Selection of defective components in unknown backgrounds

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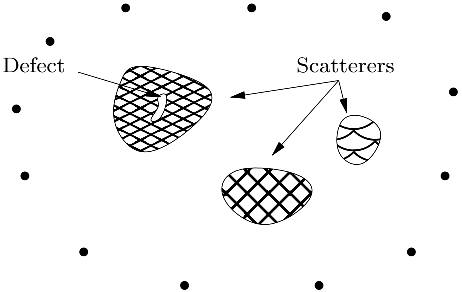
BREMEN, 2015

Model problem



Goal: Determine defects or defective components in a complex and unknown medium from multi-static measurements of scattered waves at a given frequency.

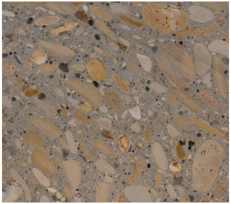
Constraints:

- The background is unknown and cannot be accurately reconstructed.
- The background components diameters are comparable to the wavelength.

But: We have access to differential measures: measures with and without defects.

The original motivation

Detection of defects in a concrete like material using ultrasounds



An example of a concrete structure

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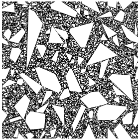
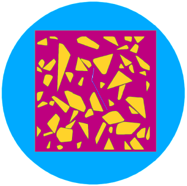
A simulation using the Linear Sampling Method (without differential measurements)



An example of synthetic background



A filtered background + crack

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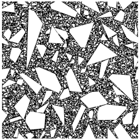

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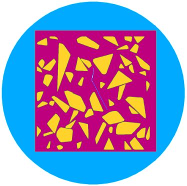

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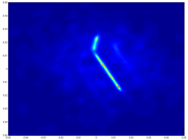

Reconstructed crack using
the exact background

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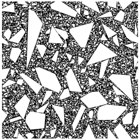

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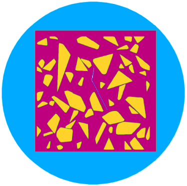

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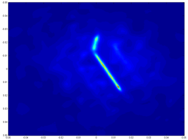

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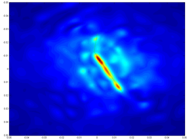

Reconstructed crack using a homogeneous background: weak perturbation

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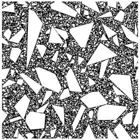

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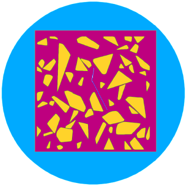

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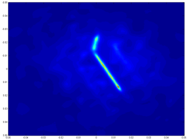

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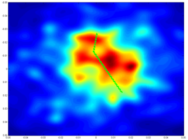

Reconstructed crack using a homogeneous background: strong perturbation

Outline

- A model problem (not for cracks)
- The Linear Sampling Method revisited
- Application to the case of differential measurements
- Numerical results and perspectives

A simple model problem

Scalar acoustic equation for inhomogeneous media

The background index n_0 : $n_0 = 1$ in $\mathbb{R}^d \setminus D_0$ and $\mathbb{R}^d \setminus D_0$ is connected.

The modified index n : $n = 1$ in $\mathbb{R}^d \setminus D$ and $\mathbb{R}^d \setminus D$ is connected.

The total fields $u_0 \in H^1_{loc}(\mathbb{R}^d)$ and $u \in H^1_{loc}(\mathbb{R}^d)$

$$\Delta u_0 + k^2 n_0 u_0 = 0 \text{ and } \Delta u + k^2 n u = 0 \text{ in } \mathbb{R}^d$$

We assume that the field is generated by incident plane waves:

$$u^i(\theta, x) := e^{ikx\cdot\theta} \quad \theta \in \mathbb{S}^{d-1}$$

The scattered fields

$$u_0^s(\theta, \cdot) = u_0 - u^i(\theta, \cdot) \text{ and } u^s(\theta, \cdot) = u - u^i(\theta, \cdot) \quad \text{in } \mathbb{R}^d,$$

satisfy the Sommerfeld radiation condition.

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Our data is formed by (noisy measurements of) so-called farfield patterns

$$u_0^\infty(\theta, \hat{x}) \text{ and } u^\infty(\theta, \hat{x}) \text{ for all } (\theta, \hat{x}) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$$

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Recall that with $\hat{x} := x/|x|$,

$$u_0^s(\theta, x) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}}(u_0^\infty(\theta, \hat{x}) + O(1/|x|))$$

$$u^s(\theta, x) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}}(u^\infty(\theta, \hat{x}) + O(1/|x|))$$

as $|x| \to \infty$.

