Iterative Solution of an Inverse Source Problem

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joint work with

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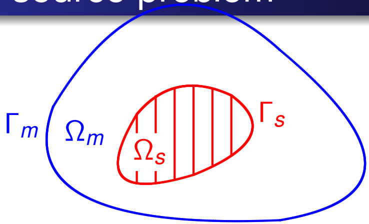
The inverse source problem

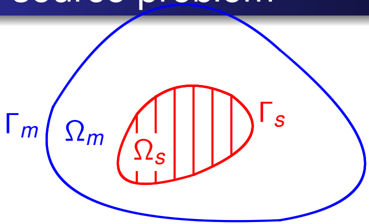


Determine the shape Γ_s of an extended source Ω_s of homogeneous strength within a conducting medium from one pair of Cauchy data $(u|_{\Gamma_m}, \partial u/\partial\nu|_{\Gamma_m})$ of a solution $u \in H^1(\Omega_m)$ to

$$\Delta u = \chi_s \quad \text{in } \Omega_m$$

with χ_s the characteristic function of the source Ω_s.

The inverse source problem



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Model problem for some imaging methods, e.g. bioluminescence tomography (BLT)

Main idea: Nonlinear integral equation for Γ_s

Outline

1. The inverse problem
2. Nonlinear integral equations for inverse boundary value problems
3. Equivalent boundary value problem with nonlocal impedance condition
4. Nonlinear integral equation for inverse source problem
5. Numerical examples

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5. Numerical examples

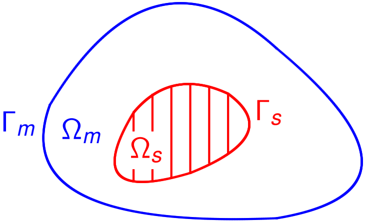
For simplicity only two dimensions and

$$\Delta u = \chi_s.$$

Immediate extensions to three dimensions and to

$$\Delta u + k^2 u = \chi_s.$$

Only one Cauchy pair



For a solution u of

$$\Delta u = \chi_s \text{ in } \Omega_m, \quad u = f \text{ on } \Gamma_m$$

the difference $v = u - u_f$ where

$$\Delta u_f = 0 \text{ in } \Omega_m, \quad u_f = f \text{ on } \Gamma_m$$

satisfies

$$\Delta v = \chi_s \text{ in } \Omega_m, \quad v = 0 \text{ on } \Gamma_m.$$

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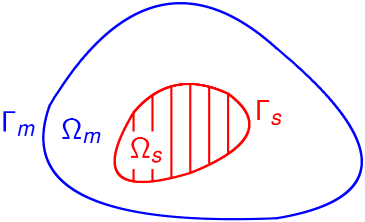

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$$\Delta v = \chi_s \text{ in } \Omega_m, \quad v = 0 \text{ on } \Gamma_m.$$

Hence, in principle, need only to consider Dirichlet data $f = 0$.

Uniqueness

Theorem

Novikov *1938*

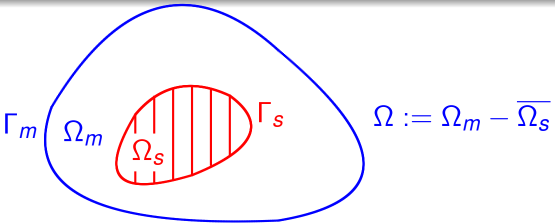
Γ_s is uniquely determined by one pair of Cauchy data provided it is starlike.

For a proof see **Isakov** 2005

Some previous work

1. **Hettlich, Rundell** 1996
   Regularized Newton iterations for boundary to data map $F : \Gamma_s \mapsto \partial u/\partial\nu|_{\Gamma_m}$ (for fixed f), that is, for the solution of $F(\Gamma_s) = g$.
2. **Hohage** 1997
   On the convergence of the above.
3. **Ring** 1995
   Derive moment problem for Fourier coefficients of radial function representing Γ_s from Poisson integral for disk plus least squares (for the case where Γ_m is a circle).
4. **Hanke, Rundell** 2011
   Derive moment problem for Fourier coefficients of radial function from reciprocity gap principle using trigonometric harmonics as trial functions plus Newton iterations.

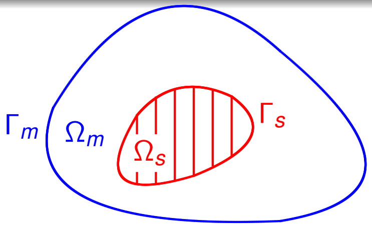
Inverse Dirichlet problem



Determine the shape Γ_s of a perfect conductor Ω_s within a conducting medium from one pair of Cauchy data $(u|_{\Gamma_m}, \partial u/\partial\nu|_{\Gamma_m})$ of a solution $u \in H^1(\Omega)$ to

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma_s.$$

Inverse Dirichlet problem



$\Omega := \Omega_m - \overline{\Omega_s}$

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Main idea: Obtain nonlinear integral equations for Γ_s from Green's integral theorem.

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Γ_s is uniquely determined by one Cauchy pair on Γ_m (with $f \neq 0$)

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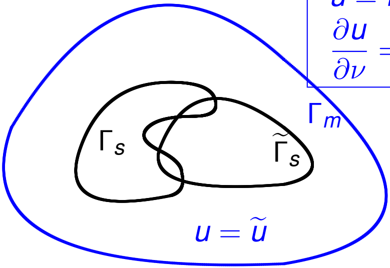
$$\boxed{\begin{array}{l} u = f = \tilde{u} \\ \dfrac{\partial u}{\partial \nu} = g = \dfrac{\partial \tilde{u}}{\partial \nu} \end{array}}$$



$$\Gamma_s \quad \leftrightarrow \quad u$$

$$\widetilde{\Gamma}_s \quad \leftrightarrow \quad \widetilde{u}$$

Uniqueness

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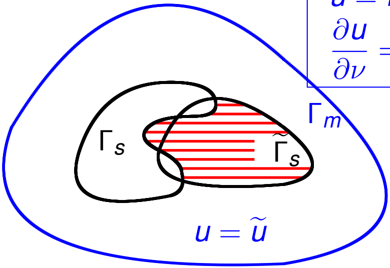

$$\Gamma_s \quad \leftrightarrow \quad u$$

$$\widetilde{\Gamma}_s \quad \leftrightarrow \quad \widetilde{u}$$

In shaded domain: $\triangle u = 0$

On boundary: $u = 0$

$\Rightarrow u = 0$ in shaded domain $\qquad \Rightarrow u = 0$ on Γ_m

Boundary integral operators

Introduce single- and double-layer potential operators

$$S_{jk} : H^{-1/2}(\Gamma_j) \to H^{1/2}(\Gamma_k) \quad \text{and} \quad K_{jk} : H^{1/2}(\Gamma_j) \to H^{1/2}(\Gamma_k)$$

by

$$(S_{jk}\varphi)(x) := 2 \int_{\Gamma_j} \Phi(x,y)\varphi(y)\, ds(y), \quad x \in \Gamma_k,$$

and

$$(K_{jk}\varphi)(x) := 2 \int_{\Gamma_j} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, \varphi(y)\, ds(y), \quad x \in \Gamma_k,$$

for $j, k = m, s$.

$$\Phi(x,y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|}\,, \quad x \neq y,$$

ν is unit normal directed into $R^2 \setminus \Omega_m$ and $R^2 \setminus \Omega_s$, respectively.

