Ivanov regularization for parameter identification in PDEs

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joint work with
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Outline

- Motovation: parameter identification in PDEs
- Tikhonov, Ivanov, Morozov regularization
- a counterexample for the equivalence of Tikhonov and Ivanov
- convergence analysis
- application to inverse problems for PDEs

motivation and problem setting

Parameter Identification in Differential Equations: Some Examples

- Identify spatially varying coefficients/source a, b, c in elliptic boundary value problem

$$-\nabla(a\nabla u) + cu = b \text{ in } \Omega\,,$$

$\Omega \subseteq \mathbb{R}^d$, $d \in \{1, 2, 3\}$, from boundary or (restricted) interior observations of u.

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- Identify parameter ϑ in

$$\dot{u}(t) = f(t, u(t), \vartheta)\ t \in (0, T)\,, \quad u(0) = u_0(\vartheta)$$

from discrete of continuous observations of u.
$y_i = g_i(u(t_i), \vartheta)$, $i \in \{1, \dots, m\}$ or $y(t) = g(t, y(t), \vartheta)$, $t \in (0, T)$

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bound constraints arise naturally
$\rightsquigarrow$ use them for regularizing the inverse problem

Problem Setting

nonlinear inverse problem

$$F(x) = y$$

$F : X \to Y, \quad X, Y$ Banach spaces,
forward operator = observation operator $\circ$ parameter-to-state map
F not continuously invertible, noisy data

$$\mathcal{S}(y, y^\delta) \le \delta$$

$\mathcal{S} : Y \times Y \to \mathbb{R}$ data misfit functional

Tikhonov, Ivanov, Morozov regularization

Tikhonov Regularization

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$\mathcal{R} : X \to \overline{\mathbb{R}}$ regularization functional
Tikhonov regularization: x_α^δ solves

$$\min_{x \in \mathcal{D}(F)} \mathcal{S}(F(x), y^\delta) + \alpha \mathcal{R}(x) \, .$$

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$\alpha \in \operatorname{argmax}\left\{\alpha \geq 0 : \text{ minimizer } x_\alpha^\delta \text{ exists and } \mathcal{S}(F(x_\alpha^\delta), y^\delta) \leq \tau\delta\right\}$

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or
α such that minimizer x_α^δ exists and $\delta < \mathcal{S}(F(x_\alpha^\delta), y^\delta) \leq \tau\delta$
where $\tau \geq 1$ is a fixed constant independent of δ.

Ivanov, Morozov

- Morozov regularization (method of the residual): x^δ_{Mo} solves

$$\min_{x\in\mathcal{D}(F)} \mathcal{R}(x) \text{ s.t. } \mathcal{S}(F(x),y^\delta)\leq\tau\delta$$

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- Ivanov regularization (method of quasi solutions): x^δ_ρ solves

$$\min_{x\in\mathcal{D}(F)} \mathcal{S}(F(x),y^\delta) \text{ s.t. } \mathcal{R}(x) \le \rho$$

where $\rho \in \{\rho^I_*, \rho^{II}_*, \rho^{III}_*\}$ (regularization parameter):
$\rho^I_* = \rho^\dagger = \mathcal{R}(x^\dagger)$
$\rho^{II}_* \in \text{argmin}\,\{\rho \ge 0 : \text{ minimizer } x^\delta_\rho \text{ exists and } \mathcal{S}(F(x^\delta_\rho),y^\delta) \le \tau\delta\}$
ρ^{III}_* such that minimizer $x^\delta_{\rho^{III}_*}$ exists and $\delta < \mathcal{S}(F(x^\delta_{\rho^{III}_*}),y^\delta) \le \tau\delta$

