Westfälische
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Data assimilation applied to a model for price formation

joint with Martin Burger, Marie-Therese Wolfram

Inverse Problems for PDEs

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29. März 2016

Outline

Price Formation - Introduction

In a series of papers J.-M. Lasry and P.-L. Lions introduced coarse grained models for economical equilibria in markets with a large number of rational players.

Basic setup:

Consider N players, whose investment strategies follow stochastic differential equations with Brownian diffusion and drifts, determined as Nash equilibria of an appropriate cost functional.
When $N \to \infty$ tends to infinity: systems of highly non-linear PDEs are obtained, e.g.

- Viscous Hamilton-Jacobi-Bellmann systems
- One-dimensional parabolic equation with a free boundary

J.-M. Lasry and P.-L. Lions
Mean field games.
Jpn. J. Math., 2(1):229-260, 2007.

Price Formation

Large group of buyers (f^B) and vendors (f^V), formation of the agreed price $x = p(t) \in \mathbb{R}$ described by

$$
\begin{aligned}
&\frac{\partial f^B}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f^B}{\partial x^2} = \lambda(t)\delta\left(x - p(t) + a\right), \quad \text{for } x < p(t)\\
&f^B \geq 0,\ f^B(x,t) = 0 \ \text{ for } x \leq p(t)\\
&\frac{\partial f^V}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f^V}{\partial x^2} = \lambda(t)\delta\left(x - p(t) - a\right), \quad \text{for } x > p(t)\\
&f^V \geq 0,\ f^V(x,t) = 0 \ \text{ for } x \geq p(t),
\end{aligned}
$$

where $\quad f^B(p(t),t) = f^V(p(t),t) = 0,$

$$\lambda(t) = -\frac{\sigma^2}{2}\frac{\partial f^B}{\partial x}\left(p\left(t\right), t\right) = \frac{\sigma^2}{2}\frac{\partial f^V}{\partial x}\left(p\left(t\right), t\right)$$

is transaction rate, $\sigma > 0$ randomness, $2a$ bid-ask spread.

Boltzmann Model (Burger et al. 2013)

Alternative approach: Start from a kinetic model of the form

$$\partial_t f^B(x,t) = \frac{\sigma^2}{2}\partial_{xx} f^B(x,t) - kf^B(x,t)f^V(x,t) + kf^B(x+a,t)f^V(x+a,t),$$
$$\partial_t f^V(x,t) = \frac{\sigma^2}{2}\partial_{xx} f^V(x,t) - kf^B(x,t)f^V(x,t) + kf^B(x-a,t)f^V(x-a,t)$$

Main difference: Agent trade at all prices. Volume of transactions at price x and time t:

$$\mu(x,t) = kf^B(x,t)f^V(x,t).$$

This converges to the Lasry and Lions Model as $k \to \infty$ (infinite transaction rate)

Price Formation

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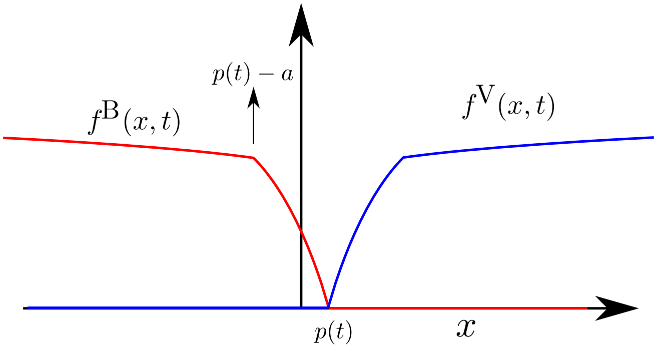
is transaction rate, $\sigma > 0$ randomness, $2a$ bid-ask spread.

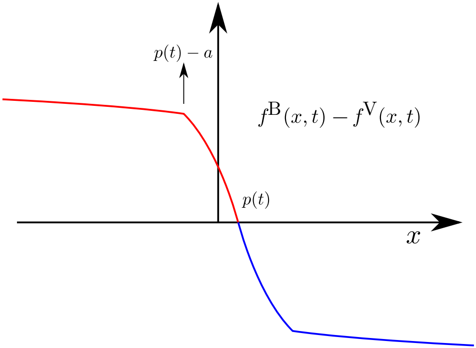
Price Formation



$$f^B(x, t = 0) > 0 \text{ for } x < p_0, f^B(x, t = 0) = 0 \text{ for } x \geq p_0$$
$$f^V(x, t = 0) > 0 \text{ for } x > p_0, f^V(x, t = 0) = 0 \text{ for } x \leq p_0.$$

