An inverse Dirichlet-to-Neumann problem with applications to acoustic tomography of moving fluid

Alexey Agaltsov
agaltsov⊗cmap.polytechnique.fr

Universität Bremen
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The differential operator

- We consider the following operator with smooth coefficients:

$$L_{A,V} = -\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j} + iA_j(x)\right)^2 + Q(x), \qquad \text{(OP)}$$

where $x = (x_1, \dots, x_d) \in D$,

$$A = (A_1, \dots, A_d), \quad A_j(x) \in M_n(\mathbb{C}), \quad Q(x) \in M_n(\mathbb{C}),$$
$$D \textit{ is an open bounded domain in } \mathbb{R}^d \textit{ with boundary } \partial D$$

- $L_{A,Q}$ acts on $\mathbb{C}^n$-valued functions in D

The Dirichlet-to-Neumann map

$$L_{A,Q} = -\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j} + iA_j(x)\right)^2 + Q(x), \qquad \text{(OP)}$$

- Suppose that $E \in \mathbb{C}$ is not a DE for $L_{A,Q}$ in D:

$$\begin{cases} L_{A,Q}\psi = E\psi & \text{in } D, \\ \psi|_{\partial D} = f, \end{cases}$$

  is uniquely solvable for any sufficiently regular f on ∂D.

- The Dirichlet-to-Neumann map $\Lambda_{A,Q} = \Lambda_{A,Q}(E)$:

$$\Lambda_{A,Q} f = \sum\nolimits_{j=1}^{d} \nu_j\left(\tfrac{\partial}{\partial x_j} + iA_j\right)\psi\big|_{\partial D}, \qquad \text{(DN)}$$

  where $\nu = (\nu_1, \ldots, \nu_d)$ is the unit exterior normal to ∂D.

Gauge invariance of the DN map

$$L_{A,Q} = -\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j} + iA_j(x)\right)^2 + Q(x), \qquad \text{(OP)}$$

$$\Lambda_{A,Q} f = \sum\nolimits_{j=1}^{d} \nu_j \left(\tfrac{\partial}{\partial x_j} + iA_j\right)\psi\big|_{\partial D}, \qquad \text{(DN)}$$

- Conjugation of $L_{A,Q}$ by a smooth $GL_n(\mathbb{C})$-valued function g:

$$\begin{cases} gL_{A,Q}g^{-1} = L_{A^g,Q^g}, \\ A_j^g = gA_jg^{-1} + i\frac{\partial g}{\partial x_j}g^{-1}, \quad j = 1,\dots,d, \\ Q^g = gQg^{-1}. \end{cases} \qquad \text{(GT)}$$

- The following formula holds:

$$\Lambda_{A^g,Q^g} = g|_{\partial D}\Lambda_{A,Q}(g|_{\partial D})^{-1}.$$

The IDN problem

$$L_{A,Q} = -\sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j} + iA_j\right)^2 + Q, \qquad \text{(OP)}$$

$$\Lambda_{A,Q}(\psi|_{\partial D}) = \sum_{j=1}^{d} \nu_j \left(\frac{\partial}{\partial x_j} + iA_j\right)\psi\Big|_{\partial D}, \; L_{A,Q}\psi = E\psi, \qquad \text{(DN)}$$

$$\begin{cases} gL_{A,Q}g^{-1} = L_{A^g,Q^g}, \\ \Lambda_{A^g,Q^g} = \Lambda_{A,Q}, \\ g \text{ is smooth } GL_n(\mathbb{C})\text{-valued, } g|_{\partial D} = \mathrm{Id} \end{cases} \qquad \text{(GT)}$$

The inverse Dirichlet-to-Neumann problem

Given $\Lambda_{A,Q}$ at fixed E, find $L_{A,Q}$ modulo (GT).

$$\boxed{\text{D-to-N map}} \xrightarrow{\text{inverse Dirichlet-to-Neumann problem}} \boxed{\begin{array}{c}\text{conjugacy class}\\ \text{of Schrödinger}\\ \text{operators}\end{array}}$$

The IDN problem: scalar case

- A_j, Q are scalar functions, $d \in \{2,3\}$, $A = (A_1, \dots, A_d)$

$$L_{A,Q} = -(\nabla + iA)^2 + Q, \qquad \text{(OP)}$$

$$\Lambda_{A,Q} f = (\nu \cdot (\nabla + iA))\psi\big|_{\partial D}, \qquad \text{(DN)}$$

$$\begin{cases} e^{i\varphi} L_{A,Q} e^{-i\varphi} = L_{A^\varphi, Q^\varphi}, \\ A^\varphi = A + \nabla\varphi, \\ Q^\varphi = Q \end{cases} \qquad \text{(GT)}$$