Nonlinear integral equations

Apply Green's integral formula for

$$u = 0 \quad \text{and} \quad \varphi := \frac{\partial u}{\partial \nu} \quad \text{on } \Gamma_s$$

$$\begin{aligned} u(x) &= -\int_{\Gamma_s} \Phi(x,\cdot)\varphi\, ds \\ &\quad + \underbrace{\int_{\Gamma_m} \left\{ \Phi(x,\cdot) g - \frac{\partial \Phi(x,\cdot)}{\partial \nu} f \right\} ds,}_{w(x)/2 :=} \quad x \in \Omega. \end{aligned}$$

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$$\int_{\Gamma_s} \Phi(x,\cdot)\varphi \, ds = \int_{\Gamma_m} \left\{ \Phi(x,\cdot)g - \frac{\partial \Phi(x,\cdot)}{\partial \nu} f \right\} ds - \frac{1}{2} f(x), \quad x \in \Gamma_m.$$

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$$S_{sm}\varphi = w|_{\Gamma_{m,+}}$$

$$S_{ss}\varphi = w|_{\Gamma_s}$$

Linearization with respect to Γ_s and φ
(plus iteration and regularization).
K., Rundell 2005

Parametrized operators

$$\Gamma_j = \{z_j(t) : t \in [0, 2\pi]\}, \quad j = m, s$$

Introduce parameterized single-layer potential operators

$$\widetilde{S}_{sk} : C^2[0, 2\pi] \times H^{-1/2}[0, 2\pi] \to H^{1/2}[0, 2\pi]$$

by

$$\widetilde{S}_{sk}(z_s, \psi)(t) := \frac{1}{\pi} \int_0^{2\pi} \ln \frac{1}{|z_s(t) - z_k(\tau)|}\, \psi(\tau)\, d\tau, \quad k = s, m$$

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Need to solve

$$\begin{aligned} \widetilde{S}_{sm}(z_s, \psi) &= w \circ z_m \\ \widetilde{S}_{ss}(z_s, \psi) &= w \circ z_s \end{aligned}$$

where $\psi = |z_s'| \varphi \circ z_s$.

Fréchet derivatives

$$\widetilde{S}_{sk}(z_s,\psi)(t) := \frac{1}{\pi}\int_0^{2\pi} \ln\frac{1}{|z_s(t)-z_k(\tau)|}\,\psi(\tau)\,d\tau$$

Fréchet derivatives are given explicitly by

$$d\widetilde{S}_{ss}[z_s,\psi;\zeta](t) = \frac{1}{\pi}\int_0^{2\pi} \frac{[z_s(t)-z_s(\tau)]\cdot[\zeta(t)-\zeta(\tau)]}{|z_s(\tau)-z_s(t)|^2}\,\psi(\tau)\,d\tau$$

$$d\widetilde{S}_{sm}[z_0,\psi;\zeta](t) = \frac{1}{\pi}\int_0^{2\pi} \frac{[z_m(t)-z_s(\tau)]\cdot\zeta(\tau)}{|z_m(t)-z_s(\tau)|^2}\,\psi(\tau)\,d\tau$$

Simultaneous linearization

Need to solve

$$\widetilde{S}_{sm}(z_s, \psi) = w \circ z_m$$

$$\widetilde{S}_{ss}(z_s, \psi) = w \circ z_s$$

Given an approximate solution z_s, ψ solve the linear system

$$d\widetilde{S}_{sm}[z_s, \psi; \zeta] + \widetilde{S}_{sm}(z_s, \eta) = w \circ z_m - \widetilde{S}_{sm}(z_s, \psi)$$

$$d\widetilde{S}_{ss}[z_s, \psi; \zeta] + \widetilde{S}_{ss}(z_s, \eta) - \zeta \cdot (\operatorname{grad} w) \circ z_s = w \circ z_s - \widetilde{S}_{ss}(z_s, \psi)$$

for ζ, η to update z_s, ψ into $z_s + \zeta, \psi + \eta$

Nonlinear integral equations

$$
\begin{aligned}
S_{sm}\varphi &= S_{mm}g - K_{mm}f - f \\
S_{ss}\varphi &= S_{ms}g - K_{ms}f
\end{aligned}
$$

Linearization both equations with respect to Γ_s and φ
(plus regularized iteration).

K., Rundell 2005 Dirichlet condition
Ivanyshyn, K. 2006, . . . Neumann cond. and inverse scattering
Cakoni, K., Schuft 2010 Impedance condition
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Linearize only one equation (in inverse scattering).
Johansson, Sleeman 2007 Linearize data equation
K., Serranho 2003, ... Linearize field equation

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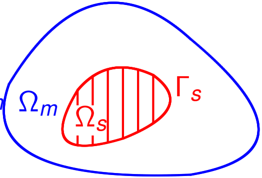
Related to, but different from regularized Gauss–Newton iterations for data to boundary map $F : \Gamma_s \mapsto \partial u/\partial\nu|_{\Gamma_m}$

Equivalent nonlocal impedance condition

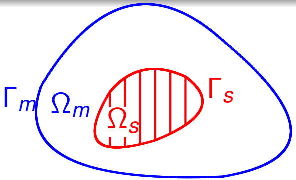


$$u_0(x) := \frac{1}{4}|x|^2 \quad \text{solves} \quad \Delta u_0 = 1 \quad \text{in } R^2$$

For the solution $u \in H^1(\Omega_m)$ of the source problem the difference $u - u_0$ is harmonic in Ω_s and therefore

$$\left.\frac{\partial(u - u_0)}{\partial\nu}\right|_{\Gamma_s} = A_s(u - u_0)|_{\Gamma_s}.$$

$A_s : H^{1/2}(\Gamma_s) \to H^{-1/2}(\Gamma_s)$ Dirichlet-to-Neumann map for Ω_s



$$\Omega := \Omega_m - \overline{\Omega_s}$$

$$\left.\frac{\partial(u-u_0)}{\partial\nu}\right|_{\Gamma_s} = A_s(u-u_0)|_{\Gamma_s}$$

Theorem

Solving

$$\Delta u = \chi_s \ in\ \Omega_m, \quad u = f \ on\ \Gamma_m$$

is equivalent to solving

$$\Delta u = 0 \ in\ \Omega, \quad u = f \ on\ \Gamma_m, \quad \frac{\partial u}{\partial \nu} - A_s u|_{\Gamma_s} = \frac{\partial u_0}{\partial \nu} - A_s u_0|_{\Gamma_s} \ on\ \Gamma_s$$

Nonlinear integral equation

For

$$\frac{\partial u}{\partial \nu} - \lambda u = 0 \quad \text{on } \Gamma_s$$

we have the two equations

$$\begin{aligned} K_{sm}\varphi - S_{sm}\lambda\varphi &= f + K_{mm}f - S_{mm}g \\ -\varphi + K_{ss}\varphi - S_{ss}\lambda\varphi &= K_{ms}f - S_{ms}g \end{aligned}$$

for the unknowns Γ_s and $\varphi := u|_{\Gamma_s}$.

