Westfälische
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Münster

Diffuse Interface Methods for Inverse Problems

Case Study for an elliptic Cauchy Problem

joint work with Martin Burger (WWU) & Ole Elvetun (NMBU)

Inverse Problems for PDEs
ZeTeM
University of Bremen

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General Inverse Problems

- $F : \mathcal{X} \to \mathcal{Y}$ forward operator, f (noisy) data. Determine u from
$$F(u) = f$$
- Variational regularization
$$J(u) = \|F(u) - f\|_{\mathcal{Y}}^q + \alpha \|u - u_*\|_{\mathcal{X}}^r$$
with $q, r \geq 1$, u_* a prior for u.
- Diffuse interface variational regularization
$$J^\epsilon(u) = \|F^\epsilon(u) - f^\epsilon\|_{\mathcal{Y}^\epsilon}^q + \alpha \|u - u_*\|_{\mathcal{X}^\epsilon}^r$$
  - ϵ-dependence on $F^\epsilon : \mathcal{X}^\epsilon \to \mathcal{Y}^\epsilon$
  - ϵ-dependence on f
  - ϵ-dependent topologies

Sharp Interfaces

Typical terms in variational regularization

$$\int_D h(x, u, \nabla u)\,\mathrm{d}x, \qquad \int_{\partial D} h(x, u, \nabla u)\,\mathrm{d}\sigma$$

- Domain D has complex geometry, but with $C^{1,1}$ boundary
- $\int_D \,\mathrm{d}x$ and $\int_{\partial D} \,\mathrm{d}\sigma$ need resolution of D and ∂D
- Idea: replace $\int_D \,\mathrm{d}x$ and $\int_{\partial D} \,\mathrm{d}\sigma$ by more convenient expressions

Phase-Field Function

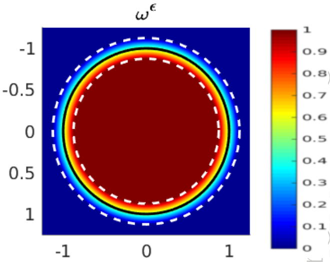
- Idea: "Smooth" ∂D via weighted averaging
- Oriented distance function $d_D(x) = \mathrm{dist}(x, D) - \mathrm{dist}(x, \mathbb{R}^n \setminus D)$
- $S(t) = t/|t|$ for $|t| \geq 1$, $S(t) = t$ for $|t| < 1$
- $S(t/\epsilon) \to sign(t)$ as $\epsilon \to 0$
- Set $\varphi^\epsilon(x) = S(-d_D(x)/\epsilon))$,

$$\varphi^\epsilon(x) \to \begin{cases} 1, & x \in D \\ -1 & x \notin D \end{cases} \quad \text{as } \epsilon \to 0$$

- Phase-field (masking) function

$$\omega^\epsilon = \frac{1}{2}(1 + \varphi^\epsilon) \approx \chi_D$$

- $\epsilon \approx$ width of diffuse layer



Diffuse Approximation of Integrals

- Diffuse volume integral[1]

$$\int_D h \,\mathrm{d}x = \int_\Omega h\chi_D \,\mathrm{d}x \approx \int_\Omega h\omega^\epsilon \,\mathrm{d}x = \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\int_{\{d_D<t\}} h \,\mathrm{d}x \,\mathrm{d}t$$

  - Error=$O(\epsilon^{1+k-\frac{1}{p}})$ for $h \in W^{k,p}(\Omega)$, $k \in \{0,1\}$

- Diffuse boundary integral

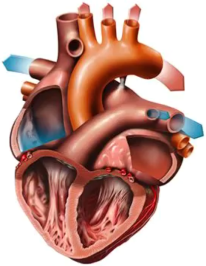
$$\int_{\partial D} h \,\mathrm{d}\sigma \approx \int_\Omega h|\nabla\omega^\epsilon| \,\mathrm{d}x = \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\int_{\{d_D=t\}} h \,\mathrm{d}\sigma \,\mathrm{d}t$$

  - Error=$O(\epsilon^{1+k-\frac{1}{p}})$ for $h \in W^{1+k,p}(\Omega)$

[1] Burger Elvetun MS: Analysis of the diffuse domain method for 2[nd] order elliptic boundary value problems. Found Comp Math (2015)

The ECG Problem

Electric currents on the surface of the heart and on the surface of the body give rise to a body-potential.