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$\rho_*^I=\rho^\dagger=\mathcal{R}(x^\dagger)$
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[Lorenz& Worliczek'13, Ivanov'62,'63, Dombrovskaja& Ivanov'65, Seidman& Vogel'89, Ivanov& Vasin& Tanana'02, Neubauer& Ramlau'14]

a counterexample for the equivalence of Tikhonov and Ivanov

very similar to counerexample in [Lorenz& Worliczek'13] in 1-d:
X=Y= $\mathbb{R}$, $\quad f(x) = (x - x_0)^3 + y, \quad y^\delta = y + \delta, \quad r(x) = |x|.$

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$X=Y=\mathbb{R}$, $\quad f(x)=(x-x_0)^3+y$, $\quad y^\delta=y+\delta$, $\quad r(x)=|x|$.
$x^\dagger=x_0$ solves $f(x)=y$
$x^\dagger=x_0$ solves Ivanov $\min_{x\in\mathbb{R}}|f(x)-y^\delta|$ s.t. $r(x)\leq\rho$ with $\rho=r(x^\dagger)$

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However, for any $\alpha > 0$, a Tikonov minimizer
$x_\alpha^\delta \in \mathrm{argmin}\{\frac{1}{2}|f(x) - y^\delta|^2 + \alpha r(x)\}$ differs from x_0:

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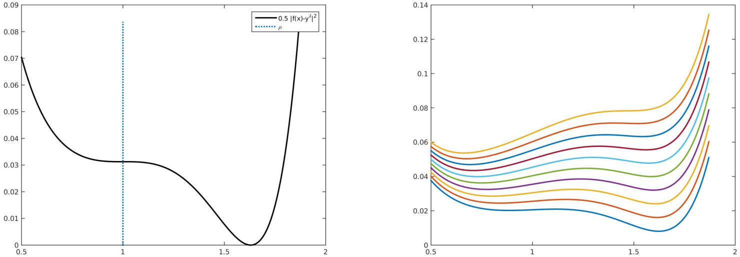

Figure : Ivanov (left) and Tikhonov (right; for different values of α) $\delta = 0.25$

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Can be lifted to an ill-posed function space setting
$X = L^\infty(\Omega)$, $Y = L^2(\Omega)$, $\mathcal{S}(y_1, y_2) = \frac{1}{2}\|y_1 - y_2\|^2_{L^2}$, $\mathcal{R}(x) = \frac{1}{2}\|x\|^2_{L^\infty}$

$$F(x)(t) = \int_\Omega \Phi(t - s) f(x(s))\, ds\,, \quad t \in \Omega$$

with $\Phi : \Omega - \Omega \to [0, \infty)$ with $\int_\Omega \Phi(t - s)\, ds = 1$ for all $t \in \Omega$.
If Φ is weakly singular then F is compact.
If Φ is the Green's function of a differential operator D then
$F(x) = y$ is equivalent to

$$Dy = f(x)$$

(a nonlinear inverse source problem for a PDE)

convergence analysis

Assumptions

Assumption

There exist topologies $\mathcal{T}_X$, $\mathcal{T}_Y$ on X, Y such that

1. *F is $\mathcal{T}_X$-$\mathcal{T}_Y$-sequentially closed*
2. $\mathcal{R}$, $\mathcal{S}(\cdot, y^\delta)$ *are lower semicontinuous wrt.* $\mathcal{T}_X$ *and* $\mathcal{T}_Y$.
3. $\forall\ C > 0$: $M_C^{\mathcal{R}} = \{x \in \mathcal{D}(F) \,:\, \mathcal{R}(x) \leq C\}$ *is* $\mathcal{T}_X$*-compact.*
4. $\forall\ C > 0$, $y^\delta \in Y$: $M_C^{\mathcal{S}} = \{y \in F(\mathcal{D}(F)) \,:\, \mathcal{S}(y, y^\delta) \leq C\}$ $\mathcal{T}_Y$*-comp.*
5. *There exists a solution $x^\dagger$ of $Fx = y$ such that $\mathcal{R}(x^\dagger) < \infty$.*
6. $\delta < \mathcal{S}(F(x_{\rho_0}^\delta), y^\delta)$.
7. $\mathcal{S}(\tilde{y}, y) = 0$ *implies* $\tilde{y} = y$
8. *There exists a constant $C_S > 0$ such that for all $y_1, y_2, y_3 \in Y$*

$$\mathcal{S}(y_1, y_3) \leq C_S(\mathcal{S}(y_1, y_2) + \mathcal{S}(y_3, y_2))$$