Nonlinear integral equation

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$$\frac{\partial u}{\partial \nu} - \lambda u = 0 \quad \text{on } \Gamma_s$$

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for the unknowns Γ_s and $\varphi := u|_{\Gamma_s}$.
For nonlocal and inhomogeneous impedance condition we have

$$\begin{aligned} K_{sm}\varphi - S_{sm}A_s\varphi &= f + K_{mm}f - S_{mm}g + S_{sm}\eta \\ -\varphi + K_{ss}\varphi - S_{ss}A_s\varphi &= K_{ms}f - S_{ms}g + S_{ss}\eta \end{aligned}$$

for the unknowns Γ_s and $\varphi := u|_{\Gamma_s}$ with $\eta := \left.\frac{\partial u_0}{\partial \nu}\right|_{\Gamma_s} - A_s u_0|_{\Gamma_s}$

Nonlinear integral equation

Green's integral formula yields the two equations

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$$
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\end{aligned}
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for the unknowns Γ_s and $\varphi := u|_{\Gamma_s}$ with $\eta := \left.\dfrac{\partial u_0}{\partial \nu}\right|_{\Gamma_s} - A_s u_0|_{\Gamma_s}$

Eliminate the slip variable:

$$
\Delta v = 0 \quad \text{in } \Omega_s, \qquad v = \varphi \quad \text{on } \Gamma_s
$$

$$
v(x) = \int_{\Gamma_s} \left\{ \Phi(x,\cdot)A_s\varphi - \frac{\partial \Phi(x,\cdot)}{\partial \nu}\,\varphi \right\} ds, \quad x \in \Omega_s
$$

$$
\Rightarrow \quad \varphi = S_{ss}A_s\varphi - K_{ss}\varphi
$$

Nonlinear integral equation

Theorem

The inverse source problem is equivalent to solving the nonlinear integral equation

$$-\frac{1}{2}\,(K_{sm} - S_{sm}A_s)\Big(K_{ms}f - S_{ms}g$$

$$+S_{ss}\frac{\partial u_0}{\partial \nu}\Big|_{\Gamma_s} - K_{ss}u_0|_{\Gamma_s} - u_0|_{\Gamma_s}\Big)$$

$$-S_{sm}\Big(\frac{\partial u_0}{\partial \nu}\Big|_{\Gamma_s} - A_s u_0|_{\Gamma_s}\Big) = f + K_{mm}f - S_{mm}g$$

for the unknown Γ_s.

Nonlinear integral equation

Theorem

The inverse source problem is equivalent to solving the nonlinear integral equation

$$-\frac{1}{2}\left(K_{sm} - S_{sm}A_s\right)\Big(K_{ms}f - S_{ms}g + S_{ss}\frac{\partial u_0}{\partial \nu}\Big|_{\Gamma_s} - K_{ss}u_0|_{\Gamma_s} - u_0|_{\Gamma_s}\Big)$$

$$-S_{sm}\left(\frac{\partial u_0}{\partial \nu}\Big|_{\Gamma_s} - A_s u_0|_{\Gamma_s}\right) = f + K_{mm}f - S_{mm}g$$

for the unknown Γ_s.

Solve by regularized Newton iterations

Linearization

Fréchet derivatives of double-layer operators K_{jk} are analogous to those of single-layer operators S_{jk}.

Recall

$$S_{ss}A_s - K_{ss} = I.$$

$$\Rightarrow \quad A_s = S_{ss}^{-1}(I + K_{ss})$$

Invertibility of single-layer operator S_{ss} requires a geometric assumption.

Relation with boundary-to-data map

Newton iterations for the nonlinear integral equation are equivalent to Newton iterations for

$$W(F\Gamma_s - g) = 0$$

where

$$W := S_{mm} + \frac{1}{2}\,[K_{sm} - S_{sm}A_s]S_{ms}$$

Numerical examples

Approximations starlike with radial function a trigonometric polynomial of degree $m = 6$.
Initial guess is a circle with radius

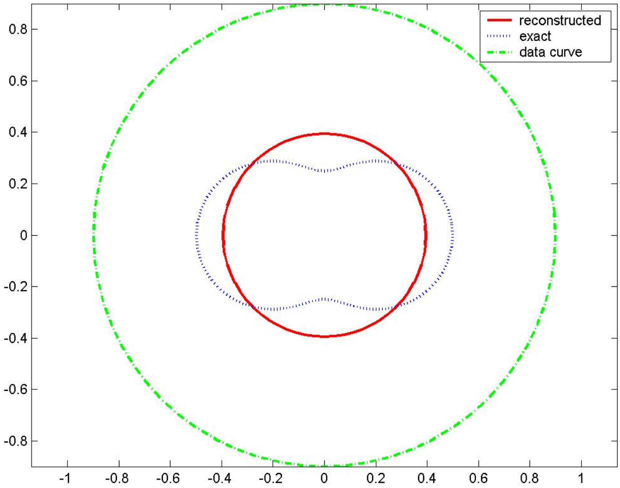
$$\pi\rho^2 = \int_{\Gamma_m} g\, ds,$$

since

$$\int_{\Omega_m} \chi_s\, dx = \int_{\Gamma_m} g\, ds.$$

Same regularization parameter in all examples.
64 quadrature points for discretization of integral operators.

Numerical example with exact data



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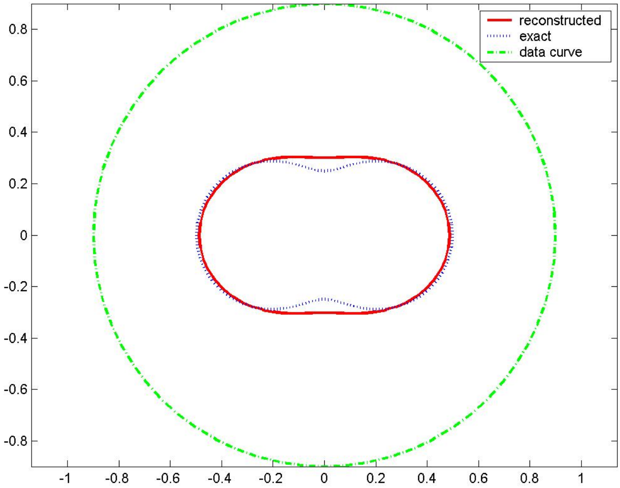

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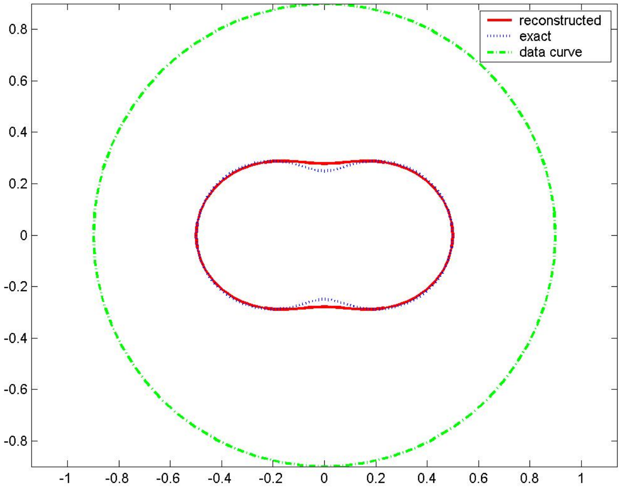

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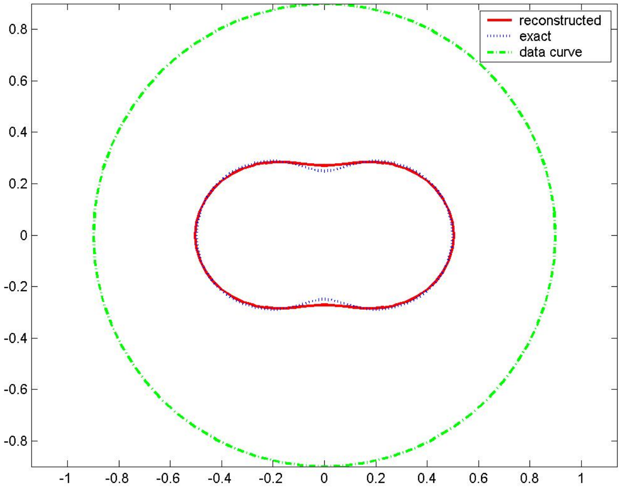