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Goal: Assuming that $D_0 \subset D$ and would like to reconstruct (an approximation of)

$$\Omega \equiv \mathrm{supp}(n - n_0)$$

without knowing (or approximating) n and n_0.

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Algorithm: introduce a filtered difference between the indicator functions provided by a modified version of the Linear Sampling Method (LSM) applied to each set of data separately.

A generalized version of LSM

- A version based on a new exact characterization of the scatterer geometry in terms of the farfields.
- A version capable of answering the imaging problem for differential measure: explicit link with solutions of the interior transmission problem.
- A flexible setting that can be generalized to limited aperture or/and near-field data (ongoing).

Related reference: L. Audibert - H. Haddar, Inverse Problems, 2014

Outline of LSM

Farfield Operator: $F : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$, defined by

$$Fg(\hat{x}) := \int_{\mathbb{S}^{d-1}} u^\infty(\theta, \hat{x}) g(\theta) ds(\theta).$$

Let us define for $\psi \in L^2(D)$, the unique function $w \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying

$$\begin{cases} \Delta w + nk^2 w = k^2(1-n)\psi \text{ in } \mathbb{R}^d, \\ \lim\limits_{r\to\infty} \int\limits_{|x|=r} \left| \frac{\partial w}{\partial r} - ikw \right|^2 ds = 0. \end{cases} \tag{1}$$

Remark

$$\psi = u^i(\theta, \cdot) \Rightarrow w = u^s(\theta, \cdot) \Rightarrow w^\infty = u^\infty(\theta, \cdot)$$

$\Rightarrow$ Fg is nothing but w^∞ for w solution of (1) with $\psi = v_g$ in D, where

$$v_g(x) := \int_{\mathbb{S}^{d-1}} u^i(\theta, x) g(\theta) ds(\theta),\ g \in L^2(\mathbb{S}^{d-1}),\ x \in \mathbb{R}^d.$$

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$\Rightarrow$ Considering the (compact) operator $H : L^2(\mathbb{S}^{d-1}) \to L^2(D)$ defined by

$$Hg := v_g|_D, \tag{2}$$

and the (compact) operator $G : \overline{\mathcal{R}(H)} \subset L^2(D) \to L^2(\mathbb{S}^{d-1})$ defined by

$$G\psi := w^\infty,$$

then clearly:

$$\boxed{F = G \circ H}$$

Main ingredient of LSM

Theorem: Assume that ITP is well posed. With $\phi_z^\infty(\hat{x}) = e^{-ik\hat{x}\cdot z}$ we have: $\phi_z^\infty \in \mathcal{R}(G)$ if and only if $z \in D$.

Main ingredients of the proof:

- $\overline{\mathcal{R}(H)} = \{v \in L^2(D); \Delta v + k^2 v = 0 \text{ in } D\}$.
- ϕ_z^∞ is the farfield of $\Phi(\cdot, z)$, radiating solution of $\Delta\Phi + k^2\Phi = -\delta_z$.

Interior Transmission Problem (ITP): $(u, v) \in L^2(D) \times L^2(D)$ such that $u - v \in H^2(D)$ and

$$\begin{cases} \Delta u + k^2 n u = 0 \quad \text{in } D, \\ \Delta v + k^2 v = 0 \quad \text{in } D, \\ (u - v) = f \quad \text{on } \partial D, \\ \frac{\partial}{\partial \nu}(u - v) = g \quad \text{on } \partial D, \end{cases} \tag{3}$$

for given $f \in H^{\frac{3}{2}}(\partial D)$ and $g \in H^{\frac{1}{2}}(\partial D)$.

Remark: A well posed ITP requires $n \not\equiv 1$ in any neighborhood of ∂D.

Main theorem of LSM

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$$Fg(\hat{x}) := \int_{\mathbb{S}^{d-1}} u^\infty(\theta, \hat{x}) g(\theta) ds(\theta).$$

Theorem: Assume that ITP is well posed. Then the operator F is injective with dense range. Moreover, the following holds.

- If $z \in D$ then there exists g_z^ϵ such that $\|Fg_z^\epsilon - \phi_z\|_{L^2(\mathbb{S}^{d-1})} \leq \epsilon$ and $\limsup_{\epsilon \to 0} \|Hg_z^\epsilon\|_{L^2(D)} < \infty$.
- If $z \notin D$ then for all g_z^ϵ such that $\|Fg_z^\epsilon - \phi_z\|_{L^2(\mathbb{S}^{d-1})} \leq \epsilon$, $\lim_{\epsilon \to 0} \|Hg_z^\epsilon\|_{L^2(D)} = \infty$.

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$\Rightarrow$ Gives a "characterization" of D in terms of a nearby solutions of

$$Fg_z^\epsilon \simeq \phi_z.$$

Problems: This is not constructive...

