Inverse scattering in half-space with random boundary conditions

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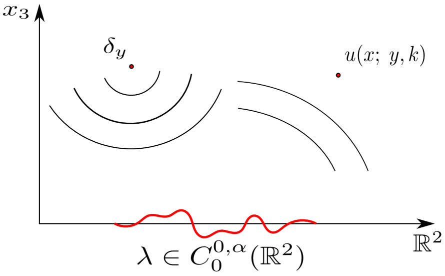
Scattering with random Robin boundary



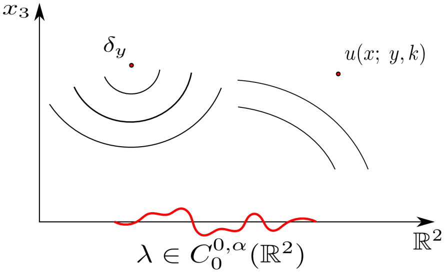
We study acoustic scattering in half-space

$$\begin{aligned} (\Delta + k^2)u &= \delta_y, \quad x_3 > 0, \\ \left(\frac{\partial}{\partial x_3} + \lambda\right) u &= 0, \quad x_3 = 0, \end{aligned}$$

where λ is a Gaussian random field and the full wave $u = u(x; y, k)$ satisfies the Sommerfeld radiation condition.

Here, λ is assumed to have a bounded support D and to be real which indicates that the boundary is non-absorbing.

Scattering with random Robin boundary



Consider backscattering, that is, $x = y$, in a bounded subset $U \subset \mathbb{R}^3_+$.

Measurement $=$ backscattered amplitude $|u_s(x; x, k)|^2$ averaged over frequency.

What information can we recover from the statistics of the Robin coefficient λ?

T.H., M. Lassas and L.Päivärinta, *Inverse acoustic scattering problem in half-space with anisotropic random impedance*, arXiv:1407.2481

Some background in literature

Lassas, Päivärinta and Saksman (2009): time-harmonic Schrödinger $\left(\Delta - q + k^2\right) u = \delta_y$ in $\mathbb{R}^2$

- q microlocally isotropic Gaussian random field
- averaged backscattering data
- recover principal symbol of C_q

Deterministic direct problem in half-space geometry has been studied by Chandler-Wilde, Nédélec, Karamyan and others.

Deterministic inverse problem?

The stochastic inverse problem has been considered by e.g. Bal and Jing (2011) in homogenization framework.

Classical symbols and the principal symbol

If the symbol $\sigma \in \mathcal{S}^{\rho}_{1,0}$ has an asymptotic expansion

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \sigma_{\rho-j}(x,\xi)$$

for smooth σ_{ρ_j}, homogeneous in ξ of degree $\rho - j$ for $\xi \geq 1$ and

$$\sigma(x,\xi) - \sum_{j=0}^{N} \sigma_{\rho-j}(x,\xi) \in S^{\rho-N-1}_{1,0}$$

for all N then we say that σ is classical and σ_ρ is its principal symbol.

Stochastic model: isotropic

We assume that λ is a zero-centered generalized Gaussian random field on $\mathbb{R}^2$ with covariance operator C_λ such that $\mathrm{supp}(\lambda) \subset D$, where $D \subset \mathbb{R}^2$ is bounded.

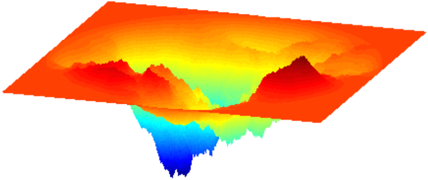
Moreover, λ is microlocally isotropic, if the principal symbol of C_λ satisfies

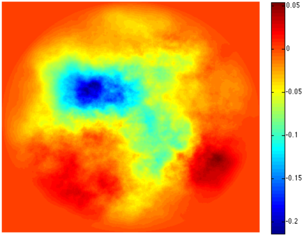
$$\sigma^p(x, \xi) = \frac{b(x)}{|\xi|^{2\epsilon+2}}$$

for some $\epsilon > 0$ and some bounded function $b \in C^\infty(\mathbb{R}^2)$ supported on D.

It follows that $\lambda \in C^{0,\alpha}(\mathbb{R}^2)$ almost surely for any $\alpha < \epsilon$.

