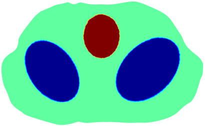
Direct reconstructions from partial-boundary data in electrical impedance tomography

Andreas Hauptmann
joint with Matteo Santacesaria and Samuli Siltanen



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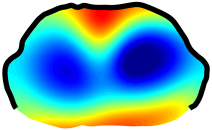

Inverse Problems for PDEs
University of Bremen
March 30, 2016

Outline

Electrical impedance tomography (EIT)

The D-bar algorithm

The partial-boundary problem

Error analysis

Computational verification

Reconstructions

Conclusions

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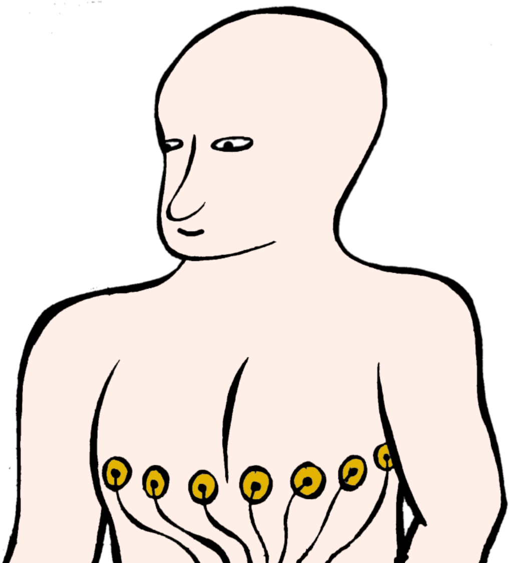
Electrical impedance tomography (EIT) is an emerging medical imaging technique

Feed electric currents through electrodes. **Measure** the resulting voltages. Repeat the measurement for several current patterns.

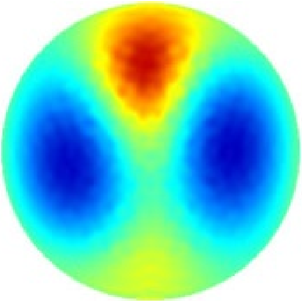
Reconstruct distribution of electric conductivity inside the patient. Different tissues have different conductivities, so EIT gives an image of the patient's inner structure.

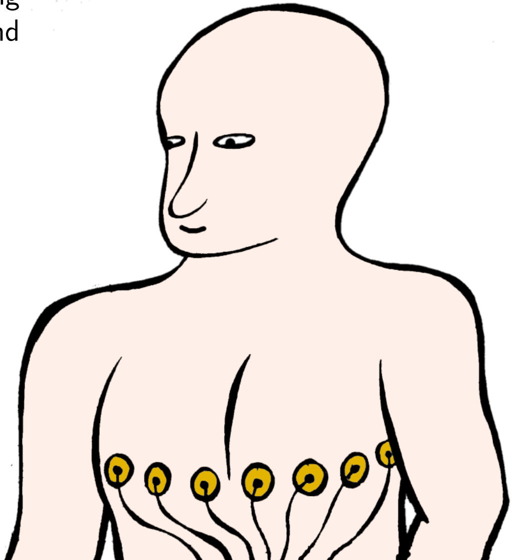
EIT is a harmless and painless imaging method suitable for long-term monitoring.



This talk concentrates on applications of EIT to chest imaging

Medical applications: monitoring cardiac activity, lung function, and pulmonary perfusion.





The mathematical model of EIT is the inverse conductivity problem introduced by Calderón

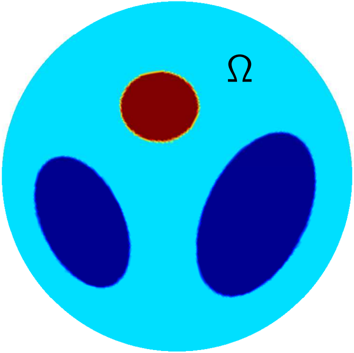
Let $\Omega \subset \mathbb{R}^2$ be the unit disc and $\sigma : \Omega \to \mathbb{R}$ the conductivity with $0 < c \leq \sigma(z) \leq C$ for $z \in \Omega$.

Injecting current φ at the boundary $\partial\Omega$ leads to the elliptic PDE

$$\begin{cases} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\ \sigma \frac{\partial u}{\partial \nu} &= \varphi \text{ on } \partial\Omega. \end{cases}$$

For uniqueness we require $\int_{\partial\Omega} \varphi = 0$ and $\int_{\partial\Omega} u = 0$.
Boundary measurements are modelled by the Neumann-to-Dirichlet map

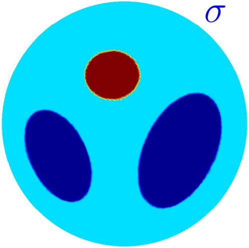
$$\mathcal{R}_\sigma : \varphi \mapsto u|_{\partial\Omega}.$$



Calderón's problem is to recover σ from the knowledge of $\mathcal{R}_\sigma$. It is a nonlinear and ill-posed inverse problem.

We illustrate the measurement process in EIT

Given the phantom



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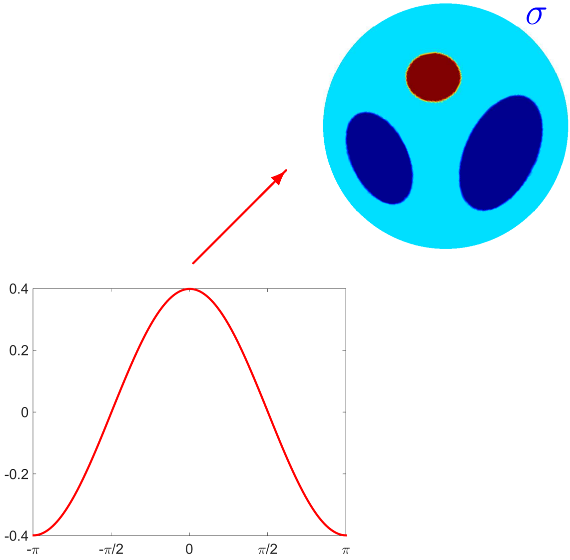
We apply currents on the boundary



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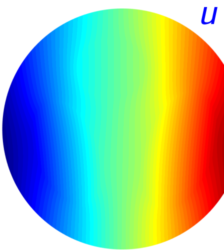
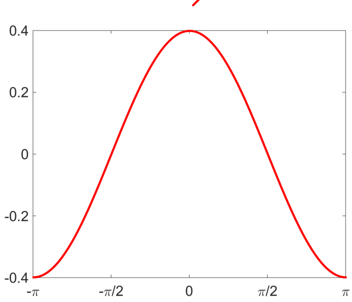
Creating the voltage distribution





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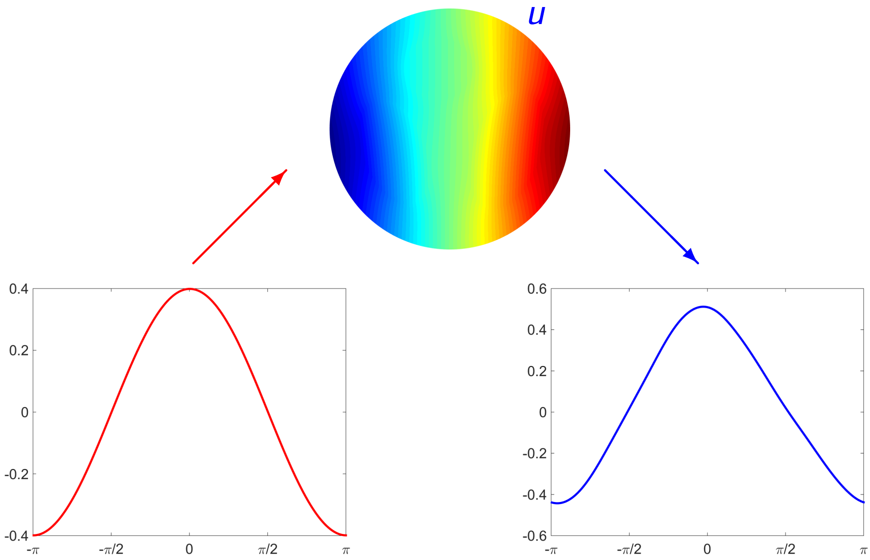
We can measure the boundary trace