- We do not know how to construct g_z^ϵ. In practice we use a regularization scheme.
- We cannot compute $\|Hg_z^\epsilon\|_{L^2(D)}$. In practice we use $\|g_z^\epsilon\|_{L^2(\mathbb{S}^{d-1})}$.

A robust formulation of LSM

Idea: Reconstruct a nearby solution of the LSM by using a least squares misfit functional with a penalty term that controls $\|Hg_z^\epsilon\|^2_{L^2(D)}$.

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We exploit the (second) Factorization:

$$w^\infty(\hat{x}) = -\int_D e^{-iky.\hat{x}}(1-n)k^2(\psi(y)+w(y))dy,$$

$\Rightarrow G = H^*T\psi$ where $H^* : L^2(D) \to L^2(\mathbb{S}^{d-1})$ is the adjoint of H given by

$$H^*\varphi(\hat{x}) := \int_D e^{-iky.\hat{x}}\varphi(y)dy,\ \varphi \in L^2(D),\ \hat{x} \in \mathbb{S}^{d-1},$$

and where T: $L^2(D) \to L^2(D)$ is defined by

$$T\psi := -k^2(1-n)(\psi + w), \tag{4}$$

$$\boxed{F = H^* \circ T \circ H}$$

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$$\boxed{F = H^* \circ T \circ H}$$

Theorem: Assume that (ITP) is well posed and there exists $n_0 > 0$ and $\alpha > 0$ such that

$1 - \Re(n(x)) + \alpha\Im(n(x)) \geq n_0$ for a.e. $x \in D$
or
$\Re(n(x)) - 1 + \alpha\Im(n(x)) \geq n_0$ for a.e. $x \in D$.

Then: $\quad |(T\psi, \psi)_{L^2(D)}| \geq c\|\psi\|^2_{L^2(D)}$ for all $\psi \in \mathcal{R}(H)$.

$\Rightarrow |(Fg, g)_{L^2(\mathbb{S}^{d-1})}| \geq c\|Hg\|^2_{L^2(D)}$

$\Rightarrow |(Fg, g)_{L^2(\mathbb{S}^{d-1})}|$ is equivalent to $\|Hg\|^2_{L^2(D)}$

Abstract setting for a Generalized LSM (GLSM)

We consider two bounded linear operators $F : X \to X^*$ and $B : X \to X^*$

$$\boxed{F = GH \quad \text{and} \quad B = H^* TH}$$

$H : X \to Y$, $T : Y \to Y^*$ and $G : \overline{\mathcal{R}(H)} \subset Y \to X^*$ are bounded.

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$H : X \to Y$, $T : Y \to Y^*$ and $G : \overline{\mathcal{R}(H)} \subset Y \to X^*$ are bounded.
For $\alpha > 0$ be a given parameter and $\phi \in X^*$ we consider:

$$J_\alpha(\phi; g) := \alpha |\langle Bg, g \rangle| + \|Fg - \phi\|^2 \quad \forall g \in X.$$

Remark This functional has not a minimizer in general!

Assume that F has dense range. Then for all $\phi \in X^*$,

$$j_\alpha(\phi) := \inf_{g \in X} J_\alpha(\phi; g) \to 0 \text{ as } \alpha \to 0.$$

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$$j_\alpha(\phi) := \inf_{g \in X} J_\alpha(\phi; g) \to 0 \text{ as } \alpha \to 0.$$

$\Rightarrow$ Nearby solutions (of the farfield equation) are given by $g_\alpha \in X$ such that

$$J_\alpha(\phi; g_\alpha) \leq j_\alpha(\phi) + p(\alpha).$$

where $p(\alpha) > 0$ is such that $p(\alpha) \to 0$ as $\alpha \to 0$

Main theorem of GLSM (for noise free)

$F : X \to X^*$, $B : X \to X^*$ and $F = GH$ and $B = H^* T H$

$$J_\alpha(\phi; g) := \alpha |\langle Bg, g\rangle| + \|Fg - \phi\|^2 \quad \forall g \in X.$$

Theorem: We assume in addition that

- G is compact and $F = GH$ has dense range.
- T satisfies: $|\langle T\varphi, \varphi\rangle| > \mu \|\varphi\|^2 \quad \forall \varphi \in \mathcal{R}(H)$.

Consider for $\alpha > 0$ and $\phi \in X^*$, $g_\alpha \in X$ such that

$$\boxed{J_\alpha(\phi; g_\alpha) \le j_\alpha(\phi) + p(\alpha) \text{ and } p(\alpha) \le C\alpha.}$$

- $\phi \in \mathcal{R}(G) \Rightarrow \limsup_{\alpha \to 0} |\langle Bg_\alpha, g_\alpha\rangle| < \infty$.
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Application: $\mathcal{R}(G)$ characterizes the inclusion $D \Rightarrow F$ and B uniquely determine D.