- $F = \operatorname{curl} A$ and Q are gauge invariant and are uniquely determined by $\Lambda_{A,V}(E)$, see [9] ($d \geq 3$) and [8] ($d = 2$)
- $(A - (A \cdot \nu)\nu)|_{\partial D}$ is uniquely determined by $\Lambda_{A,V}(E)$, see [5]
- If A, V are real-valued and D is simply connected, $\Lambda_{A,V}(E)$ uniquely determines A and V, see [2]

Acoustic scattering: main equation

$$L_{A,Q} = -(\nabla + iA)^2 + Q \tag{OP}$$

- Time-harmonic acoustic pressure ψ in moving fluid with sound speed $c = c(x)$, density $\rho = \rho(x)$, velocity $\mathbf{v} = \mathbf{v}(x)$ and absorption coefficient $\alpha = \omega^{\zeta(x)}\alpha_0(x)$ at fixed frequency ω satisfies $L_\omega\psi = 0$, where

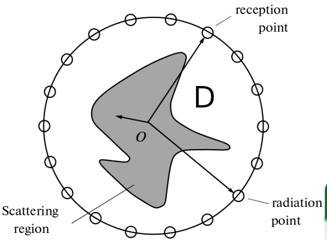
$$L_\omega = -\Delta - 2i\left(\frac{\omega\mathbf{v}}{c^2} + \frac{i}{2}\nabla\ln\rho\right)\cdot\nabla - \frac{\omega^2}{c^2} - 2i\omega\frac{\alpha}{c} \tag{AC}$$

- For simplicity, $\mathbf{v}$, $c - c_0$, $\nabla\rho$ and α are supported in D, where c_0 is the background sound speed. The DN map:

$$\Lambda_\omega(\psi|_{\partial D}) = \frac{\partial\psi}{\partial\nu}, \quad L_\omega\psi = 0.$$

Acoustic scattering: measurable data

$$L_\omega = -\Delta - 2i\left(\frac{\omega \mathbf{v}}{c^2} + \frac{i}{2}\nabla \ln \rho\right)\cdot\nabla - \frac{\omega^2}{c^2} - 2i\omega\frac{\alpha}{c} \qquad \text{(AC)}$$



(illustration from [6])

Data from point sources: $G_\omega|_{X\times Y}$, $\omega \in \Omega$, where X, $Y \subset \partial D$, $\Omega \subset \mathbb{R}_+$,

$$\begin{cases} L_\omega G_\omega(x,y) = -\delta_y(x), & x \in \mathbb{R}^d, \\ G_\omega \text{ satisfies radiation condition,} \end{cases}$$

Acoustic tomography problem

Given $G_\omega|_{X\times Y}$ for $\omega \in \Omega$ and c_0, find c, $\mathbf{v}$, $\nabla\rho$ and α in D

data from point sources $\Longrightarrow$ acoustic tomography problem $\Longrightarrow$ fluid parameters

Acoustic scattering: reduction to the IDN problem

Using the second Green formula, one can obtain (see [6]):

$$G_\omega(x,y) - G^0_\omega(x,y) = \int\limits_{\partial D}\int\limits_{\partial D} G^0_\omega(x,z)(\Lambda_\omega - \Lambda^0_\omega)(z,w)G_\omega(w,y)dy\,dw$$

where G^0_ω, Λ^0_ω correspond to $\mathbf{v}=0$, $\nabla\rho=0$, $c=c_0$, $\alpha=0$.

$$\begin{tikzcd}
\boxed{\text{data from point sources}} \arrow[rr, Rightarrow, "\text{acoustic tomography problem}"] \arrow[d, "\text{second Green formula}"] & & \boxed{\text{fluid parameters}} \\
\boxed{\text{D-to-N map(s)}} & &
\end{tikzcd}$$

- A_j, V are $M_n(\mathbb{C})$-valued functions in D

$$L_{A,Q} = -\sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j} + iA_j(x)\right)^2 + Q(x), \qquad \text{(OP)}$$

