

Computing Interior Eigenvalues of Domains from Far Fields

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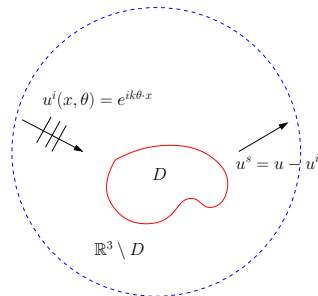
An example

Scattering problem

$$\begin{aligned}\Delta u + k^2 u &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \frac{\partial u}{\partial \nu} \Big|_{\partial D} + \tau u|_{\partial D} &= 0 && \text{on } \partial D,\end{aligned}$$

Interior Robin eigenvalue problem $v := u^i$

$$\begin{aligned}\Delta v + k^2 v &= 0 && \text{in } D, \\ \frac{\partial v}{\partial \nu} \Big|_{\partial D} + \tau v|_{\partial D} &= 0 && \text{on } \partial D.\end{aligned}$$



Scattering from a Robin obstacle.

An example

Scattering problem

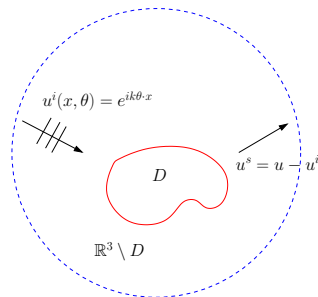
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Radiating scattered field

$$u^s(x, \theta) = \frac{\exp(ik|x|)}{4\pi|x|} \left(u^\infty(\hat{x}, \theta) + \mathcal{O}\left(\frac{1}{|x|}\right) \right) \text{ as } |x| \rightarrow \infty.$$



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An example

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Interior Robin eigenvalue problem $v := u^i$

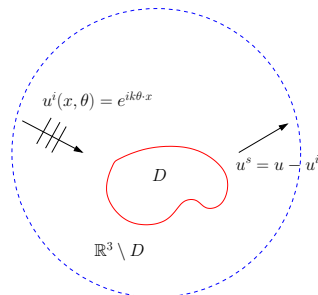
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Radiating scattered field

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Far field operator

$$F : L^2(\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2), \quad Fg(\hat{x}) := \int_{\mathbb{S}^2} u^\infty(\hat{x}, \theta) g(\theta) \, dS(\theta), \quad \hat{x} \in \mathbb{S}^2.$$



Scattering from a Robin obstacle.

Inside-outside duality

- Spectral decomposition: $Fg = \sum_{j \in \mathbb{N}} \lambda_j(g, g_j)g_j \quad \forall g \in L^2(\mathbb{S}^2)$ with $\lambda_j \rightarrow 0$ as $j \rightarrow \infty$.
- λ_j are on the circle $|z - i8\pi^2/k| = 8\pi^2/k$.
- $\lambda_j = r_j \exp(i\vartheta_j)$ with $r_j \geq 0$ and $\vartheta_j \in [0, \pi)$. $\operatorname{Re}(\lambda_j) > 0$ for all $j \in \mathbb{N}$ large enough. Since $\lambda_j \rightarrow 0$ as $j \rightarrow \infty$, $\vartheta_j \rightarrow 0$ as $j \rightarrow \infty$. Hence $\vartheta^* = \max_{j \in \mathbb{N}} \vartheta_j$ is well defined and attained by some $\lambda^* \neq 0$.

Theorem (Lechleiter-Peters 2014)

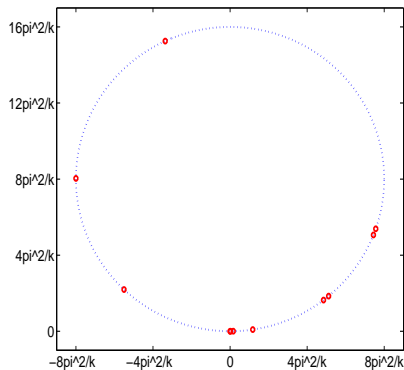
If $\lim_{i \rightarrow \infty} \vartheta^(k_i) = \pi$ for some sequence $\{k_i\}_{i \in \mathbb{N}}$ with $k_i > k$ and $k_i \rightarrow k > 0$ as $i \rightarrow \infty$, then k^2 is an interior Robin eigenvalue of $-\Delta$ in D . If $k^2 > 0$ is a Robin eigenvalue of $-\Delta$ in D then $\lim_{i \rightarrow \infty} \vartheta^*(k_i) = \pi$ for any sequence $\{k_i\}_{i \in \mathbb{N}}$ with $k_i > k$ that tends to k from above.*