Price Formation

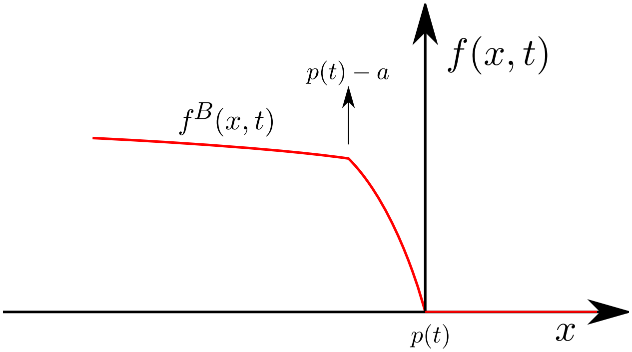
Introduce $f = f^B - f^V$



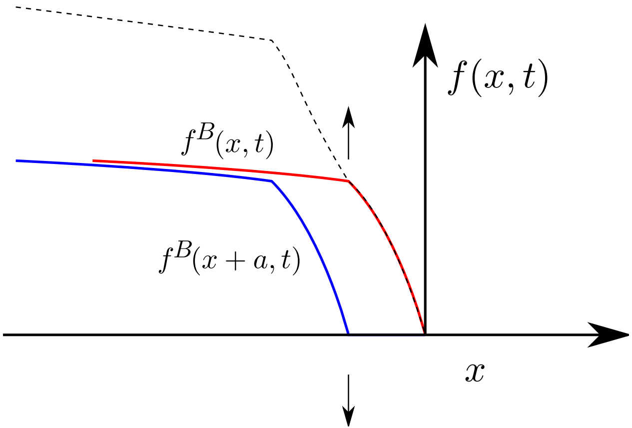
System reduces to single equation

$$f(x, t = 0) > 0 \text{ for } x < p_0, \ f(x, t = 0) < 0 \text{ for } x > p_0.$$

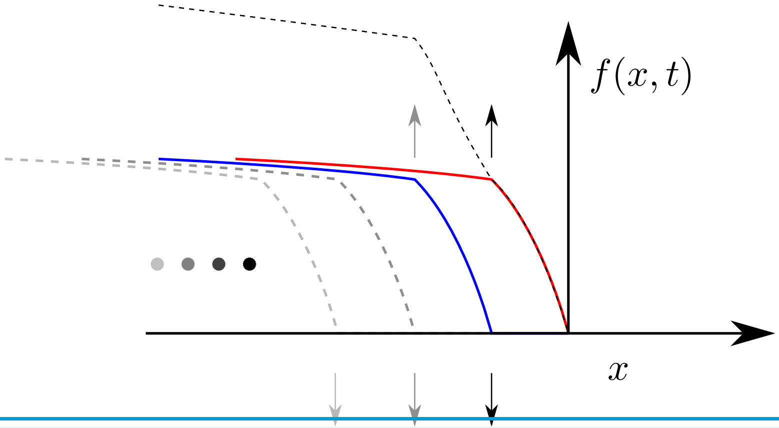
Global existence



Global existence



Global existence



Global existence

- Apply construction to positive and negative part, i.e.

$$F(x,t) = \begin{cases} \sum_{n=0}^{\infty} f^B(x+na,t), & x < p(t), \\ -\sum_{n=0}^{\infty} f^V(x-na,t), & x > p(t). \end{cases}$$

$\Rightarrow$ Then, F fulfills, in the sense of distributions

$$\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0,$$

with initial datum

$$F_I(x) = \begin{cases} \sum_{n=0}^{\infty} f_I^B(x+na), & x < p_0, \\ -\sum_{n=0}^{\infty} f_I^V(x-na), & x > p_0, \end{cases}$$

The free boundary $p = p(t)$ is the zero-level set of the solution F of the heat equation

Global existence

- Let F be the solution of the heat equation with initial datum F_I as constructed above.
- Then $f = f(x,t)$ given by

$$f(x,t) = \begin{cases} F(x,t) - F(x+a), & x < p(t), \\ -F(x,t) + F(x-a), & x > p(t). \end{cases}$$

is a solution to the FBP.

Theorem (Global Existence)

There exists a unique smooth solution $f = f(x,t)$ of the FBP for $t \in [0,\infty)$. Furthermore, $p \in \mathcal{C}([0,\infty))$.

Bounded domain

- Additional boundary conditions (non-local):

$$\partial_x F(-L,t) = \partial_x F(-L+a,t),$$
$$\partial_x F(L,t) = \partial_x F(L-a,t)$$

- Existence as long as $-L+a < p(t) < L-a$:

Theorem

The BVP has global solution conserving total mass of buyers and vendors iff $p(t) \in (-L+a, L-a)$ for all $t > 0$. Then $p(t)$ converges to $p_\infty \in (-L+a, L-a)$ as $t \to \infty$. Furthermore

$$|p(t) - p(0)| \leq C(\underline{\Lambda}, F^0)\sqrt{t}$$

$\rightarrow$ Not a restriction for inverse approach: $p(t)$ given !