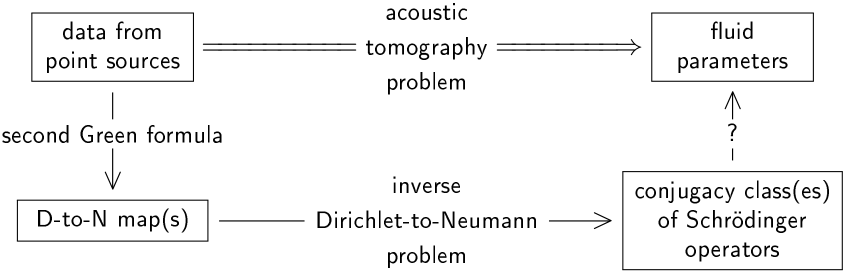
$$\Lambda_{A,Q}(\psi|_{\partial D}) = \sum_{j=1}^{d} \nu_j \left(\frac{\partial}{\partial x_j} + iA_j\right)\psi\Big|_{\partial D}, \ L_{A,Q}\psi = E\psi, \qquad \text{(DN)}$$

$$\begin{cases} gL_{A,Q}g^{-1} = L_{A^g,Q^g}, \\ \Lambda_{A^g,Q^g} = \Lambda_{A,Q}, \\ g \text{ is smooth } GL_n(\mathbb{C})\text{-valued, } g|_{\partial D} = \mathrm{Id} \end{cases} \qquad \text{(GT)}$$

The inverse Dirichlet-to-Neumann problem

Given $\Lambda_{A,Q}$ at fixed E, find $L_{A,V}$ modulo (GT).

- *Question.* Suppose that we know how to solve the IDN problem. How to complete the following diagram?



Gauge fixing

$$L_\omega = -\Delta - 2i\left(\frac{\omega \mathbf{v}}{c^2} + \frac{i}{2}\nabla \ln \rho\right)\cdot\nabla - \frac{\omega^2}{c^2} - 2i\omega\frac{\alpha}{c} \qquad \text{(AC)}$$

- functions F and q_ω are invariants of the conjugacy class:

$$F = \operatorname{curl}\tfrac{\mathbf{v}}{c^2},$$
$$q_\omega = f_1 - \omega^2 f_2 + i\omega f_3 - 2i\omega^{1+\zeta}\alpha_0,$$
$$f_1 = \rho^{\frac{1}{2}}\Delta\rho^{-\frac{1}{2}}, \quad f_2 = \tfrac{1}{c^2} + \tfrac{\mathbf{v}}{c^2}\tfrac{\mathbf{v}}{c^2}, \quad f_3 = \nabla\cdot\left(\tfrac{\mathbf{v}}{c^2}\right) - \tfrac{\mathbf{v}}{c^2}\cdot\nabla\ln\rho$$

- The fluid parameters can be recovered as follows:

$$\begin{array}{ccccc}
\boxed{\Lambda_{\omega_1},\ldots,\Lambda_{\omega_N}} & \longrightarrow & \boxed{\begin{array}{c}\text{conjugacy classes}\\ \text{of } L_{\omega_1},\ldots,L_{\omega_N}\end{array}} & \longrightarrow & \boxed{F,\ q_{\omega_1},\ \ldots,\ q_{\omega_N}} \\
\Downarrow & & & & \downarrow \\
\boxed{\mathbf{v},\ c,\ \rho,\ \zeta,\ \alpha_0} & \longleftarrow & & & \boxed{F,\ f_1,\ f_2,\ f_3,\ \zeta,\ \alpha_0}
\end{array}$$

Gauge fixing: an example

Example. Consider the case $\alpha_0 \equiv 0$.

- Λ_ω at fixed ω uniquely determines F and q_ω, where

$$F = \operatorname{curl} \frac{\mathbf{v}}{c^2},$$
$$q_\omega = f_1 - \omega^2 f_2 + i\omega f_3,$$
$$f_1 = \rho^{\frac{1}{2}} \Delta \rho^{-\frac{1}{2}}, \quad f_2 = \frac{1}{c^2} + \frac{\mathbf{v}}{c^2}\frac{\mathbf{v}}{c^2}, \quad f_3 = \nabla \cdot \left(\frac{\mathbf{v}}{c^2}\right) - \frac{\mathbf{v}}{c^2} \cdot \nabla \ln \rho$$