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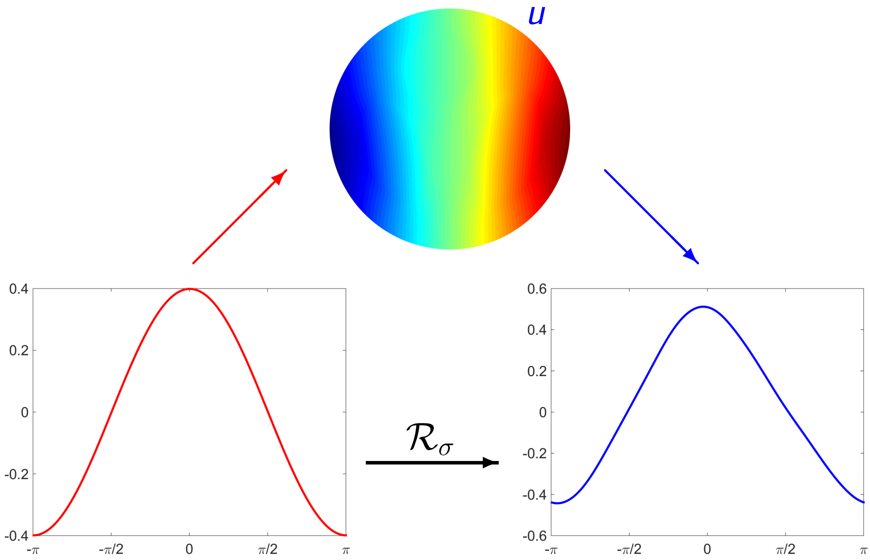
This is the current-to-voltage measurement



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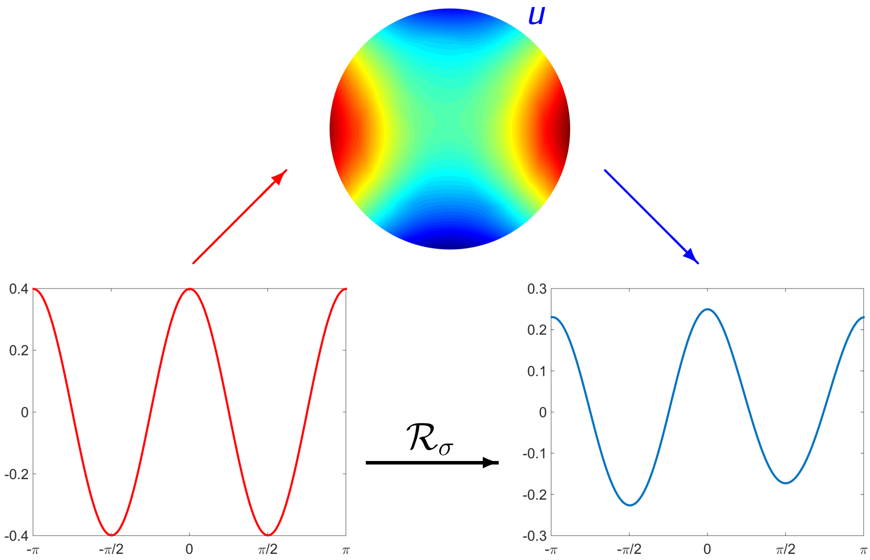
We do this for different current patterns



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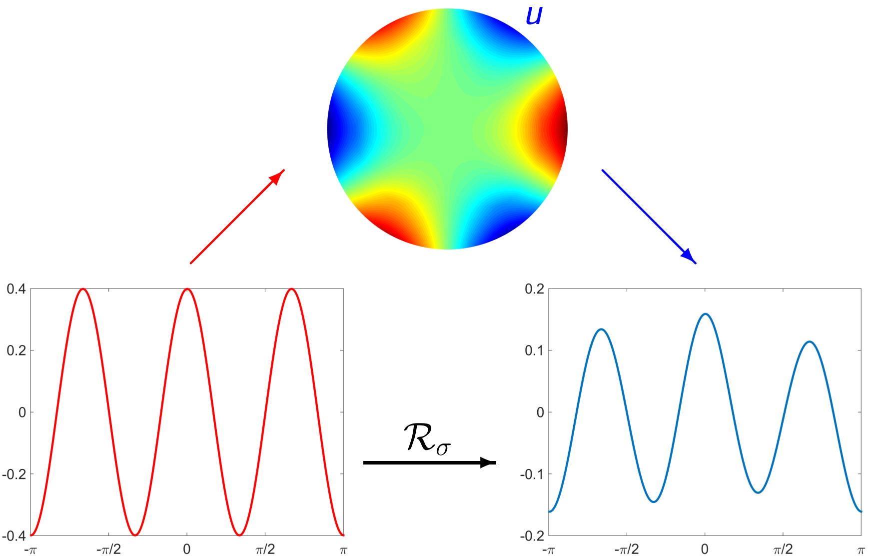

Representing the measurement

- For a fixed conductivity σ, the Neumann-to-Dirichlet map (ND map), also know as current-to-voltage map, is the linear operator that maps every possible Neumann data to the corresponding Dirichlet data.

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- We measure the ND map with respect to an orthonormal basis φ_n (the applied current), and obtain a matrix approximation $\mathbf{R}_\sigma$ by the inner product

$$(\mathbf{R}_\sigma)_{i,j} = (\mathcal{R}_\sigma \varphi_i, \varphi_j) = (u_i|_{\partial\Omega}, \varphi_j).$$

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A (very) brief history of the two-dimensional D-bar method for EIT important for our work

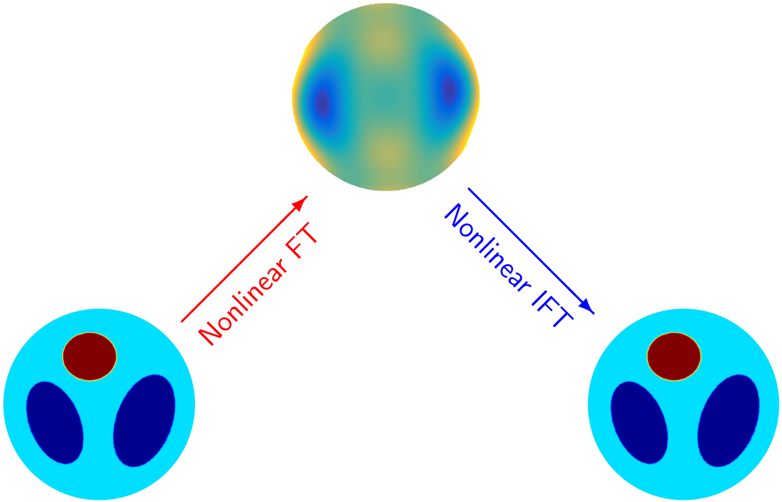
1980 Calderón: Introduces the inverse conductivity problem

1988 Novikov & **1996 Nachman**: Uniqueness and reconstruction for 2D EIT with C^2 conductivities and infinite-precision data