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Similar asymptotic characterizations: Inf-Criterion (Nachman-Päivärinta-Teirilä (2007), Kirsch-Grinberg (2008)), Probe Method (Ikehata (2005), Erhard-Potthast (2006)).

Application with the natural choice: $B = F$

For $z \in \mathbb{R}^d$ we consider $g_\alpha^z \in L^2(S^{d-1})$ such that

$$\boxed{J_\alpha(\phi_z^\infty; g_\alpha^z) \le j_\alpha(\phi_z^\infty) + p(\alpha) \text{ and } p(\alpha) \le C\alpha.}$$

$\phi_z^\infty(\hat{x}) = e^{-ik\hat{x}\cdot z}$

Theorem: Assume that there exists $n_* > 0$ and $\alpha > 0$ such that

$1 - \Re(n(x)) + \alpha\Im(n(x)) \ge n_*$ for a.e. $x \in D$ or
$\Re(n(x)) - 1 + \alpha\Im(n(x)) \ge n_*$ for a.e. $x \in D$.

Then, except for a countable set of k (without finite accumulation points),

- $z \in D \Rightarrow \limsup_{\alpha\to 0} |\langle F g_\alpha^z, g_\alpha^z\rangle| < \infty.$
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$\Rightarrow$ An indicator of some approximation of D is given by

$$z \to 1/|\langle F g_\alpha^z, g_\alpha^z\rangle|.$$

Application with the natural choice: $B = F$

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Remarks

- In this case

$$\lim_{\alpha \to 0} |\langle F g_\alpha^z, g_\alpha^z \rangle| = \lim_{\alpha \to 0} |\langle \phi_z^\infty, g_\alpha^z \rangle| = \lim_{\alpha \to 0} |v_{g_\alpha^z}(z)|$$

  $\Rightarrow$ We obtain a similar indicator function as the one proposed by Arens (2004), Arens-Lechleiter (2009), to justify LSM using the $(F^* F)^{1/4}$ of Kirsch (1997) in the case $\Im n = 0$.

- However this turns out to be a bad indicator function for noisy measurements.

On other possible choices for B

Under more restrictive assumptions on the refractive index

- If $\Im n > n_0 > 0$ in D then we can use

$$B = \Im(F) = \frac{1}{2i}(F - F^*)$$

- If $Re(e^{it}(n-1)) > n_0|n-1] > n_1 > 0$ in D for some t

$$B = F_{\#} := |e^{it}F + e^{-it}F^*| + \Im(F)$$

(Using the Factorization theorem of Kirsch-Grinberg)

Remarks

- For these cases the functional J_α is convex.
- In these cases we also have (Kirsch-Grinberg)

$$z \in D \text{ iff } \phi_z \in \mathcal{R}(B^{1/2}).$$

Main theorem of GLSM for noisy operators

B^δ and F^δ compact operators corresponding with noisy measurements

$$\left\|F^\delta - F\right\| \le \delta \|F^\delta\| \quad \text{and} \quad \left\|B^\delta - B\right\| \le \delta \|B^\delta\|$$

for some $\delta > 0$.

Remark:

$$|\langle Bg,\, g\rangle| \le |\left\langle B^\delta g,\, g\right\rangle| + \delta \|B^\delta\| \, \|g\|^2$$

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Remark:

$$|\langle Bg,\, g\rangle| \leq |\left\langle B^\delta g,\, g\right\rangle| + \delta \|B^\delta\| \, \|g\|^2$$

$\Rightarrow$ We consider the functional:

$$J_\alpha^\delta(\phi; g) := \alpha(|\left\langle B^\delta g,\, g\right\rangle| + \delta \|B^\delta\| \, \|g\|^2) + \left\|F^\delta g - \phi\right\|^2 \quad \forall\, g \in X,$$

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Theorem: Let g_α^δ be the minimizer of $J_\alpha^\delta(\phi; \cdot)$ for $\alpha > 0$, $\delta > 0$ and $\phi \in X^*$. Then

- $\phi \in \mathcal{R}(G) \Rightarrow \limsup_{\alpha \to 0} \limsup_{\delta \to 0} \left(\left|\langle B^\delta g_\alpha^\delta, g_\alpha^\delta \rangle\right| + \delta \|B^\delta\| \left\|g_\alpha^\delta\right\|^2 \right) < \infty$
- $\phi \notin \mathcal{R}(G) \Rightarrow \lim_{\alpha \to 0} \liminf_{\delta \to 0} \left(\left|\langle B^\delta g_\alpha^\delta, g_\alpha^\delta \rangle\right| + \delta \|B^\delta\| \left\|g_\alpha^\delta\right\|^2 \right) = \infty$