Bounded domain

- Additional boundary conditions (non-local):

$$\partial_x F(-L,t) = 0,$$
$$\partial_x F(L,t) = 0$$

- Existence as long as $-L + a < p(t) < L - a$:

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$\rightarrow$ Approximate by Neumann bonditions in this talk

Outline

On a One-Dimensional Price Formation Model

Inverse Problem / Data assimilation

Numerical examples

Data assimilation

Inverse Problem

Given measurements of the price $p(t)$ and the transaction rate $\Lambda(t)$ for some interval $[0, T]$, can we predict the price for $t > T$?

- Classical data assimilation approach:
  Use the measurements to reconstruct $F(x, 0)$. Then, use the model to predict $p(t)$ for $t > T$
- Puel[1]: Use the measurements to recontruct $F(x, T)$. Then, use the model to predict $p(t)$ for $t > T$
  Main ingredient: Duality estimate + optimal control problem

[1] J.P. Puel, A nonstandard approach to a data assimilation problem and Tychonov regularization revisited, SICON, 48(2), 2009

Duality estimate

Duality estimate

$$\int_{-L}^{L} F(T,x)\Psi(x)\,dx = \int_{-L}^{L} F_0(x)\Phi(0,x)\,dx + \int_0^T \Lambda(t)v(t)\,dt$$

For Φ s.t. $(x \in [-L, L],\ t \in [T, 0])$

$$-\partial_t \Phi(x,t) - \partial_{xx}\Phi(x,t) = \delta_{p(t)} u(t) + \delta'_{p(t)} v(t)$$
$$\Phi(x, T) = \Psi(x).$$

Backward in time, u, v: controls

Note: Measurement of the price $p(t)$ does not appear

Duality estimate

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Backward in time, u, v: controls

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Adjoint equation

Theorem (Adjoint equation)

For any $\psi \in L^2([-L, L])$, u, $v \in L^2(0, T)$ *and any* $T > 0$, *the exists a weak solution*

$$\Phi \in L^2(0, T; H^{1/2-\varepsilon}([-L, L]))$$

to the equation

$$-\partial_t \Phi(x,t) - \partial_{xx}\Phi(x,t) = \delta_{p(t)} u(t) + \delta'_{p(t)} v(t),$$
$$\Phi(x,T) = \psi(x), \ x \in [-L, L]$$

Note: $\delta' \in H^{-3/2-\varepsilon}([-L, L])$ (1d)

Duality estimate

Duality estimate

$$\int_{-L}^{L} F(T,x)\Psi(x)\,dx = \int_{-L}^{L} F_0(x)\Phi(0,x)\,dx + \int_0^T \Lambda(t)v(t)\,dt$$

Idea

Chose u, v such that $\Phi(0,x) = 0$ (or at least $\Phi(0,x) \approx 0$) - optimal control problem

Issues

- Regularity of quantities involved
- Analysis of optimal control problem

Optimal control problem

Structure: Heat operator and control on a curve $p = p(t)$:

$$(Adj) \quad \begin{cases} -\partial_t \Phi - \partial_{xx} \Phi = \delta_{p(t)} u(t) + \delta'_{p(t)} v(t), \\ \Phi(x, T) = \Psi(x) \end{cases}$$

Controls can only act on $p(t)$ by means of δ and δ'

Exact null-controllability

For every $T > 0$ and every $\Psi \in L^2(-L, L)$ there exist $u,\ v \in L^2(0, T)$ such that $\Phi(x, T)$ satisfies the equation above and that

$$\Phi(x, 0) = 0.$$

Sketch of proof

- Similar to proofs of boundary controllability (e.g. Fattorini and Russel, '71)
- Assume $p(t) = p_0$ for all t
- Expand everything in terms of eigenfunction of Laplace operator (w/ Neumann b.c.):

$$\Psi(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x) \ \rightarrow \ \Phi(x,t) = e^{-(T-t)\lambda_n} b_n(t)\varphi_n(x), \ b_n(0) = a_n$$

- Exponential moment equation (similar to duality estimate)

$$e^{-T\lambda_n} a_n = \int_0^T e^{\lambda_n s} \varphi_n(p_0) u(s)\, ds + \int_0^T e^{\lambda_n s} \partial_x \varphi_n(p_0) v(s)\, ds$$

with $\lambda_n = n\pi/(2L)$ and $\varphi_n(x) = \cos(\frac{n\pi}{2L} x)$.

