A Sampling and Reconstruction Framework for Solving PDE-Driven Inverse Problems

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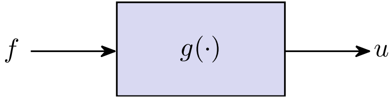
Outline

Inverse Problems

Related to measuring an *effect* with an intent to determine the *cause* from obtained measurements.



- Effect: u is measured (with errors)
- Cause: one of g or f is usually known

We consider *physics-driven* Inverse Problems where the system g is known (– some physical law).

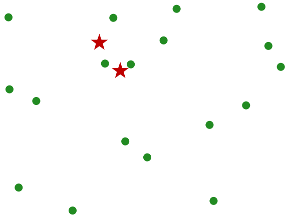
Inverse Problems in Physics: Diffusion

Diffusion

Stochastic movement of a collection of particles from regions of high concentration to regions of lower concentration (until an equilibrium is established).

Sensor networks measure:

- Leakages in/from factories,
- Temperature in server rooms,
- Nuclear fallouts (Fukushima).



The field $u(\mathbf{x}, t)$ induced by a source distribution $f(\mathbf{x}, t)$ satisfies:

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) - \mu \nabla^2 u(\mathbf{x}, t) = f(\mathbf{x}, t). \tag{1}$$

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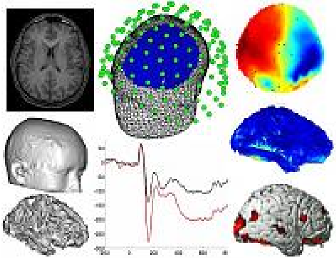
Inverse Problems in Physics: Wave

Wave

A disturbance that travels through a medium from one location to another (transferring energy).

Such fields arise in acoustics, electromagnetics, fluid dynamics and so on. Sensor networks measure:

- Bioelectric neural currents in neurons of cerebral cortex (EEG/MEG),
- Pressure waves from a speaker/acoustic source.



$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(\mathbf{x},t) - \nabla^2 u(\mathbf{x},t) = f(\mathbf{x},t).$$

Sensor Networks and Inverse Problems

Other PDEs: Laplace's Equation, Advection-/Convection-Diffusion Equation, Helmholtz and many more.
Given these (spatiotemporal) measurements we may wish to find:

- source of factory leakage, detect plume sources
- find hot/cold spots in server clusters
- predict nuclear fallout concentration elsewhere
- center of mass of active regions
- acoustic source localization

Sources can be localized or non-localized $\longrightarrow$ Parameterize sources f.

Problem Formulation: Field Sources

	Instantaneous	Non-Instantaneous
Point	$f(\mathbf{x},t)=\sum_{m=1}^{M} c_m \delta(\mathbf{x}-\boldsymbol{\xi}_m, t-\tau_m)$	$f(\mathbf{x},t)=\sum_{m=1}^{M} c_m e^{\alpha_m(t-\tau_m)} \delta(\mathbf{x}-\boldsymbol{\xi}_m) H(t-\tau_m)$
Line	$f(\mathbf{x},t)=cL(\mathbf{x})\delta(t-\tau)$	$f(\mathbf{x},t)=cL(\mathbf{x})e^{\alpha(t-\tau)}H(t-\tau)$
Polygonal	$f(\mathbf{x},t)=cF(\mathbf{x})\delta(t-\tau)$	$f(\mathbf{x},t)=cF(\mathbf{x})e^{\alpha(t-\tau)}H(t-\tau)$

Where,

- $L(\mathbf{x}) \in \Omega$ describes a line with endpoints $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$.
- $F(\mathbf{x}) \in \Omega$ describes a convex polygon with vertices $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_M\}$.
- $\alpha_m, c_m, \boldsymbol{\xi}_m$ and τ_m is the release rate, intensity, location and activation time of m-th source.