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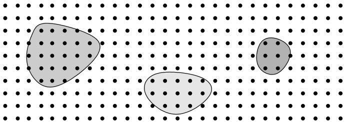
$\Rightarrow$ From the numerical perspective this theorem indicates that a criterion to localize the object would be

$$1 / \left(\left| \left\langle B^\delta g_\alpha^\delta, \, g_\alpha^\delta \right\rangle \right| + \delta \|B^\delta\| \left\| g_\alpha^\delta \right\|^2 \right)$$

for small values of α.

On the numerical implementation

$$J_\alpha^\delta(\phi; g) := \alpha(|\langle B^\delta g,\, g\rangle| + \delta\|B^\delta\|\,\|g\|^2) + \left\|F^\delta g - \phi\right\|^2$$



For each z in the sampling grid, compute

$$g_z = \arg\min J_\alpha^\delta(\phi_z^\infty; g),$$

then plot:

$$z \mapsto 1/\left(|\langle B^\delta g_z,\, g_z\rangle| + \delta\|B^\delta\|\,\|g_z\|^2\right)$$

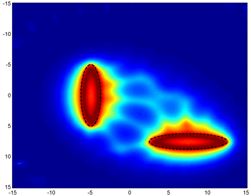
Initialization: we use the Tikhonov-Morozov regularized solution

$$(\eta(\delta) + (F^\delta)^* F^\delta) g_z^0 = (F^\delta)^* \phi_z^\infty$$

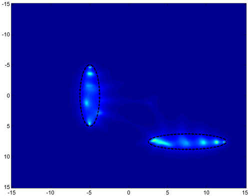
We choose: $\alpha = \eta(\delta)/(\|F^\delta\| + \delta)$.

Numerical results

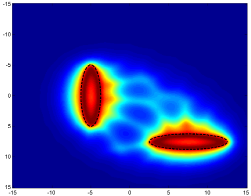
Without optimization, Noise $\delta = 0\%$



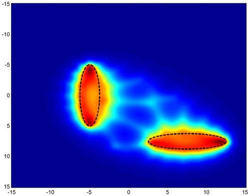
$1/|\langle B^\delta g_z^0,\, g_z^0 \rangle|$



$1/\|g_z^0\|^2$



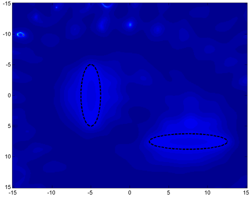
$1/\left(|\langle B^\delta g_z^0,\, g_z^0 \rangle| + \delta \|B^\delta\|\, \|g_z^0\|^2\right)$



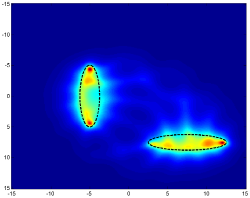
$F_\sharp$ method

Numerical results

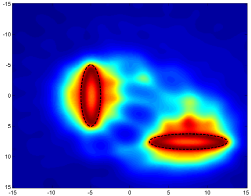
Without optimization, Noise $\delta = 1\%$



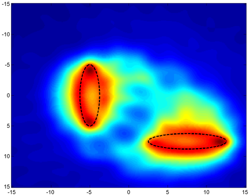
$1/|\langle B^\delta g_z^0,\, g_z^0 \rangle|$



$1/\|g_z^0\|^2$



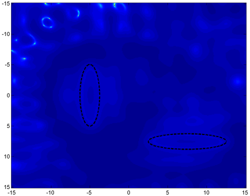
$1/\left(|\langle B^\delta g_z^0,\, g_z^0 \rangle| + \delta \|B^\delta\|\,\|g_z^0\|^2\right)$



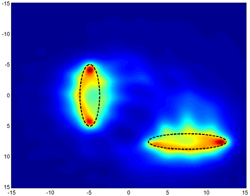
$F_\sharp$ method

Numerical results

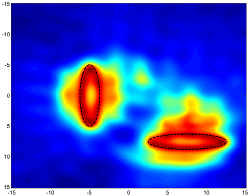
Without optimization, Noise $\delta = 5\%$



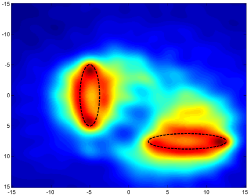
$1/|\langle B^{\delta} g_z^0,\, g_z^0 \rangle|$