Regularized optimal control problem

$$\min_{u(t),v(t)} \frac{1}{2}\int_{-L}^{L} \Phi(x,0)^2 dx + \frac{\alpha}{2}\int_0^T \left[u(t)^2 + v(t)^2\right] \, dt,$$

subject to (Adj)

Denote by $\bar{u}$, $\bar{v}$ solutions to exact problem:

$$J(u,v) \leq J(\bar{u},\bar{v}) = \int_0^T \left[\frac{\alpha}{2}\bar{u}(t)^2 + \frac{\alpha}{2}\bar{v}(t)^2\right],$$

$$\Rightarrow \quad \int_{-L}^{L} \Phi(x,0)^2 dx \leq C(\Psi)\alpha$$

Stability / Convergence

Question

Given two measurements $p_1(t)$, $p_2(t)$ and $\Lambda_1(t)$, $\Lambda_2(t)$,
$\rightarrow$ estimate differences in $p_1(t > T)$ and $p_2(t > T)$?

$$\int_{-L}^{L} (F_1(x,T) - F_2(x,T))\psi(x)\,dx = \int_{-L}^{L} (F_1^0(x)\varphi_1(x,0) - F_2^0(x)\varphi_2(x,0))\,dx + \int_0^T \Lambda_1(t)v_1(t) - \Lambda_2(t)v_2(t)\,dt$$

- Red term: Handle by exact controllability
- Question: What happens to the other term?

Stability / Convergence

Question

Given two measurements $p_1(t)$, $p_2(t)$ and $\Lambda_1(t)$, $\Lambda_2(t)$,
$\rightarrow$ estimate differences in $p_1(t > T)$ and $p_2(t > T)$?

$$\int_{-L}^{L}(F_1(x,T) - F_2(x,T))\psi(x)\,dx = \int_{-L}^{L}(F_1^0(x)\varphi_1(x,0) - F_2^0(x)\varphi_2(x,0))\,dx$$
$$+ \int_0^T \Lambda_1(t)(v_1(t) - v_2(t)) + v_2(t)(\Lambda_1(t) - \Lambda_2(t))\,dt$$

- Red term: Handle by exact controllability
- Question: What happens to the other term?

Stability of optimal controls

For two different prices $p_1(t)$ and $p_2(t)$ such that $p_1(0) = p_2(0)$, we have the following stability estimate for the controls $u_{1,2}$ and $v_{1,2}$, respectively:

$$
\begin{aligned}
&\int_{-L}^{L} (\varphi_1(x,0) - \varphi_2(x,0))^2 \, dx \\
&+ \alpha \int_0^T (u_1(t) - u_2(t))^2 + (v_1(t) - v_2(t))^2 \, dt \\
&\leq C(T) \|p_1 - p_2\|_{C^0([0,T])}^{1-\gamma} \|\psi\|_{H^{-3/2-\varepsilon}([-L,L])}
\end{aligned}
$$

for every $\gamma > 0$

Proof: Use optimality system.

Technical issue: Regularity of solutions Φ

Work in progress

$$\int_{-L}^{L}(F_1(x,T)-F_2(x,T))\psi(x)\,dx = \underbrace{\int_{-L}^{L}(F_1^0(x)\varphi_1(x,0)-F_2^0(x)\varphi_2(x,0))\,dx}_{\leq\alpha}$$

$$+\int_0^T \underbrace{\Lambda_1(t)(v_1(t)-v_2(t))}_{\leq\frac{\delta^{1-\gamma}}{\alpha}}+\underbrace{v_2(t)(\Lambda_1(t)-\Lambda_2(t))}_{\leq\delta}\,dt$$

→ Obtain almost $\sqrt{\delta}$ rate

→ Source condition: Implicit in regularity needed for optimal control problem

Outline

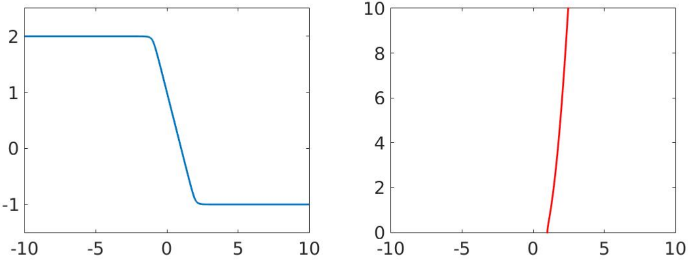
On a One-Dimensional Price Formation Model