Problem Formulation: Field PDE Model

Let $u(\mathbf{x}, t)$ denote the field induced by a source distribution $f(\mathbf{x}, t)$ then a physics-driven system, in general, has the Green's function solution:

$$u(\mathbf{x}, t) = (f * g)(\mathbf{x}, t) = \int_{\mathbf{x}' \in \mathbb{R}^2} \int_{t' \in \mathbb{R}} g(\mathbf{x}', t') f(\mathbf{x} - \mathbf{x}', t - t') \, \mathrm{d}t' \mathrm{d}\mathbf{x}' \quad (2)$$

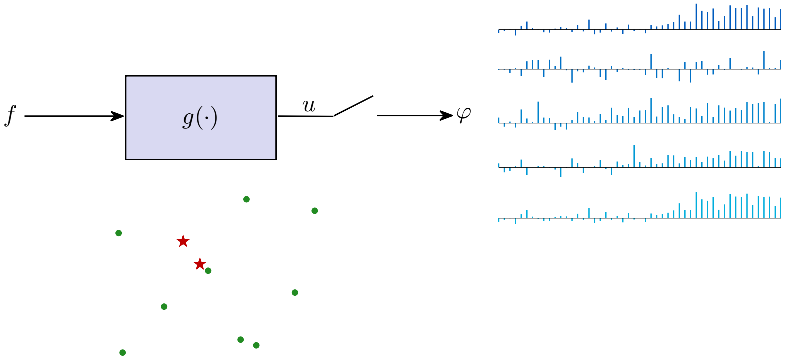
where $g(\mathbf{x}, t)$ is the Green function of the field.
For e.g.,

- **2D diffusion field:** $\frac{\partial}{\partial t} u(\mathbf{x}, t) - \mu \nabla^2 u(\mathbf{x}, t) = f(\mathbf{x}, t)$, has $g(\mathbf{x}, t) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{\|\mathbf{x}\|^2}{4\mu t}} H(t)$, where $H(t)$ is the step function.

Problem Formulation: Field Measurements

Aim

Estimate $f(\mathbf{x}, t)$ from spatiotemporal samples $\{\varphi_{n,l} = u(\mathbf{x}_n, t_l)\}_{n,l}$ for $n = 1, \ldots, N$ and $l = 0, \ldots, L$, of the measured field.



Recall that

$$\begin{aligned} u(\mathbf{x}, t) &= \int_{\mathbf{x}' \in \mathbb{R}^2} \int_{t' \in \mathbb{R}} g(\mathbf{x}', t') f(\mathbf{x} - \mathbf{x}', t - t') \, \mathrm{d}t' \mathrm{d}\mathbf{x}' \\ &= \langle f(\mathbf{x}', t'), g(\mathbf{x} - \mathbf{x}', t - t') \rangle_{\mathbf{x}', t'} . \end{aligned}$$

Mathematically the spatiotemporal sample $\varphi_{n,l}$ is

$$\begin{aligned} \varphi_{n,l} &= u(\mathbf{x}_n, t_l) \\ &= \langle f(\mathbf{x}, t), g(\mathbf{x}_n - \mathbf{x}, t_l - t) \rangle_{\mathbf{x}, t} \end{aligned} \tag{3}$$

Consider a weighted-sum of the samples $\{\varphi_{n,l}\}_{n,l}$:

$$\sum_{n=1}^{N}\sum_{l=0}^{L} w_{n,l}\varphi_{n,l} = \sum_{n=1}^{N}\sum_{l=0}^{L} w_{n,l} \left\langle f(\mathbf{x},t), g(\mathbf{x}_n - \mathbf{x}, t_l - t)\right\rangle_{\mathbf{x},t}$$
$$= \left\langle f(\mathbf{x},t), \underbrace{\sum_{n=1}^{N}\sum_{l=0}^{L} w_{n,l} g(\mathbf{x}_n - \mathbf{x}, t_l - t)}_{=\Psi_k(\mathbf{x})\Gamma(t)} \right\rangle, \qquad (4)$$

where $w_{n,l} \in \mathbb{C}$ are some arbitrary weights (to be determined).
We wish to find $f(\mathbf{x},t)$:

- For our source types, can we choose functions $\Psi_k(\mathbf{x})$ and $\Gamma(t)$ that makes this problem tractable? — YES!