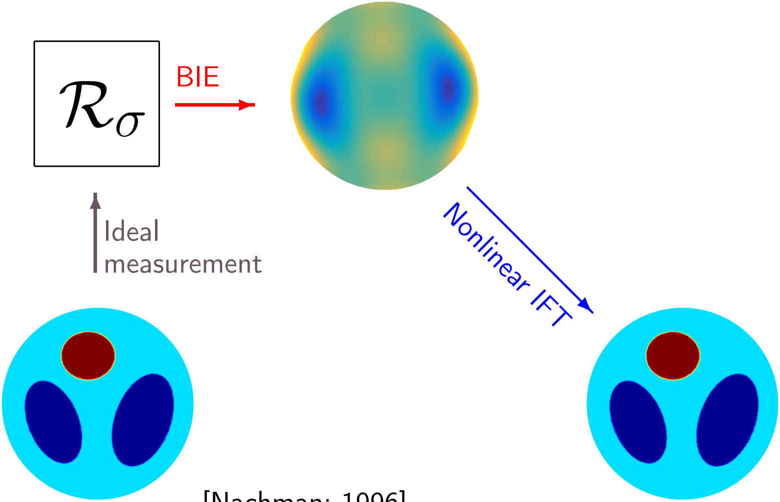
2000 Siltanen, Mueller and Isaacson: Numerical implementation of Nachman's method

2009 Knudsen, Lassas, Mueller and Siltanen: Regularized EIT

There exists a nonlinear Fourier transform adapted to electrical impedance tomography

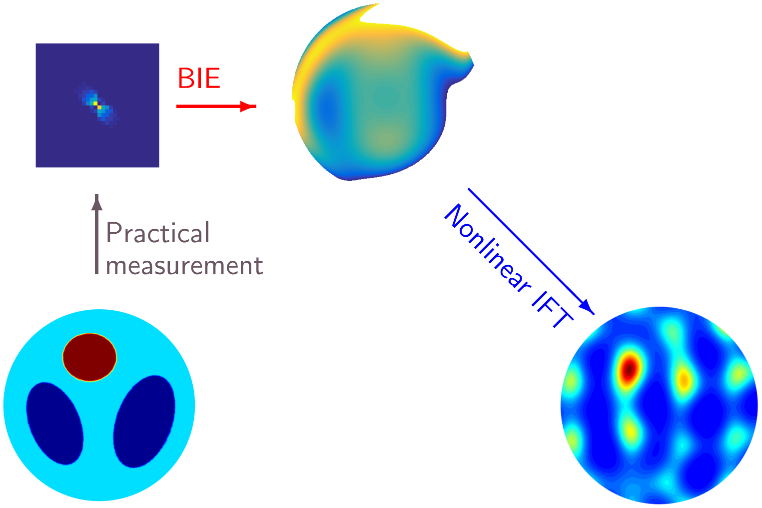


The nonlinear Fourier transform can be recovered from infinite-precision EIT measurements

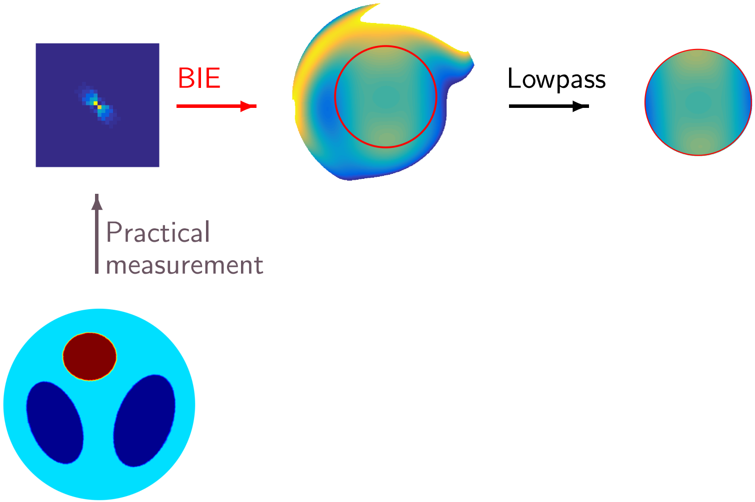


[Nachman; 1996]

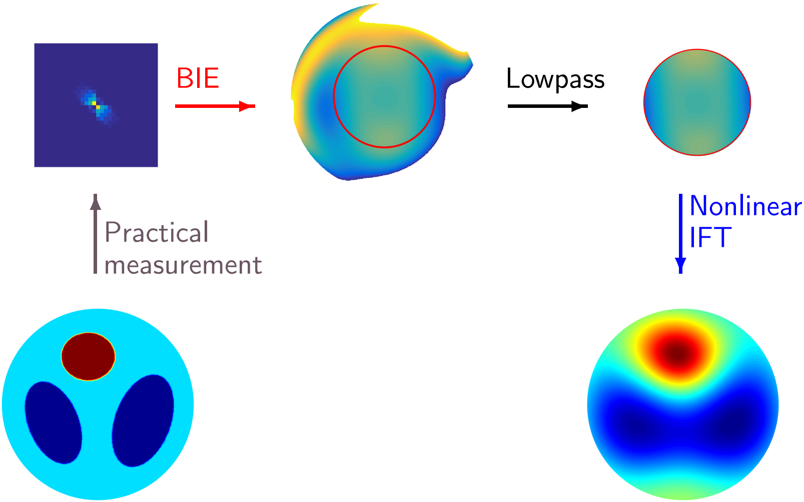
Measurement noise prevents the recovery of the nonlinear Fourier transform at high frequencies



We truncate away the bad part in the transform; this is a nonlinear low-pass filter



There is currently only one regularized method for reconstructing the full conductivity distribution



[Knudsen, Lassas, Mueller & Siltanen; 2009]

Scattering transform by a Born approximation

We can compute the scattering transform from the ND map by evaluating

$$\mathbf{t}(k) = \int_{\partial\Omega} \partial_\nu e^{i\bar{k}\bar{\zeta}} (\mathcal{R}_1 - \mathcal{R}_\sigma) \partial_\nu \psi(\zeta, k) ds(\zeta).$$

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We approximate the complex geometric optics solutions by their asymptotic behaviour

$$\psi(\zeta, k) \approx e^{ikz}.$$

Then we get (on the unit disk)

$$\begin{aligned}
\mathbf{t}^{\exp}(k) &= \int_{\partial\Omega} \partial_\nu e^{i\bar{k}\bar{\zeta}} (\mathcal{R}_1 - \mathcal{R}_\sigma) \partial_\nu e^{ik\zeta} ds(\zeta) \\
&= \int_{\partial\Omega} i\bar{k}\bar{\zeta} e^{i\bar{k}\bar{\zeta}} (\mathcal{R}_1 - \mathcal{R}_\sigma) ik\zeta e^{ik\zeta} ds(\zeta)
\end{aligned}$$

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Which we use to solve the D-bar equation to obtain the reconstruction of σ!

What is the central idea of our research

- Introduce a framework for current patterns applied on a part of the boundary by defining a partial-boundary Neumann-to-Dirichlet map
- Analyse the error compared to full-boundary current patterns
- Restricted measurement on part of the boundary recovered by extrapolation
- Use approximation to compute reconstructions from the partial Neumann-to-Dirichlet map

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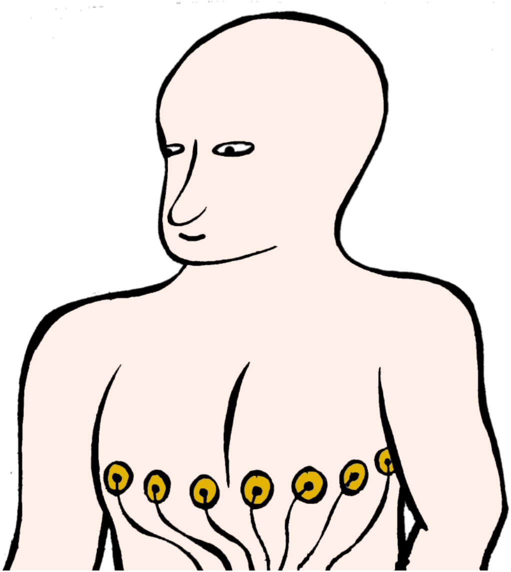
Why partial boundary?