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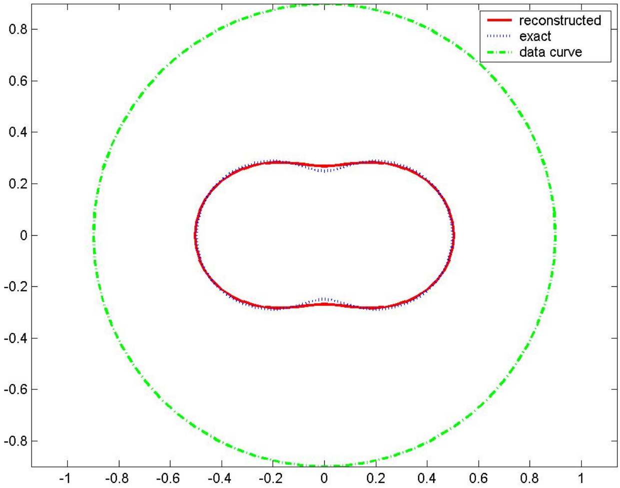

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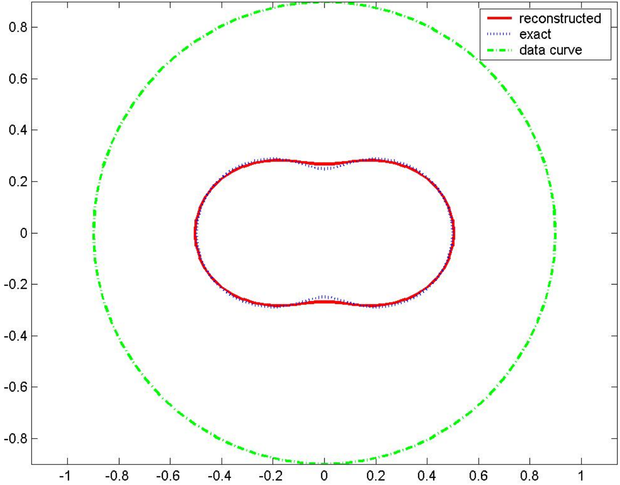
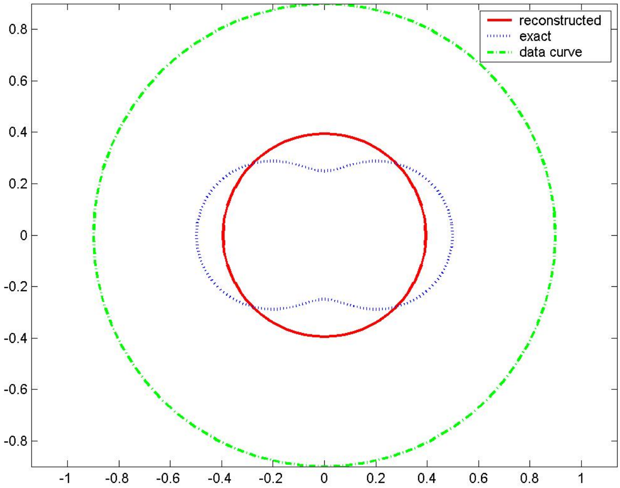


Repeat

Numerical example with noisy data



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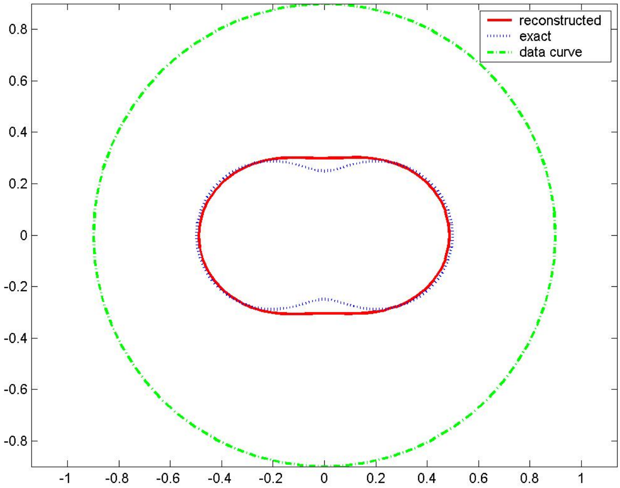

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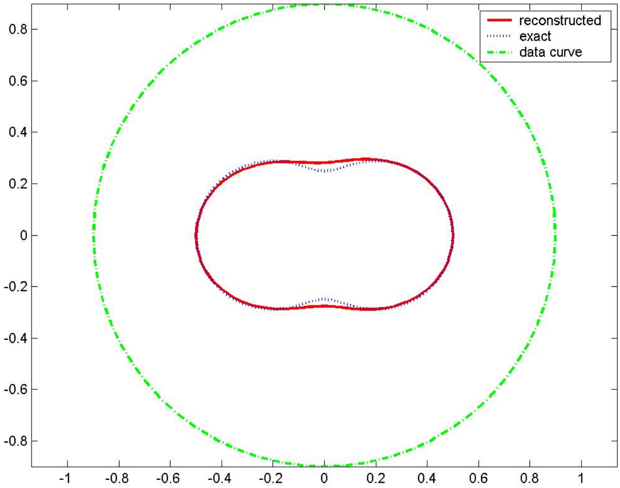

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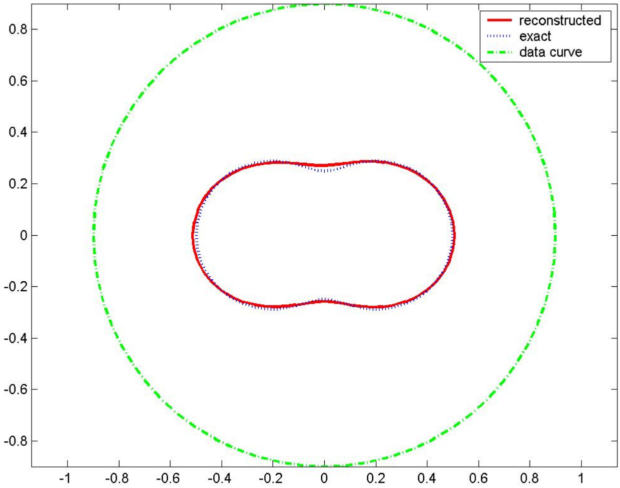

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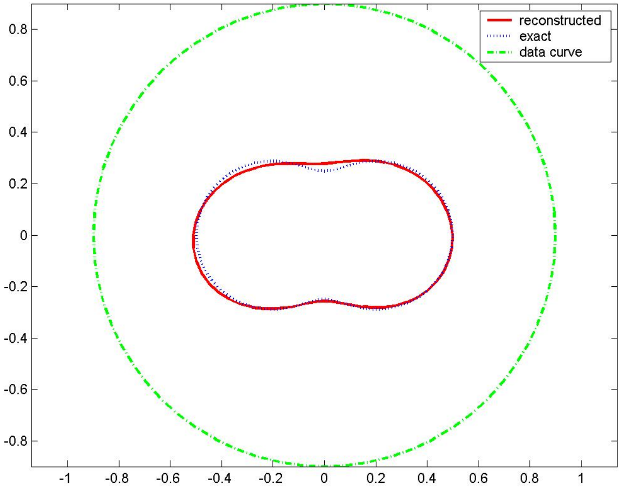
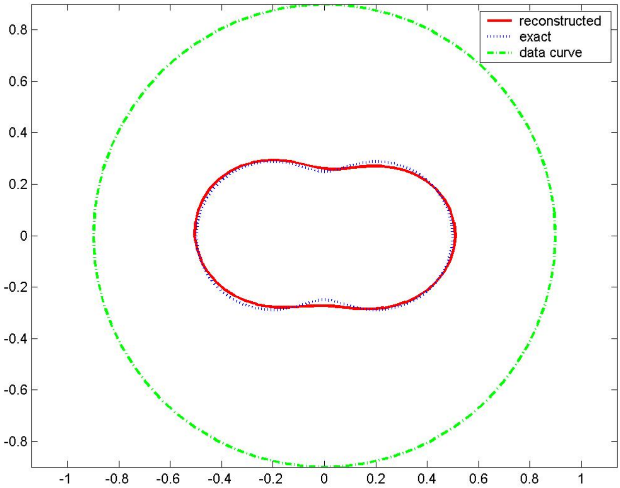
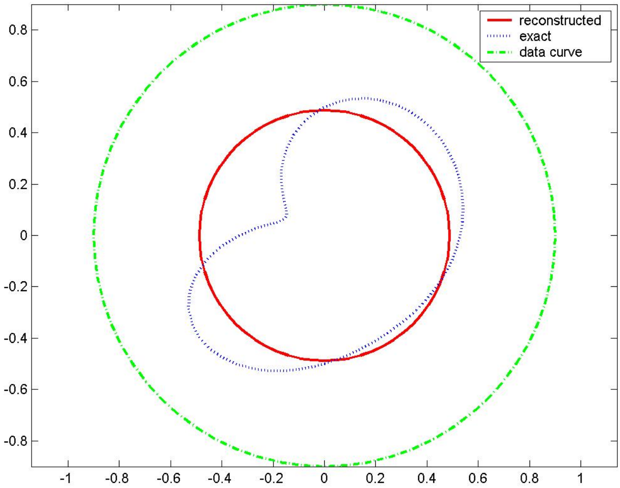