Finding parameters

1. Find f_1, f_2, f_3, separating real and imaginary parts of q_ω and solving a linear system (requires two frequencies)
2. Find ρ from f_1
3. Find $\frac{\mathbf{v}}{c^2}$ from $\operatorname{curl} \frac{\mathbf{v}}{c^2}$, ρ and f_3
4. Find c and $\mathbf{v}$ from $\frac{\mathbf{v}}{c^2}$ and f_2

Gauge fixing: summary

$$L_\omega = -\Delta - 2i\left(\frac{\omega \mathbf{v}}{c^2} + \frac{i}{2}\nabla \ln \rho\right)\cdot\nabla - \frac{\omega^2}{c^2} - 2i\omega^{1+\zeta}\frac{\alpha_0}{c}, \qquad \text{(AC)}$$

$$\Lambda_\omega(\psi|_{\partial D}) = \frac{\partial\psi}{\partial\nu}\Big|_{\partial D}, \quad L_\omega\psi = 0.$$

- $\rho \equiv \rho_0,\ \alpha_0 \equiv 0 \implies \Lambda_\omega$ at fixed ω determines $\mathbf{v}$, c
- $\alpha_0 \equiv 0 \implies \Lambda_\omega$ at 2 ω's determines $\mathbf{v}$, c, ρ
- $\zeta \neq 0 \implies \Lambda_\omega$ at 3 ω's determines $\mathbf{v}$, c, ρ, ζ, α_0
- There is an explicit example of non-uniquenes when $\zeta \equiv 0$

For more general results of these types see [2] and [4]

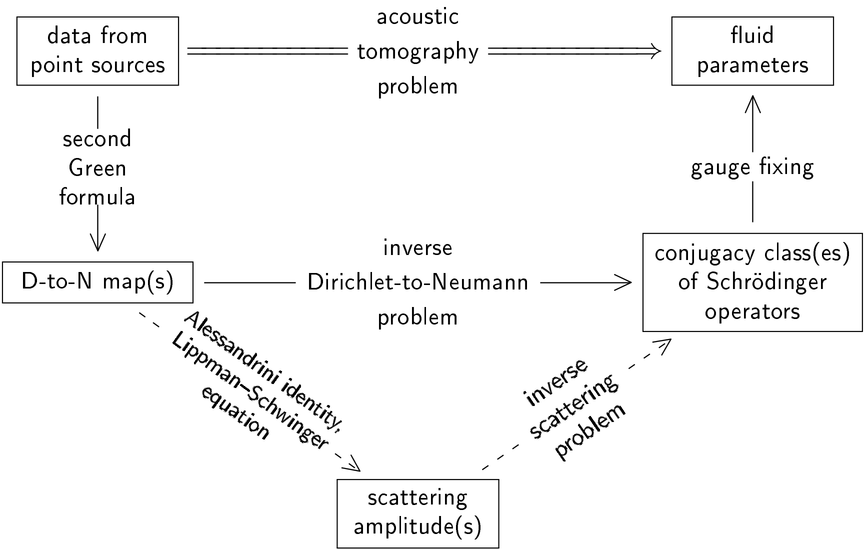
Solving the IDN problem

- So far we have the following scheme with vertical arrows being explicit algorithms:

$$\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\text{data from point sources}} \arrow[rr, Rightarrow, "\text{acoustic tomography problem}"] \arrow[d, "\text{second Green formula}"] & & \boxed{\text{fluid parameters}} \\
\boxed{\text{D-to-N map(s)}} \arrow[rr, "\text{inverse Dirichlet-to-Neumann problem}"] & & \boxed{\text{conjugacy class(es) of Schrödinger operators}} \arrow[u, "\text{gauge fixing}"']
\end{tikzcd}$$

- *Question.* How to solve constructively the inverse Dirichlet-to-Neumann problem?

Solving the IDN problem



Direct and inverse scattering

$$L_{A,Q} = -\sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j} + iA_j\right)^2 + Q, \tag{OP}$$

$$A_j,\ Q \text{ are smooth } M_n(\mathbb{C})\text{-valued in } D$$

- Set A, Q equal to zero outside of D
- Consider functions $\psi^+(\cdot, k)$, $k \in S^1_{\sqrt{E}} = \{\kappa \in \mathbb{R}^d \mid \kappa^2 = E\}$:

$$\begin{cases} L_{A,Q}\psi^+(x,k) = E\psi^+(x,k), \quad x \in \mathbb{R}^d, \\ \psi^+(x,k) = e^{ikx}\mathrm{Id}_n + \psi^+_{\mathrm{sc}}(x,k), \\ \left(\frac{\partial}{\partial r} - i\sqrt{E}\right)\psi^+_{\mathrm{sc}}(x,k) = o(r^{-(d-1)/2}), \quad r = |x| \to \infty. \end{cases}$$

