On the Direct and Inverse Scattering Problem for Periodic Media

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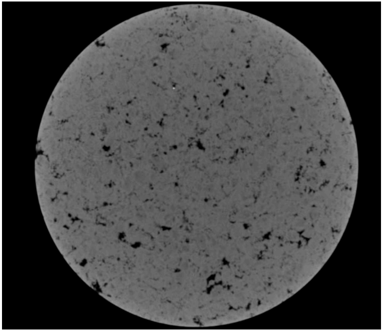
Introduction

- Composite materials are at the foundation of many contemporary engineering designs. Typically such materials have periodic structure.
- Non-destructive testing



Important question: What kind of information about the microstructure can we detect from scattering data?

Introduction

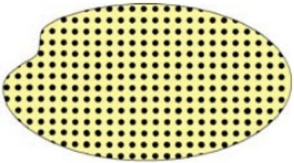
1. Mathematical understanding of wave propagation through periodic media of bounded support.

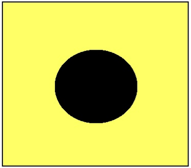
   Joint work with B. Guzina and S. Moskow

2. Transmission eigenvalues for a periodic medium and their use to obtain information about the effective material properties of the medium.

   Joint work with H. Haddar and I. Harris

Scattering by a Periodic Media





Y

$Y := [0, 1]^d$, $\epsilon > 0$ small compared to ka. The wave number k is fixed. $x \in D$ is referred to as the slow variable and $x/\epsilon \in Y$ as the fast variable. Let $A(y) = (a_{ij}(y))$ be a symmetric matrix and $n(y)$ s.th.

- $a_{ij}(y)$ and $n(y)$ are periodic functions with period Y
- $\inf_{y \in Y} \inf_{|\xi|=1} \bar{\xi} \cdot A(y)\xi = A_{min} > 0$, $\sup_{y \in Y} \sup_{|\xi|=1} \bar{\xi} \cdot A(y)\xi = A_{max} < \infty$

  $\inf_{y \in Y} n(y) = n_{min} > 0$ and $\sup_{y \in Y} n(y) = n_{max} < \infty$

The material properties of the inhomogeneity D are represented by

$A_\epsilon := A(x/\epsilon)$ and $n_\epsilon := n(x/\epsilon)$ for $x \in D$.

Scattering by an Inhomogeneous Media

The scattering problem for a given incident wave u^i by the periodic media D reads:

$$\begin{aligned}
\nabla_x \cdot A(x/\epsilon)\nabla_x w_\epsilon + k^2 n(x/\epsilon) w_\epsilon &= 0 \quad \text{in } D \\
\Delta_x u_\epsilon + k^2 u_\epsilon &= 0 \quad \text{in } \mathbb{R}^d \setminus \overline{D} \\
u_\epsilon - w_\epsilon &= f \quad \text{on } \partial D \\
\nu \cdot \nabla_x u_\epsilon - \nu \cdot A(x/\epsilon)\nabla_x w_\epsilon &= g \quad \text{on } \partial D
\end{aligned}$$

where the scattered field u_ϵ satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u_\epsilon}{\partial r} - iku_\epsilon \right) = 0, \qquad r = |x| \to \infty$$

$f := -u^i$ and $g := -\nu \cdot \nabla_x u^i$ on the interface ∂D.

Homogenization Approach

BENSOUSSAN-LIONS-PAPANICOLAOU (1978), KESAVAN (1979), MOSKOW-VOGELIUS (1993), ALLAIR (2002), KENIG-LIN-SHEN, (2012)-(2013) . . .

- Start with the ansatz

$$w_\epsilon(x) \approx w_0(x,y) + \epsilon w_1(x,y) + \epsilon^2 w_2(x,y) + \cdots$$

$$u_\epsilon(x) \approx u_0(x,y) + \epsilon u_1(x,y) + \epsilon^2 u_2(x,y) + \cdots$$