Numerical example with noisy data

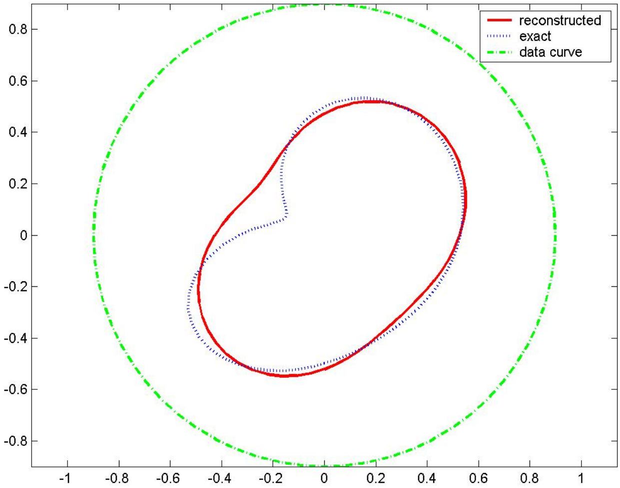


Repeat

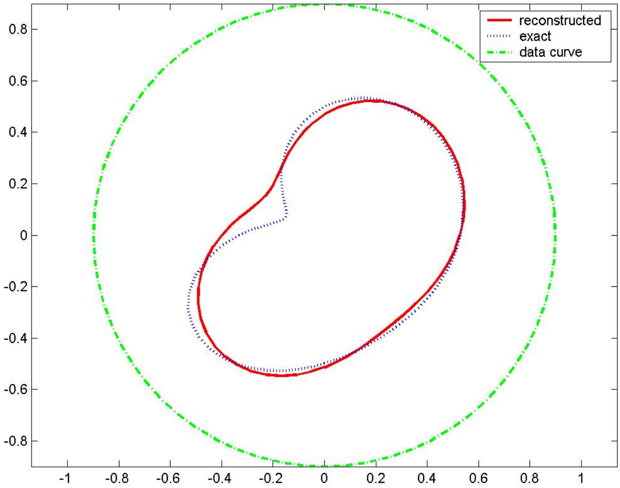
Numerical example with exact data



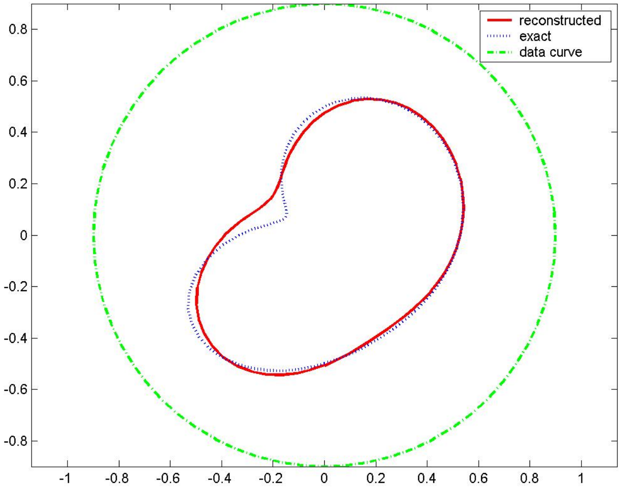
Numerical example with exact data



Numerical example with exact data



Numerical example with exact data



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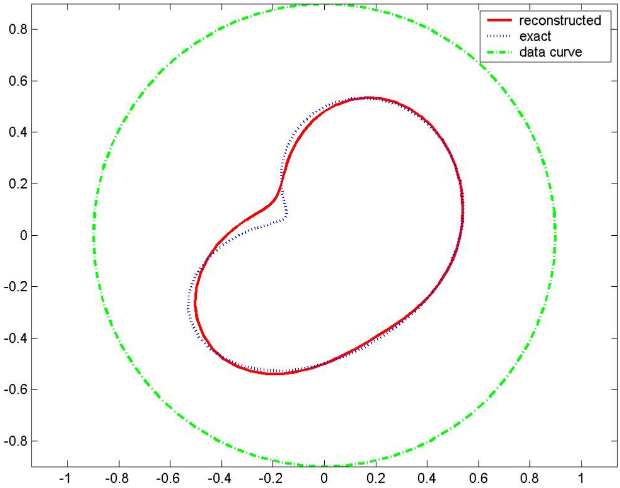
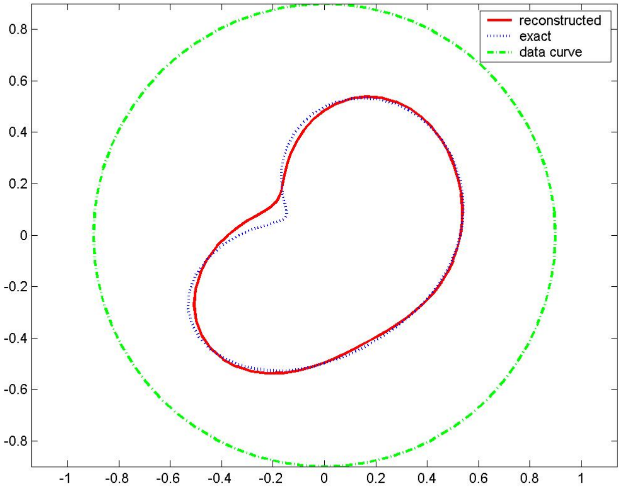