Let these (new) *generalized measurements* be

$$\mathcal{R}(k) = \sum_{n=1}^{N} \sum_{l=0}^{L} w_{n,l}\varphi_{n,l} = \langle f(\mathbf{x}, t), \Psi_k(\mathbf{x})\Gamma(t)\rangle$$
$$= \int_{\Omega}\int_{t\in[0,T]} \Psi_k(\mathbf{x})\Gamma(t)f(\mathbf{x}, t)\mathrm{d}t\mathrm{d}V,$$

where $\Psi_k(\mathbf{x})$ for $k \in \mathbb{Z}^d$, $d = \{1, 2\}$, and $\Gamma(t)$ a family of properly chosen *spatial* and *temporal sensing functions*, respectively.
Proper choice $\Longrightarrow$ solvability & stability of new problem.

- As an example, take the **instantaneous** source distribution $f(\mathbf{x}, t) = \sum_{m=1}^{M} c_m\delta(\mathbf{x} - \boldsymbol{\xi}_m, t - \tau_m)$, then:

$$\mathcal{R}(k) = \sum_{m=1}^{M} c_m\Psi_k(\boldsymbol{\xi}_m)\Gamma(\tau_m).$$

Choice of Sensing Functions: 2D Case

For $\mathbf{x} \in \mathbb{R}^2$, we may choose

- $\Gamma(t) = e^{-\mathrm{j}t/T}$, and
- $\Psi_k(\mathbf{x}) = e^{-k(x_1 + \mathrm{j}x_2)}$, for $k = 0, 1, \ldots, K$.

Then,

$$\mathcal{R}(k) = \sum_{m=1}^{M} c_m e^{-\mathrm{j}\tau_m/T} e^{-k(\xi_{1,m} + \mathrm{j}\xi_{2,m})}$$
$$= \sum_{m=1}^{M} c'_m v_m^k.$$

Can be solved to jointly recover $c'_m = c_m e^{-\mathrm{j}\tau_m/T}$ and $v_m = e^{-(\xi_{1,m} + \mathrm{j}\xi_{2,m})}$ for $m = 1, \ldots, M$ from $\{\mathcal{R}(k)\}_{k=0}^{K}$, using Prony's method providing $K \geq 2M - 1$.

Choice of Sensing Functions: 3D Case

For $\mathbf{x} \in \mathbb{R}^3$, we may choose

- $\Gamma(t) = e^{-\mathrm{j}t/T}$, and
- $\Psi_k \to \Psi_{k_1,k_2}(\mathbf{x}) = e^{-k_1(x_1+\mathrm{j}x_2)-\mathrm{j}k_2x_3}$, for $k_1, k_2 \in \{0, 1, \ldots, K\}$.

Consequently,

$$\begin{aligned}\mathcal{R}(k_1, k_2) &= \sum_{m=1}^{M} c_m e^{-\mathrm{j}\tau_m/T} e^{-k_1(\xi_{1,m}+\mathrm{j}\xi_{2,m})-\mathrm{j}k_2\xi_{3,m}} \\ &= \sum_{m=1}^{M} c'_m \alpha_m^{k_1} \beta_m^{k_2}.\end{aligned}$$

Algebraically Coupled Matrix Pencil (ACMP) applied on sequence $\{\mathcal{R}(k_1, k_2)\}_{k_1,k_2}$ can recover jointly all unknowns.

Non-localized Source

The Instantaneous Line Source: $f(\mathbf{x},t) = cL(\mathbf{x})\delta(t-\tau)$, thus $\mathcal{R}(k)$ reduces to:

$$\begin{aligned}
\mathcal{R}(k) &= \int_\Omega \int_t \Psi_k(\mathbf{x})\Gamma(t) f(\mathbf{x},t)\,\mathrm{d}t\,\mathrm{d}V \\
&= c\Gamma(\tau)\int_\Omega \Psi_k(\mathbf{x}) L(\mathbf{x})\,\mathrm{d}V \\
&= c\Gamma(\tau)\int_{L(\mathbf{x})} \Psi_k(\mathbf{x})\,\mathrm{d}S \\
&= \frac{1}{k} c\ell(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\Gamma(\tau)\sum_{m=1}^{2}(-1)^m \Psi_k(\boldsymbol{\xi}_m)
\end{aligned}$$