- Use the multi-scale differentiation

$$\nabla = \nabla_x + \frac{1}{\epsilon}\nabla_y$$

- Compare the powers of ϵ

Note: For an arbitrary function $\phi(x,y)$, $x \in D$ and $y \in Y$, $\bar{\phi}(x)$ denotes the mean value

$$\bar{\phi}(x) := \int_Y \phi(x,y)\,dy.$$

Homogenization Approach

- $u_\epsilon = u_0(x)$ in the exterior of D
- w_0 depends only on x
- the homogenized (formal) limiting problem is

$$\begin{aligned} \nabla \cdot A_h \nabla w_0 + k^2 n_h w_0 = 0 &\quad \text{in} \quad D \\ \Delta u_0 + k^2 u_0 = 0 &\quad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \\ u_0 - w_0 = f &\quad \text{on} \quad \partial D \\ \nu \cdot \nabla u_0 - \nu \cdot A_h \nabla w_0 = g &\quad \text{on} \quad \partial D. \end{aligned}$$

$$\text{where} \quad n_h = \overline{n} \quad \text{and} \quad (A_h)_{ij} = \int_Y \left(a_{ij}(y) - a_{ik}(y) \frac{\partial \chi^j}{\partial y_k}(y) \right) dy$$

and the cell functions $\chi^j(y)$ are $H^1(Y)$ periodic solutions to

$$\nabla_y \cdot A(y) \nabla_y \chi^j(y) = \nabla_y \cdot A(y) y_j = \frac{\partial}{\partial y_i} a_{ij}(y)$$

with zero-mean. (We use the Einstein summation notation).

Homogenization Approach

- Furthermore

$$w_1(x,y) = -\chi^j(y)\frac{\partial w_0}{\partial x_j} + \overline{w}_1(x) \text{ referred to as bulk correction}$$

- It can be shown that $\overline{w}_1(x)$ satisfies

$$\nabla\cdot A_h\nabla\overline{w}_1 + k^2 n_h\overline{w}_1 = \left(\overline{a_{ki}\chi^j} - \overline{a_{kl}\frac{\partial\chi^{ij}}{\partial y_l}}\right)\frac{\partial^3 w_0}{\partial x_i\partial x_j\partial x_k} - k^2\left(\overline{a_{ki}\frac{\partial\beta}{\partial y_i}} - \overline{n\chi^k}\right)\frac{\partial w_0}{\partial x_k}$$

  where β is the unique zero-mean Y-periodic solution to

$$\nabla_y\cdot A_h\nabla_y\beta(y) = \overline{n} - n(y)$$

  The source term can be shown to be zero! Hence it suffices to take $\overline{w}_1 = 0$, in D.

- $w_0 + \epsilon w_1$ and u_0 do not satisfy the transmission conditions (even approximately to the correct order). Hence we need to correct for the boundary.

Homogenization Approach

Our boundary corrector is the radiating solution to

$$\begin{aligned}
\nabla\cdot A(x/\epsilon)\nabla\theta_\epsilon + k^2 n(x/\epsilon)\theta_\epsilon &= 0 && \text{in } D\\
\Delta\theta_\epsilon + k^2\theta_\epsilon &= 0 && \text{in } \mathbb{R}^d\setminus\overline{D}\\
\theta_\epsilon^+ - \theta_\epsilon^- &= -\chi^j(y)\frac{\partial w_0}{\partial x_j} && \text{on } \partial D\\
(\nabla\theta_\epsilon\cdot\nu)^+ - (A(x/\epsilon)\nabla\theta_\epsilon\cdot\nu)^- &= \big(\mathrm{rot}(q) + k^2 A_h\nabla_y\beta(y)w_0\big)\cdot\nu && \text{on } \partial D
\end{aligned}$$

where $q(x,y)$ is the Y-periodic solution to (note that $v_0 - A_h\nabla w_0$ has Y-average zero and zero divergence)

$$\mathrm{rot}_y(q) = v_0 - A_h\nabla w_0 \qquad \text{with}$$

$$(v_0(x,y))_i = \left(a_{ij}(y) - a_{ik}(y)\frac{\partial\chi^j}{\partial y_k}(y)\right)\frac{\partial w_0}{\partial x_j}.$$

Alternatively, the Neumann transmission condition read

$$\left(\frac{v_0-\overline{v_0}}{\epsilon} + \mathrm{rot}_x(q) + k^2 A_h\nabla_y\beta(y)w_0\right)\cdot\nu$$

Homogenization Approach

Let $U_\epsilon \in H^1_{loc}(\mathbb{R}^d)$ is such $U_\epsilon = u_\epsilon$ in $\mathbb{R}^d \setminus \overline{D}$ and $U_\epsilon = w_\epsilon$ in D for $\epsilon \geq 0$.