Assumptions

. . . like for Tikhonov regularization

Well-posedness

Theorem (Clason& BK& Klassen'16)

Let $y^\delta \in Y$, $\tau \geq 1$, $\delta > 0$ be fixed and let for the two functionals $\mathcal{R} : X \to R_0^+$, $\mathcal{S} : Y \times Y \to R_0^+$, conditions 1.–5. of Assumption 1 hold, with $x^\dagger$ an $\mathcal{R}$-minimizing solution of $Fx = y$.
Then x_{Mo}^δ and x_ρ^δ with $\rho = \rho_*^I$ or with $\rho = \rho_*^{II}$ or, if additionally Assumption 1.6. holds, with $\rho = \rho_*^{III}$, are well defined and

$$\mathcal{R}(\hat{x}^\delta) \leq \mathcal{R}(x^\dagger) \quad \mathcal{S}(F(\hat{x}^\delta), y^\delta) \leq \tau\delta.$$

holds for $\hat{x}^\delta \in \{x_{Mo}^\delta,\ x_{\rho_*^I}^\delta,\ x_{\rho_*^{II}}^\delta,\ x_{\rho_*^{III}}^\delta\}$.

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Then x^δ_{Mo} and x^δ_ρ with $\rho = \rho^I_*$ or with $\rho = \rho^{II}_*$ or, if additionally Assumption 1.6. holds, with $\rho = \rho^{III}_*$, are well defined and

$$\mathcal{R}(\hat{x}^\delta) \leq \mathcal{R}(x^\dagger) \quad \mathcal{S}(F(\hat{x}^\delta), y^\delta) \leq \tau\delta.$$

holds for $\hat{x}^\delta \in \{x^\delta_{Mo},\ x^\delta_{\rho^I_*},\ x^\delta_{\rho^{II}_*},\ x^\delta_{\rho^{III}_*}\}$.
For any $\rho_1, \rho_2 \geq \rho^{II}_*$ and any two minimizers $x^\delta_{\rho_i}$, $i \in \{1, 2\}$

$$\rho_1 \leq \rho_2 \ \Rightarrow \mathcal{S}(F(x^\delta_{\rho_1}), y^\delta) \geq \mathcal{S}(F(x^\delta_{\rho_2}), y^\delta)$$

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$$\rho_1 \leq \rho_2 \ \Rightarrow \mathcal{S}(F(x_{\rho_1}^\delta), y^\delta) \geq \mathcal{S}(F(x_{\rho_2}^\delta), y^\delta)$$

If F is linear, $\mathcal{R} = \|\cdot\|$, and $y^\delta \in \overline{F(X)} \subseteq Y$, then for any $\rho \in (0, \rho(y^\delta) := \inf\left\{\rho > 0 : y^\delta \in Q_\rho\right\} \in [0, \infty])$, any Ivanov minimizer lies on the boundary of the feasible set, i.e., $\|x_\rho^\delta\| = \rho$

Convergence

Theorem (Clason& BK& Klassen'16)

Let $y \in F(\mathcal{D}(F))$ and let $(y^\delta)_{\delta>0}$ be a family of noisy data satisfying $\mathcal{S}(y, y^\delta) \leq \delta$ such that (with $y^0 := y$) for all $\delta \geq 0$, and for two functionals $\mathcal{R} : X \to \bar{R}_0^+$, $\mathcal{S} : Y \times Y \to \bar{R}_0^+$, Assumption 1.1.–1.5. and 1.7., 1.8. hold with $x^\dagger$ an $\mathcal{R}$-minimizing solution of $Fx = y$.
Then we have $\mathcal{T}_X$-subsequential convergence as $\delta \to 0$ to $x^\dagger$ for x_{Mo}^δ and for x_ρ^δ with $\rho \in \{\rho_*^I, \rho_*^{II}, \rho_*^{III}\}$ (imposing Assumption 1.6. for the latter).