Non-localized source

From $\mathcal{R}(k) = \frac{1}{k} c\ell(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)\Gamma(\tau) \sum_{m=1}^{2} (-1)^m \Psi_k(\boldsymbol{\xi}_m)$ and the usual choice for sensing functions $\Gamma(t) = e^{-\mathrm{j}t/T}$ and $\Psi_k(\mathbf{x}) = e^{-k(x_1 + \mathrm{j}x_2)}$, then:

$$\begin{aligned}\mathcal{R}'(k) \triangleq k\mathcal{R}(k) &= c\ell(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)\Gamma(\tau) \sum_{m=1}^{2} (-1)^m \Psi_k(\boldsymbol{\xi}_m) \\ &= c\ell(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) e^{-\mathrm{j}\tau/T} \sum_{m=1}^{2} (-1)^m e^{-k(\xi_{1,m} + \mathrm{j}\xi_{2,m})}\end{aligned}$$

Can again recover c, τ and the endpoints $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ of the line source using Prony's method from $\{\mathcal{R}'(k)\}_{k=1}^{K}$ (providing $K \geq 4$).

- **Polygonal sources** (complex analysis):
  surface integral $\rightarrow$ line integral $\rightarrow$ Ψ_k evaluated at vertices

Computing $\mathcal{R}(k)$ reliably from sensor Measurements?

$$\mathcal{R}(k) = \sum_{n=1}^{N} \sum_{l=0}^{L} w_{n,l} \varphi_{n,l}$$

Thus computing $\mathcal{R}(k)$ is equivalent to finding the weights $w_{n,l}$. These weights may be found:

1. By formulating a linear system (**explicit**)
   - Inversion of large matrices.
   - Conditioning and stability considerations.
   - Uniform/non-uniform samples.
2. Using results universal sampling/FRI theory (**explicit**)
   - Approximate Strang-Fix theory.
   - Exponential reproduction.
   - Uniform samples.
3. Using Green's second identity (**implicit**)
   - For 2D diffusion field.
   - Uniform/non-uniform samples.

Explicit computation I: Linear System

We desire $\{w_{n,l}\}_{n,l}$, so that $\sum_{n=1}^{N}\sum_{l=0}^{L} w_{n,l}g(\mathbf{x}_n - \mathbf{x}, t_l - t) = \Psi_k(\mathbf{x})\Gamma(t)$, where g, Ψ_k and Γ are known.

For e.g. the 2D heat problem $g(\mathbf{x}, t) = \frac{1}{4\pi t} e^{-\frac{\|\mathbf{x}\|^2}{4\mu t}} H(t)$, also $\Gamma(t) = e^{-\mathrm{j}t/T}$ and $\Psi_k(\mathbf{x}) = e^{-k(x_1+\mathrm{j}x_2)}$.

1. Can formulate a **linear system**:

$$\begin{bmatrix} g(\mathbf{x}_1 - \mathbf{x}'_1, t_l - t'_j) & \cdots & g(\mathbf{x}_N - \mathbf{x}'_1, t_l - t'_j) \\ \vdots & & \vdots \\ g(\mathbf{x}_1 - \mathbf{x}'_I, t_l - t'_j) & \cdots & g(\mathbf{x}_N - \mathbf{x}'_I, t_l - t'_j) \end{bmatrix} \begin{bmatrix} w_{1,l} \\ \vdots \\ w_{N,l} \end{bmatrix} = \begin{bmatrix} \Psi_k(\mathbf{x}'_1)\Gamma(t'_j) \\ \vdots \\ \Psi_k(\mathbf{x}'_I)\Gamma(t'_j) \end{bmatrix}$$

$$\mathbf{G}_{l,j}\mathbf{w}_l = \mathbf{p}_j$$

$$\Rightarrow \begin{bmatrix} \mathbf{G}_{0,1} & \cdots & \mathbf{G}_{L,1} \\ \vdots & & \vdots \\ \mathbf{G}_{0,J} & \cdots & \mathbf{G}_{L,J} \end{bmatrix} \begin{bmatrix} \mathbf{w}_0 \\ \vdots \\ \mathbf{w}_L \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_J \end{bmatrix}.$$

$$\mathbf{G}\mathbf{w} = \mathbf{p}$$

2. Solve $\mathbf{G}\mathbf{w} = \mathbf{p}$, where $\mathbf{G} \in \mathbb{R}^{N(L+1)\times IJ}$, $\mathbf{w} \in \mathbb{R}^{N(L+1)}$ and $\mathbf{p} \in \mathbb{R}^{IJ}$.