- Inverse scattering: [Colton-Kress, Kirsch-Grinberg](#)
- Interior eigenvalues: [Cakoni-Colton-Monk 07](#), [Cakoni-Gintides-Haddar 10](#).
- Inside-outside duality:
 - Obstacles with Dirichlet, Neumann or Robin boundary condition: [Eckmann-Pillet 95](#), [Lechleiter-Peters 14](#).
 - Isotropic or anisotropic penetrable acoustic media: [Kirsch-Lechleiter 13](#), [Lechleiter-Peters 15](#).
 - Electromagnetic scattering from penetrable anisotropic dielectric media: [Lechleiter-Rennoch 15](#).

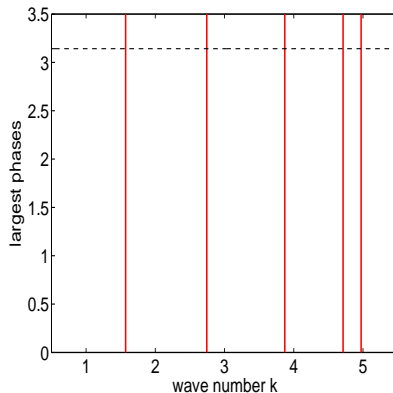
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.500



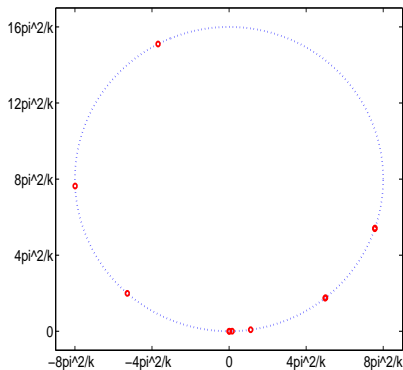
Largest phases.

Robin, $\tau=1$, unit ball

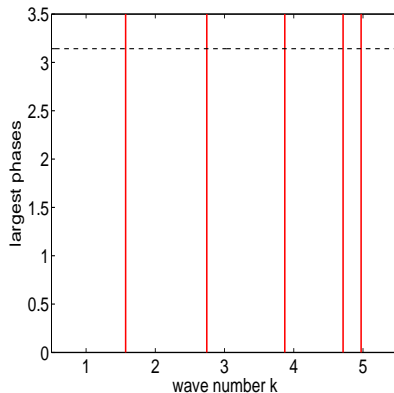
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.475



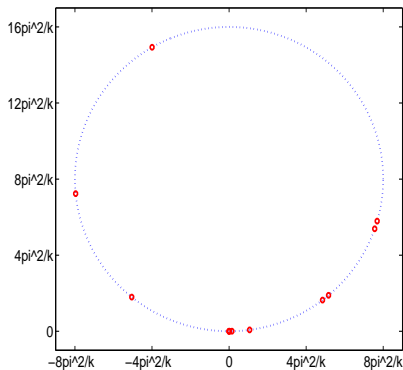
Largest phases.

Robin, $\tau=1$, unit ball

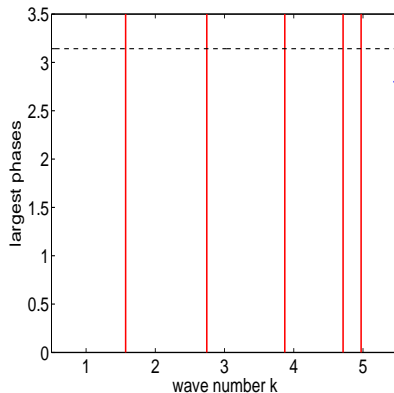
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.450



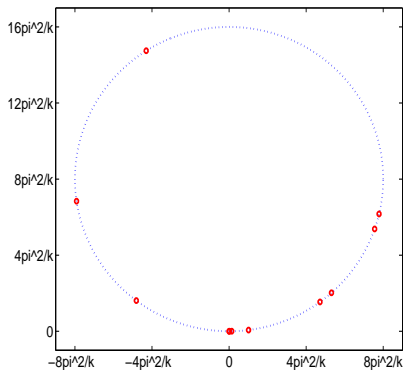
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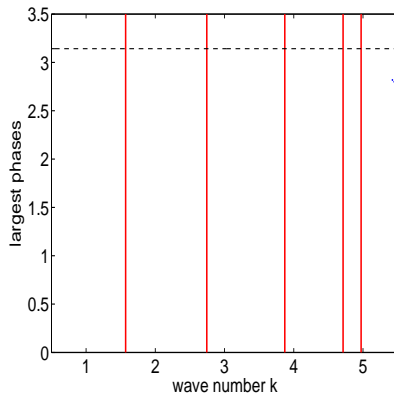
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.425