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Then we have $\mathcal{T}_X$-subsequential convergence as $\delta \to 0$ to $x^\dagger$ for x^δ_{Mo} and for x^δ_ρ with $\rho \in \{\rho^I_*, \rho^{II}_*, \rho^{III}_*\}$ (imposing Assumption 1.6. for the latter).
Under a variational source condition for $\phi : \mathbb{R}^+ \to \mathbb{R}^+$

$$\forall \tilde{x} \in \left\{ x \in \mathcal{D}(F) : \mathcal{R}(\tilde{x}) \le \mathcal{R}(x^\dagger) \right\} \; : \; E(\tilde{x}, x^\dagger) \le \phi(\mathcal{S}(F(\tilde{x}), F(x^\dagger)))$$

$\hat{x}^\delta \in \{x^\delta_{Mo}, x^\delta_{\rho^I_*}, x^\delta_{\rho^{II}_*}, x^\delta_{\rho^{III}_*}\}$ satisfy the rates

$$E(\hat{x}^\delta, x^\dagger) = O(\phi(\delta))$$

application to inverse problems for PDEs

Parameter Identification in Elliptic PDEs

$$\mathcal{S}(y_1, y_2) = \|y_1 - y_2\|_Y = \|y_1 - y_2\|_{L^p(\Omega)}, \; p \in [1, \infty], \quad \mathcal{R} = \|\cdot\|_X$$

- Identify the diffusion coefficent $a = a(x)$ in

$$-\nabla(a \nabla u) = f \text{ in } \Omega$$

from measurements of u, given f, $u|_{\partial\Omega} = g$. $\quad X = BV(\Omega)$

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- Identify the potential c in
$$-\Delta u + cu = f \text{ in } \Omega$$
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Hence $\hat{a}^\delta, \hat{b}^\delta, \hat{c}^\delta \in \{x^\delta_{Mo}, x^\delta_{\rho^I_*}, x^\delta_{\rho^{II}_*}, x^\delta_{\rho^{III}_*}\}$ are well-defined and converge weakly* in X to $a^\dagger, b^\dagger, c^\dagger$ as $\delta \to 0$.

Inverse Source Problem

Identify the source term $b = b(x)$ in

$$-\Delta u = b \text{ in } \Omega$$

from measurements of u, given $u|_{\partial\Omega} = g$, $\Omega \subseteq \mathbb{R}^d$.

$\mathcal{S}(y_1, y_2) = \|y_1 - y_2\|_{L^p(\Omega)}\,,\ p \in [1, \infty], \quad \mathcal{R}(b) = \|b\|_{L^\infty}$

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Variational source condition

$$\forall \tilde{b} \in \left\{ b \in X : \mathcal{R}(\tilde{b}) \leq \mathcal{R}(b^\dagger) \right\} \ : \ E(\tilde{b}, b^\dagger) \leq \phi(\mathcal{S}(F(\tilde{b}), F(b^\dagger)))$$

holds with

$E(b_1, b_2) = \|b_1 - b_2\|_{W^{-1,p}(\Omega)}$, $\phi(t) = \sqrt{t}$ or

$E(b_1, b_2) = \|b_1 - b_2\|_{BV(\Omega)^*}$, $\phi(t) = t^{(1-\frac{d}{p}-\epsilon)/2}$, $\;p > d$, $\epsilon \in (0, 1 - \frac{d}{p}]$

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from measurements of u, given $u|_{\partial\Omega} = g$, $\Omega \subseteq \mathbb{R}^d$.