Explicit computation II: Approx. Strang-Fix

Alternatively, can obtain a closed-form expression for $w_{n,l}$

- **Task:** compute $\{w_{n,l}\}_{n,l}$ such that,

$$\sum_{n}\sum_{l=0}^{L} w_{n,l}\, g(\mathbf{x}_n - \mathbf{x}, t_l - t) = e^{-k(x_1+\mathrm{j}x_2)} e^{-\mathrm{j}t/T}.$$

- Our problem is multi-dimensional so in 2D, for e.g., $\mathbf{x}_n = (n_1\Delta_{x_1}, n_2\Delta_{x_2})$ and $t_l = l\Delta_t$, we actually desire[1]:

$$\sum_{n_1,n_2,l} w_{n_1,n_2,l}(k)\, g(n_1\Delta_{x_1} - x_1, n_2\Delta_{x_2} - x_2, l\Delta_t - t) = e^{-k(x_1+\mathrm{j}x_2)} e^{-\mathrm{j}t/T}.$$

[1] A lexicographic ordering of $\{(n_1\Delta_{x_1}, n_2\Delta_{x_2})\}_{n_1,n_2}$ gives the usual $n = 1, \ldots, N$, where $N = N_1 N_2$.

Explicit computation II: Approx. Strang-Fix

Consider classical 1D exponential reproduction problem:

$$\sum_{n\in\mathbb{Z}} w_n(k)g(x-n) = e^{\mathrm{j}\omega_k x},$$

for $k \in \mathbb{Z}$.

1. **Strang-Fix Conditions:** Above equation holds iff

$$G(\omega_k) \neq 0 \ \text{ and } \ G(\omega_k + 2\pi\ell) = 0 \ \ \forall \ell \in \mathbb{Z}\backslash\{0\},$$

   where $G = \mathcal{F}(g)$ is the Fourier transform of g (Poisson Summation).

2. **Approximate Strang-Fix Conditions:** For approximate exponential reproduction, i.e. $\sum_{n\in\mathbb{Z}} w_n(k)g(x-n) \approx e^{\mathrm{j}\omega_k x}$, then

$$w_n(k) = \frac{1}{G(\omega_k)} e^{\mathrm{j}\omega_k n}.$$

   The *constant-least squares* coefficients.

Explicit computation II: Approx. Strang-Fix

Can extend to multiple dimensions using Poisson summation formula for lattices and the multi-dimensional Fourier Transform of $g(\mathbf{x}, t)$:

$$G(\omega_{x_1}, \omega_{x_2}, \omega_t) = \int_{t \in \mathbb{R}} \int_{\mathbf{x} \in \mathbb{R}^2} g(\mathbf{x}, t) e^{-\mathrm{j}(\omega_{x_1} x_1 + \omega_{x_2} x_2 + \omega_t t)} \mathrm{d}\mathbf{x} \mathrm{d}t.$$

The desired coefficients for problem in 2D space and time is:

$$w_{n_1, n_2, l}(k) = \frac{1}{G(-\mathrm{j}k, k, 1/T)} e^{kn_1} e^{\mathrm{j}kn_2} e^{\mathrm{j}l/T}.$$