- Quantity of interest: electric current on the surface of the heart
- Measurable quantity: potential on the surface of the body

Forward Problem: $F : u \to v_{|\partial B}$

Given $u \in L^2(\partial H)$, determine the potential $v_{|\partial B}$ where $v \in H^1_\diamond(D)$ satisfies

$$\int_D M\nabla v \cdot \nabla w \,\mathrm{d}x = \int_{\partial H} uw \,\mathrm{d}\sigma \quad \forall w \in H^1_\diamond(D) = \{w \in H^1(D) : \int_{\partial H} w \,\mathrm{d}\sigma = 0\}.$$

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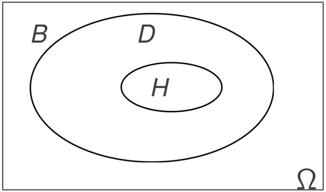
- Complex geometry (time-dependent/implicit)
- Extend the problem to “simple” domain Ω ($\rightsquigarrow$ “simple” grids)



Schematic model of the heart and the torso

Diffuse Forward Problem

New functions spaces

- $L^2(\partial H) \rightsquigarrow \mathcal{U}^\epsilon = \{u \in L^2(\Omega) : \int_\Omega |u|^2 \gamma_H |\nabla \omega^\epsilon| dx < \infty\}$
- $L^2(\partial B) \rightsquigarrow \mathcal{M}^\epsilon = \{v \in L^2(\Omega) : \int_\Omega |v|^2 \gamma_B |\nabla \omega^\epsilon| dx < \infty\}$
- $H^1_\diamond(D) \rightsquigarrow$

$$\mathcal{H}^\epsilon_\diamond = \{w \in L^2(\Omega) | \int_\Omega (|\nabla w|^2 + |w|^2) \omega^\epsilon \,\mathrm{d}x < \infty, \int_\Omega w |\nabla \omega^\epsilon| \gamma_H = 0\}$$

Given $u \in \mathcal{U}^\epsilon$, determine $v_{|\mathrm{supp}(\gamma_B)} \in \mathcal{M}^\epsilon$ where $v \in \mathcal{H}^\epsilon_\diamond$ satisfies

$$\int_\Omega M \nabla v \cdot \nabla w \omega^\epsilon \,\mathrm{d}x = \int_\Omega u w \gamma_H |\nabla \omega^\epsilon| \,\mathrm{d}x \quad \forall w \in \mathcal{H}^\epsilon_\diamond. \qquad (FP^\epsilon)$$

Extension of Data and Perturbation Analysis

- Forward operator

$$F^\epsilon : \mathcal{U}^\epsilon \to \mathcal{M}^\epsilon, \quad F^\epsilon(u) = v_{|\mathrm{supp}(\gamma_B|\nabla\omega^\epsilon|)} \quad \text{s.t. } v \text{ solves } (FP^\epsilon)$$

- Extension of data $E_B : L^2(\partial B) \to \mathcal{M}^\epsilon$
  - $\tilde{f}(x) := E_B f(x) = f(x - d_D(x)n(x)), \quad x \in \mathrm{supp}(\gamma_B)$
  - $\|E_B f - E_B f^\delta\|_{\mathcal{M}^\epsilon} \leq C(\epsilon)\|f - f^\delta\|_{L^2(\partial B)}$, $C(\epsilon) \to 1$ as $\epsilon \to 0$.
- Let v^ϵ solve (FP^ϵ) for data $E_H u^\dagger$, then[2]

$$\|v^\dagger - v^\epsilon\|_{\mathcal{H}^\epsilon} \leq C\epsilon^{3/2}\|v^\dagger\|_{W^{3,\infty}(\Omega)}$$

[2] Burger Elvetun MS: Diffuse Interface Methods for Inverse Problems: Case Study for an Elliptic Cauchy Problem. Inv Prob (2015)

Diffuse Inverse Problem and Tikhonov Regularization

Diffuse Inverse Problem

Given $f^\delta \in L^2(\partial B)$, determine $u^\epsilon_\delta \in \mathcal{U}^\epsilon$ s.t. $F^\epsilon(u^\epsilon_\delta) \approx E_B f^\delta$.