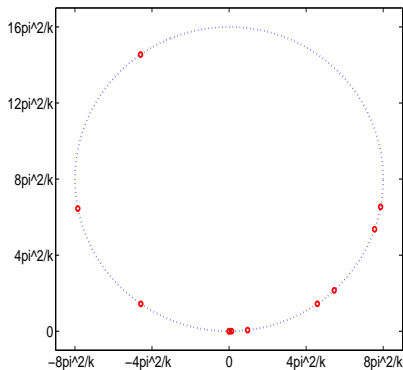
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Robin, $\tau=1$, unit ball

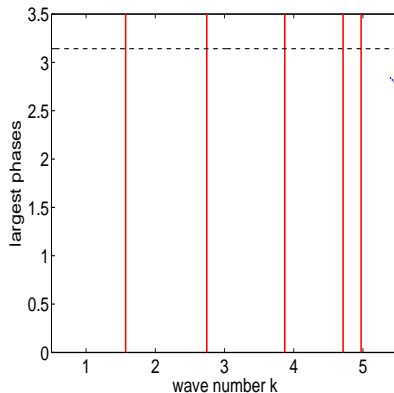
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.400



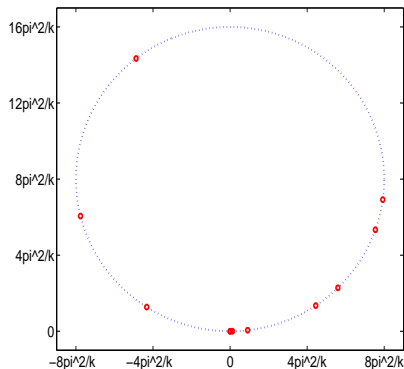
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Robin, $\tau=1$, unit ball

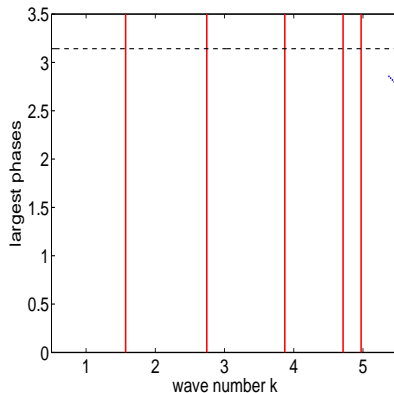
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.375



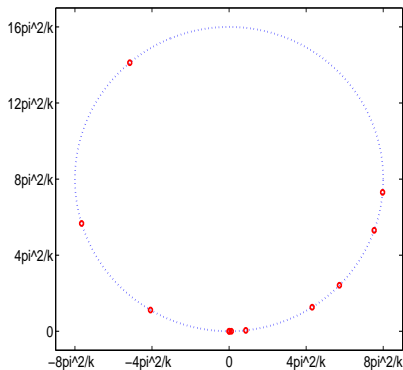
Largest phases.

Robin, $\tau=1$, unit ball

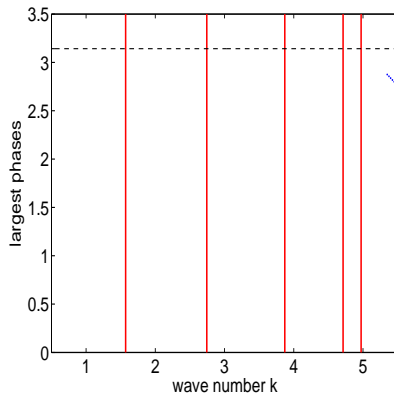
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.350



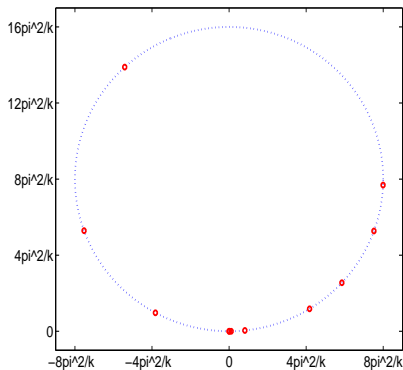
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Robin, $\tau=1$, unit ball