$1/\|g_z^0\|^2$

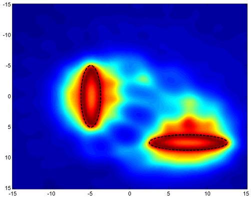


$1/\left(|\langle B^{\delta} g_z^0,\, g_z^0 \rangle| + \delta \|B^{\delta}\| \, \|g_z^0\|^2 \right)$

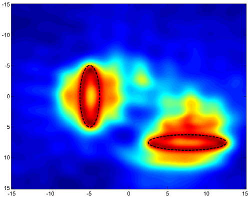


$F_\sharp$ method

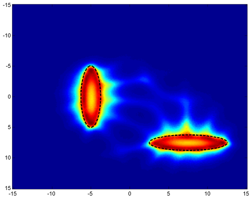
Numerical results Optim GLSM



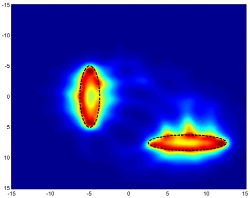
GLSM without optim, $\delta = 1\%$



GLSM without optim, $\delta = 5\%$



GLSM with optim, $\delta = 1\%$



GLSM with optim, $\delta = 5\%$

Towards applications to differential measurements

Main ingredient: exploit the link between GLSM and ITP.

Towards applications to differential measurements

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$F : X \to X^*$ and $B : X \to X^*$

$$F = GH \quad \text{and} \quad B = H^* TH$$

$$J_\alpha(\phi; g) := \alpha |\langle Bg,\, g\rangle| + \|Fg - \phi\|^2 \quad \forall g \in X. \tag{5}$$

Theorem: We assume in addition that

$$\varphi \mapsto |\langle T\varphi,\, \varphi\rangle| \text{ is uniformly convex}$$

$$J_\alpha(\phi; g_\alpha) \leq j_\alpha(\phi) + p(\alpha) \text{ with } \frac{p(\alpha)}{\alpha} \to 0 \text{ as } \alpha \to 0. \tag{6}$$

If $\phi \in \mathcal{R}(G)$ then $Hg_\alpha \to \varphi$ such that $G(\varphi) = \phi$.

This is a consequence of Tikhonov applied to $G(\varphi) = \phi$.

Towards application to differential measurements

Main ingredient: exploit the link between GLSM and ITP.

Corollary: with $F = F$ and $B = F_\sharp$

$$J_\alpha(\phi_z^\infty; g_z^\alpha) \leq j_\alpha(\phi_z^\infty) + p(\alpha) \text{ with } \frac{p(\alpha)}{\alpha} \to 0 \text{ as } \alpha \to 0.$$

If $z \in D$ then $Hg_z^\alpha \to v_z$ strongly in $L^2(D)$ where v_z is such that there exists $u_z \in L^2(D)$ for which (u_z, v_z) is a solution of ITP with $(f, g) = (\Phi(z, \cdot), \partial_\nu \Phi(z, \cdot))$

Notation for $\mathrm{ITP}(D, n, f, g)$: $(u, v) \in L^2(D) \times L^2(D)$ such that $u - v \in H^2(D)$ and

$$\begin{cases} \Delta u + k^2 n u = 0 \quad \text{in } D, \\ \Delta v + k^2 v = 0 \quad \text{in } D, \\ (u - v) = f \quad \text{on } \partial D, \\ \frac{\partial}{\partial \nu}(u - v) = g \quad \text{on } \partial D. \end{cases}$$

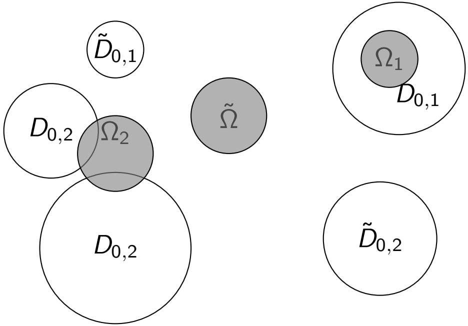
Application to differential measurements

Assumptions on the geometry:

$$D_0 \subset D \quad \overline{D} = \overline{\Omega} \cup \overline{D}_0 \quad \text{ and } n = n_0 \text{ in } D_0 \setminus \Omega$$

$$D_0 = \bigcup_i \tilde{D}_{0,i} \cup \bigcup_i D_{0,i}.$$

$D_{0,i}$, $i = 1, \ldots, M$ the components of D_0 that intersect with Ω.