$\mathcal{S}(y_1, y_2) = \|y_1 - y_2\|_{L^p(\Omega)}$, $p \in [1, \infty]$, $\quad \mathcal{R}(b) = \|b\|_{L^\infty}$

$\hat{b}^\delta \in \{x^\delta_{Mo}, x^\delta_{\rho^I_*}, x^\delta_{\rho^{II}_*}, x^\delta_{\rho^{III}_*}\}$ converges weakly* in $L^\infty(\Omega)$ to $b^\dagger$ as $\delta \to 0$.

Variational source condition

$$\forall \tilde{b} \in \left\{ b \in X : \mathcal{R}(\tilde{b}) \leq \mathcal{R}(b^\dagger) \right\} \; : \; E(\tilde{b}, b^\dagger) \leq \phi(\mathcal{S}(F(\tilde{b}), F(b^\dagger)))$$

holds with

$E(b_1, b_2) = \|b_1 - b_2\|_{W^{-1,p}(\Omega)}$, $\phi(t) = \sqrt{t}$ or

$E(b_1, b_2) = \|b_1 - b_2\|_{BV(\Omega)^*}$, $\phi(t) = t^{(1-\frac{d}{p}-\epsilon)/2}$, $p > d$, $\epsilon \in (0, 1 - \frac{d}{p}]$

Hence $\hat{b}^\delta \in \{x^\delta_{Mo}, x^\delta_{\rho^I_*}, x^\delta_{\rho^{II}_*}, x^\delta_{\rho^{III}_*}\}$ satisfy the rates

$$\|\hat{b}^\delta - b^\dagger\|_{W^{-1,p}(\Omega)} = O(\sqrt{\delta}), \quad \|\hat{b}^\delta - b^\dagger\|_{BV(\Omega)^*} = O(\delta^{(1-\frac{d}{p}-\epsilon)/2})$$

Numerical Experiments

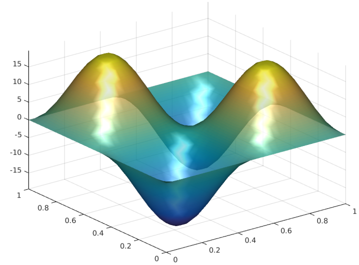
$X = L^\infty(\Omega)$, $Y = L^2(\Omega)$, $g = 0$, $\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ $\rightsquigarrow$
Consider Tikhonov, Morozov, Ivanov as PDE constrained optimization problems with pointwise control bounds:

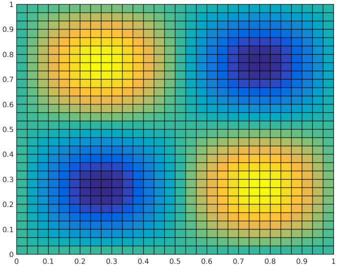
$$\min \tfrac{1}{2}\|u - y^\delta\|^2_{L^2(\Omega)} + \alpha r \quad \text{s.t.} \quad -\Delta u = b\,, \quad |b(x)| \le r,\ x \in \Omega$$

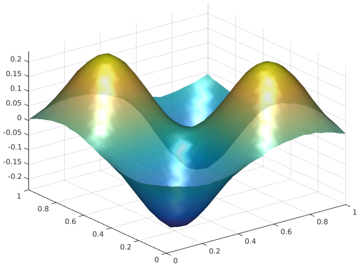
$$\min r \quad \text{s.t.} \quad -\Delta u = b\,, \quad \tfrac{1}{2}\|u - y^\delta\|^2_{L^2(\Omega)} \le \tau\delta\,, \quad |b(x)| \le r,\ x \in \Omega$$

$$\min \tfrac{1}{2}\|u - y^\delta\|^2_{L^2(\Omega)} \quad \text{s.t.} \quad -\Delta u = b\,, \quad |b(x)| \le \rho,\ x \in \Omega$$