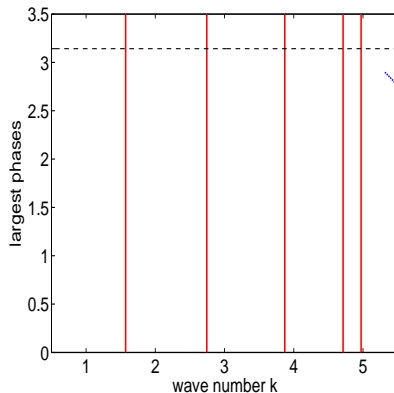
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.325



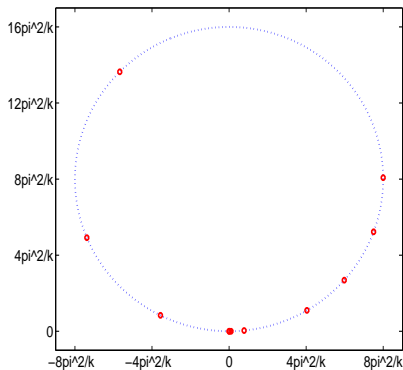
Largest phases.

Robin, $\tau=1$, unit ball

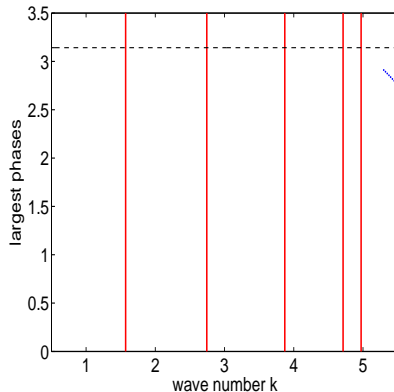
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.300



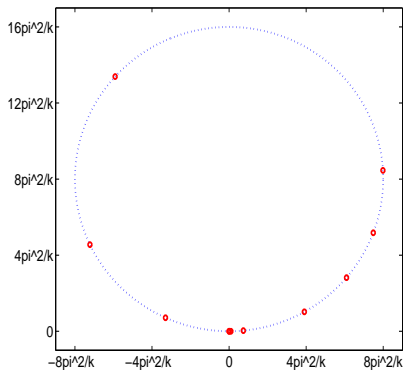
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Robin, $\tau=1$, unit ball

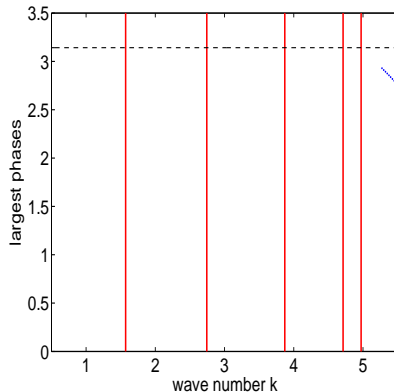
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.275



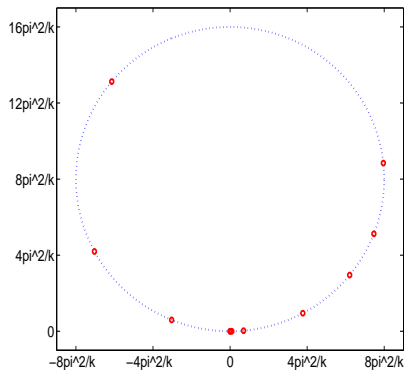
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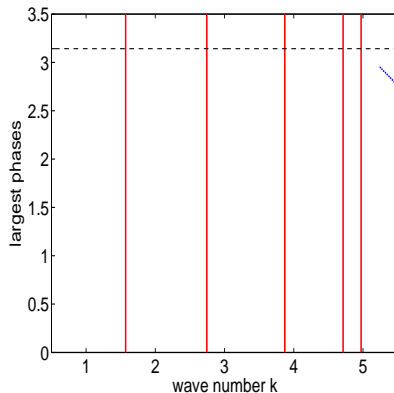
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.250



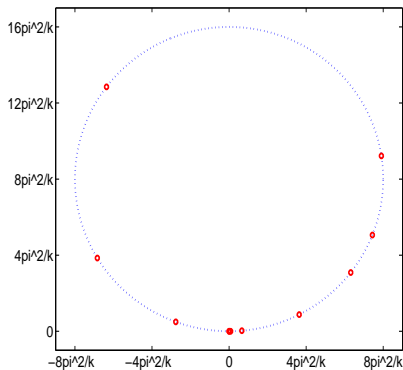
Largest phases.