Numerical example with exact data



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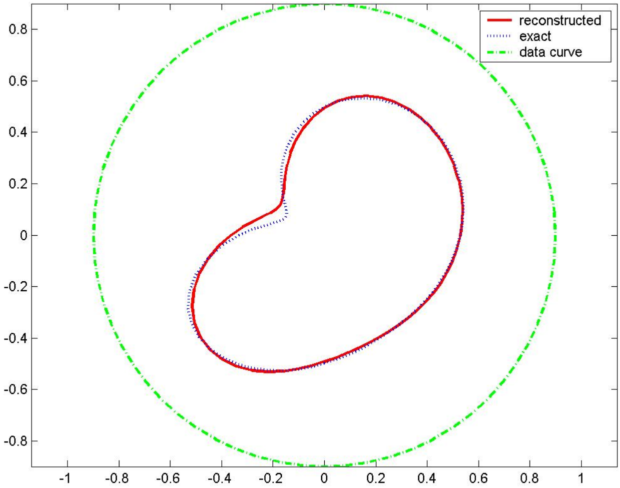
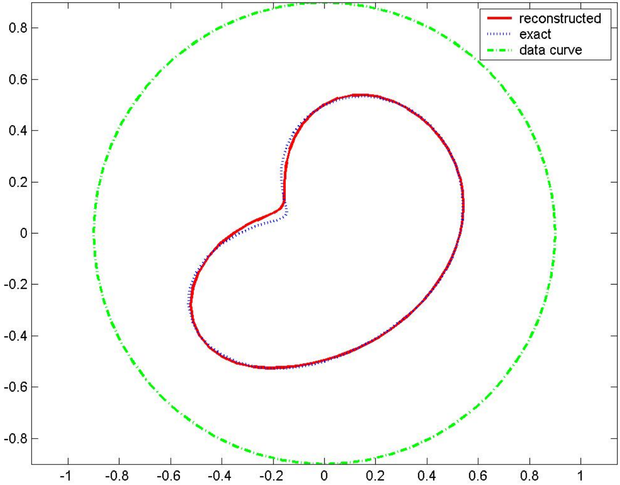
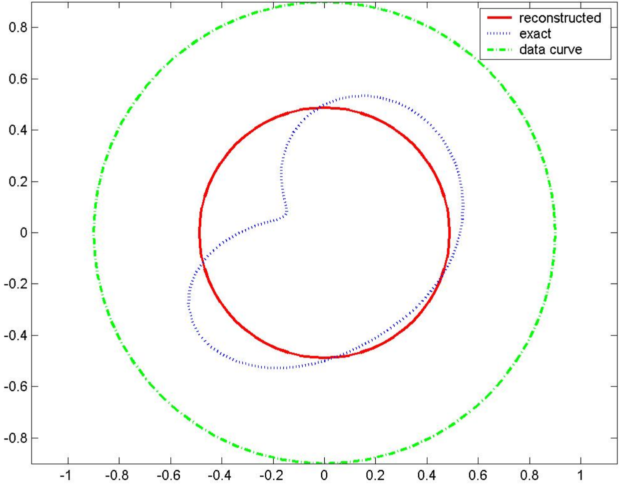