- Tikhonov regularization

$$J^\epsilon(u) = \frac{1}{2}\|F^\epsilon(u) - E_B f^\delta\|^2_{\mathcal{M}^\epsilon} + \frac{\alpha}{2}\|u\|^2_{\mathcal{U}^\epsilon} \to \min_{u\in\mathcal{U}^\epsilon}!$$

Diffuse Inverse Problem and Tikhonov Regularization

Diffuse Inverse Problem

Given $f^\delta \in L^2(\partial B)$, determine $u_\delta^\epsilon \in \mathcal{U}^\epsilon$ s.t. $F^\epsilon(u_\delta^\epsilon) \approx E_B f^\delta$.

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- Topologies depend on ϵ! Standard theory is not applicable.
- How to obtain compactness of minimizers $\{u^\epsilon_{\alpha,\delta}\}$?

Diffuse Inverse Problem and Tikhonov Regularization

Diffuse Inverse Problem

Given $f^\delta \in L^2(\partial B)$, determine $u_\delta^\epsilon \in \mathcal{U}^\epsilon$ s.t. $F^\epsilon(u_\delta^\epsilon) \approx E_B f^\delta$.

- Tikhonov regularization

$$J^\epsilon(u) = \frac{1}{2}\|F^\epsilon(u) - E_B f^\delta\|^2_{\mathcal{M}^\epsilon} + \frac{\alpha}{2}\|u\|^2_{\mathcal{U}^\epsilon} \to \min_{u\in\mathcal{U}^\epsilon}!$$

$\Rightarrow$ PDE-constrained Tikhonov regularization

$$\tilde{J}^\epsilon(u,v) = \frac{1}{2}\|v - E_B f^\delta\|^2_{\mathcal{M}^\epsilon} + \frac{\alpha}{2}\|u\|^2_{\mathcal{U}^\epsilon} \to \min_{u\in\mathcal{U}^\epsilon}! \quad \text{s.t. } v \text{ solves } FP^\epsilon$$

Lagrangian Formulation of the Regularized Problem

Consider Lagrangian

$$L^\epsilon(u,v,p) = \tilde{J}^\epsilon(u,v) + \langle M\nabla v, \nabla p\rangle_{\omega^\epsilon} - \langle u,p\rangle_{\mathcal{U}^\epsilon}.$$

- Saddle-point problem: Find $(u,v,p) \in \mathcal{U}^\epsilon \times \mathcal{H}^\epsilon_\diamond \times \mathcal{H}^\epsilon_\diamond$ s.t.

$$\begin{array}{lcll} a^\epsilon(u,v;q,w) + b^\epsilon(q,w;p) & = & f^\epsilon(q,w) & \forall\,(q,w) \in \mathcal{U}^\epsilon \times \mathcal{H}^\epsilon_\diamond \\ b^\epsilon(u,v;r) & = & 0 & \forall\, r \in \mathcal{H}^\epsilon_\diamond \end{array}$$

- $a^\epsilon : (\mathcal{U}^\epsilon \times \mathcal{H}^\epsilon_\diamond) \times (\mathcal{U}^\epsilon \times \mathcal{H}^\epsilon_\diamond) \to \mathbb{R}$

$$a^\epsilon(u,v;q,w) = \langle v,w\rangle_{\mathcal{M}^\epsilon} + \alpha\langle u,q\rangle_{\mathcal{U}^\epsilon}$$

- $b^\epsilon : (\mathcal{U}^\epsilon \times \mathcal{H}^\epsilon_\diamond) \times \mathcal{H}^\epsilon_\diamond \to \mathbb{R}$

$$b^\epsilon(u,v;p) = \langle M\nabla v, \nabla p\rangle_{\omega^\epsilon} - \langle u,p\rangle_{\mathcal{U}^\epsilon}$$

Existence of Saddle-Points: Properties of a^ϵ and b^ϵ

- $\|(u,v)\|_\alpha^2 = \alpha(\|u\|_{\mathcal{U}^\epsilon}^2 + \|\nabla v\|_{L^2(\omega^\epsilon)}^2) + \|v\|_{\mathcal{M}^\epsilon}^2$