Robin, $\tau=1$, unit ball

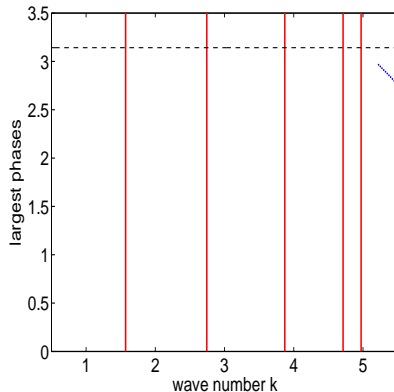
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.225



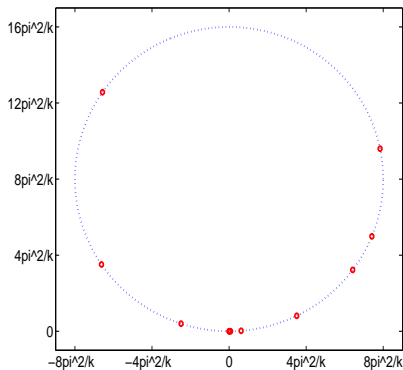
Largest phases.

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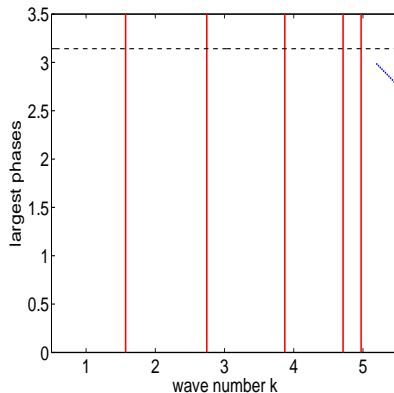
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.200



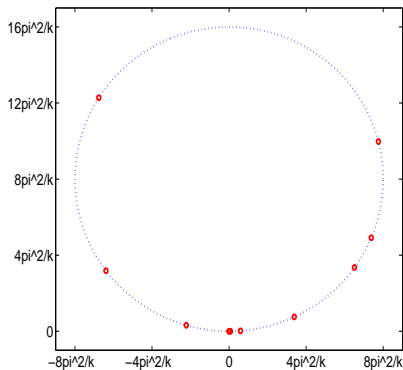
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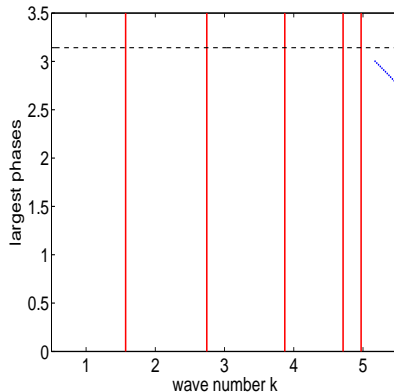
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.175



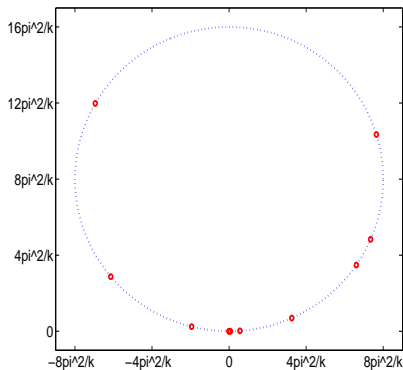
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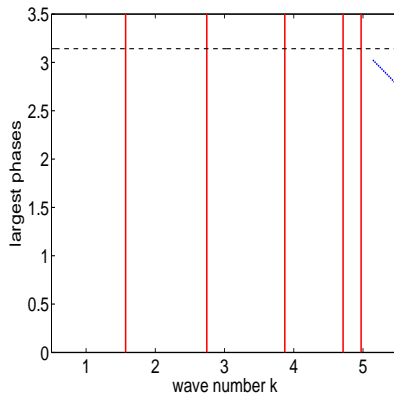
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.150



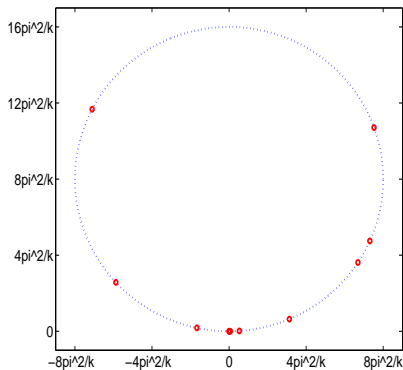
Largest phases.