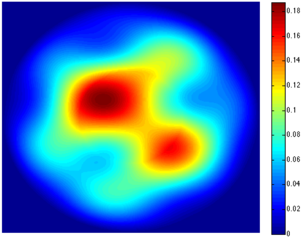
Example: fractional Brownian field





Top: Fractional Brownian field with Hurst index $H = 0.8$

Right: The isotropic strength $b(x)$. In this case $\epsilon = H$.



Stochastic model: anisotropic

We call λ microlocally anisotropic, if the principal symbol of C_λ satisfies

$$\sigma^p(x,\xi) = \frac{b\left(x, \frac{\xi}{|\xi|}\right)}{|\xi|^{2\epsilon+2}}$$

for some $\epsilon > 0$ and some bounded function $b \in C^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$

Likewise, here it follows that $\lambda \in C^{0,\alpha}(\mathbb{R}^2)$ almost surely for any $\alpha < \epsilon$.

Stochastic model: some assumptions

Regarding the local strength $b \in C^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$:

- $b\left(x, \frac{\xi}{|\xi|}\right) = b\left(x, -\frac{\xi}{|\xi|}\right)$ for any $\xi \in \mathbb{R}^2$ and $\xi \neq 0$ and
- for some $s > 0$, $b = b(x, \xi)$ can be extended to a function in $\mathbb{R}^2 \times \mathbb{R}^2$ which is s-homogeneous and real-analytic w.r.t. ξ.

The key example of anisotropy is the case

$$b(x, \xi) = \langle \xi, A(x)\xi \rangle,$$

where $A : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ is smooth and symmetric.

Stochastic model: Gaussian potential field

Define a Gaussian random process Y on $\mathbb{R}^2$ by $\mathbb{E}Y = 0$ and

$$c_Y(z_1, z_2) = |z_1 - z_2|^{2+\epsilon} + r(z_1, z_2)$$

for $\epsilon > 0$, where r is smooth. Then write

$$q = D_v Y := (v(x) \cdot \nabla)\, Y$$

for a vector field $v \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$ such that $\mathrm{supp}(v) \subset D$. Now $C_q = D_v^* C_Y D_v : \mathcal{D}(\mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2)$. We obtain $\sigma(C_q) \in S_{1,0}^{-2-\epsilon}(\mathbb{R}^2 \times \mathbb{R}^2)$ and the principal symbol of q satisfies

$$\sigma^p(C_q) \propto (v(x) \cdot \xi)^2 |\xi|^{-4-\epsilon} = \left\langle \frac{\xi}{|\xi|}, A(x)\frac{\xi}{|\xi|} \right\rangle |\xi|^{-2-\epsilon},$$

where $A(x) = v(x)v(x)^\top$.

Measurement

Denote $u = u_{in} + u_s$, where

$$u_{in}(x; y, k) = \frac{exp(ik|x-y|)}{4\pi|x-y|} + \frac{exp(ik|\tilde{x}-y|)}{4\pi|\tilde{x}-y|}$$

The measurement is defined by

$$m(x, y, \omega) = \lim_{K\to\infty} \frac{1}{K-1} \int_1^K k^{2(1+\epsilon+p)} |u_s(x; y, k, \omega)|^2 dk$$

One of the main results is to show that m is statistically stable, that is, there exists $m_0 = m_0(x, y)$ such that

$$m(x, y, \omega) = m_0(x, y)$$

almost surely.

Main result

The Robin parameter satisfies $\lambda(x,k) = \lambda(x)k^{-p}$, $p > \epsilon + \frac{1}{2}$.
Without this assumption, all results hold for Born approximation.

Theorem

Under some technical assumptions, the backscattering data $m_0(x,x)$, $x \in U$, uniquely determines values

$$(\mathcal{F}b)\left(\xi, \frac{\xi}{|\xi|}\right) \quad \text{and} \quad (\mathcal{F}b)\left(\xi, \frac{\xi^\perp}{|\xi|}\right)$$

for all $\xi \in \mathbb{R}^2$, where $\mathcal{F} = \mathcal{F}_{x\to\xi}$ and $\xi^\perp = (\xi_2, -\xi_1)$. In particular, in the isotropic case if $b(x,\xi) = b(x)$, the data uniquely determines b everywhere.

Main result continued

Theorem

Suppose that $b(x,\xi') = \langle \xi', A(x)\xi' \rangle$ as earlier. Given the backscattering data $m(x,x)$, $x \in U$, the trace $\mathrm{tr}(A)$ can be uniquely determined everywhere. Moreover, suppose that one of the three coefficient functions a_j from

$$A(x) = \begin{pmatrix} a_1(x) & a_3(x) \\ a_3(x) & a_2(x) \end{pmatrix}$$

is known, then the data uniquely determines the other two everywhere.