Numerical Experiments: Smooth Source



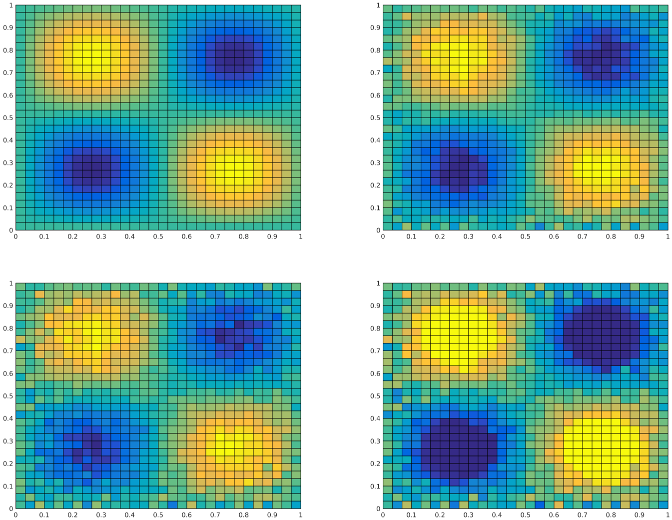




Numerical Experiments: Smooth Source; $\delta = 1\%$

exact

$\rho = \|x^\dagger\|$

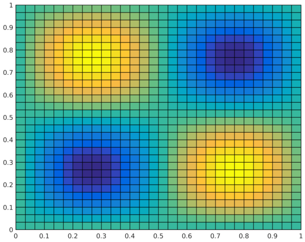


$\rho = 1.1\|x^\dagger\|$

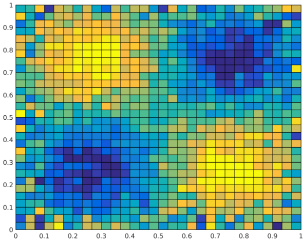
ρ by discr.princ. $< \|x^\dagger\|$

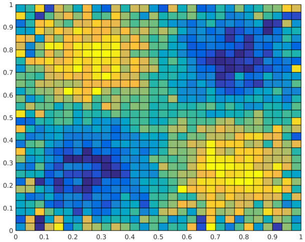
Numerical Experiments: Smooth Source; $\delta = 3\%$

exact

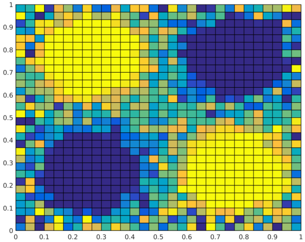


$\rho = \|x^\dagger\|$





$\rho = 1.1\|x^\dagger\|$

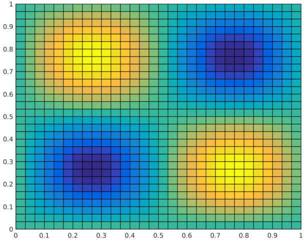


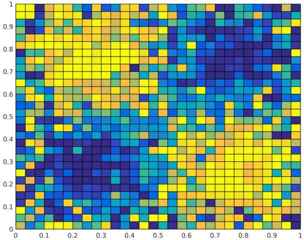
ρ by discr.princ. $< \|x^\dagger\|$

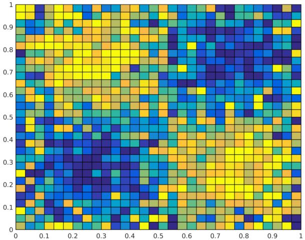
Numerical Experiments: Smooth Source; $\delta = 10\%$

