Reconstruction of the stored energy function of a certain class of hyperelastic materials from Cauchy data

joined work with **Thomas Schuster**, Frank Binder, F. Schöpfer and Arne Wöstehoff

Julia Piontkowski

Saarland University
Department of Mathematics
66123 Saarbrücken, Germany

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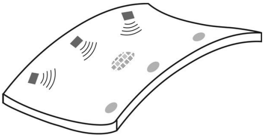
Overview

Overview

Aims of Structural Health Monitoring (SHM)

Idea: Monitoring of construction elements in carbon fibre reinforced composites by analyzing guided waves emitted by integrated piezo-ceramic actors

$\rightsquigarrow$ Structural Health Monitoring system (SHM) to detect defects in carbon fibre reinforced composites



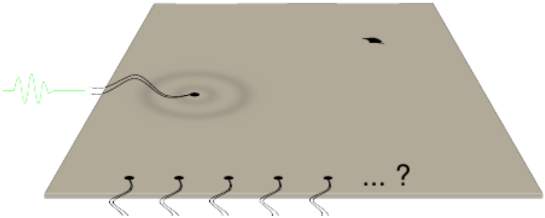
Sketch of experimental setup for an SHM system

- detection of defects
- localization of defects
- categorization of defects
- expansion of defects

Relevance of SHM in industry

- early detection of delaminations and cracks
- safety enhancement
- reduction of maintenance intervals
- cost saving by optimal assembly of sensors and actuators

Idea of SHM for anistropic plates



- Signal emitted at piezo-ceramic actuator
- Defects lead to reflection, attenuation and mode conversion
- Signals measured at piezo-ceramic sensors
- Inverse problem of damage localization from signal measurements

Identification of material parameters

Idea: Reconstruction of (spatial varying) material parameters to get information on defects and to visualize them

Some references:

- NAKAMURA, UHLMANN, 1994: Global uniqueness for an inverse boundary problem arising in elasticity
- NAKAMURA, UHLMANN, 1995: Inverse problems at the boundary for an elastic medium
- HÄHNER, 2002: Inverse reconstruction of mass density
- BONNET, CONSTANTINESCU, 2005: Inverse problems in elasticity
- KALTENBACHER, LORENZI, 2007: Reconstruction of material tensor under certain requirements
- SEDIPKOV, 2011: Inverse problems in inhomogeneous, elastic media
- BOURGEOIS ET AL., 2011: Linear sampling for elastic waveguides
- IMANUVILOV, YAMAMOTO, 2011: Reconstruction of Lamé coefficients in 2D
- IMANUVILOV, UHLMANN, YAMAMOTO, 2013: Uniquenes of Lamé coefficients from partial Cauchy data in 3D
- SCHUSTER, WÖSTEHOFF, 2014: On the identifiability of the stored energy function of hyperelastic materials from sensor data at the boundary
- BAL, MONARD, UHLMANN, 2015: Reconstruction of a fully anisotropic elasticity tensor from knowledge of displacement fields

Overview

1. Motivation and mathematical setup
   - Motivation
   - Mathematical foundations of elasticity
2. Unique solvability and stability of the IBVP
3. The inverse identification problem
   - Uniqueness of the inverse identification problem
   - Recovering the stored energy function as a conic combination
   - Ongoing research
4. Conclusions

Conservation and constitutive law

Conservation of mass:

$$\int_{U_t} \frac{\partial \rho}{\partial t}(t,x)dx = -\int_{\partial U_t} \rho \dot{u} \cdot n dA$$

Conservation of momentum:

$$\frac{d}{dt}\int_{U_t} \rho \dot{u} dx = \int_{\partial U_t} \tau(t,\psi,n)dA + \int_{U_t} f(t,x)dx$$

Conservation of angular momentum:

$$\frac{d}{dt}\int_{U_t} x \times \rho \dot{u} dx = \int_{\partial U_t} \psi \times \tau(t,\psi,n)dA + \int_{U_t} x \times f(t,x)dx$$

Constitutive law:

$$P(t,x) = \hat{P}(x, \nabla u(t,x))$$

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Wave propagation in anisotropic materials