Lemma

For any ball B_R containing D,

$$\|U_\epsilon - (U_0 + \epsilon U_1 + \epsilon\theta_\epsilon)\|_{H^1(B_R)} \leq C\epsilon\|w_0\|_{H^2(D)}$$

where the constant C is independent of ϵ and w_0.

The proof is done by a duality argument to bound

$$\int_{B_R} (U_\epsilon - U_0 - \epsilon U_1 - \epsilon\theta_\epsilon)\,\phi\,dx$$

by $\|\phi\|_{H^{-1}(B_R)}$ independently of ϵ.

Homogenization Approach

The analysis of the boundary corrector function θ_ϵ

$$\|\theta_\epsilon\|_{H^1(D)} + \|\theta_\epsilon\|_{H^1(B_R\setminus\overline{D})} \leq C_R\epsilon^{-1/2}\|w_0\|_{H^2(D)}$$

$$\|\theta_\epsilon\|_{L^2(B_R)} \leq C_R\|w_0\|_{H^2(D)}.$$

Note The L^2-estimate is proven only for the corrector θ_ϵ using duality argument and the previous lemma. L^2-L^2 general estimates for the transmission problem are not available (see AVELLANEDA-LIN and KENIG-LIN-SHEN for the Dirichlet case.)

Unfortunately, finding the limit of the boundary corrector θ_ϵ as $\epsilon \to 0$ is a very hard problem in general

There is a vast literature in the case of the Dirichlet or Neumann boundary value problems (BENSOUSSAN-LIONS-PAPANICOLAOU, ALLAIR, MOSKOW-VOGELIUS, GÉRARD VARET-MASMOUDI . . .)
For the boundary corrector at a straight interface between a periodic and a homogeneous half space (CLAY-FLISS-VINOLES)

Homogenization Approach

Theorem

Let u_ϵ, w_ϵ be the solution to the original problem, u_0, w_0 the solution to the homogenized problem, and $w_1 = -\chi^j(y)\frac{\partial w_0}{\partial x_j}$ the bulk correction. Then

$$\|u_\epsilon - u_0\|_{H^1(B_R\setminus D)} + \|w_\epsilon - (w_0 + \epsilon w_1)\|_{H^1(D)} \leq C\epsilon^{1/2}\|w_0\|_{H^2(D)}$$

$$\|u_\epsilon - u_0\|_{L^2(B_R\setminus D)} + \|w_\epsilon - w_0\|_{L^2(D)} \leq C\epsilon\|w_0\|_{H^2(D)}.$$

Theorem

In addition let θ_ϵ be the boundary corrector. Then

$$\|w_\epsilon - (w_0 + \epsilon w_1 + \epsilon\theta_\epsilon)\|_{L^2(B_R)} \leq C\epsilon^2\|w_0\|_{H^4(D)}$$

$$\|w_\epsilon - (w_0 + \epsilon w_1 + \epsilon\theta_\epsilon)\|_{H^1(D)} + \|u_\epsilon - (u_0 + \epsilon\theta_\epsilon)\|_{H^1(B_R\setminus D)} \leq C\epsilon^{3/2}\|w_0\|_{H^4(D)}$$

Homogenization Approach

Remarks

- While the bulk correction is necessary for H^1 convergence, it does not in general improve upon the L^2 estimate. Unless the boundary correction goes to zero (in general not the case)
- Our convergence approach can be iteratively used to obtain higher order estimates.The boundary correctors have the same structure.
- Interesting is that starting at the order ϵ^2, the mean of the scattered field outside D is affected not only by the boundary layer but also by the mean of the bulk correction. This is referred to as "dispersion effect" ALLAIRE-AMAR 1999, WAUTIER-GUZINA 2014.

  Such an effect could be important for the inverse problem of detecting microstructure effects in the measured (mean) scattered field outside the periodic media.