Inverse Problem / Data assimilation

Numerical examples

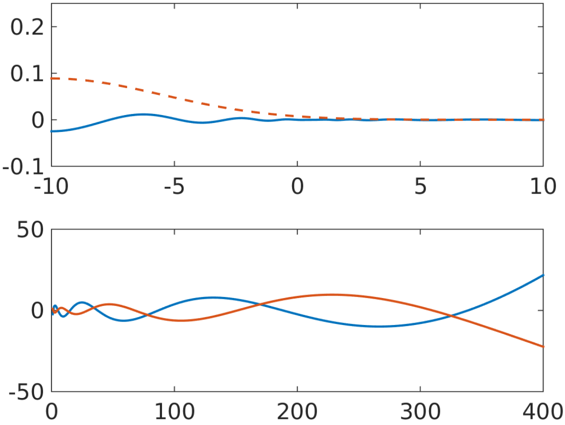
Numerical examples

- Discretize direct and adjoint problem by P1-FEM in 1d
- Time discretization using implicit Euler scheme
- Approximate δ and δ' distributions
- Solve optimal control problem by means of conjugate gradients (CG)



Numerical examples

Above $\Phi(x, 0)$ with and without control, below: u and v



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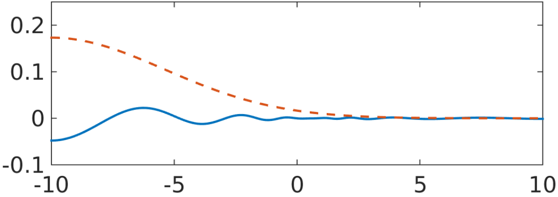

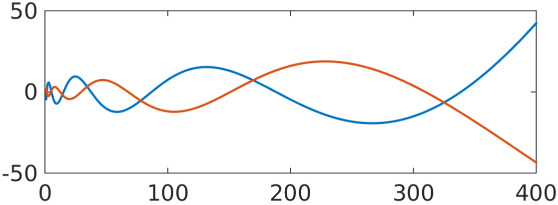

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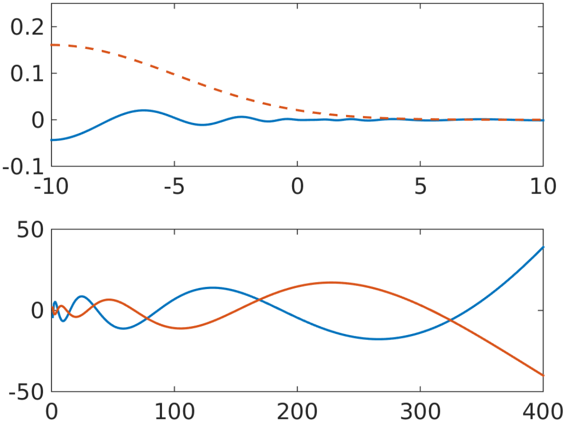

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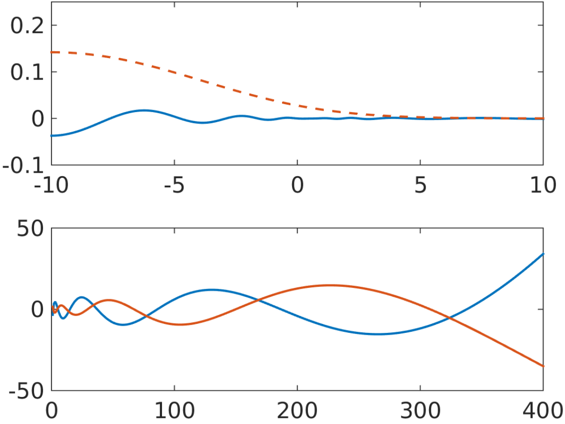

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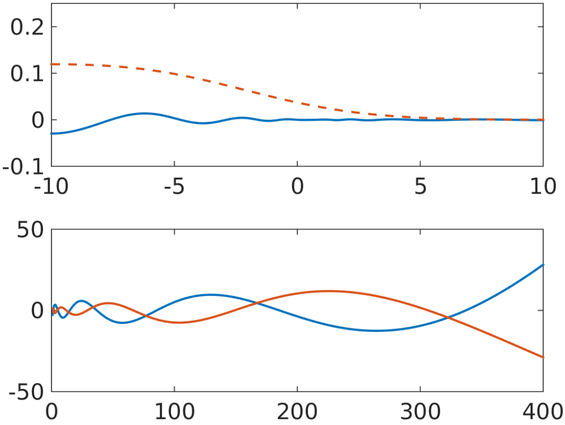