Wave propagation in a domain $\Omega \subset \mathbb{R}^3$ is governed by the equation

$$\rho(x)\ddot{u}(t,x) - \mathrm{div}\hat{P}(x, \nabla u(t,x)) = f(t,x)$$

with

$u : [0,T] \times \Omega \to \mathbb{R}^3$	displacement field
$f : [0,T] \times \Omega \to \mathbb{R}^3$	volume force
$\rho : \Omega \to \mathbb{R}$	mass density
$P : [0,T] \times \Omega \to \mathbb{R}^{3\times 3}$	Piola-Kirchhoff stress tensor
$\hat{P} : \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$	response function

Hyperelasticity

A material is called hyperelastic, if there is a stored energy function $C: \ \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ with

$$\hat{P}(x, Y) = \nabla_Y C(x, Y) = (\partial_{Y_{ij}} C(x, Y))_{i,j}$$

Physical requirements:

1. $C(x, 0) = 0$ for almost all $x \in \Omega$
2. $\nabla_Y C(x, 0) = 0$ for almost all $x \in \Omega$

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Stored energy function as conic combination

Assumption: The function $C(x, Y)$ has a representation

$$C(x, Y) = \sum_{K=1}^{N} \alpha_K C_K(x, Y)$$

with $\alpha_K \geq 0$ and fixed $C_K : \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R}$

Compare:
KALTENBACHER, LORENZI, 2007
SCHUSTER, WÖSTEHOFF, 2014

IBVP for waves in hyperelastic, anisotropic materials

From all these assumptions we obtain the

Hyperelastic wave equation

$$\rho(x)\ddot{u}(t,x) - \sum_{K=1}^{N} \alpha_K \mathrm{div}[\nabla_Y C_K(x, \nabla u(t,x))] = f(t,x)$$
$$u(0,x) = u_0(x), \quad x \in \Omega$$
$$\dot{u}(0,x) = u_1(x), \quad x \in \Omega$$
$$u(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega$$

Measurements

We assume to have N piezo sensors measurements of (weighted mechanical) stresses on parts of the boundary

$$\begin{aligned}\delta_L \quad &:= \quad \int_{\partial\Omega} \varphi_L(x)^\top \hat{P}(x, \nabla u(T_L, x)) \cdot \nu(x) d\sigma(x) \\ &= \quad \sum_{K=1}^{N} \alpha_K \int_{\partial\Omega} \varphi_L(x)^\top [\nabla_Y C_K(x, \nabla u(T_L, x))] \cdot \nu(x) d\sigma(x)\end{aligned}$$

$L = 1, ..., N$ with

$\varphi_L \in L^2(\partial\Omega, \mathbb{R}^3)$	weight functions
$T_L \in (0, T)$	instants of time
$\nu : \ \partial\Omega \to \mathbb{R}^3$	outer unit normal field of $\partial\Omega$

Overview

Admissible solutions and coefficient vectors

$$
\begin{aligned}
&E(M_0, M_1, M_2, M_3, (\kappa^{[a]})_{a=1,2}, (\mu^{[b]})_{b=1,\dots,7}) := \\
&\Big\{(u,\alpha) \in [L^\infty((0,T)\times\Omega,\mathbb{R}^3) \cap W^{1,\infty}((0,T),H^1(\Omega,\mathbb{R}^3))] \times \mathbb{R}_+^N : \\
&\|\partial_{x_l}\partial_{x_j} u\|_{L^\infty((0,T),L^2(\Omega,\mathbb{R}^3))} \le M_0, \|\partial_{x_l}\dot{u}\|_{L^\infty((0,T),L^\infty(\Omega,\mathbb{R}^3))} \le M_1, \\
&\|\partial_{x_l}\partial_{x_j}\dot{u}\|_{L^\infty((0,T),L^\infty(\Omega,\mathbb{R}^3))} \le M_2, \|\partial_{x_l}\partial_{x_j} u\|_{L^\infty((0,T),L^\infty(\Omega,\mathbb{R}^3))} \le M_3, \\
&\sum_{K=1}^N \alpha_K \kappa_K^{[a]} \ge \kappa^{[a]}, \sum_{K=1}^N \alpha_K \mu_K^{[b]} \le \mu^{[b]}, a = 1,2, b = 1,\dots,7\Big\}
\end{aligned}
$$

Continuous dependence on data

Theorem (Wöstehoff, Schuster, 2013)