The Limit of the Boundary Layer

Example of a Boundary Corrector Limit

- Consider $D := [0,\,1] \times [0,\,1]$ (more generally it can be a convex polygonal domain with sides of rational or infinite slope) and by linearity compute $\lim_{\epsilon\to 0}\theta_\epsilon$ with non-zero transmission data only on one side at a time. Let us fix the side $x_1 = 1$, where

$$\theta_\epsilon^+ - \theta_\epsilon^- = \chi^1(x/\epsilon)\frac{\partial w_0}{\partial x_1} \quad \text{on } \partial D \cap \{x_1 = 1\}$$

$$(\nabla\theta_\epsilon \cdot \nu)^+ - (a(x/\epsilon)\nabla\theta_\epsilon \cdot \nu)^- = \frac{1}{\epsilon} g_1(x/\epsilon)\frac{\partial w_0}{\partial x_1} \quad \text{on } \partial D \cap \{x_1 = 1\}$$

where

$$g_1(x/\epsilon) = a_{11}(x/\epsilon) - a_{1k}(x/\epsilon)\frac{\partial \chi^1}{\partial y_k}(x/\epsilon) - A_{11}$$

The Limit of the Boundary Layer

Example of a Boundary Corrector Limit

The above transmission data depends on the choice of ϵ. If, for example $\epsilon_k = 1/k$ for k an integer, the boundary layer problem would see only the boundary slice of the periodic functions

$$\chi^1(y) \text{ and } g_1(y).$$

Hence one can expect different limits of θ_ϵ for different sequences of ϵ going to zero.

We assume that ϵ_k is a sequence going to zero for which the boundary cutoff is fixed, i.e. the fractional part of $1/\epsilon_k$ is constant, i.e.

$$\frac{1}{\epsilon_k} - \lfloor \frac{1}{\epsilon_k} \rfloor = \delta, \qquad \text{for all } k$$

and the boundary data are now

$$\chi^1(\delta, y_2) \text{ and } g_1(\delta, y_2).$$

The Limit of the Boundary Layer

Need the solution of a strip problem

$$G^+ = \{y_1 > 0; y_2 \in [0,1]\} \quad \text{and} \quad G^- = \{y_1 < 0; y_2 \in [0,1]\}$$

$$\begin{aligned}
\nabla_y \cdot a(y_1, y_2)\nabla w^- &= 0 && y_1 < 0, \ -\infty < y_2 < +\infty \\
\Delta_y w^+ &= 0 && y_1 > 0, \ -\infty < y_2 < +\infty \\
w^+(0, y_2) - w^-(0, y_2) &= \chi^1(\delta, y_2) && -\infty < y_2 < +\infty \\
\partial_{y_1} w^+(0, y_2) - a_{1i}(\delta, y_2)\partial_{y_i} w^-(0, y_2) &= g_1(\delta, y_2) && -\infty < y_2 < +\infty
\end{aligned}$$

such that w^+, w^- are periodic in y_2 and $e^{-\gamma y_1}\nabla w^- \in L^2(G^-)$ and $e^{\gamma y_1}\nabla w^+ \in L^2(G^+)$ for some $\gamma > 0$.

This problem has a unique solution up to an arbitrary constant (the proof adapts the approach in J.L. LIONS (1981))

Set $d^+ = \lim_{y_1 \to \infty} w$ and $d^- = \lim_{y_1 \to -\infty} w$ and define

$$\chi_1^* = d^+ - d^-$$

The Limit of the Boundary Layer

We can prove that $\theta_{\epsilon_k} \to \theta^*$ strongly in $L^2_{loc}(\mathbb{R}^2)$ where θ^* solves

$$\begin{aligned}
\nabla \cdot A \nabla \theta^* + k^2 \overline{n} \theta^* &= 0 \quad \text{in } D \\
\Delta \theta^* + k^2 \theta^* &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D} \\
(\theta^*)^+ - (\theta^*)^- &= \chi_1^* \frac{\partial w_0}{\partial x_1} \quad \text{on } \partial D \cap \{x_1 = 1\} \\
(\theta^*)^+ - (\theta^*)^- &= 0 \quad \text{on } \partial D \setminus \{x_1 = 1\} \\
(\nabla \theta^* \cdot \nu)^+ - (A \nabla \theta^* \cdot \nu)^- &= \overline{a_{12}(\delta, y_2) w(0, y_2)^-} \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \quad \text{on } \partial D \cap \{x_1 = 1\} \\
(\nabla \theta^* \cdot \nu)^+ - (A \nabla \theta^* \cdot \nu)^- &= 0 \quad \text{on } \partial D \setminus \{x_1 = 1\}
\end{aligned}$$

where A is the homogenized matrix, and $\overline{a_{12}(\delta, y_2) w(0, y_2)^-}$ denotes the average in the y_2 direction of $a_{12} w$ at $y_1 = 0$ coming from the left side of the strip.