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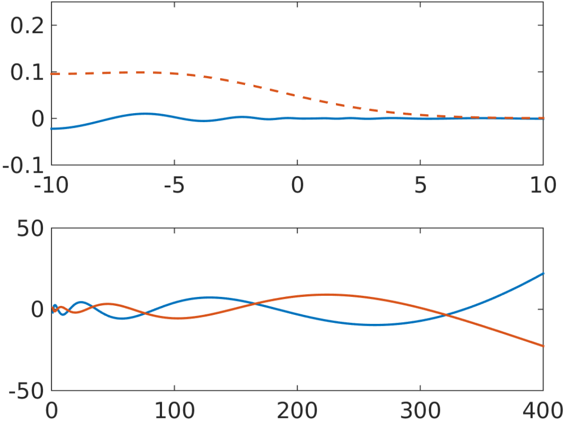

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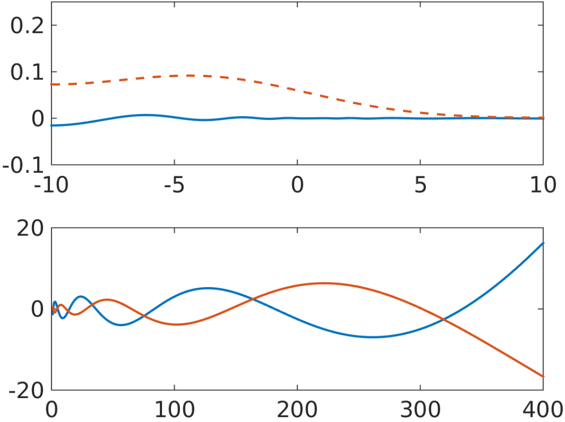

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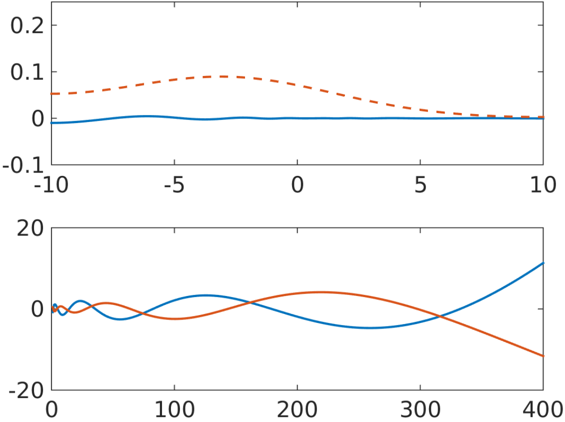

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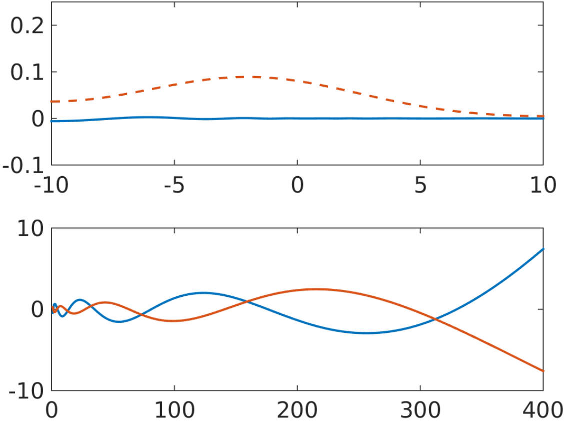

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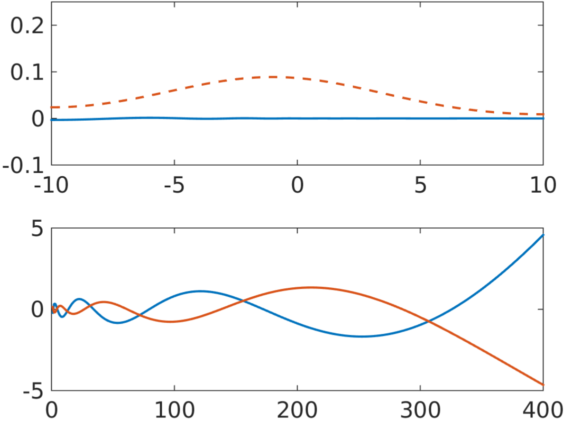

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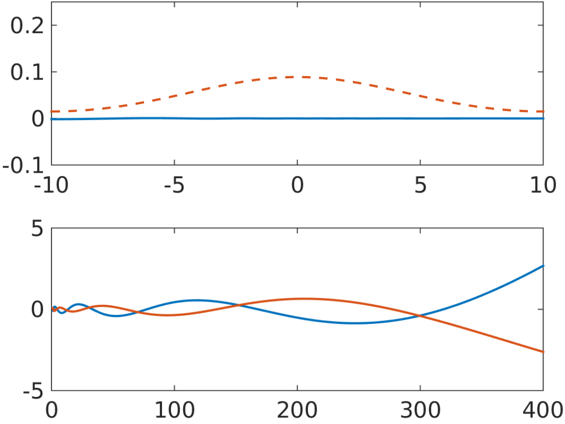