Implicit computation of weights $\{w_{n,l}\}_{n,l}$

1. **Green's second identity:** Let $u(\mathbf{x}, t)$ and $\Psi_k(\mathbf{x})$ be scalar functions in $\mathcal{C}^2$, over $\Omega \in \mathbb{R}^2$, then:
$$\oint_{\partial\Omega} (\Psi_k \nabla u - u \nabla \Psi_k) \cdot \hat{\mathbf{n}}_{\partial\Omega} \mathrm{d}S = \int_{\Omega} \left(\Psi_k \nabla^2 u - u \nabla^2 \Psi_k\right) \mathrm{d}V,$$
where $\hat{\mathbf{n}}_{\partial\Omega}$ is the outward pointing unit normal to the boundary $\partial\Omega$.
2. Substitute (inhomogenous) PDE and choose Ψ_k to satisfy $\frac{\partial \Psi_k}{\partial t} + \mu \nabla^2 \Psi_k = 0$, thus:
$$\int_{\Omega} \frac{\partial}{\partial t}(u\Psi_k)\mathrm{d}V - \mu \oint_{\partial\Omega} (\Psi_k \nabla u - u \nabla \Psi_k) \cdot \hat{\mathbf{n}}_{\partial\Omega} \mathrm{d}S = \int_{\Omega} \Psi_k f \mathrm{d}V.$$
3. Multiply through by $\Gamma(t)$ and integrate over $t = [0, T]$:

$$\underbrace{\int_0^T \Gamma \int_{\Omega} \Psi_k \frac{\partial u}{\partial t} + u \frac{\partial \Psi_k}{\partial t} \mathrm{d}V - \mu \oint_{\partial\Omega} (\Psi_k \nabla u - u \nabla \Psi_k) \cdot \hat{\mathbf{n}}_{\partial\Omega} \mathrm{d}S \mathrm{d}t}_{= \mathcal{R}(k)} = \int_{\Omega} \int_0^T \Psi_k \Gamma f \, \mathrm{d}t \, \mathrm{d}V$$

Implicit computation of weights $\{w_{n,l}\}_{n,l}$

From:

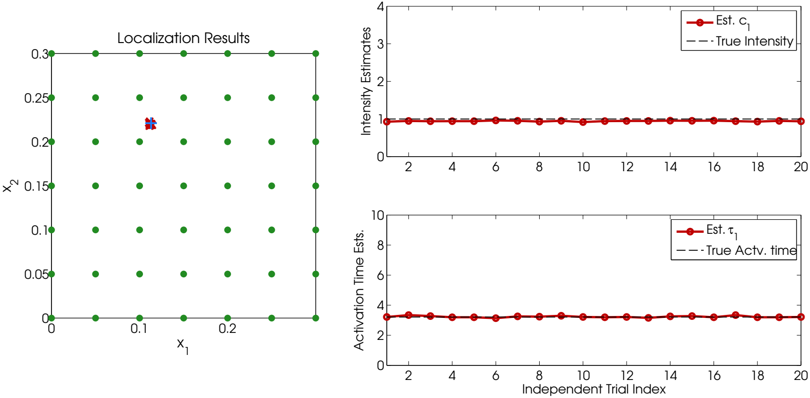
$$\underbrace{\int_0^T \Gamma \int_\Omega \Psi_k \frac{\partial u}{\partial t} + u \frac{\partial \Psi_k}{\partial t} \mathrm{d}V - \mu \oint_{\partial\Omega} (\Psi_k \nabla u - u \nabla \Psi_k) \cdot \hat{\mathbf{n}}_{\partial\Omega} \mathrm{d}S \mathrm{d}t}_{=\mathcal{R}(k)} = \int_\Omega \int_0^T \Psi_k \Gamma f \,\mathrm{d}t \,\mathrm{d}V$$

$$\Rightarrow \mathcal{R}(k) = \langle f(\mathbf{x}, t), \Psi_k(\mathbf{x}) \Gamma(t) \rangle$$

As such we can obtain $\{\mathcal{R}(k)\}$ by **approximating the integrals** from the spatiotemporal samples using standard quadrature schemes.

- Mesh required.
- Integral simply a linear combination of field samples.
- Distributed computation (consensus-based estimation).