- The scattering amplitude $f_{A,Q}$ on $\mathcal{M}_E = S^1_{\sqrt{E}} \times S^1_{\sqrt{E}}$:

$$\psi^+_{\mathrm{sc}}(x,k) = C(d)\frac{|k|^{\frac{d-3}{2}} e^{i|x||k|}}{|x|^{\frac{(d-1)}{2}}} f_{A,Q}\left(k, \frac{|k|}{|x|}x\right)\left(1 + o(1)\right), \quad |x| \to \infty.$$

Direct scattering problem

$$L_{A,Q} = -\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j} + iA_j\right)^2 + Q \qquad \text{(OP)}$$

Direct scattering problem

Given $L_{A,Q}$, find $f_{A,Q}$.

- $\psi^+(\cdot, k)$ satisfies the Lippmann-Schwinger equation:

$$\psi^+(x,k) = e^{ikx}\mathrm{Id}_n + \int_D G^+(x-y,k)\big(L_{A,Q} - L_{0,0}\big)\psi^+(y,k)\,dy, \quad \text{(LS)}$$

$$G^+(x,k) = -(2\pi)^{-d}\int_{\mathbb{R}^d}\frac{e^{i\xi x}d\xi}{\xi^2 - k^2 - i0} \text{“}\simeq\text{”} \frac{1}{E - L_{0,0}}$$

- The scattering amplitude $f_{A,Q}$ can be found from:

$$f_{A,Q}(k,l) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-ilx}\big(L_{A,Q} - L_{0,0}\big)\psi^+(x,k)\,dx \quad \text{(SA)}$$

Gauge invariance

$$L_{A,Q} = -\sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j} + iA_j\right)^2 + Q, \qquad \text{(OP)}$$

$$f_{A,Q}(k,l) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ilx} (L_{A,Q} - L_{0,0}) \psi^+(x,k)\, dx \qquad \text{(SA)}$$

- Conjugation of $L_{A,Q}$ by a $GL_n(\mathbb{C})$-valued function g:

$$\begin{cases} gL_{A,Q}g^{-1} = L_{A^g,Q^g}, \\ A_j^g = gA_jg^{-1} + i\frac{\partial g}{\partial x_j}g^{-1}, \quad j = 1,\dots,d, \\ Q^g = gQg^{-1} \end{cases} \qquad \text{(GT)}$$

- The amplitude is gauge invariant:

$$f_{A^g,V^g}(k,l) = f_{A,V}(k,l),$$

if $g \to \mathrm{Id}$ at ∞ sufficiently fast

The inverse scattering problem

$$L_{A,Q} = -\sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j} + iA_j\right)^2 + Q, \tag{OP}$$

$$f_{A,Q}(k,l) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ilx}(L_{A,Q} - L_{0,0})\psi^+(x,k)\,dx \tag{SA}$$

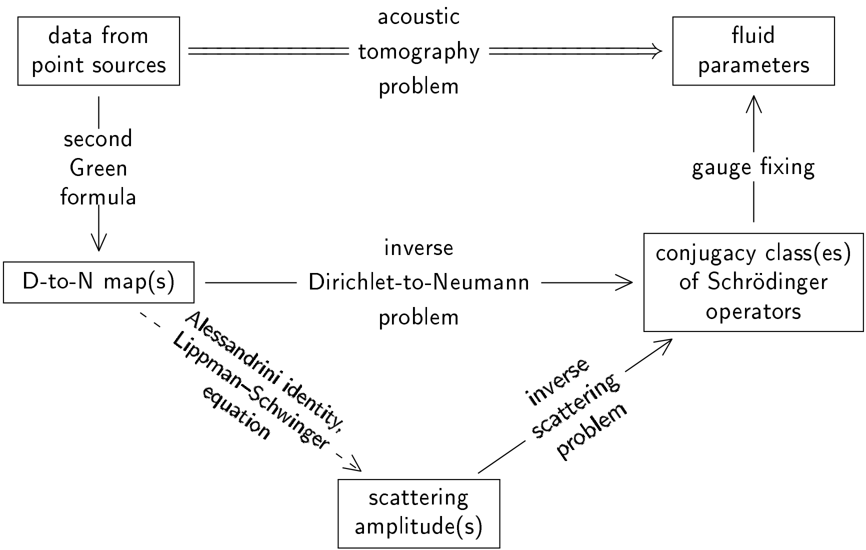
$$\begin{cases} gL_{A,Q}g^{-1} = L_{A^g,Q^g}, \\ f_{A^g,Q^g} = f_{A,Q}, \\ g \text{ is smooth } GL_n(\mathbb{C})\text{-valued}, \\ g \to \mathrm{Id} \text{ at } \infty \text{ sufficiently fast} \end{cases} \tag{GT}$$