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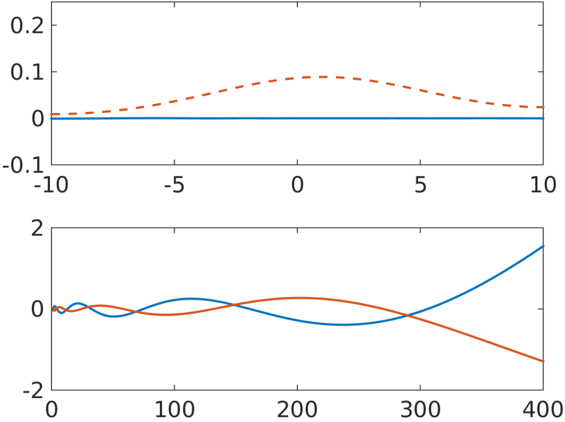

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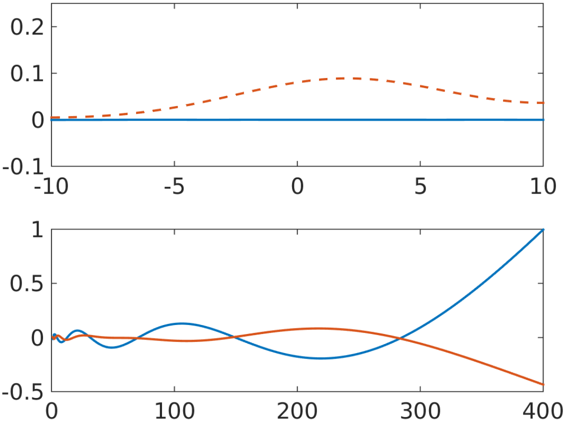

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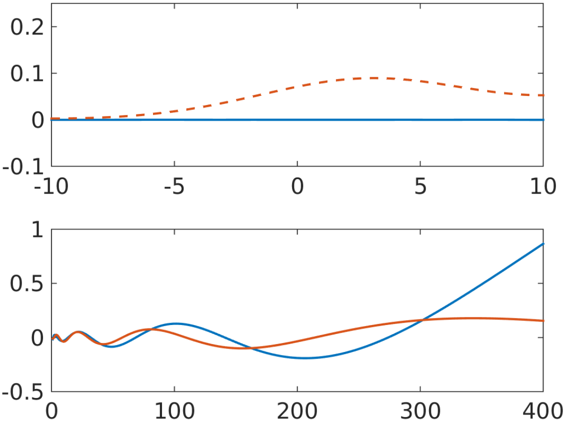

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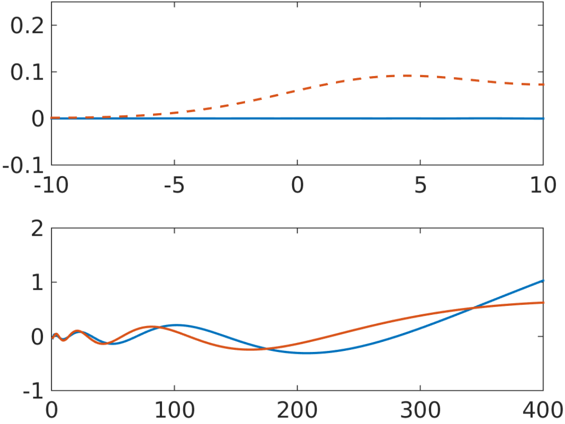

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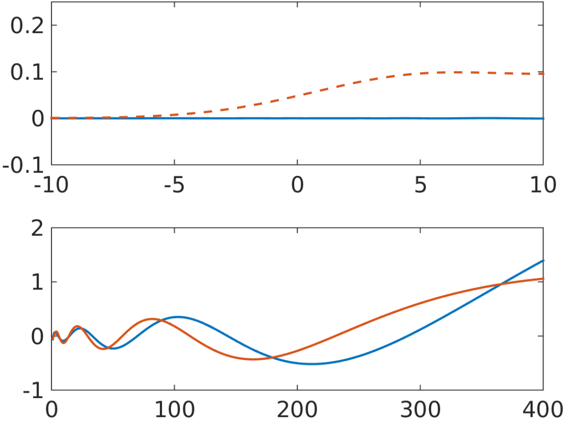