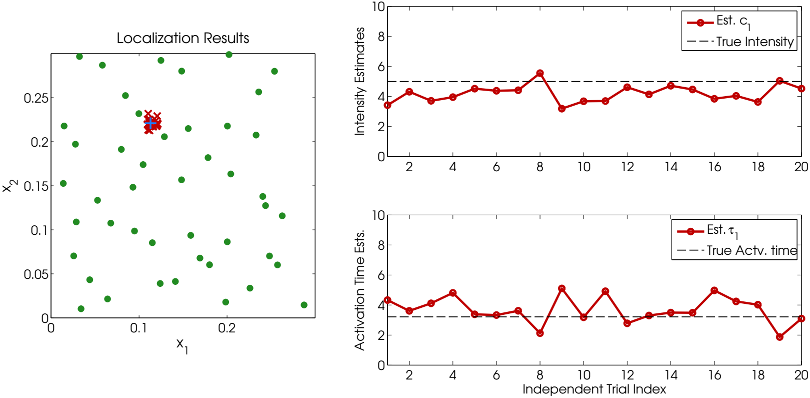
Synthetic data: Point Diffusion Source



(a) Uniform spatial sampling ($N = 49$)

Centralized estimation for $M = 1$ diffusion source in 2D, field is sampled for $T_{end} = 10s$ at $\frac{1}{\Delta t} = 1Hz$. Here $K = 10$.

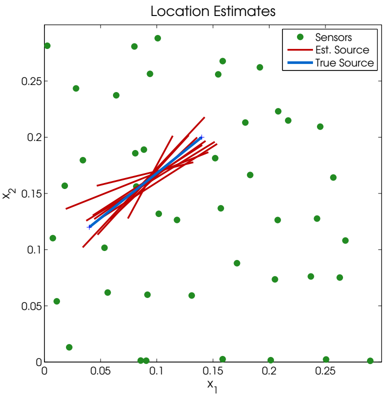
Synthetic data: Point Diffusion Source

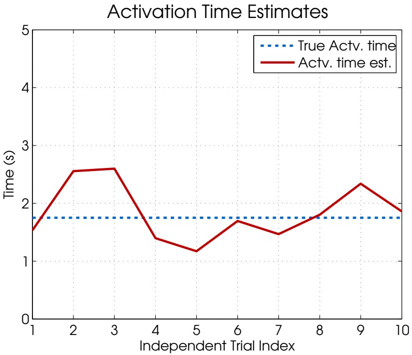


(b) Non-uniform spatial sampling ($N = 45$)

Centralized estimation for $M = 1$ diffusion source in 2D, field is sampled for $T_{end} = 10s$ at $\frac{1}{\Delta t} = 1Hz$. Here $K = 10$.

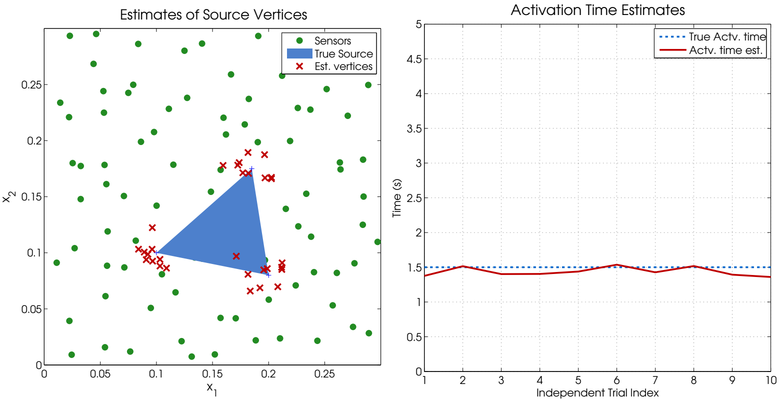
Synthetic data: Line Diffusion Source





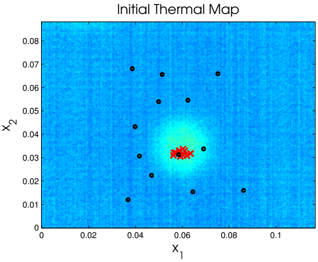
$N = 45$ arbitrarily placed sensors, field sampled at $10Hz$ for $T = 10s$ with measurement SNR$= 20$dB. $K = 6$ and $R = 5$.

Synthetic data: Triangular Diffusion Source

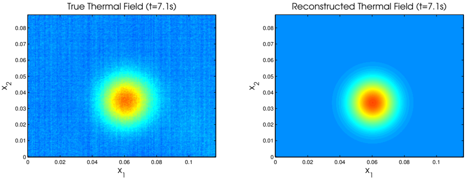


$N = 90$ arbitrarily placed sensors, field sampled at $10Hz$ for $T = 10s$ with measurement SNR$= 35$dB. $K = 6$ and $R = 5$.