In stationary monitoring, one might not have access to part of the patient, i.e. patient lying on back.

Devices and electrodes are bulky and restrict available space.

Especially three dimensional: Full boundary is simply impractical.



A note on partial-boundary data

- For full-boundary data, the Dirichlet and Neumann problems are essentially equivalent.
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  - Voltages will distribute on the whole boundary

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  - Partial zero boundary condition means insulation, a physical body is normally not
  - Voltages will distribute on the whole boundary
  - **Not practical**

A note on partial-boundary data

- For full-boundary data, the Dirichlet and Neumann problems are essentially equivalent.
  - This is not the case for partial-boundary data! (Different subsets of the Cauchy data)
- The Neumann problem => Current input
  - Partial zero boundary condition means just no current is injected
  - BUT: Voltages will still distribute on the whole boundary

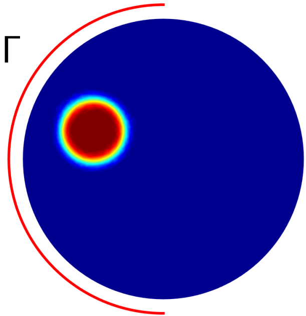
The partial-boundary problem setting

Applying a current f at part of the boundary $\Gamma \subset \partial\Omega$ leads to the elliptic PDE

$$\begin{cases} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\ \sigma \frac{\partial u}{\partial \nu} &= f \text{ on } \Gamma \\ \sigma \frac{\partial u}{\partial \nu} &= 0 \text{ on } \Gamma^c = \partial\Omega \backslash \Gamma. \end{cases}$$

With the zero mean conditions

$$\int_\Gamma f \ ds = 0 \text{ and } \int_{\partial\Omega} u \ ds = 0.$$



For the Neumann problem Francis Chung (2014) has shown uniqueness for $\sigma \in C^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$ and $n \geq 3$.

Modelling the boundary functions

We use a space of functions supported on Γ as a subspace of $\widetilde{H}^{-1/2}(\partial\Omega)$ (zero mean $H^{-1/2}$ functions) defined by

$$\widetilde{H}_{\Gamma}^{-1/2}(\partial\Omega) := \{\varphi \in \widetilde{H}^{-1/2}(\partial\Omega) : \text{supp } (\varphi) = \Gamma \text{ and } \int_{\Gamma} \varphi = 0\}.$$

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Given a linear and bounded operator $\mathcal{I} : \widetilde{H}^{-1/2}(\partial\Omega) \to \widetilde{H}_{\Gamma}^{-1/2}(\partial\Omega)$, we define the partial ND map as

$$\widetilde{\mathcal{R}}_{\sigma} := \mathcal{R}_{\sigma} \, \mathcal{I} \, .$$

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$$\widetilde{\mathcal{R}}_{\sigma} := \mathcal{R}_{\sigma}\,\mathcal{I}\,.$$

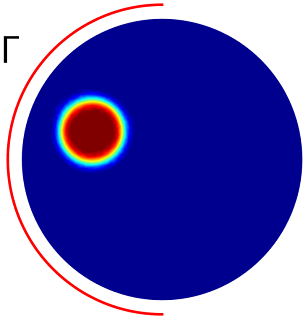
Let the basis functions on Γ be produced by $\mathcal{I}$, i.e. $\widetilde{\varphi} = \mathcal{I}\,\varphi$ for $\varphi \in \widetilde{H}^{-1/2}(\partial\Omega)$, then the central identity follows

$$\mathcal{R}_{\sigma}\,\widetilde{\varphi} = \mathcal{R}_{\sigma}\,\mathcal{I}\,\varphi = \widetilde{\mathcal{R}}_{\sigma}\varphi.$$

Reformulation of the problem setting

Given the current $\mathcal{I}\varphi = \widetilde{\varphi} \in \widetilde{H}_{\Gamma}^{-1/2}(\partial\Omega)$ consider the Neumann problem

$$\begin{cases} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\ \sigma \frac{\partial u}{\partial \nu} &= \widetilde{\varphi} \text{ on } \partial\Omega \end{cases}$$



The inverse problem is then: Given measurement of

$$\widetilde{\mathcal{R}}_{\sigma} : \widetilde{H}^{-1/2}(\partial\Omega) \to \widetilde{H}^{1/2}(\partial\Omega)$$

what can we deduce about σ?

Why this formulation?

- Calculating the partial ND matrix with respect to an orthonormal basis:

$$(\widetilde{\mathsf{R}}_\sigma)_{i,j} = (\widetilde{\mathcal{R}}_\sigma \varphi_i, \varphi_j) = (\mathcal{R}_\sigma \, \widetilde{\varphi}_i, \varphi_j)$$

- For the error analysis to the full-boundary ND map we only need knowledge of $\mathcal{I}$.

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We want to represent the Neumann-to-Dirichlet map by boundary layer potentials. For that we multiply the conductivity equation with a smooth test function and integrate over Ω

$$\int_{\Omega} \nabla(\sigma \nabla u) v dx = 0.$$

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$$\int_{\Omega} \nabla(\sigma \nabla u) v dx = 0.$$

Applying partial differentiation twice we obtain Green's identity:

$$\int_{\Omega} \nabla(\sigma \nabla u) v dx = \int_{\partial\Omega} \sigma\, \partial_{\nu}\, uvds - \int_{\partial\Omega} \sigma\, \partial_{\nu}\, vuds + \int_{\Omega} \nabla(\sigma \nabla v) u dx.$$

Choosing a suitable Green's function

Let $G_\sigma(x, y)$ be the Green's functions of the conductivity equation with Neumann boundary conditions, that is

$$\begin{aligned} -\nabla(\sigma \nabla G_\sigma(x, y)) &= \delta(x - y), \qquad \text{for } x, y \in \Omega, \\ \partial_\nu G(x, y) &= 0, \qquad \text{for } y \in \partial\Omega, x \in \Omega. \end{aligned}$$

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$$\partial_\nu G(x,y) = 0, \qquad \text{for } y\in\partial\Omega, x\in\Omega.$$

We can set $v(y) = G_\sigma(x,y)$ and insert in

$$\int_\Omega \nabla(\sigma\nabla u)v dy = \int_{\partial\Omega}\sigma\,\partial_\nu\, uvds - \int_{\partial\Omega}\sigma\,\partial_\nu\, vuds + \int_\Omega \nabla(\sigma\nabla v)udy$$

to obtain

$$u(x) = \int_{\partial\Omega}\partial_\nu\, u(y)G_\sigma(x,y)ds_y, \quad \forall x\in\Omega.$$

Representation by single layer potentials

Taking the limit $x \to \partial\Omega$, we obtain the identity

$$u(x) = (S_\sigma \, \partial_\nu \, u)(x) \quad , \quad \forall x \in \partial\Omega,$$

where $S_\sigma : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is the single layer potential given by

$$S_\sigma \varphi(x) = \int_{\partial\Omega} G_\sigma(x, y)\varphi(y) ds_y.$$

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$$S_\sigma \varphi(x) = \int_{\partial\Omega} G_\sigma(x,y)\varphi(y) ds_y.$$

Thus, in this representation, the Neumann-to-Dirichlet map coincides with the single layer operator restricted to the space $\widetilde{H}^{-1/2}(\partial\Omega)$:

$$\mathcal{R}_\sigma = S_\sigma : \widetilde{H}^{-1/2}(\partial\Omega) \to \widetilde{H}^{1/2}(\partial\Omega).$$

Basis of the error analysis

We can use this representation to define the partial ND map, by using the central identity

$$\widetilde{\mathcal{R}}_\sigma \varphi = \mathcal{R}_\sigma \mathcal{I} \varphi = S_\sigma(\mathcal{I} \varphi) = u|_{\partial\Omega},$$