Transmission Eigenvalues

Values of $k_\epsilon \in \mathbb{C}$ for which the transmission eigenvalue problem

$$\begin{aligned}
\Delta v + k^2 v = 0 & \qquad \text{in} \qquad D \\
\nabla \cdot A_\epsilon \nabla w + k^2 n_\epsilon w = 0 & \qquad \text{in} \qquad D \\
w = v & \qquad \text{on} \qquad \partial D \\
\nu \cdot A_\epsilon \nabla w = \nu \cdot \nabla v & \qquad \text{on} \qquad \partial D
\end{aligned}$$

has a non trivial solution are called transmission eigenvalues.

- If either $A_{min} - 1 > 0$ or $A_{max} - 1 < 0$ or $A \equiv I$ and either $n_{min} - 1 > 0$ or $n_{max} - 1 < 0$ or $n \equiv 1$, there exists infinitely many real transmission eigenvalues (Päivärinta-Sylvester (2008), Cakoni-Gintides-Haddar (2010), Cakoni-Kirsch (2010)).
- At a transmission eigenvalue, there is an incident wave that produces an arbitrarily small scattered wave.

Transmission Eigenvalues

First step is to study asymptotic of the resolvent: For $f, g \in L^2(D)$

$$\begin{aligned}
\Delta v_\epsilon + k^2 v_\epsilon &= f && \text{in} \quad D \\
\nabla \cdot A_\epsilon \nabla w_\epsilon + k^2 n_\epsilon w_\epsilon &= g && \text{in} \quad D \\
w_\epsilon &= v_\epsilon && \text{on} \quad \partial D \\
\nu \cdot A_\epsilon \nabla w_\epsilon - \nu \cdot \nabla v_\epsilon & && \text{on} \quad \partial D
\end{aligned}$$

The limiting problem is

$$\begin{aligned}
\Delta v_0 + k^2 v_0 &= f && \text{in} \quad D \\
\nabla \cdot A_h \nabla w_0 + k^2 n_h w_0 &= g && \text{in} \quad D \\
w_0 &= v_0 && \text{on} \quad \partial D \\
\nu \cdot A_h \nabla w_0 - \nu \cdot \nabla v_0 & && \text{on} \quad \partial D
\end{aligned}$$

Recall the bulk correction given by

$$w_1(x, y) = -\chi^j(y) \frac{\partial w_0}{\partial x_j}$$

with χ^j the solution of same cell problem as for the forward problem.

Transmission Eigenvalues

To present the idea assume that $A_{min} > 1$. We introduce

$$X(D) := \{(w,v) : \ w,v \in H^1(D) \,|\, w - v \in H^1_0(D)\}.$$

$$a_\epsilon\big((w_\epsilon, v_\epsilon);(\varphi_1,\varphi_2)\big) = \int_D A_\epsilon \nabla w_\epsilon \cdot \nabla\overline{\varphi}_1 + A_{min} w_\epsilon \overline{\varphi}_1 \, dx - \int_D \nabla v_\epsilon \cdot \nabla\overline{\varphi}_2 + v_\epsilon \overline{\varphi}_2 \, dx$$

$$b_\epsilon\big((w_\epsilon, v_\epsilon);(\varphi_1,\varphi_2)\big) := -\int_D (k^2 n_\epsilon + A_{min}) w_\epsilon \overline{\varphi}_1 \ - (k^2+1) v_\epsilon \overline{\varphi}_2 \, dx$$

Define the isomorphism $\mathbb{T} : X(D) \to X(D)$ such that $(w,v) \mapsto (w, -v+2w)$ and the operators $\mathbb{A}_\epsilon, \mathbb{B}_\epsilon : X(D) \to X(D)$ by

$$\big(\mathbb{A}_\epsilon(w_\epsilon, v_\epsilon); \mathbb{T}(w_\epsilon, v_\epsilon)\big) := a_\epsilon\big((w_\epsilon, v_\epsilon); \mathbb{T}(w_\epsilon, v_\epsilon)\big), \quad \text{coercive}$$

$$\big(\mathbb{B}_{\epsilon,k}(w_\epsilon, v_\epsilon); (w_\epsilon, v_\epsilon)\big) := b_\epsilon\big((w_\epsilon, v_\epsilon); (w_\epsilon, v_\epsilon)\big), \quad \text{compact}$$

For $\epsilon = 0$, A_ϵ, n_ϵ are replaced by A_h, n_h.