Lemma (Continuity[3])

Let $0 < \alpha \le \alpha_0$. *Then* $\exists C_c > 0$ *independent of* ϵ *and* α *such that*

$$|a^\epsilon(u,v;q,w)| \le C_c \|(u,v)\|_\alpha \|(q,w)\|_\alpha$$
$$|b^\epsilon(u,v;p)| \le \frac{1}{\sqrt{\alpha}} C_c \|(u,v)\|_\alpha \|p\|_{\mathcal{H}^\epsilon}$$

for $(u,v),(q,w) \in \mathcal{U}^\epsilon \times \mathcal{H}_\diamond^\epsilon$, $p \in \mathcal{H}_\diamond^\epsilon$.

[3] Burger Elvetun MS: Diffuse Interface Methods for Inverse Problems: Case Study for an Elliptic Cauchy Problem. Inv Prob (2015)

Lemma (Kernel ellipticity[4])

Let $0 < \alpha \leq \alpha_0$. $\exists C_e > 0$ *independent of* ϵ *and* α *such that*

$$a^\epsilon(u, v; u, v) \geq C_e \|(u, v)\|_\alpha^2$$

for all $(u, v) \in \mathcal{U}^\epsilon \times \mathcal{H}_\diamond^\epsilon$ *such that* $b^\epsilon(u, v; v) = 0$.

Technical ingredients[5]:

- Poincaré-Friedrichs inequality

$$\|v\|_{\mathcal{H}^\epsilon}^2 \leq C(\|\nabla v\|_{L^2(\omega^\epsilon)}^2 + \|v\|_{\mathcal{M}^\epsilon}^2) \quad \forall v \in \mathcal{H}^\epsilon$$

- Trace inequality

$$\|v\|_{\mathcal{U}^\epsilon} \leq C\|v\|_{\mathcal{H}^\epsilon}$$

[4] Burger Elvetun MS: Diffuse Interface Methods for Inverse Problems: Case Study for an Elliptic Cauchy Problem. Inv Prob (2015)

[5] Burger Elvetun MS: Analysis of the diffuse domain method for 2^{nd} order elliptic boundary value problems. Found Comp Math (2015)

Existence of Saddle-Points: Properties of a^ϵ and b^ϵ

Lemma (Inf-sup stability[6])

Let $0 < \alpha \le \alpha_0$. $\exists C_i > 0$ *independent of* ϵ *and* α *such that*

$$\sup_{(u,v)\in\mathcal{U}^\epsilon\times\mathcal{H}^\epsilon_\diamond} \frac{b^\epsilon(u,v;p)}{\|(u,v)\|_\alpha} \ge C_i\|p\|_{\mathcal{H}^\epsilon} \quad \forall\, p \in \mathcal{H}^\epsilon_\diamond.$$

Construction in the proof: For $p \in \mathcal{H}^\epsilon_\diamond$, choose $u = -p$ and $v = p$.

[6] Burger Elvetun MS: Diffuse Interface Methods for Inverse Problems: Case Study for an Elliptic Cauchy Problem. Inv Prob (2015)

Existence of saddle-points

Theorem (Existence[7] (Brezzi'74))

Let $0 < \alpha \leq \alpha_0$. *Then for each* $f^\epsilon \in (\mathcal{U}^\epsilon \times \mathcal{H}^\epsilon_\diamond)'$ *there exist a unique saddle-point* $(u^\epsilon, v^\epsilon, p^\epsilon) \in \mathcal{U}^\epsilon \times \mathcal{H}^\epsilon_\diamond \times \mathcal{H}^\epsilon_\diamond$ *of* L^ϵ *and a constant* C_E *independent of* ϵ *and* α *such that*

$$\alpha\big(\|u^\epsilon\|^2_{\mathcal{U}^\epsilon} + \|\nabla v^\epsilon\|^2_{L^2(\omega^\epsilon)}\big) + \|v^\epsilon\|^2_{\mathcal{M}^\epsilon} + \|p^\epsilon\|^2_{\mathcal{H}^\epsilon} \leq C_E \|f^\epsilon\|^2_{(\mathcal{U}^\epsilon \times \mathcal{H}^\epsilon_\diamond)'}.$$