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and the difference of ND maps can then be simply expressed by

$$(\widetilde{\mathcal{R}}_\sigma - \mathcal{R}_\sigma)\varphi = \mathcal{R}_\sigma(\widetilde{\varphi} - \varphi) = S_\sigma(\widetilde{\varphi} - \varphi).$$

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Further the error of ND maps is

$$\|(\widetilde{\mathcal{R}}_\sigma - \mathcal{R}_\sigma)\varphi\|_{L^2(\partial\Omega)} = \|S_\sigma(\mathcal{I} - 1)\varphi\|_{L^2(\partial\Omega)}.$$

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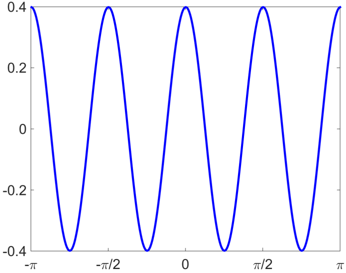
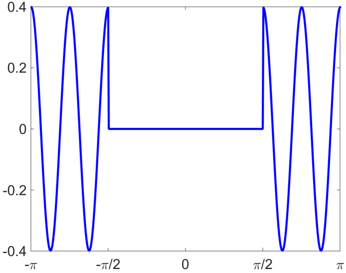
$\Longrightarrow$ By explicit knowledge of $\mathcal{I}$ we can give an asymptotic estimate dependent on $|\Gamma|$.

Choices for $\mathcal{I}$ - scaling

The functions φ are scaled on the partial-boundary Γ. We parametrize the boundary $\partial\Omega$ by an angle $\theta \in [0, 2\pi]$. Then denote $\Gamma = [\theta_1, \theta_2] \subset [0, 2\pi]$. The partial-boundary projection is then given by

$$\mathcal{I}^s \varphi(\theta) = \begin{cases} \varphi\left(\frac{\theta-\theta_1}{r}\right) & \text{if } \theta \in \Gamma, \\ 0 & \text{else }, \end{cases} \quad \text{with} \quad r = \frac{|\Gamma|}{|\partial\Omega|}.$$



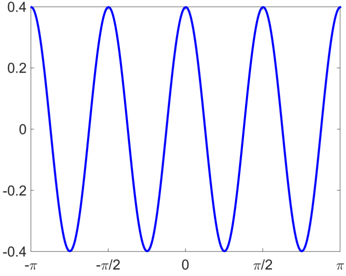


Choices for $\mathcal{I}$ - cut-off

The second option is a straight forward cut-off, such that

$$\widetilde{\varphi}(\theta) = \mathcal{I}^c \varphi(\theta) = \begin{cases} \varphi(\theta) - \frac{1}{|\Gamma|}\int_\Gamma \varphi(\tau) d\tau & \text{if } \theta \in \Gamma, \\ 0 & \text{else }, \end{cases}$$

We subtract the mean to ensure $\widetilde{\varphi} \in \widetilde{H}_\Gamma^{-1/2}(\partial\Omega)$

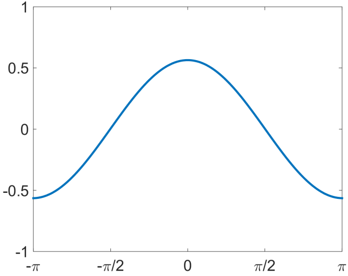


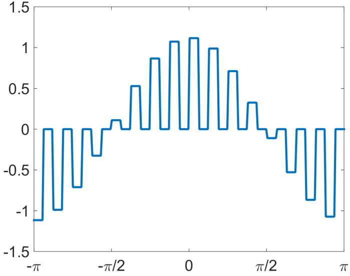
Choices for $\mathcal{I}$ - electrode projection

A more realistic approach is given by the nonorthogonal projection for a set of electrodes E_m, $m = 1, \ldots, M$, introduced in [Hyvönen09] by

$$\mathcal{I}^e \varphi(\theta) = \sum_{m=1}^{M} \frac{\chi_m(\theta)}{|E_m|} \int_{\bar{E}_m} \varphi(\tau)\, d\tau,$$

where $\bar{E}_m$ are the so-called extended electrodes.





Data error

Proposition (Error of the partial ND map)

Let $\Omega \subset \mathbb{R}^2$ be the unit disk, $\sigma \in L^\infty(\Omega)$ be a conductivity with $0 < \sigma_0 \leq \sigma(x)$ and $\sigma \equiv 1$ close to $\partial\Omega$.
Denote the partial ND maps by $\widetilde{\mathcal{R}}_\sigma^c = \mathcal{R}_\sigma \mathcal{I}^c$ and $\widetilde{\mathcal{R}}_\sigma^s = \mathcal{R}_\sigma \mathcal{I}^s$.
Let the basis functions be $\varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}$ for $n \neq 0$, and $\Gamma \subset \partial\Omega$ with $|\Gamma^c| = h > 0$. Then, for $0 \leq h \leq \pi$, there is a constant $C > 0$ independent on n such that:

$$\|(\widetilde{\mathcal{R}}_\sigma^c - \mathcal{R}_\sigma)\varphi_n\|_{L^2(\partial\Omega)} \leq Ch,$$
$$\|(\widetilde{\mathcal{R}}_\sigma^s - \mathcal{R}_\sigma)\varphi_n\|_{L^2(\partial\Omega)} \leq Cn^2 h.$$

Data error: Idea of proof

Starting with

$$\|(\widetilde{\mathcal{R}}_\sigma - \mathcal{R}_\sigma)\varphi_n\|_{L^2(\partial\Omega)} = \|S_\sigma(\mathcal{I} - 1)\varphi_n\|_{L^2(\partial\Omega)}.$$

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where $\|\varphi\|^2_{H^{-1}(\partial\Omega)} = \sum_{k\in\mathbb{Z}}(1+k^2)^{-1}|\widehat{\varphi}(k)|^2$ and

$$\widehat{\varphi}(k) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\theta)e^{-ik\theta}d\theta.$$

Thus we are left to calculate the Fourier coefficients for each choice of $\mathcal{I}$.

Error for difference data

The estimates immediately extend to difference data.

Corollary

Under the assumptions of the Proposition we have

$$\|(\widetilde{\mathcal{R}}^{c}_{\sigma,1} - \mathcal{R}_{\sigma,1})\varphi_n\|_{L^2(\partial\Omega)} \le Ch,$$
$$\|(\widetilde{\mathcal{R}}^{s}_{\sigma,1} - \mathcal{R}_{\sigma,1})\varphi_n\|_{L^2(\partial\Omega)} \le Cn^2 h,$$

where $\mathcal{R}_{\sigma,1} := \mathcal{R}_\sigma - \mathcal{R}_1$, $\widetilde{\mathcal{R}}_{\sigma,1} := \widetilde{\mathcal{R}}_\sigma - \widetilde{\mathcal{R}}_1$.