Transmission Eigenvalues

The interior transmission problem reads:

$$\left(\mathbb{A}_\epsilon + \mathbb{B}_{\epsilon,k}\right)(w_\epsilon, v_\epsilon) = \ell \qquad \text{in} \qquad X(D)$$

or

$$\left(\mathbb{I} + \mathbb{A}_\epsilon^{-1}\mathbb{C}_\epsilon + k^2\mathbb{A}_\epsilon^{-1}\mathbb{K}_\epsilon\right)(w_\epsilon, v_\epsilon) = \ell$$

with $\mathbb{C}_\epsilon$ and $\mathbb{K}_\epsilon$ compact operators independent of k^2

Theorem

If k^2 is not a transmission eigenvalue then

- (w_ϵ, v_ϵ) converge to (w_0, v_0) weakly in $X(D)$ (strongly in $L^2(D) \times L^2(D)$)
- If in addition $w_0 \in H^2(D)$ then

$$(w_\epsilon - w_0 - \epsilon w_1(x, x/\epsilon), v_\epsilon - v_0) \to (0, 0)$$

strongly in $X(D)$. The rate of convergence is in general $\epsilon^{1/2}$.

Transmission Eigenvalues

- The monotonicity result for a sequence of real transmission eigenvalues

$$k^j(A_{max}, n_{min}, D) \leq k^j_\epsilon < k^j(a_{min}, n_{max}, D)$$

  if $A_{min} > 1$ and $0 < n_{max} < 1$, implies that each of these real eigenvalues is bounded with respect to ϵ.
- Then the above analysis allow us to prove that each k^j_ϵ converges to an eigenvalue of the homogenized problem. In particular this is true for the first transmission eigenvalue.
- It is possible to actually obtain the convergence rates as well as the correction term for simple eigenvalues. The approach is based on the non-linear version of a result by OSBORN, SPECTRAL APPROXIMATIONS FOR COMPACT OPERATORS, (1975).
- Numerical implementations support the theory on the rate of convergence which is ϵ^2 if $\chi^j = 0$ and of order ϵ in general.

Determination of Transmission Eigenvalues

Real transmission eigenvalues can be determined from the scattered data

We consider the far field operator $F_\epsilon : L^2(\Omega) \to L^2(\Omega)$ defined by

$$(F_\epsilon g)(\hat{x}) := \int_\Omega u_\epsilon^\infty(\hat{x}, d, k) g(d) ds_d$$

where $u_\epsilon^\infty(\hat{x}, d, k)$ is the far field pattern due to an incident plane wave $u^i := e^{ikx \cdot d}$, $d \in \Omega$ a unit vector.

We look for the regularized solution $g_{z,k,\epsilon}$ of the far field equation

$$(F_\epsilon g)(\hat{x}) = \Phi_\infty(\hat{x}, z, k), \quad \textit{for} \quad g \in L^2(\Omega), \qquad \hat{x} \in \Omega, \ z \in D$$

where $\Phi_\infty(\hat{x}, z, k)$ is the far field pattern of the free space fundamental solution $\Phi(x, z, k)$ of the Helmholtz equation.

Solution of Far-Field Equation

If $$v_g(x) := \int_\Omega g(d) e^{ikx \cdot d} ds_d$$ then

- For k not a transmission eigenvalue, the regularized solution $g := g_{z,k,\epsilon}$ of the far field equation is such that

$$\lim_{\delta \to 0} \|v_g\|_{\mathcal{H}(D)} \quad \text{exists}$$

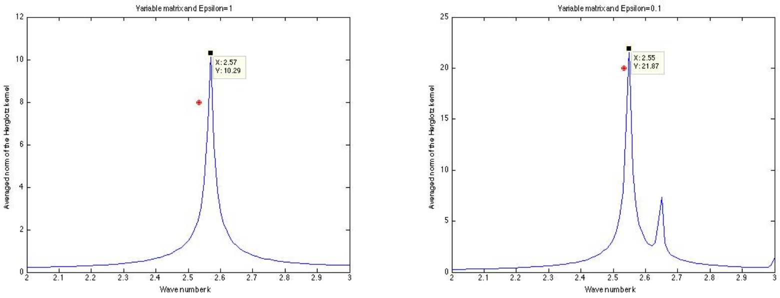
- For k a transmission eigenvalue, the regularized solution $g := g_{z,k,\delta}$ of the far field equation is such that

$$\lim_{\delta \to 0} \|v_g\|_{\mathcal{H}(D)} = \infty$$

ARENS-LECHLEITER (2006), CAKONI-COLTON-HADDAR (2010)

Remark: Under restrictive conditions on A and n, KIRSCH-LECHLEITER (2010), LECHLEITER-PETERS (2015) have characterized the first few real transmission eigenvalues in terms of the behavior of the eigenvalues of the normal far field operator.