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Regularization Properties

Lagrangian $L^{\epsilon}(u,v,p) = \tilde{J}^{\epsilon}(u,v) + \langle M\nabla v, \nabla p\rangle_{\omega^{\epsilon}} - \langle u, p\rangle_{\mathcal{U}^{\epsilon}}$

- Existence of saddle-points for $\alpha > 0$ and $\epsilon \geq 0$
- Stability: For $\alpha > 0$ and $f_1, f_2 \in \mathcal{M}^{\epsilon}$

$$\|u_1^{\epsilon} - u_2^{\epsilon}\|_{\mathcal{U}^{\epsilon}} + \|v_1^{\epsilon} - v_2^{\epsilon}\|_{\mathcal{H}^{\epsilon}} \leq \frac{C}{\sqrt{\alpha}}\|f_1 - f_2\|_{\mathcal{M}^{\epsilon}}$$

- Convergence for $\delta, \alpha(\delta), \epsilon(\delta) \to 0$ with δ^2/α and ϵ^3/α bounded

$$\lim_{\delta\to 0} \|u_{\alpha,\delta}^{\epsilon} - E_H u^{\dagger}\|_{(\mathcal{H}_{\diamond}^{\epsilon})'} = 0$$

$$\|v_{\alpha,\delta}^{\epsilon} - E_B f^{\dagger}\|_{\mathcal{M}^{\epsilon}} \leq C\sqrt{\alpha}$$

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Key Ingredient for Convergence

Consider a sequence $\{(u^\epsilon, v^\epsilon)\} \subset \mathcal{U}^\epsilon \times \mathcal{H}^\epsilon_\diamond$ s.t.

- $b^\epsilon(u^\epsilon, v^\epsilon; r) = 0$ for all $r \in \mathcal{H}^\epsilon_\diamond$
- $\|u^\epsilon\|_{\mathcal{U}^\epsilon} \leq C$ for some $C > 0$

Then there exists a seq. $\epsilon_k \to 0$ and $v \in H^1(\Omega)$ s.t.

$$\sqrt{\omega^{\epsilon_k}} \nabla v^{\epsilon_k} \to \chi_D \nabla v \quad \text{in } L^2(\Omega) \text{ as } k \to \infty$$
$$v^{\epsilon_k} \to v \quad \text{in } H^1(D) \text{ as } k \to \infty$$

Source Condition and Existence of Saddle-Points

- Recall: $u^\dagger$ is the $\|.\|^2_{L^2(\partial H)}$-minimizing solution of $Fu = f^\dagger$
- Associated Lagrangian

$$L(u, v, \lambda, p) = \|u\|^2_{L^2(\partial\mathcal{H})} - \langle v - f^\dagger, \lambda\rangle_{\partial B} + b(u, v; p)$$

- Optimality system: Saddle-points $(u^\dagger, v^\dagger, \lambda^\dagger, p^\dagger)$ satisfy

$$\langle u^\dagger, h_u\rangle_{\partial H} - \langle h_u, p^\dagger\rangle_{\partial H} = 0 \quad \text{for all } h_u \in L^2(\partial H) \qquad (1)$$

$$-\langle h_v, \lambda^\dagger\rangle_{\partial B} + \langle M\nabla h_v, \nabla p^\dagger\rangle_D = 0 \quad \text{for all } h_v \in H^1_\diamond(D) \qquad (2)$$

$$\langle v^\dagger - f^\dagger, h_\lambda\rangle_{\partial B} = 0 \quad \text{for all } h_\lambda \in L^2(\partial B) \qquad (3)$$

$$b(u^\dagger, v^\dagger; h_p) = 0 \quad \text{for all } h_p \in H^1_\diamond(D) \qquad (4)$$

- Eq. (1) $\implies$ $u^\dagger = p^\dagger$ on ∂H $\implies$ $u^\dagger = F^*\lambda^\dagger$
- $u^\dagger = F^*\lambda^\dagger$ $\implies$ (1)–(2), and $(u^\dagger, v^\dagger, \lambda^\dagger, p^\dagger)$ is a saddle-point