Operator estimates for the cut-off

For the cut-off case, this is equivalent to the operator estimates

$$\|\widetilde{\mathcal{R}}_{\sigma}^{c} - \mathcal{R}_{\sigma}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \leq Ch.$$

and

$$\|\widetilde{\mathcal{R}}_{\sigma,1}^{c} - \mathcal{R}_{\sigma,1}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \leq Ch.$$

Error of scattering transform

Given the operator estimates, we can directly prove:

Proposition

Let $\sigma \in C^2(\Omega)$ be bounded by $0 < c \leq \sigma(x) \leq C$ for all $x \in \Omega$ and $\sigma \equiv 1$ close to $\partial\Omega$.
Let $\Omega \subset \mathbb{R}^2$ be the unit disk and $\Gamma \subset \partial\Omega$ with $|\Gamma^c| = h$.
For a fixed cut-off radius $0 < R < \infty$, let $\mathbf{t}_R^{\mathrm{exp}}$ be computed from $\mathcal{R}_{1,\sigma}$ and $\widetilde{\mathbf{t}}_R^{\mathrm{exp}}$ from $\widetilde{\mathcal{R}}_{1,\sigma}^c$.
Then for $0 \leq h \leq \pi$, there are constants C_1, C_2 such that

$$\left| \frac{\mathbf{t}_R^{\mathrm{exp}}(k) - \widetilde{\mathbf{t}}_R^{\mathrm{exp}}(k)}{\bar{k}} \right| \leq C_1 h |k| e^{C_2 |k|}, \quad \text{for } |k| \leq R.$$

Reconstruction error

Theorem (Reconstruction error)

Under the assumptions of the last proposition. Additionally let σ_R and $\widetilde{\sigma}_R$ be reconstructed from $\mathbf{t}_R^{\mathrm{exp}}$, $\widetilde{\mathbf{t}}_R^{\mathrm{exp}}$ respectively. Then there exists a constant $C > 0$ depending only on R, such that

$$\|\sigma_R - \widetilde{\sigma}_R\|_{L^2(\Omega)} \leq Ch.$$

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The proof uses continuous dependence of solutions of the D-bar equation on the error in scattering transform and is based on: [Knudsen, Lassas, Mueller and Siltanen 2009].

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Measurement extension

The previous estimates all assume measurements on the full boundary. This is of course unrealistic.

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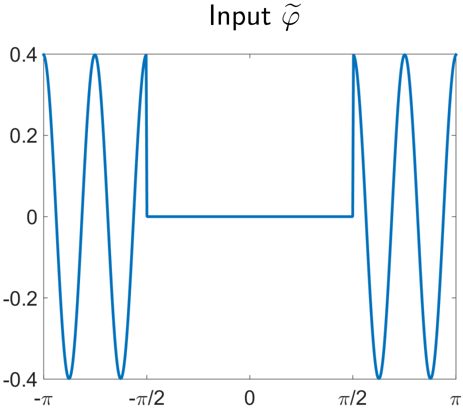
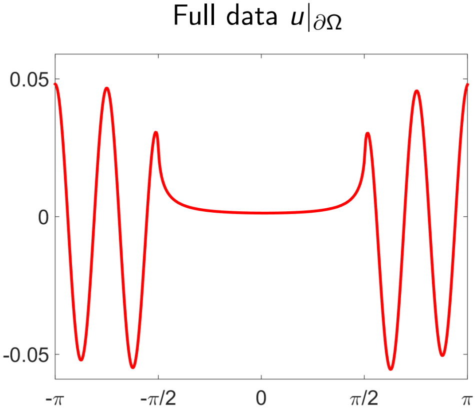
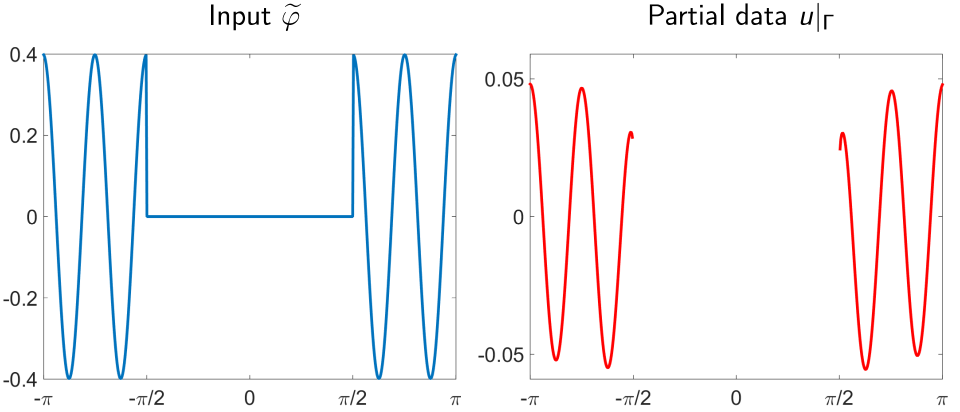
$\Longrightarrow$ The error is governed by restricting the input current to Γ, not by restriction of the measurement!

The optimal measurement

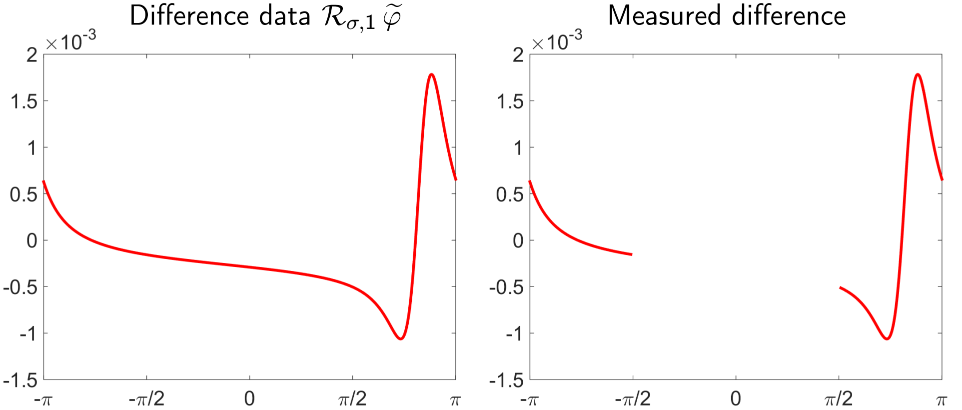




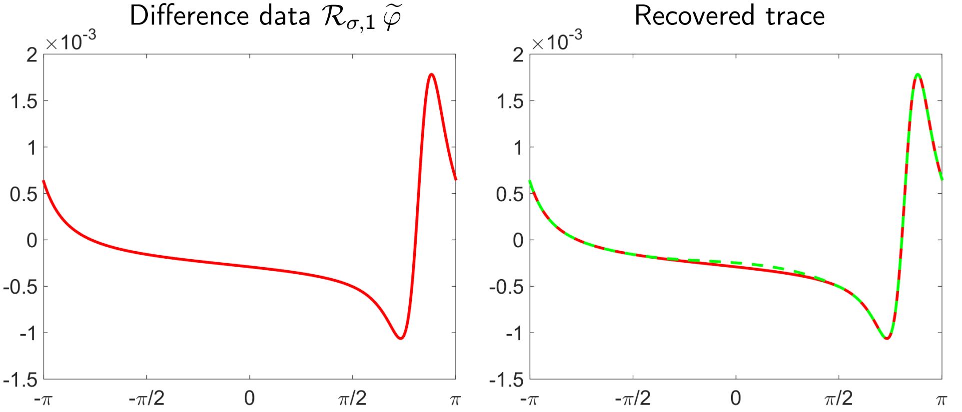
The real measurement on the partial-boundary



Difference data is smoother



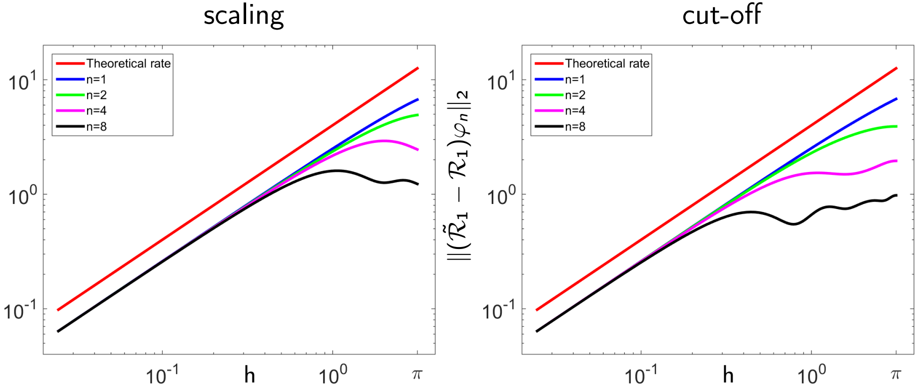
Recovering the measurement by cubic splines