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Comparison of ITP solutions:

Theorem: Assume that $\Re(n) > \Re(n_0) > 1$ or $\Re(n) < \Re(n_0) < 1$ in Ω. Let $z \in D$ and consider $(u, v) \in L^2(D) \times L^2(D)$ (resp. $(u_0, v_0) \in L^2(D_0) \times L^2(D_0)$) solutions of $\mathrm{ITP}(D, n, \Phi_z, \frac{\partial \Phi_z}{\partial \nu})$ (resp. $\mathrm{ITP}(D_0, n_0, \Phi_z, \frac{\partial \Phi_z}{\partial \nu})$). Then, except for a countable set of values of k,

- If $z \in \tilde{D}_{0,i}$ then $v = v_0$ in D_0.
- If $z \in D_{0,i}$, then $v \neq v_0$ in $D_{0,i}$ and $v = v_0$ in $D_0 \setminus D_{0,i}$.

Application to differential measurements

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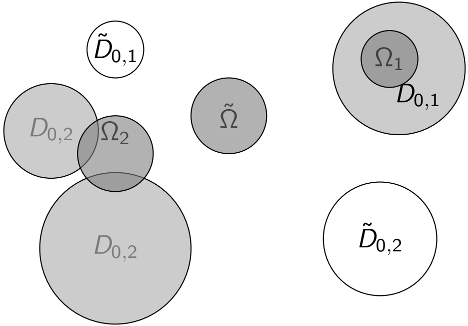
$$D_0 = \bigcup_i \tilde{D}_{0,i} \cup \bigcup_i D_{0,i}.$$

$D_{0,i}$, $i = 1, \ldots, M$ the components of D_0 that intersect with Ω.

We use this to obtain characterizations of $\tilde{\Omega}$ and $\Omega_0 := \Omega \cup_i D_{0,i}$.



Characterization of Ω_0 in terms of F and F_0

F farfield associated with D and n. $B = F_\#$
F_0 farfield associated with D_0 and n_0. $B_0 = F_{0,\#}$.

We introduce

$$\mathcal{D}(g, g_0) := |\langle B_0(g - g_0),\, g - g_0\rangle|.$$

Corollary: Under previous assumptions on D, D_0, n, n_0 and k and for g_z^α and $g_{0,z}^\alpha$ the minimizing sequences associated resp with (F, B) and (F_0, B_0)

- If $z \in \bigcup_i \tilde{D}_{0,i}$ then $\lim_{\alpha\to 0} \mathcal{D}(g_z^\alpha, g_{0,z}^\alpha) = 0$.
- If $z \in \tilde{\Omega}$ then $\lim_{\alpha\to 0} \mathcal{D}(g_z^\alpha, g_{0,z}^\alpha) = \infty$.
- If $z \in \bigcup_i D_{0,i}$ then $\lim_{\alpha\to 0} \mathcal{D}(g_z^\alpha, g_{0,z}^\alpha) < \infty$.

Characterization of Ω_0 in terms of F and F_0

The noise free case:

$$\mathcal{I}(g, g_0) := |\langle Bg,\, g\rangle| \left(1 + |\langle Bg,\, g\rangle| / |\langle B_0(g - g_0),\, g - g_0\rangle|\right).$$

Corollary: Under previous assumptions on D, D_0, n, n_0 and k. For g_z^α and $g_{0,z}^\alpha$ the minimizing sequences associated resp with (F, B) and (F_0, B_0)

If $z \notin \Omega_0$ then $\quad \lim_{\alpha \to 0} \mathcal{I}(g_z^\alpha, g_{0,z}^\alpha) = \infty.$

If $z \in \Omega_0$ then $\quad \lim_{\alpha \to 0} \mathcal{I}(g_z^\alpha, g_{0,z}^\alpha) < \infty.$

Therefore, the limit as $\alpha \to 0$ of

$$z \mapsto 1/\mathcal{I}(g_z^\alpha, g_{0,z}^\alpha) \text{ is an indicator for } \Omega_0 = \tilde{\Omega} \cup \bigcup_i D_{0,i}$$

Characterization of Ω_0 in terms of F and F_0

The noisy case:

For a fixed parameter $\eta \in (0,1)$, we define

$$g_{0,z}^{\alpha,\delta} = \arg\min_g \alpha \left(\left\langle B_0^\delta g,\, g \right\rangle + \alpha^{-\eta}\delta \|B_0^\delta\| \, \|g\|^2 \right) + \left\| F_0^\delta g - \phi_z^\infty \right\|^2$$

$$g_z^{\alpha,\delta} = \arg\min_g \alpha \left(\left\langle B^\delta g,\, g \right\rangle + \alpha^{-\eta}\delta \|B^\delta\| \, \|g\|^2 \right) + \left\| F^\delta g - \phi_z^\infty \right\|^2$$