Direct problem - uniqueness

Consider the homogeneous direct problem

$$\begin{aligned} (\Delta + k^2)u &= 0, \quad \text{in } \mathbb{R}^3_+, \\ \frac{\partial u}{\partial x_3} + \lambda u &= 0 \quad \text{on } \mathbb{R}^3_0, \end{aligned}$$

where u satisfies the Sommerfeld radiation condition and $\lambda \in C^{0,\alpha}(D)$ for some $\alpha > 0$.

Uniqueness follows from bounded support of λ: A symmetrized solution $\tilde{u}(x', x_3) = u(x', |x_3|)$, $x = (x', x_3) \in \mathbb{R}^3$, can be shown to satisfy $\lim_{R\to\infty} \int_{\mathbb{S}^2(R)} |\tilde{u}|^2 dS(x) = 0$ and the Rellich theorem yields $\tilde{u} = 0$ outside a neighbourhood of D. Finally, UCP yields the uniqueness.

Direct problem - Incoming and outgoing

The total wave is divided into an incoming and outcoming waves $u = u_{in} + u_s$:

$$(\Delta + k^2)u_{in} = \delta_y \qquad\qquad (\Delta + k^2)u_s = 0$$

$$\frac{\partial}{\partial x_3} u_{in} = 0 \qquad\qquad \left(\frac{\partial}{\partial x_3} + \lambda\right) u_s = -\lambda u_{in}$$

The homogeneous Helmholtz equation on the right can be solved by any solution generated by the single-layer potential

$$u_s(x; y, k) = S_k^+ \phi = \int_{\mathbb{R}_0^3} \frac{\exp(2\pi i|x-y|)}{4\pi|x-y|}\phi(y)dy, \quad x \in \mathbb{R}_+^3,$$

for $\phi \in C^{0,\alpha}(\mathbb{R}_0^3)$.

Direct problem - the solution as a Born series

A solution can be found by utilizing single-layer potential $u = S_k^+ \phi + u_{in}$, where ϕ is the unique solution to the problem

$$\left(\frac{1}{2} - \lambda S_k^B \right) \phi = \lambda u_{in}$$

in $L^2(D)$ for almost every realization of λ.

Set up an iterative scheme: $\phi_1 = 2\lambda u_{in}$ and for each $n \geq 1$ define

$$\phi_{n+1} = 2\lambda S_k^B(\phi_n) \quad \text{and} \quad u_n = S_k^+ \phi_n.$$

Then the Born series

$$u(x; y, k) = u_{in}(x; y, k) + u_1(x; y, k) + u_2(x; y, k) + \ldots$$

converges pointwise for any $x, y \in U$.

Connection of scattered field and the Robin parameter

The effect of the Robin parameter is visible in the Born approximation by

$$\begin{aligned} u_1(x; y, k) &= 2S_k^+(\lambda_k u_{in}(\cdot; y, k))(x) \\ &= \frac{c}{k^p} \int_{R_0^3} \frac{\exp(ik(|x-z| + |y-z|))}{|x-z||y-z|} \lambda(z) dz. \end{aligned}$$

Notice that u_1 is a Gaussian random variable since it is obtained from λ by a linear operator.

Inverse problem - correlation

Correlation between Born approximations satisfies

$$
\begin{aligned}
&\mathbb{E}\left(u_1(x,y,k_1)\overline{u_1(x,y,k_2)}\right) \\
&\propto \int_{\mathbb{R}^3_0\times\mathbb{R}^3_0} \frac{\exp(i\,(k_1\phi(z_1;x,y)-k_2\phi(z_2;x,y)))}{|x-z_1|\,|z_1-y|\,|x-z_2|\,|z_2-y|} c_\lambda(z_1,z_2)\,dz_1 dz_2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq C_N|k_1-k_2|^{-N}
\end{aligned}
$$

for any $N>0$, where $\phi(z;x,y)=|x-z|+|z-y|$. For $k=k_1=k_2$ we can show that

$$
\mathbb{E}\left(u_1(x,y,k)\overline{u_1(x,y,k)}\right) = R(x,y)k^{-2-2\epsilon-2p}+\mathcal{O}(k^{-3-2p}),
$$

where R is a known smooth function in $U\times U$.

