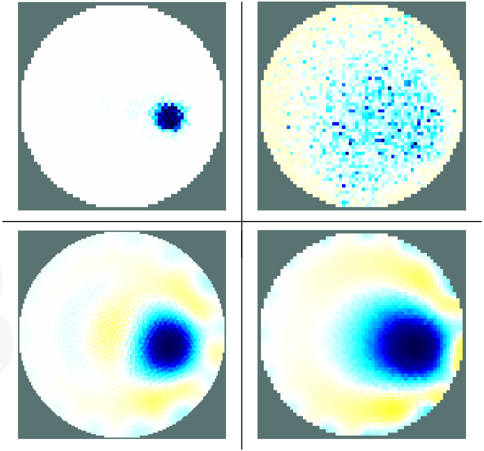


(Left-to-right) First row: True conductivity, Reference conductivity, NOSER solver

Second row: Our method, Monotonicity-based imaging, EIDORS default solver

Numerical experiment: iirc phantom data

Tank: diameter 20cm. Rod: diameter 2cm



First row: Our method (left), Monotonicity-based imaging (right)

Second row: NOSER solver (left), EIDORS default solver (right)

Conclusions

- EIT is a highly ill-posed, non-linear inverse problem.
- Generic solvers for non-linear inverse problems
  - Very flexible, real-time implementation, good enough reconstruction image
  - Ringing artifacts, convergence unclear
- Improving residuum-based methods for EIT shape reconstruction
  - allow fast, rigorous, globally convergent implementations.
  - work in any dimensions $n \geq 2$, full or partial boundary data.
  - can enhance standard residual-based methods.
  - yield rigorous resolution guarantees for realistic settings.
  - need definiteness assumption.
- Future work: Method applicable without definiteness assumption.