Computation of Real TE



$$n_\epsilon = \sin^2(2\pi x_1/\epsilon) + 2 \text{ and } A_\epsilon = \frac{1}{3}\begin{pmatrix} \sin^2(x_2/\epsilon) + 1 & 0 \\ 0 & \cos^2(x_1/\epsilon) + 1 \end{pmatrix}.$$

D is the ball of radius $r = 2$. The effective material properties are $A_h = \frac{1}{2}I$ and $n_h = \frac{3}{2}$ and the corresponding first transmission eigenvalue is $k_h^1 = 2.5340$, for $\epsilon = 0.5$ and $\epsilon = 0.1$

Thanks to JIGUANG SUN for providing the code to compute the transmission eigenvalues.

Determination of Effective Material Properties

The measured first transmission eigenvalue $k_\epsilon^1 \approx k_h^1$ can be used to obtain information about the effective material properties A_h and n_h.

- If $A_\epsilon = I$, it is known that k_h^1 uniquely determines n_h.

Example for D the ball of radius 2, $n_\epsilon = n(x/\epsilon) = \sin^2(2\pi x_1/\epsilon) + 2$.

ϵ	$k_{\epsilon,1}$	n_h	reconstructed n_h
0.1	5.046	2.5	2.5188

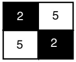
- If $n_\epsilon = 1$, k_h^1 uniquely determines a_h where $A_h = a_h I$.

Example D the ball of radius 2 and

$$A_\epsilon = \frac{1}{3}\begin{pmatrix} \sin^2(2\pi x_2/\epsilon) + 1 & 0 \\ 0 & \cos^2(2\pi x_1/\epsilon) + 1 \end{pmatrix}.$$

ϵ	k_ϵ^1	A_h	reconstructed A_h
0.1	7.349	$0.5I$	$0.4851I$

Determination of Effective Material Properties

n(y)=


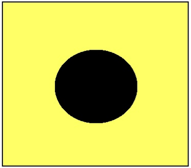
A(y)=


$k_1(n(y))$	$k_1(n_h)$	$k_1(A(y))$	$k_1(A_h)$	$k_1(n(y), A(y))$	$k_1(n_h, A_h)$
1.0930	1.0757	1.9027	1.896	0.7673	0.7139

$A(y) = I$, $n(y)$	reconstructed $n_h = 3.4123$ (exact $n_h = 3.5$)
$A(y)$, $n(y) = 1$	reconstructed $a_h = 0.4472$
$A(y)$, $n(y)$	reconstructed $n_h/a_h = 7.4704$ (gives $a_h = 0.4685$)

Determination of Effective Material Properties

An example with voids (which is not covered by our theory)



$Y = [0,1]^2$ and the is domain $D = [-3,3]^2$ and

$$n(y) = \begin{cases} 1 & \text{if} \quad (y_1 - 0.5)^2 + (y_2 - 0.5)^2 < 0.25^2 \\ 5 & \text{if} \quad (y_1 - 0.5)^2 + (y_2 - 0.5)^2 \geq 0.25^2 \end{cases}$$

which gives that $n_h = 5 - \frac{\pi}{4}$ and

$$A(y) = \begin{cases} I & \text{if} \quad (y_1 - 0.5)^2 + (y_2 - 0.5)^2 < 0.25^2 \\ 0.5I & \text{if} \quad (y_1 - 0.5)^2 + (y_2 - 0.5)^2 \geq 0.25^2 \end{cases}$$

$A_h = a_h I$, where a_h can only be computed numerically.

Determination of Effective Material Properties

An example with voids

The computed first transmission eigenvalue using far field data.

$k_1(n(y))$	$k_1(n(y), A(y))$
0.8745	0.7599

The estimated effective material properties.

$A(y) = I$, $n(y)$	reconstructed $n_h = 4.2678$ (exact $n_h = 4.2146$)
$A(y)$, $n(y)$	reconstructed $n_h/a_h = 5.0550$ (hence $a_h = 0.8337$)