We then consider

$$D(\alpha, z) = \liminf_{\delta \to 0} \left\langle B_0^\delta (g_{0,z}^{\alpha,\delta} - g_z^{\alpha,\delta}),\, g_{0,z}^{\alpha,\delta} - g_z^{\alpha,\delta} \right\rangle$$

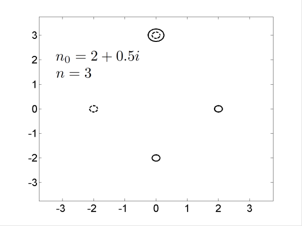
$$A(\alpha, z) = \liminf_{\delta \to 0} \left(\left\langle B^\delta g_z^{\alpha,\delta},\, g_z^{\alpha,\delta} \right\rangle + \alpha^{-\eta}\delta \|B^\delta\| \, \left\| g_z^{\alpha,\delta} \right\|^2 \right)$$

$$\mathcal{I}(\alpha, z) := A(\alpha, z) \left(1 + A(\alpha, z)/D(\alpha, z)\right).$$

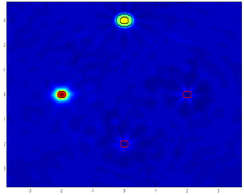
Theorem: Under previous assumptions on D, D_0, n, n_0 and k,

$$z \in \Omega_0 \text{ iff } \lim_{\alpha \to 0} \mathcal{I}(\alpha, z) < +\infty.$$

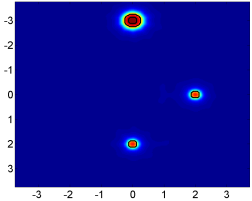
Some numerical results



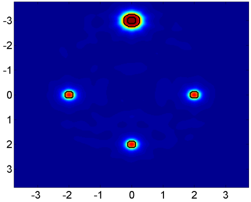
Exact configuration



Differential GLSM

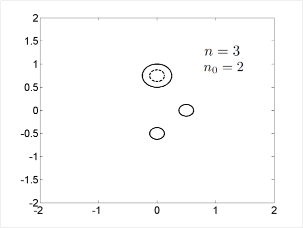


Background Reconstruction

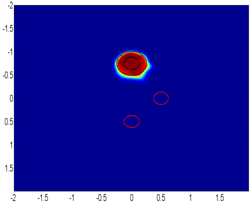


Medium Reconstruction

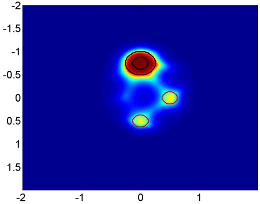
Some numerical results



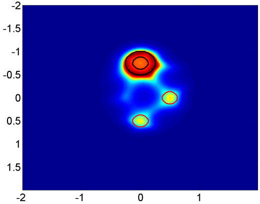
Exact configuration



Differential GLSM



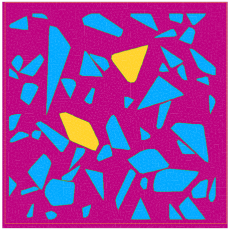
Background Reconstruction

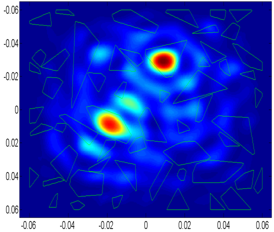


Medium Reconstruction

Some numerical results for scattered background

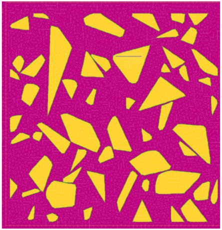
Two defects of type inhomogeneities

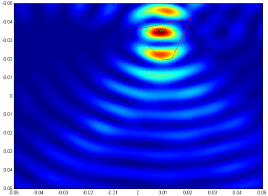




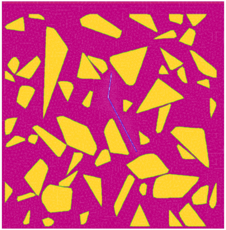
Some numerical results for scattered background

One defect of type internal crack

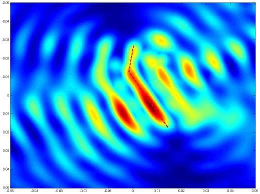




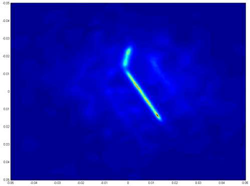
Some numerical results for scattered background



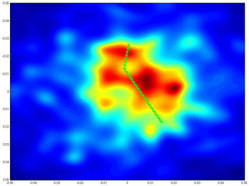
A medium size external crack



Differential GLSM



Reconstructed crack using
the exact background



Reconstructed crack using
a homogeneous background

Thank you