With Ω bounded and having a $\mathcal{C}^2$-boundary, let u, $\tilde{u}$ be two solutions to the IVBP corresponding to (α, u_0, u_1, f), $(\tilde{\alpha}, \tilde{u}_0, \tilde{u}_1, \tilde{f})$, respectively, $(u, \alpha), (\tilde{u}, \tilde{\alpha}) \in E(M_0, M_1, M_2, M_3, (\kappa^{[a]})_{a=1,2}, (\mu^{[b]})_{b=1,\dots,7})$ and certain restrictions especially to C_K we have for all $t \in (0, T)$

$$\Big[\|\rho(\dot{u} - \dot{\tilde{u}})(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \kappa(\alpha)\|(Ju - J\tilde{u})(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^{3\times 3})}$$
$$+\|\rho(\ddot{u} - \ddot{\tilde{u}})(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \kappa(\alpha)\|(J\dot{u} - J\dot{\tilde{u}})(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^{3\times 3})}$$
$$+\|(u - \tilde{u})(t,\cdot)\|^2_{H^2(\Omega,\mathbb{R}^3)}\Big]^{1/2}$$
$$\leq \bar{C}_0\Big[\mu(\alpha)\|(u_0 - \tilde{u}_0)(t,\cdot)\|^2_{H^2(\Omega,\mathbb{R}^3)} + \|(u_1 - \tilde{u}_1)(t,\cdot)\|^2_{H^1(\Omega,\mathbb{R}^3)}\Big]^{1/2}$$
$$+\bar{C}_1\|f - \tilde{f}\|_{W^{1,1}((0,T),L^2(\Omega,\mathbb{R}^3))} + \bar{C}_2\|\alpha - \tilde{\alpha}\|_\infty.$$

Overview

Admissible data

$$\begin{aligned} D(m,r) \quad := \quad & \{(f,u_0,u_1) \in W^{1,1}((0,T),L^2(\Omega,\mathbb{R}^3)) \times \\ & (H^2(\Omega,\mathbb{R}^3) \cap H_0^1(\Omega,\mathbb{R}^3)) \times H^1(\Omega,\mathbb{R}^3) : \\ & \|f\|_{W^{1,1}((0,T),L^2(\Omega,\mathbb{R}^3))} + \|u_0\|_{H^2(\Omega,\mathbb{R}^3)} + \|u_1\|_{H^1(\Omega,\mathbb{R}^3)} \leq r, \\ & \|W(u_0)^{-1}\|_\infty \leq 1/m\} \end{aligned}$$

with

$$W(u_0) = \left(\int_{\partial\Omega} \varphi_L(x)^\top [\nabla_Y C_K(x,\nabla u_0(x))] \cdot \nu(x) d\sigma(x) \right)_{K,L=1,\dots,N}$$

Uniqueness and continuous dependency

Let

$$(f, u_0, u_1), (\tilde{f}, \tilde{u}_0, \tilde{u}_1) \in D(m, r)$$

be two sets of data and let

$$(u, \alpha), (\tilde{u}, \tilde{\alpha}) \in E(M_0, M_1, M_2, M_3, \kappa^{[1]}, (\mu^{[b]})_{b=1,\dots,7})$$

be two solutions of the identification problem corresponding to

$$(f, u_0, u_1, \delta) \quad \text{and} \quad (\tilde{f}, \tilde{u}_0, \tilde{u}_1, \tilde{\delta})$$

respectively. Let

$$\bar{T} := \max\{T_1, \dots, T_N\}$$

be sufficiently small.

Uniqueness and continuous dependency

$\bar{T}$ sufficiently small:

$$\hat{K}(\bar{T}) := C(r, M_0) \max_{L=1,\dots,N} \|\varphi_L\|_{L^2(\partial\Omega,\mathbb{R}^3)} \sum_{K=1}^{N} \mu_K^{[1]} \bar{T}^{1/4} < m$$

and

$$\hat{C}(\bar{T}) := \frac{\tilde{C}}{\hat{K}(\bar{T})} \max_{L=1,\dots,N} \|\varphi_L\|_{L^2(\partial\Omega,\mathbb{R}^3)} \sum_{K=1}^{N} \alpha_K \mu_K^{[1]} \bar{T}^{1/4} < 1$$

Uniqueness and continuous dependency

Theorem (Wöstehoff, Schuster, 2013)

Then, there are constants $\hat{C}_0, \hat{C}_1 > 0$ such that

$$\begin{aligned} \|\alpha - \tilde{\alpha}\|_\infty &\leq \frac{1}{(m - \hat{K}(\bar{T}))(1 - \hat{C}(\bar{T}))} \Big\{ \|\delta - \tilde{\delta}\|_\infty + \\ &+ \hat{C}_0 \left[\|u_0 - \tilde{u}_0\|^2_{H^2(\Omega,\mathbb{R}^3)} + \|u_1 - \tilde{u}_1\|^2_{H^1(\Omega,\mathbb{R}^3)} \right]^{1/2} + \\ &+ \hat{C}_1 \|f - \tilde{f}\|_{W^{1,1}((0,T),L^2(\Omega,\mathbb{R}^3))} \Big\} \end{aligned}$$