Numerical example with exact data

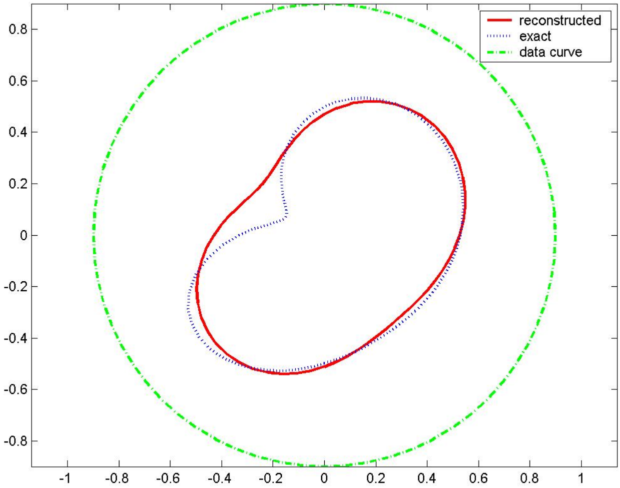


Repeat

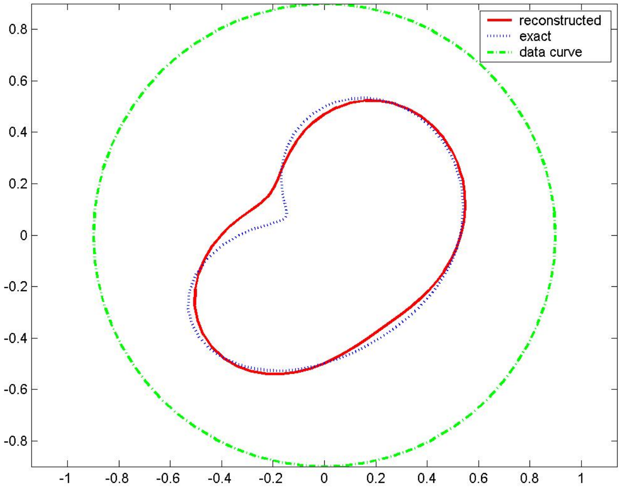
Numerical example with noisy data



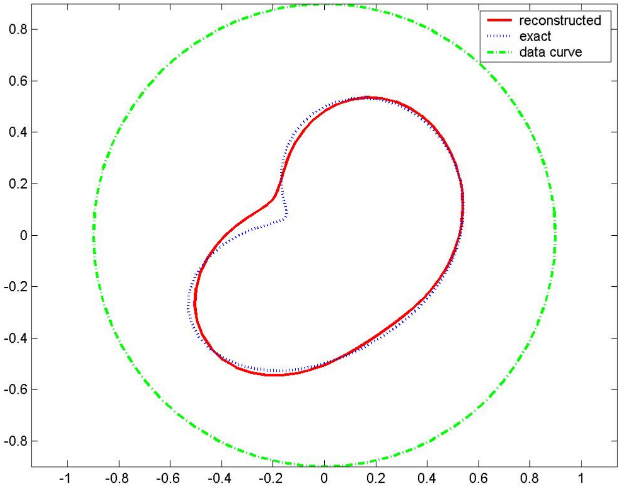
Numerical example with noisy data



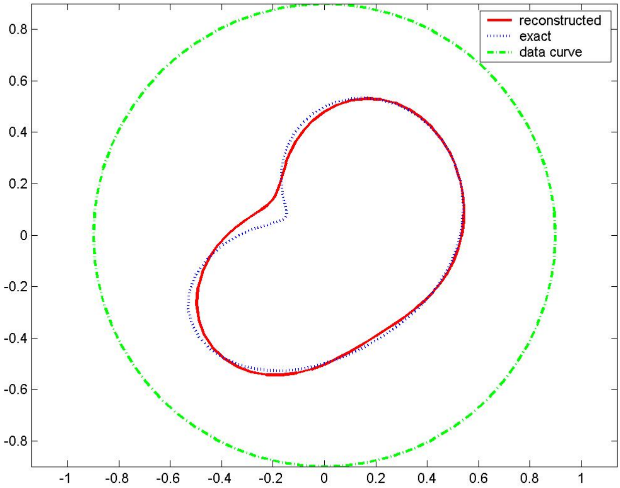
Numerical example with noisy data



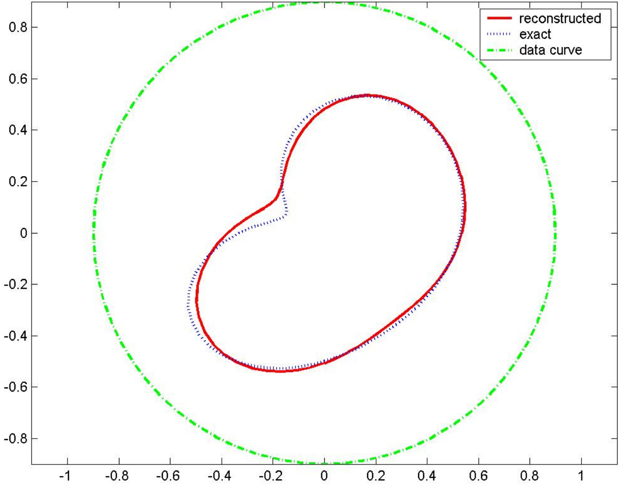
Numerical example with noisy data



Numerical example with noisy data

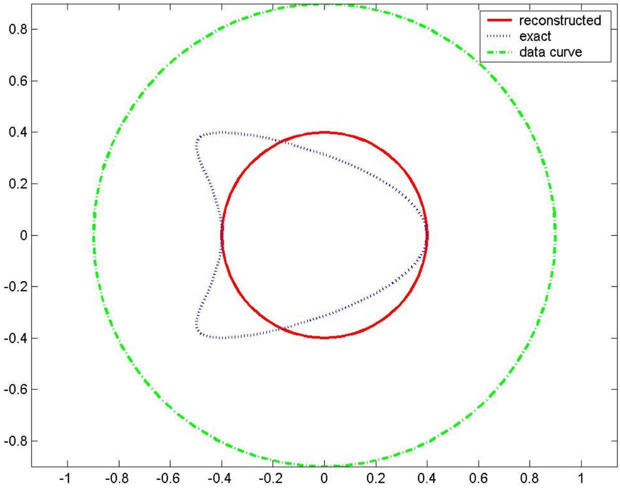


Numerical example with noisy data

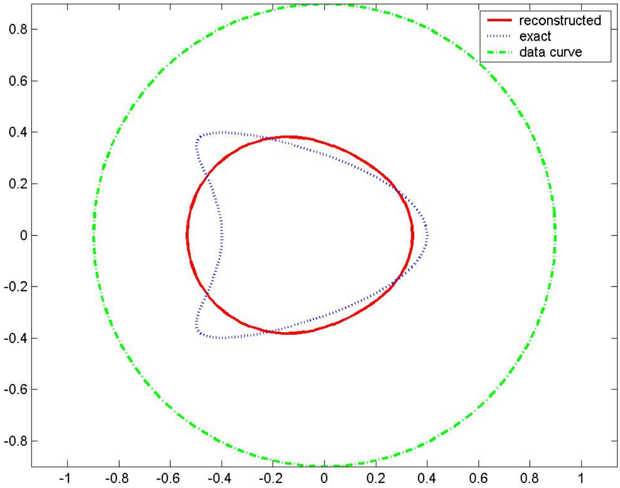


Repeat

Numerical example with exact data



Numerical example with exact data



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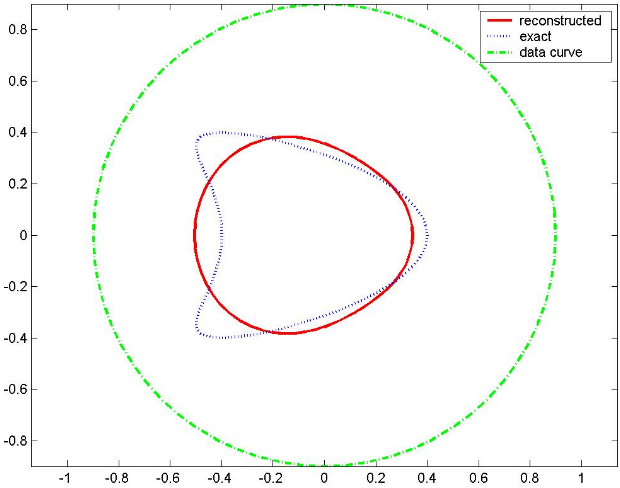
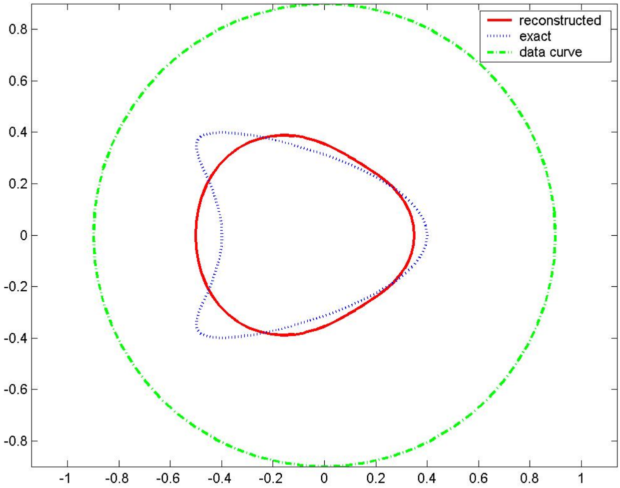
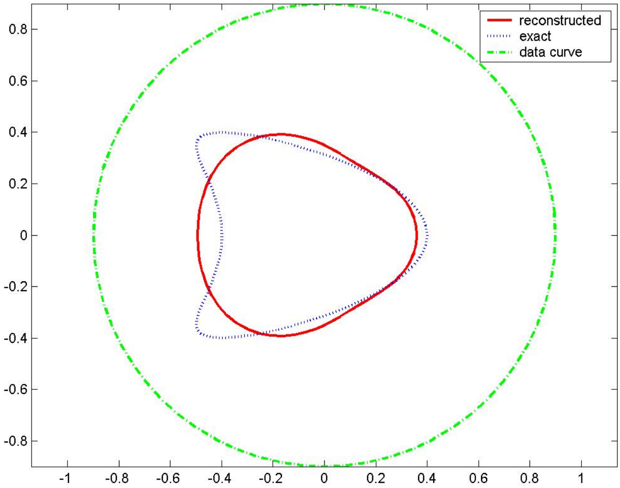


Numerical example with exact data



Numerical example with exact data



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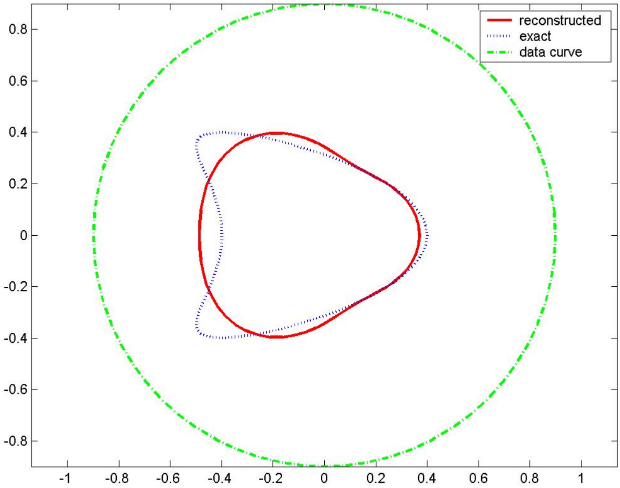
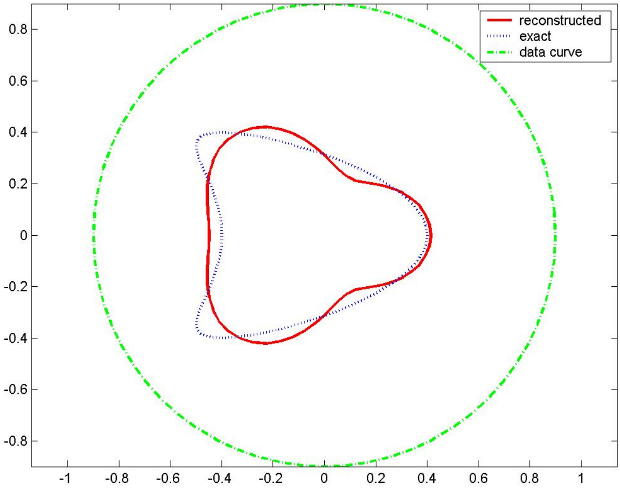