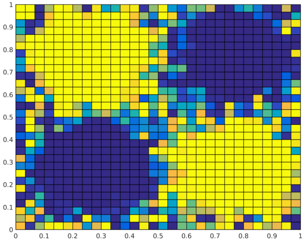
exact

$\rho = \|x^\dagger\|$





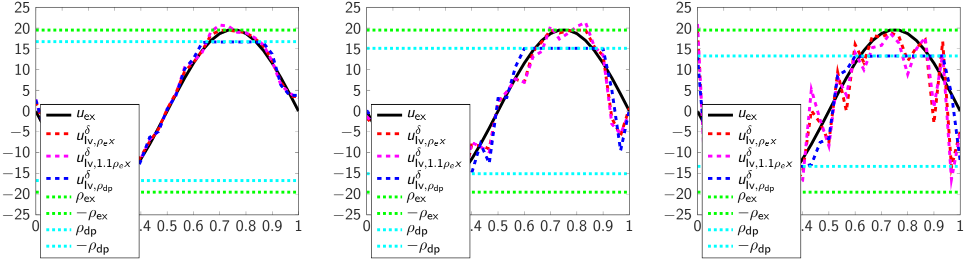




$\rho = 1.1\|x^\dagger\|$

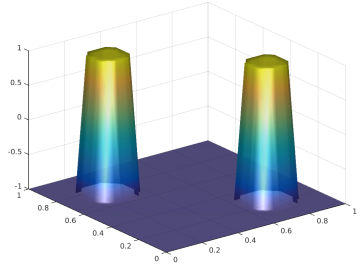
ρ by discr.princ. $< \|x^\dagger\|$

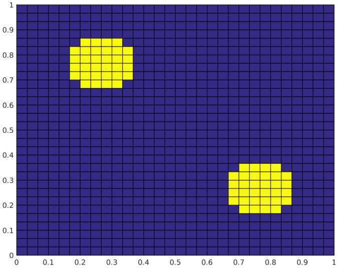
Numerical Experiments: Smooth Source; Cross Sections

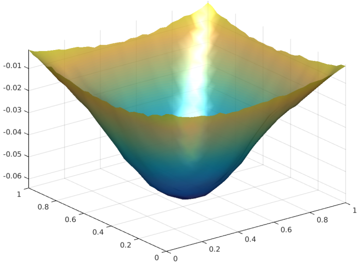


$\delta = 1\%$ $\delta = 3\%$ $\delta = 10\%$

Numerical Experiments: Bang-Bang Source

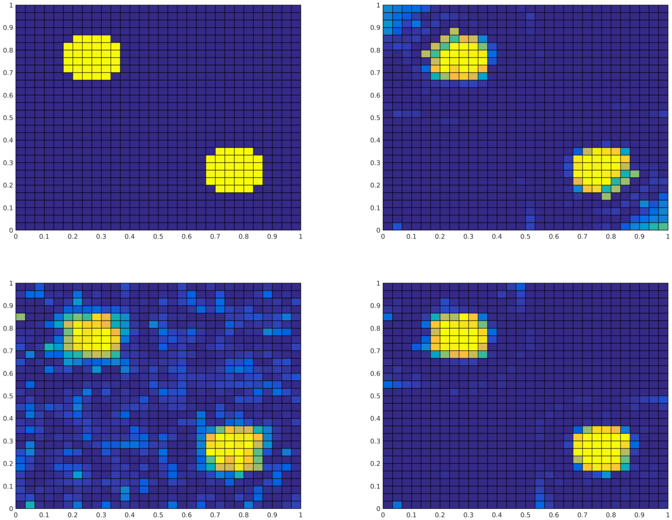






Numerical Experiments: Bang-Bang Source; $\delta = 1\%$

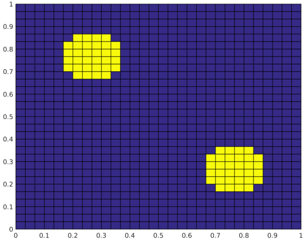
exact $\qquad\qquad$ $\rho = \|x^\dagger\|$



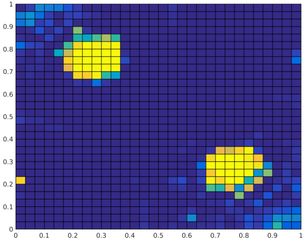
$\rho = 1.1\|x^\dagger\|$ $\qquad\qquad$ ρ by discr.princ. $< \|x^\dagger\|$