Overview

Linearization: linear, hyperelastic materials

We have:

$$P(t,x) = \hat{P}(x,Y) = \nabla_Y C(x,Y)$$

with $Y = \nabla u(t,x)$.
Linearization around $Y = I$ provides:

$$\mathbb{C}(x) = \nabla_Y \hat{P}(x,I) = \nabla_Y \nabla_Y C(x,I)$$

the **elasticity tensor**.
The stored energy function of a linear, hyperelastic material is:

$$\tilde{C}(Y) = \frac{1}{2}\epsilon(Y) : \mathbb{C} : \epsilon(Y)$$

with $\epsilon(Y) = \frac{1}{2}(Y + Y^\top)$.

Linearized model

Wave propagation in a domain $\Omega \subset \mathbb{R}^3$ of anisotropic, linear hyperelastic material is governed by the equation:

$$\rho(x)\ddot{u}(t,x) - \sum_{K=1}^{N} \alpha_K L^\top \mathbb{C}_K(x) L u(t,x) = f(t,x)$$

$L := \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{pmatrix}$	matrix differential operator
$\sum_{K=1}^{N} \alpha_K \mathbb{C}_K \in \mathbb{R}^{6\times 6}$	spd. elasticity tensor
u	displacement field
f	volume force

Set of admissible parameters

Take

$$X_3 := \{M \in P_3^{3\times 3} | M^\top = M\} \quad \text{with}$$

$$P_3 \subset \mathscr{C}(\Omega) \quad \text{a 3-dimensional subspace}$$

and the subset of admissible parameters

$$M^3_{\epsilon,s} := \{M \in X_3 | y^\top M(x) y \geq \epsilon y^\top y,$$
$$y^\top M(x) y \leq s y^\top y \; \forall y \in \mathbb{R}^3, \forall x \in \Omega\}$$

Convexity, closedness and boundedness, i.e. compactness, of $M^3_{\epsilon,s} \subset X_3$ are easy to see with respect to the norm

$$\|M\|_{X_3} := \sup_{y \in \mathbb{R}^3 \setminus \{0\}} \frac{\|y^\top M y\|_\infty}{y^\top y}$$

Decomposition of polynomial elasticity tensor

Theorem

Let $0 < \eta < \epsilon < s$. Then, there is a finite number of elements $H_1, \dots, H_N \in M^3_{\eta, s+\epsilon-\eta}$, such that

$$M^3_{\epsilon,s} \subset \mathrm{conv}\{H_1, \dots, H_N\}.$$

Proof by covering $M^3_{\epsilon,s}$ with sets $M + P$, where $M \in M^3_{\epsilon,s}$ and P is an analogue to a polytope.

Isotropic elasticity tensor

$$C = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & & & \\ \lambda & \lambda + 2\mu & \lambda & & & \\ \lambda & \lambda & \lambda + 2\mu & & & \\ & & & \mu & & \\ & & & & \mu & \\ & & & & & \mu \end{pmatrix}$$

with Lamé parameters λ, μ with $\mu > 0$ and $3\lambda + 2\mu > 0$.

$$\rho \ddot{u} - L^\top C L u = f \quad \Rightarrow \quad \rho \ddot{u} - \mu \Delta u - (\lambda + \mu) \operatorname{grad} \operatorname{div} u = f$$

(Lamé-Navier-equation)

Theorem

Let C be isotropic and $\epsilon \leq 3\lambda + 2\mu, \mu \leq s$. Then

$$C = \alpha_1 C(\epsilon, s) + \alpha_2 C(s, \epsilon)$$

with

$$\alpha_1 = \frac{3s\lambda + (2s - \epsilon)\mu}{s^2 - \epsilon^2} \geq 0, \quad \alpha_2 = \frac{-3\epsilon\lambda + (s - 2\epsilon)\mu}{s^2 - \epsilon^2} \geq 0$$

and

$$C(\epsilon, s) = \begin{pmatrix} s/3 + 2\epsilon & s/3 - \epsilon & s/3 - \epsilon & & & \\ s/3 - \epsilon & s/3 + \epsilon & s/3 & & & \\ s/3 - \epsilon & s/3 & s/3 + \epsilon & & & \\ & & & \epsilon & & \\ & & & & \epsilon & \\ & & & & & \epsilon \end{pmatrix}$$

Identifiability for isotropic, hyperelastic materials

Main result for isotropic, hyperelastic materials:

The elasticity tensor $\mathbb{C}$ may be written as a conical combination of two matrices that are elements of $M_{\epsilon,8s}$, so that $\mathbb{C}$ and hence $\tilde{C}$ is uniquely determined by two measurements of piezo sensors, given an appropriate excitation signal.