The convergence result: Laplace equation

We want to verify the convergence for the constant conductivity $\sigma \equiv 1$ and basis functions of different order:

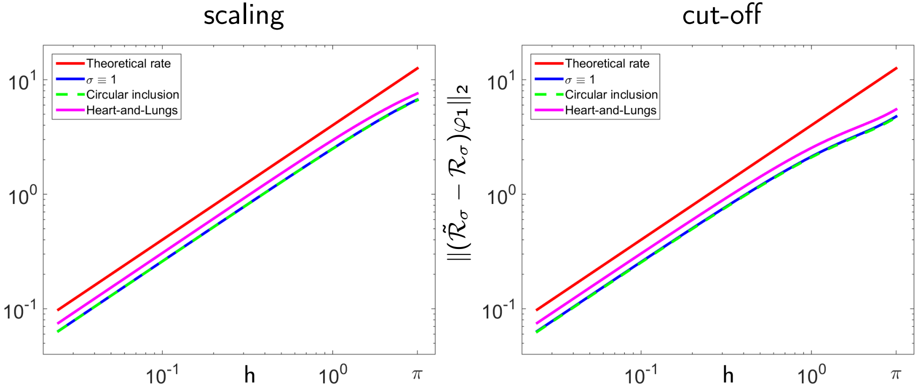
$$\|(\widetilde{\mathcal{R}}_1 - \mathcal{R}_1)\varphi_n\|_{L^2(\partial\Omega)} \leq Ch$$



The convergence result: Different conductivities

We want to check that the convergence estimate holds for different conductivities

$$\|(\widetilde{\mathcal{R}}_\sigma - \mathcal{R}_\sigma)\varphi_1\|_{L^2(\partial\Omega)} \leq Ch$$

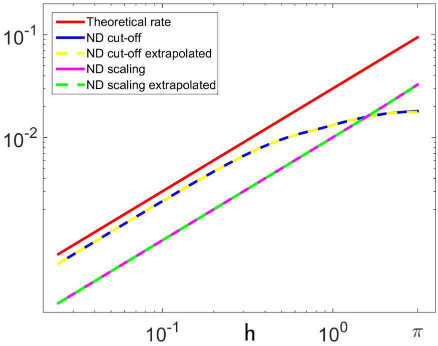


The convergence result: difference ND-matrices and extrapolation

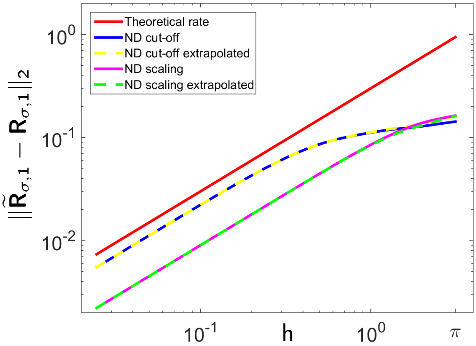
At last we check the error of difference ND-matrices with 16 basis functions and introduce the extrapolation, we should have

$$\|\widetilde{\mathbf{R}}_{\sigma,1} - \mathbf{R}_{\sigma,1}\|_2 \leq Ch.$$

Circular inclusion



Heart-and-Lungs



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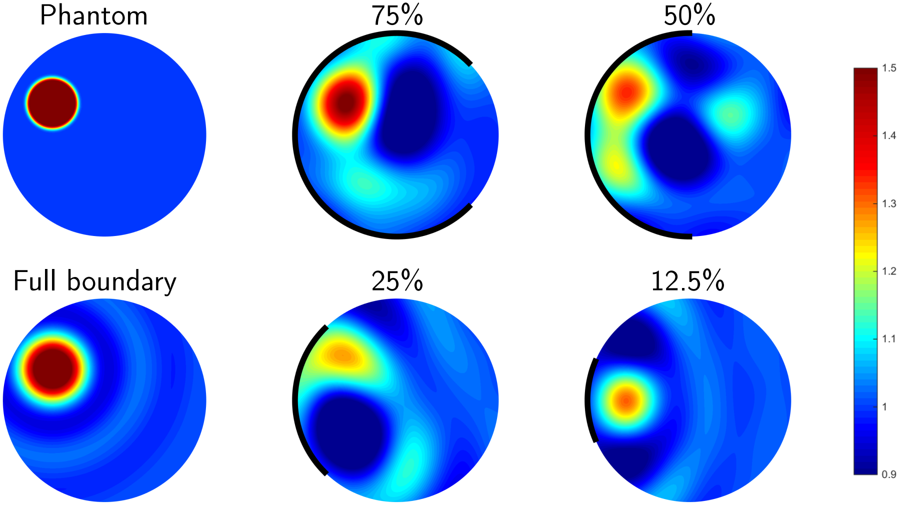
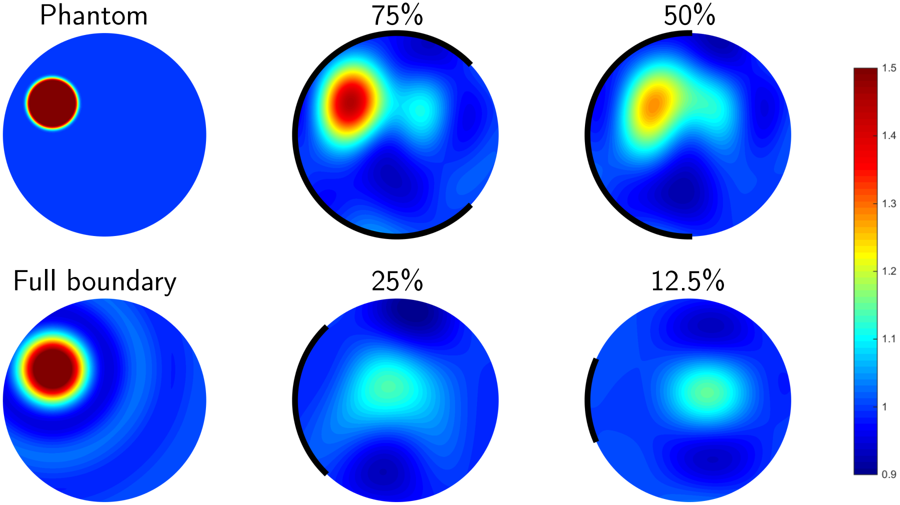
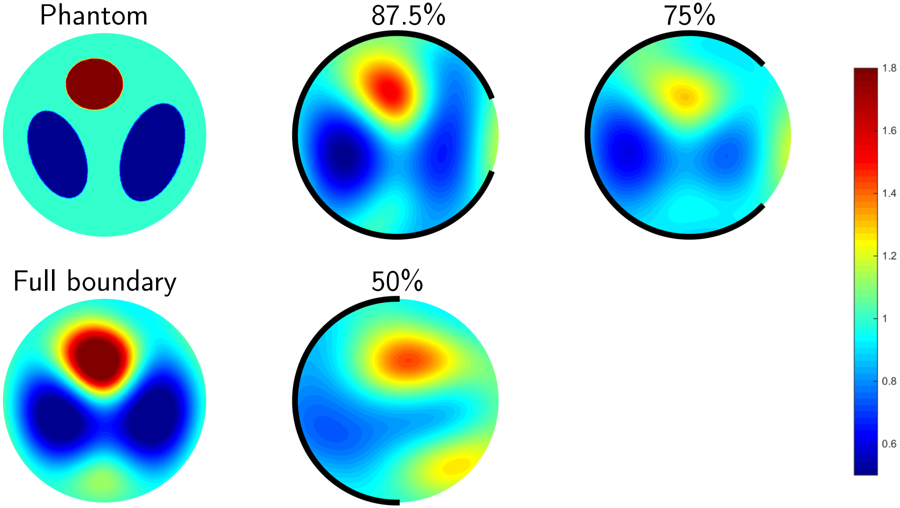
A simple circular inclusion: cut-off basis