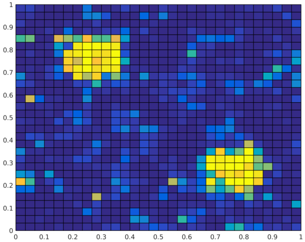
Numerical Experiments: Bang-Bang Source; $\delta = 2\%$

exact

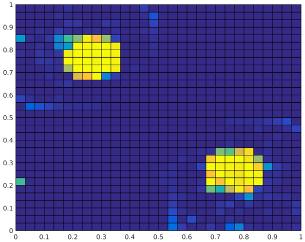


$\rho = \|x^\dagger\|$





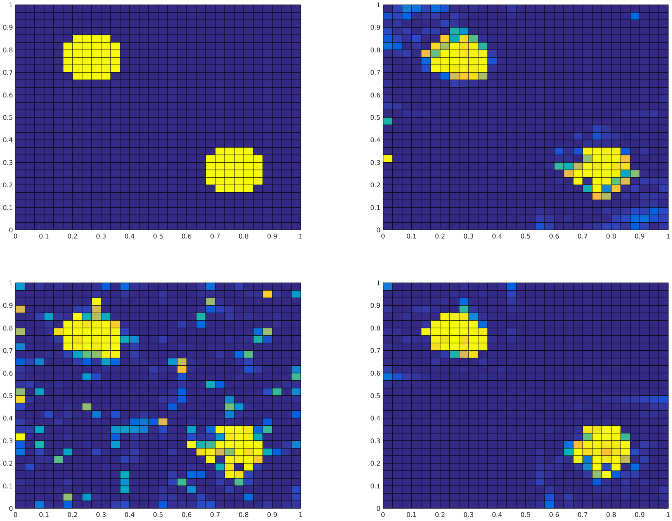
$\rho = 1.1\|x^\dagger\|$



ρ by discr.princ. $< \|x^\dagger\|$

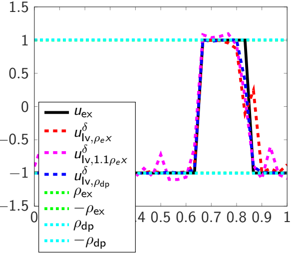
Numerical Experiments: Bang-Bang Source; $\delta = 4\%$

exact $\rho = \|x^\dagger\|$

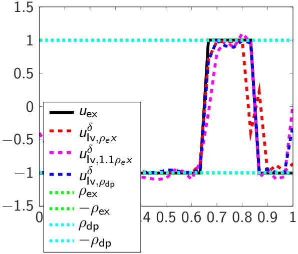


$\rho = 1.1\|x^\dagger\|$ ρ by discr.princ. $< \|x^\dagger\|$

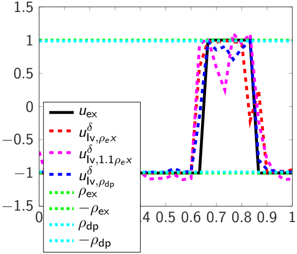
Numerical Experiments: Bang-Bang Source; Cross Sections



$\delta = 1\%$



$\delta = 2\%$



$\delta = 4\%$

Conclusions

- Ivanov regularization allows to naturally incorporate information on $\rho^I_* = \mathcal{R}(x^\dagger)$
- If ρ^I_* is known then optimal convergence rates are obtained without needing knowledge on δ and ϕ!
- underestimation of ρ^I_* still yields a stable solution, overestimation might be fatal

Conclusions

- Ivanov regularization allows to naturally incorporate information on $\rho_*^I = \mathcal{R}(x^\dagger)$
- If ρ_*^I is known then optimal convergence rates are obtained without needing knowledge on δ and ϕ!
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Thank you for your attention!

ifip tc7.2+4 workshop

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Offen im Denken

Optimal Control Meets Inverse Problems

University of Duisburg-Essen

September 5-9, 2016

Organizers:

Christian Clason
Arnd Rösch
Irwin Yousept

https://www.uni-due.de/mathematik/agclason/ifip2016