Overview

Ongoing research: Approach

Given: finite dictionary $\{C_1, ..., C_N\}$
Approach:

$$\min_{\alpha \in \mathbb{R}_+^N} J_R(\alpha) := \min_{\alpha \in \mathbb{R}_+^N} \left\{ \frac{1}{2} \| Q\mathcal{T}(\alpha) - \tilde{u}^\delta \|_2^2 + \beta R(\alpha) \right\},$$

where $\mathcal{T}$ maps α to $u(C(\alpha))$ with $C(\alpha) = \sum_{K=1}^N \alpha_K C_K$ and R is a penalty term. Examples for R:

$$R(\alpha) = \| \sum_j \alpha_j C_j \|_U^2 / 2 \quad \text{or} \quad R(\alpha) = \|\alpha\|_1$$

The optimality conditions are:

$$0 \in \mathcal{T}'(\alpha_*)^* Q^* (Q\mathcal{T}(\alpha_*) - \tilde{u}^\delta) + \beta \partial R(\alpha_*).$$

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Fréchet derivative of $\mathcal{T}$

The Fréchet derivative of $\mathcal{T}$ (with respect to α) $\mathcal{T}'(\alpha)$ is defined by $\mathcal{T}'(\alpha)h = v$, where v solves

$$\rho\ddot{v}(t,x) - \mathrm{div}[\nabla_Y\nabla_Y C_\alpha(x, Ju(t,x)) : Jv(t,x)] = \mathrm{div}[\nabla_Y C_h(x, Ju(t,x))]$$
$$v(0,x) = \dot{v}(0,x) = 0 \text{ for } x \in \Omega$$
$$v(t,x) = 0 \text{ for } x \in \partial\Omega$$

with

$$C_\alpha = \sum_{K=1}^{N} \alpha_K C_K \text{ and } C_h = \sum_{K=1}^{N} h_K C_K.$$

Fréchet derivative of $\mathcal{T}$

Lemma (P., Schuster, 2015)

The Gâteaux derivative of $\mathcal{T}$ exists and is continuous in h for all $h \in \mathbb{R}_+^N$, i.e. there is a constant $L_1 > 0$ with

$$\|\mathcal{T}'(\alpha)h\|_{L^2(0,T;H^1(\Omega,\mathbb{R}^3))} \leq L_1 \|h\|_\infty .$$

Uniform convergence

Theorem (P., Schuster, 2015)

$\mathcal{T}$ is Fréchet differentiable, i.e. there are constants $L_2 > 0$ and $\beta > 1$ with

$$\|\mathcal{T}(\alpha + h) - \mathcal{T}(\alpha) - \mathcal{T}'(\alpha)h\|_{L^2(0,T;H^1(\Omega,\mathbb{R}^3))} \leq L_2 \|h\|_\infty^\beta$$

for $\|h\|_\infty \to 0$.

Fréchet derivative of $\mathcal{T}$

The Fréchet derivative of $\mathcal{T}$ is very useful for

- optimality conditions
- linearization of $\mathcal{T}$: $\mathcal{T}(\alpha) = \mathcal{T}(\alpha_*) + \mathcal{T}'(\alpha_*)(\alpha - \alpha_*)$
- iterative solution methods (for example Landweber)

Conclusions

- Uniqueness and continuous dependency of data of the *direct problem*, when the stored energy function is a conic combination
- Uniqueness and continuous dependency of (piezo) measurement data of the *inverse problem*, i.e. identification of stored energy function in linear case if it can be represented as conical combination (also includes spatially variable energy functions)
- Conditions for conic representations of (spatially variable) elasticity tensors
- Two piezo measurements are sufficient to identify isotropic materials
- Fréchet derivative (useful for linearization, iterativ solving, optimal conditions)

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At the end

Thank you for your attention!