A simple circular inclusion: scaling basis



Heart-and-Lungs on the unit circle: cut-off basis



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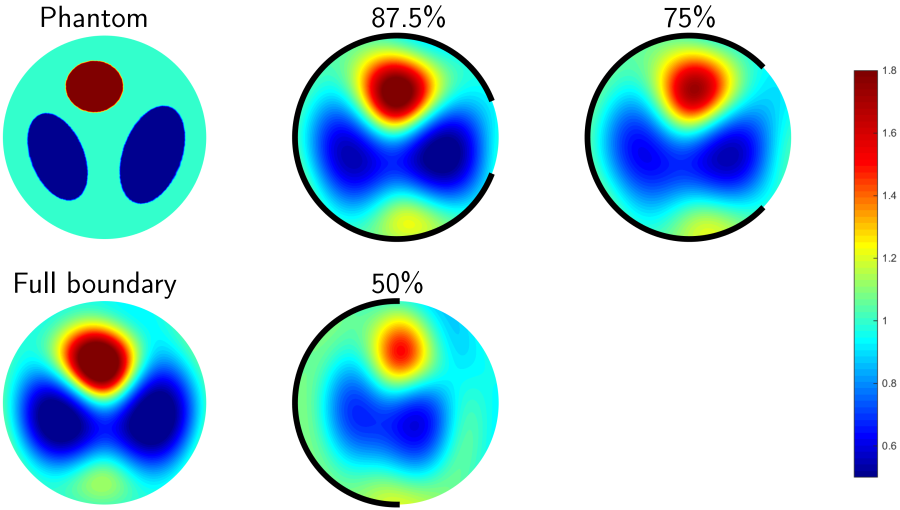
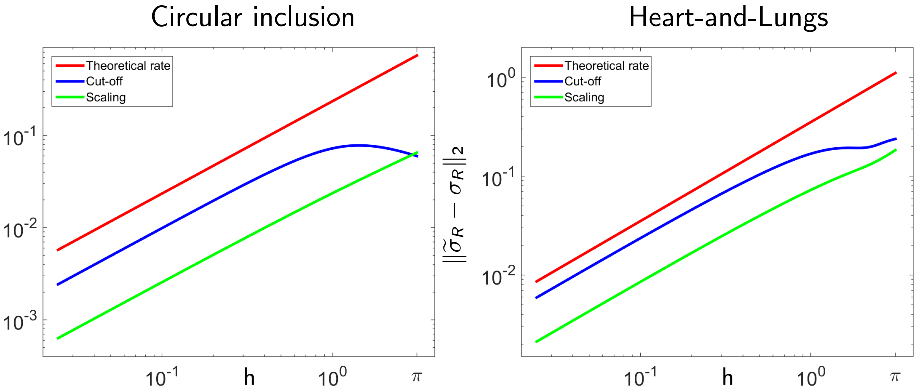


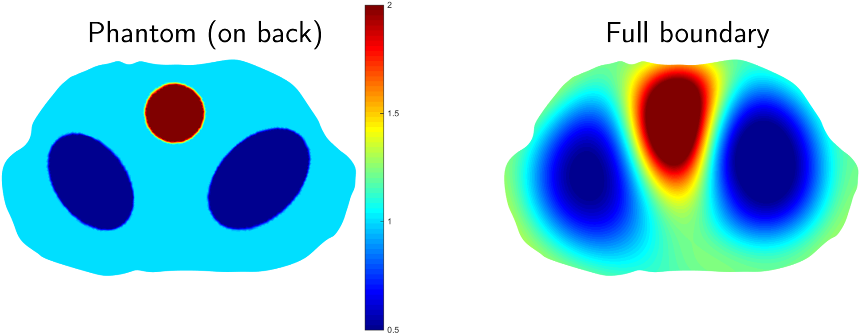
The convergence result: reconstructions

Let us check if the error estimates for the reconstructions hold as

$$\|\sigma_R - \widetilde{\sigma}_R\|_{L^2(\Omega)} \leq Ch.$$



A realistic chest phantom



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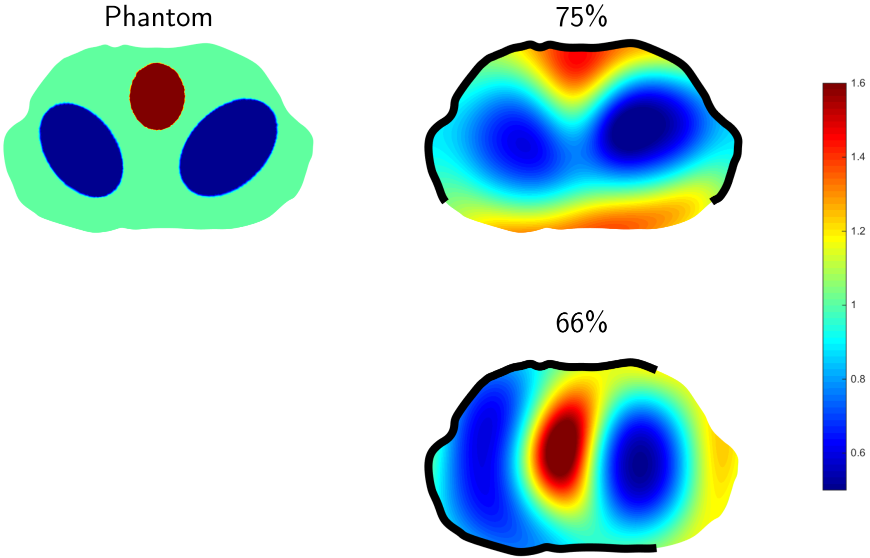

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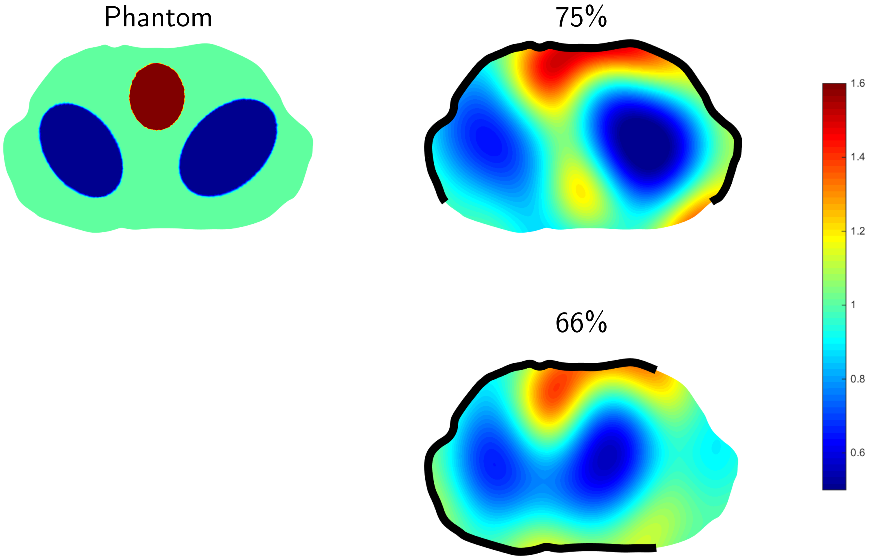

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- With the partial ND map, we have introduced a framework in which we can represent the measurement with respect to various choices of boundary mappings.
- We have established error estimates to the full-boundary case.
- By the extrapolation approach of the measurement, even more complicated phantoms can be reconstructed.
- Results will be submitted very soon

Ongoing research: A realistic approach

1. Use a realistic model (such as complete electrode model) and incorporate noise
2. Extrapolation by optimization
   - Alternating scheme to update extrapolated traces and partial ND map
3. Incorporating real measurement data

Thank you for your attention