Ergodicity of the measurement

From ergodicity theory it's known that if X_t, $t \geq 0$ is a real-valued stochastic process such that $|\mathbb{E} X_t X_{t+r}| \leq C(1+r)^{-\epsilon}$ for all $t, r \geq 0$ with some $\epsilon > 0$ then

$$\lim_{K\to\infty} \frac{1}{K} \int_1^K X_t dt = 0.$$

It follows that

$$\lim_{K\to\infty} \frac{1}{K-1} \int_1^K k^{2(1+\epsilon+p)} |u_1(x;y,k)|^2 dk \\ = \lim_{K\to\infty} \frac{1}{K-1} \int_1^K k^{2(1+\epsilon+p)} \mathbb{E}|u_1(x;y,k)|^2 dk$$

Measurement data in backscattering

The measurement can be approximated by

$$
\begin{aligned}
m_0(x,x) &= \lim_{K\to\infty} \frac{1}{K-1} \int_1^K k^{2(1+\epsilon+p)} \mathbb{E}|u_s(x;x,k,\omega)|^2 dk \\
&\approx R(x,x),
\end{aligned}
$$

which is obtained since $\left\| S_k^B \right\|_{L^2(D)\to L^2(D)} \leq Ck^{-1/2}$. In fact, $R(x,x)$ has explicit formula

$$
R(x,x) = C \int_D \frac{b\left(z, \frac{z-x}{|z-x|}\right)}{|z-x|^4} dz
$$

for any $x \in U$.

Anisotropic spherical Radon transform

Denote a transformation

$$(\mathcal{S}b)(x', r) = \int_{\mathbb{S}^1} b(x' + r\theta, \theta) d|\theta|$$

for any $x' \in \mathbb{R}^3_0$ and $r > 0$. Then we can write

$$R(x, x) \propto \frac{1}{2\pi} \int_0^\infty (\mathcal{S}b)(x', r) \frac{1}{(r^2 + x_3^2)^2} dr,$$

Trick: apply $\frac{C}{x_3}\partial_{x_3}$ repeatedly to recover any integral of type

$$\int_0^\infty \frac{(\mathcal{S}b)(x', r)}{r^2} Q\left(\frac{1}{r^2}\right) dr,$$

where $Q(t) = \sum_{j=0}^{p} a_j t^j$, $p \geq 0$.

Anisotropic spherical Radon transform continued

Trick: apply $\frac{C}{x_3}\partial_{x_3}$ repeatedly to recover any integral of type

$$\int_0^\infty \frac{(\mathcal{S}b)(x',r)}{r^2} Q\left(\frac{1}{r^2}\right) dr,$$

where $Q(t) = \sum_{j=0}^p a_j t^j$, $p \geq 0$.

The support of $r \mapsto \mathcal{S}b(x',r)$ lies in a finite interval $[a,b]$ with $a, b > 0$. Since functions of the form $Q(1/r^2)$ are dense in $C([a,b])$, we can uniquely determine $\mathcal{S}b(x',r)$ for all $r > 0$ and any $x' \in U'$, where U' is the projection of U to $\mathbb{R}^3_0$.

Analyticity assumptions on $b(x',\cdot)$ give $\mathcal{S}b$ everywhere.

How to invert S?

- Anisotropic spherical Radon transform

$$(\mathcal{S}b)(x', r) = \int_{\mathbb{S}^1} b(x' + r\theta, \theta) d|\theta|$$

  has a null-space and thus full inversion is impossible
- However, one can find b modulo $\mathcal{N}(\mathcal{S})$
- Any function $f \in \mathcal{N}(\mathcal{S})$ satisfies

$$(\mathcal{F}_{x\to\xi} f)\left(\xi, \frac{\xi}{|\xi|}\right) = (\mathcal{F}_{x\to\xi} f)\left(\xi, \frac{\xi^\perp}{|\xi|}\right) = 0$$

  for any $\xi \neq 0$.
- In consequence, $\mathcal{F}_{x\to\xi} b$ is known at same locations

Summary

- Method for reconstructing statistical properties of random Robin coefficient in half-space from backscattering data
- Stability under measurement noise
- Anisotropic models can be recovered

T.H., M. Lassas and L.Päivärinta, *Inverse acoustic scattering problem in half-space with anisotropic random impedance*, arXiv:1407.2481