Convergence Rates

- Source condition: $u^\dagger = F^* \lambda^\dagger$
- A-priori parameter choice: $\alpha \approx \delta$ and $\epsilon \approx \delta^{2/3}$

$\Rightarrow$ Convergence rate[8]

$$\|u^\epsilon_{\alpha,\delta} - E_H u^\dagger\|_{\mathcal{U}^\epsilon} + \|\nabla v^\epsilon_{\alpha,\delta} - \nabla v^\dagger\|_{L^2(\omega^\epsilon)} = O(\sqrt{\delta})$$

$$\|v^\epsilon_{\alpha,\delta} - v^\dagger\|_{\mathcal{M}^\epsilon} = O(\delta)$$

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$$\|v^\epsilon_{\alpha,\delta} - v^\dagger\|_{\mathcal{M}^\epsilon} = O(\delta)$$

- Estimate the error

$$(u^\epsilon_{\alpha,\delta} - E_H u^\dagger, v^\epsilon_{\alpha,\delta} - v^\dagger, p^\epsilon_{\alpha,\delta} - \alpha p^\dagger)$$

via saddle-point formulation

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- α can be chosen as in the sharp interface case
- Same rates as in the sharp interface case

[8] Burger Elvetun MS: Diffuse Interface Methods for Inverse Problems: Case Study for an Elliptic Cauchy Problem. Inv Prob (2015)

Example

- Heart $H = B_{0.3}(0)$. Body $B = B_1(0)$
- Conductivity tensor $M = \frac{1}{x^2+y^2}\begin{bmatrix} 0.3x^2 + y^2 & -0.7xy \\ -0.7xy & x^2 + 0.3y^2 \end{bmatrix}$
- Solution: $u^\dagger := F^* w$ for some function w
- Data: $f^\delta = Fu^\dagger + \eta$
- Preconditioned KKT-system

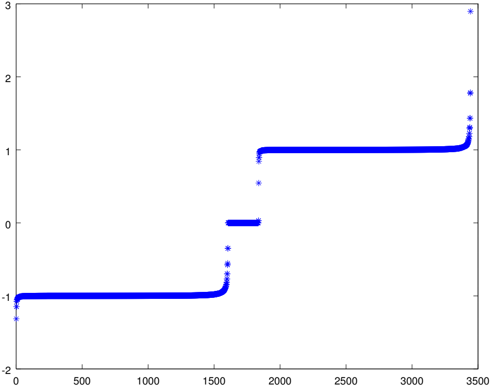
$$\underbrace{\begin{bmatrix} R^{-1}_{\mathcal{U}^\epsilon_\beta} & 0 & 0 \\ 0 & R^{-1}_{\mathcal{H}^\epsilon_\diamond} & 0 \\ 0 & 0 & R^{-1}_{\mathcal{H}^\epsilon_\diamond} \end{bmatrix} \begin{bmatrix} \alpha R_{\mathcal{U}^\epsilon_\beta} & 0 & [Q^\epsilon]' \\ 0 & T^\epsilon & [P^\epsilon]' \\ Q^\epsilon & P^\epsilon & 0 \end{bmatrix}}_{\mathcal{A}^\epsilon_\alpha} \begin{bmatrix} u^\epsilon_{\alpha,\delta} \\ v^\epsilon_{\alpha,\delta} \\ p^\epsilon_{\alpha,\delta} \end{bmatrix} = \mathcal{B}^\epsilon \begin{bmatrix} 0 \\ \tilde{T}^\epsilon f^\delta \\ 0 \end{bmatrix}.$$

- Numerical method[9]: FEM and preconditioned MINRES[10]

[9] Burger Elvetun MS: Diffuse Interface Methods for Inverse Problems: Case Study for an Elliptic Cauchy Problem. Inv Prob (2015)

[10] Nielsen Mardal: Analysis of the minimal residual method applied to ill-posed optimality systems. SIAM J. Sci. Comput. (2013)

Eigenvalue Distribution of $\mathcal{A}_\alpha^\epsilon$ for $\alpha = 10^{-4}$, $\epsilon = 1/8$



$$\mathrm{sp}(\mathcal{A}_\alpha^{\epsilon,h}) \subset [-b,-a]\cup[c\alpha, 2\alpha]\cup\{\tau_1,\tau_2,\ldots,\tau_{N(\alpha)}\}\cup[a,b], \qquad N(\alpha) = O(\ln(\alpha^{-1}))$$