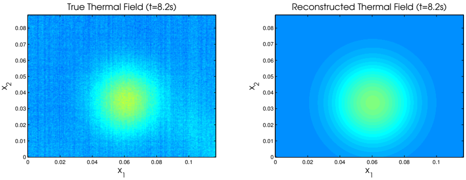
Simulation Results: Real Diffusion Data



(a) Thermal distribution (immediately after activation) and location estimates.



(b) Real field (left) and its reconstruction (right) at $t = 7.1s$.



(c) Real field (left) and its reconstruction (right) at $t = 8.2s$.

Simulation Results: Laplace - Synthetic data

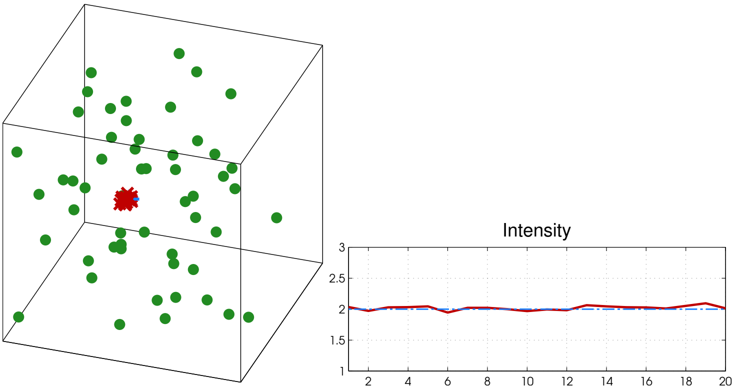


Figure 1: Single point source recovery in 3D using samples obtained by $N = 57$ sensors with $K_1 = K_2 = 1$ for spatial sensing function family. Results for 20 independent trials are given.

Conclusion

1. Reconstructing localized and non-localized sources: point, line and (convex) polygons.
   - Compute generalized measurements.
   - Use tools from complex analysis to modify $\mathcal{R}(k)$.
   - Recover location of point sources or endpoints (vertices) of line (polygonal) source.

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2. Further extensions
   - Reconstructing localized sources in bounded regions (rooms).
   - 3D source recovery.

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   - Recover location of point sources or endpoints (vertices) of line (polygonal) source.
2. Further extensions
   - Reconstructing localized sources in bounded regions (rooms).
   - 3D source recovery.
3. **Generalisation Possible**
   - Same principle can be generalized to PDE-driven fields: wave, Poisson etc.
   - Compute the analysis coefficients $\{w_{n,l}\}$.
   - Turn to Finite Rate of Innovation (FRI) theory: exponential reproduction.

Thank You.

Approximate Strang-Fix Error

From

$$w_{k,n} = \frac{1}{G(\omega_k)} e^{\mathrm{j}\omega_k n},$$

then the approximation $\hat{\psi}_k(x)$ of the exponential $\psi_k(x) = e^{\mathrm{j}\omega_k x}$ is $\hat{\psi}_k(x) = \sum_{n\in\mathbb{Z}} w_n(k) g(x-n)$.
This becomes $\hat{\psi}_k(x) = e^{\mathrm{j}\omega_k x} \frac{1}{G(\omega_k)} \sum_{\ell\in\mathbb{Z}} G(\omega_k + 2\pi\ell) e^{\mathrm{j}2\pi\ell x}$ when we substitute $w_n(k) = \frac{1}{G(\omega_k)} e^{\mathrm{j}\omega_k n}$ and apply Poisson's summation formula.
We obtain the error $\varepsilon(x) = \psi_k(x) - \hat{\psi}_k(x)$ for this approximation:

$$\varepsilon(x) = e^{\mathrm{j}\omega_k x} \left(1 - \frac{1}{G(\omega_k)} \sum_{\ell\in\mathbb{Z}} G(\omega_k + 2\pi\ell) e^{\mathrm{j}2\pi\ell x}\right).$$

Will be small if $G(\omega_k + 2\pi\ell)$ decays quickly enough to zero as $|\ell|$ increases.

- Exponential decay for Gaussian.