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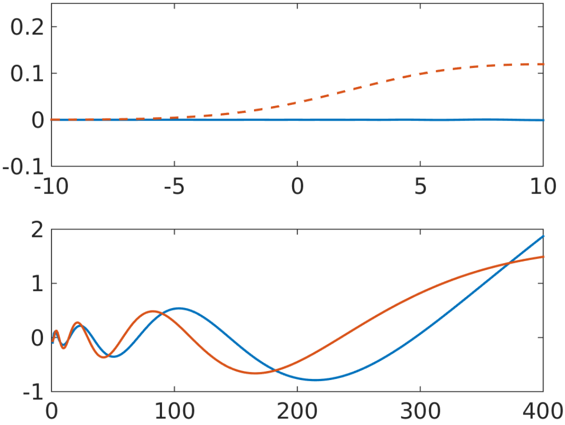

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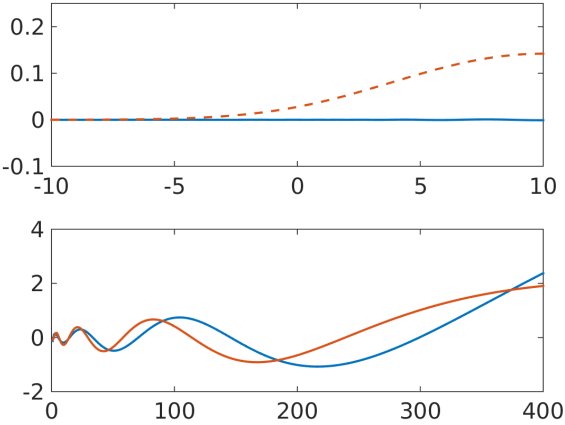

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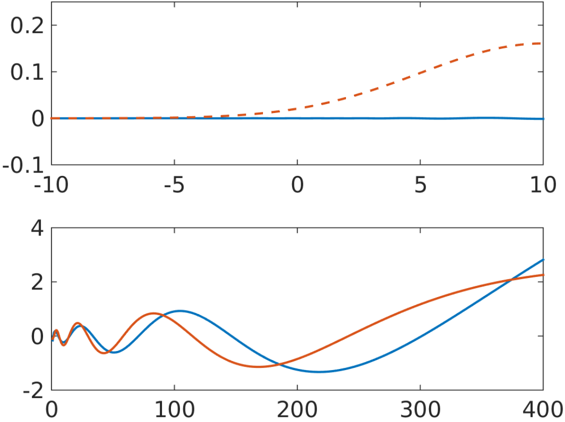

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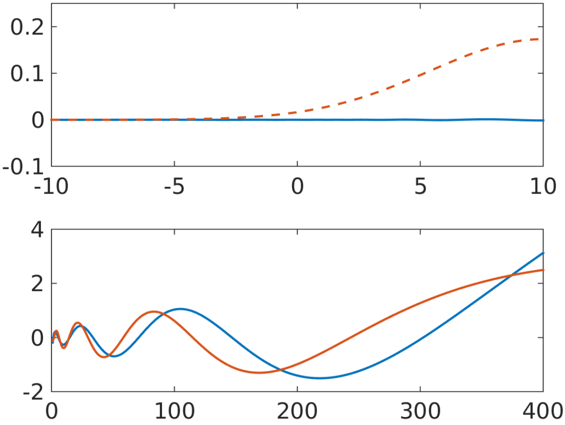
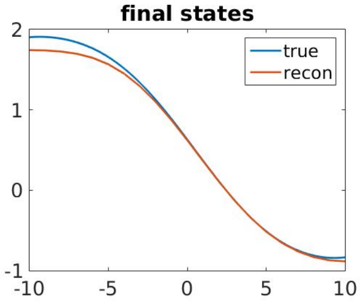
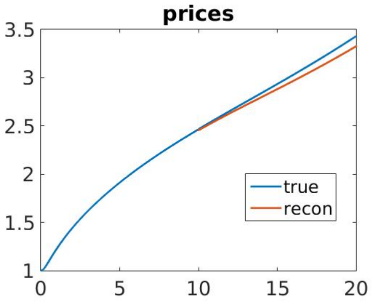


Numerical examples





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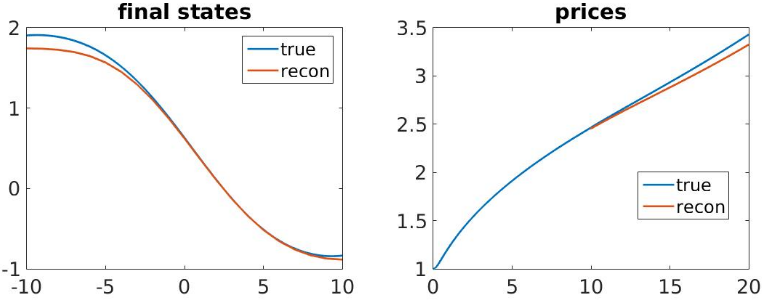

Thank you very much for your attention