Robustness of the Preconditioner

$\epsilon \backslash \alpha$	1	.1	.01	.001	.0001
2^{-2}	57	100	143	186	238
2^{-3}	57	91	126	157	195
2^{-4}	64	102	126	144	183
2^{-5}	57	83	115	143	159
2^{-6}	55	79	105	123	155

iterations $= O(\ln(\alpha^{-1}))$. No increase w.r.t. ϵ.

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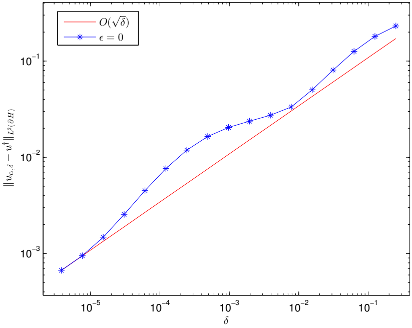
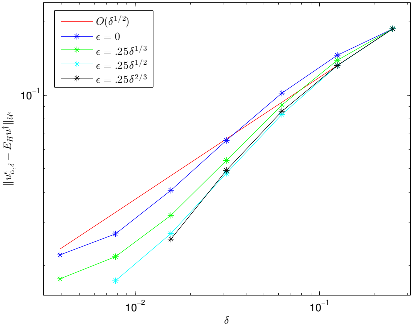
Convergence rates



(a) $\|u_{\alpha,\delta} - u^{\dagger}\|_{L^2(\partial H)}$



(b) $\|u^{\epsilon}_{\alpha,\delta} - E_H u^{\dagger}\|_{\mathcal{U}^{\epsilon}}$, with $\epsilon = \delta^{\nu}/4$, where $\nu = \{1/3, 1/2, 2/3\}$

$$\alpha \approx \sqrt{\delta}$$

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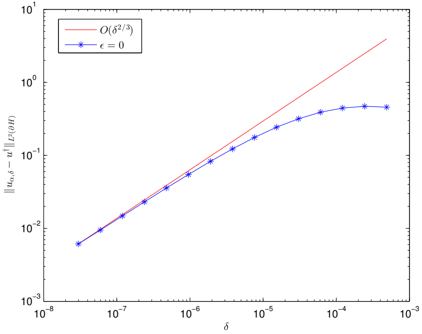
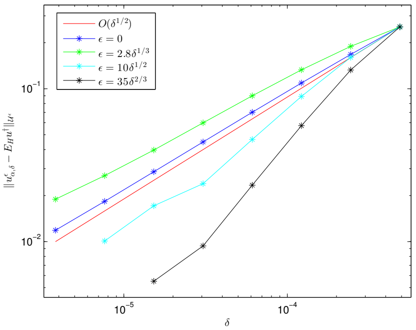
Convergence rates 2



(a) $\|u_{\alpha,\delta} - u^\dagger\|_{L^2(\partial H)}$



(b) $\|u^\epsilon_{\alpha,\delta} - E_H u^\dagger\|_{\mathcal{U}^\epsilon}$, with $\epsilon = \delta^\nu/4$, where $\nu = \{1/3, 1/2, 2/3\}$

$$\alpha \approx \delta^{2/3}$$

Burger Elvetun MS: Diffuse Interface Methods for Inverse Problems: Case Study for an Elliptic Cauchy Problem. Inv Prob (2015)

Conclusions & Extensions

- Introduced a diffuse interface method for Inverse Problems
- Implicit treatment of boundaries/interfaces
- Diffuse Tikhonov regularization constitutes a regularization method
- Under usual source condition, we obtain convergence rates $O(\sqrt{\delta})$ with a-priori choice $\epsilon = \delta^{2/3}$
- Method works for non-smooth data, but approximation (w.r.t. ϵ) is worse
- Extension to time-dependent problems
- Extension to perturbed distance functions/geometries
- Extension to geometries described by finitely many detector locations

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Variational Methods for Dynamic Inverse Problems in the Life Sciences

is gratefully acknowledged