The inverse scattering problem

Given $f_{A,Q}$ at fixed E, find $L_{A,Q}$ modulo (GT).

- In scalar case an explicit algorithm for solving the ISP was proposed in [3]

Solving the IDN problem



From D-to-N map to scattering amplitude

$$L_{A,Q} = -\sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j} + iA_j\right)^2 + Q, \qquad \text{(OP)}$$

$$\Lambda_{A,Q}(\psi|_{\partial D}) = \sum_{j=1}^{d} \nu_j\left(\frac{\partial}{\partial x_j} + iA_j\right)\psi\Big|_{\partial D},\ L_{A,Q}\psi = E\psi,$$

$$A_j,\ Q \text{ are } M_n(\mathbb{C})\text{-valued with compact support in } D \qquad \text{(DN)}$$

- *Alessandrini identity.* Let u_0 satisfy $L_{0,0}u_0 = Eu_0$ and u satisfy $L_{A,Q}u = Eu$ in D. Then the following formula holds:

$$\int_D u_0(x)\big(L_{A,Q} - L_{0,0}\big)u(x)dx = \int_{\partial D} u_0(x)\big(\Lambda_{A,Q} - \Lambda_{0,0}\big)u(x)dx \qquad \text{(AI)}$$

From D-to-N map to scattering amplitude

$$\psi^+(x,k) = e^{ikx} + \int_D G^+(x-y,k)\big(L_{A,Q} - L_{0,0}\big)\psi^+(y,k)\,dy \tag{LS}$$

$$f_{A,Q}(k,l) = (2\pi)^{-d}\int_D e^{-ilx}(L_{A,Q} - L_{0,0})\psi^+(x,k)\,dx \tag{SA}$$

$$\int_D u_0(x)\big(L_{A,Q} - L_{0,0}\big)u(x)dx = \int_{\partial D} u_0(x)\big(\Lambda_{A,Q} - \Lambda_{0,0}\big)u(x)dx \tag{AI}$$

- (LS) + (AI) imply

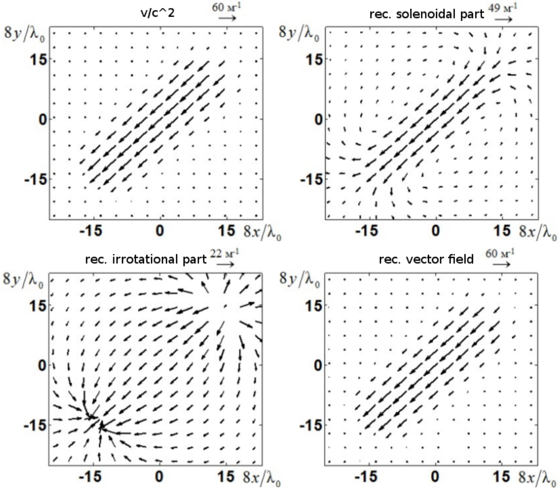
$$\psi^+(x,k) = e^{ikx} + \int_{\partial D} G^+(x-y,k)\big(\Lambda_{A,Q} - \Lambda_{0,0}\big)\psi^+(x,k)\,dx$$

- (SA) + (AI) imply

$$f_{A,Q}(k,l) = (2\pi)^{-d}\int_{\partial D} e^{-ilx}\big(\Lambda_{A,Q} - \Lambda_{0,0}\big)\psi^+(x,k)\,dx$$

- For more general case of non-zero background potentials see [1]

$$L_\omega = -\Delta - 2i\omega\frac{\mathbf{v}}{c^2}\cdot\nabla - \frac{\omega^2}{c^2} - 2i\omega^2\frac{\alpha_0}{c}, \quad \omega\in\{\omega_1,\omega_2\}$$



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