Numerical example with exact data



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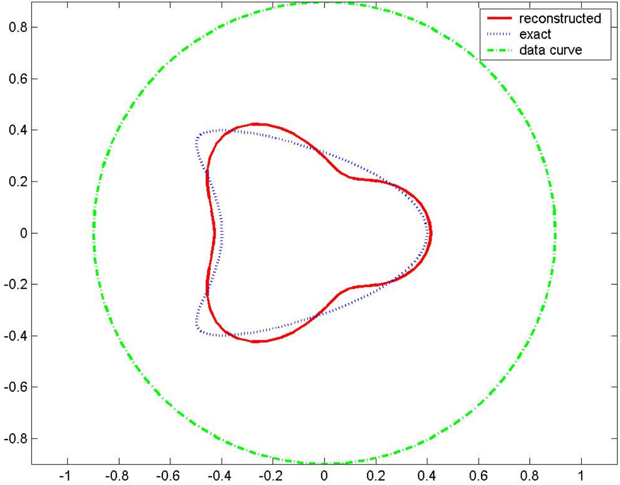
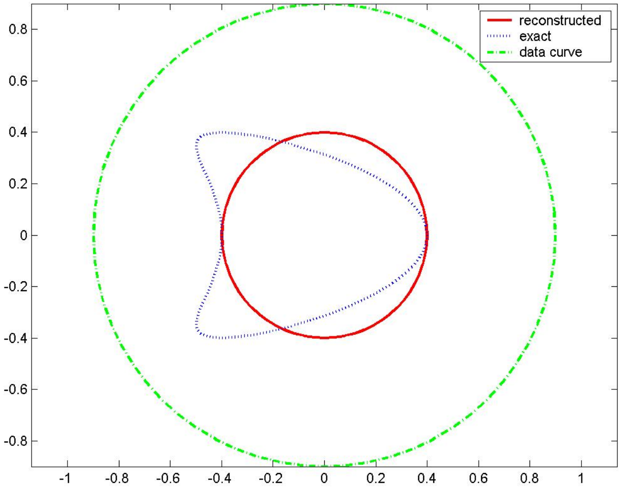
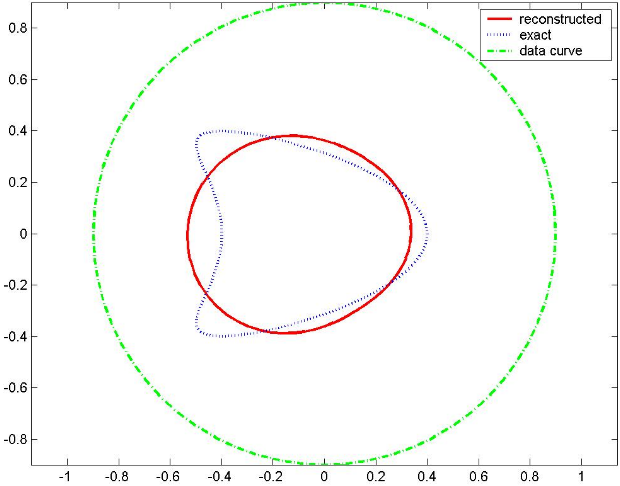


Repeat

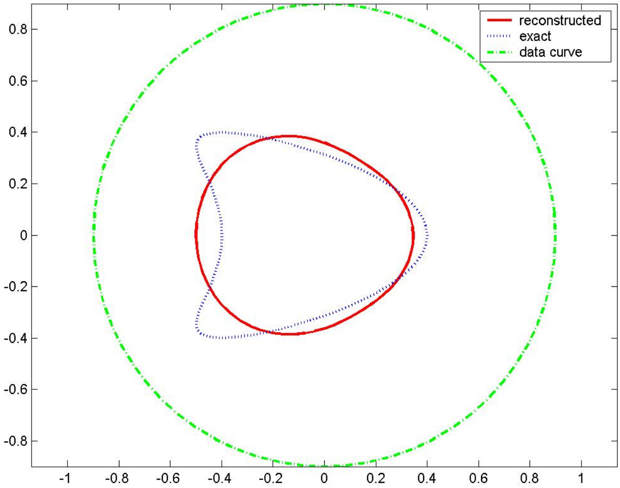
Numerical example with noisy data



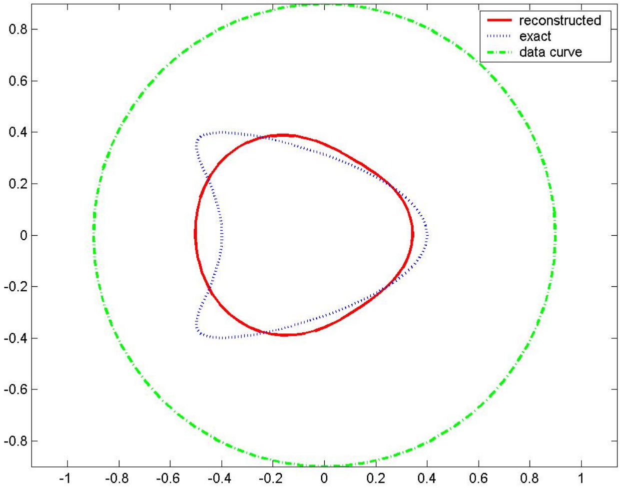
Numerical example with noisy data



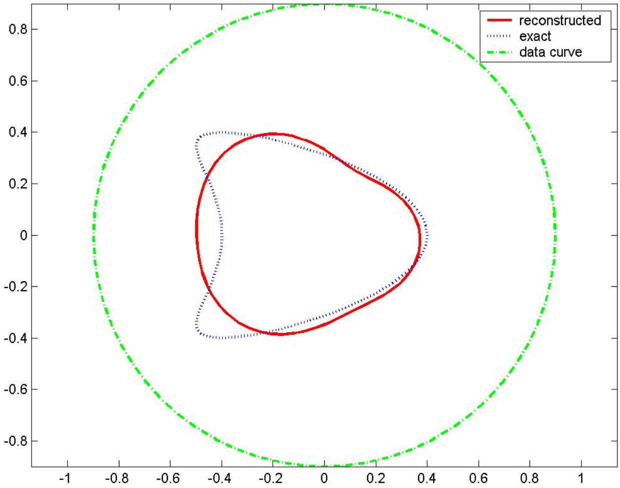
Numerical example with noisy data



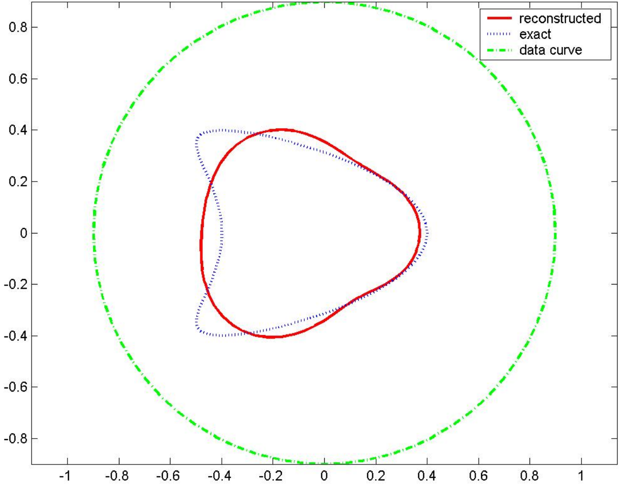
Numerical example with noisy data



Numerical example with noisy data



Numerical example with noisy data



Repeat

Future work

1. Numerics in three dimensions and for Helmholtz equation
2. Numerics for multiple inclusions
3. Uniqueness for arbitrary shape

References

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  On rational approximation methods for inverse source problems.
  Inverse Problems and Imaging **5**, 185–202 (2011).
- Hettlich, F. and Rundell, W.:
  Iterative methods for the reconstruction of an inverse potential problem.
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- Kress, R. and Rundell, W.:
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  Jour. Integral Equations and Appl., to appear

Thank you