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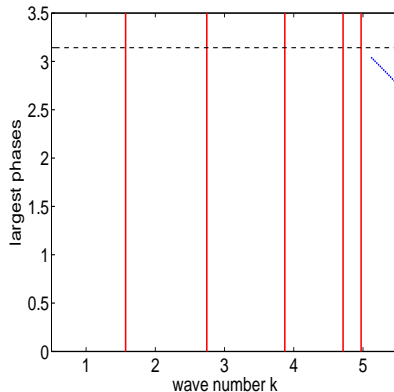
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.125



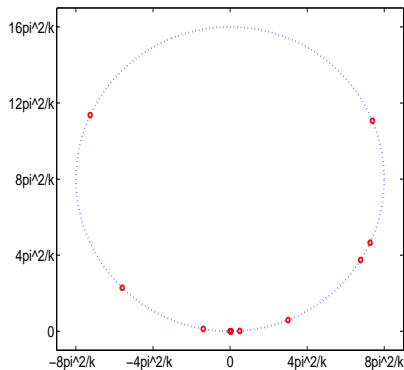
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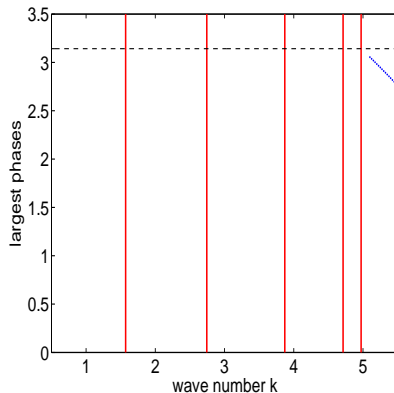
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.100



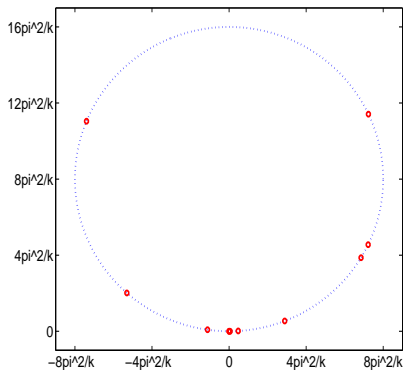
Largest phases.

Robin, $\tau=1$, unit ball

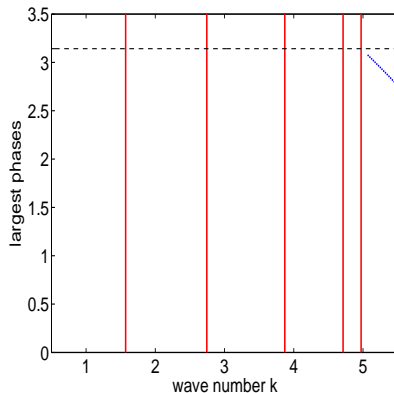
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.075



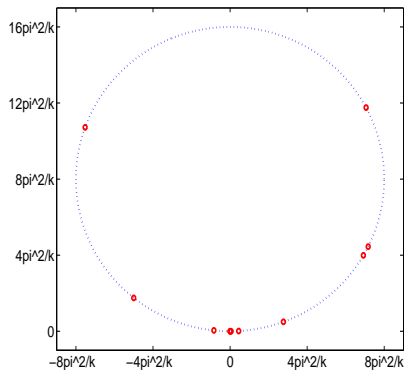
Largest phases.

Robin, $\tau=1$, unit ball

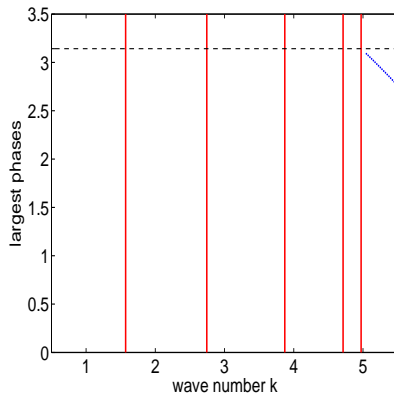
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.050



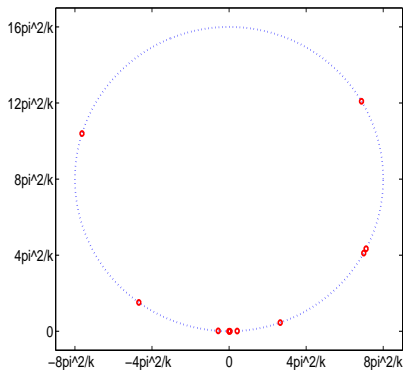
Largest phases.

Robin, $\tau=1$, unit ball

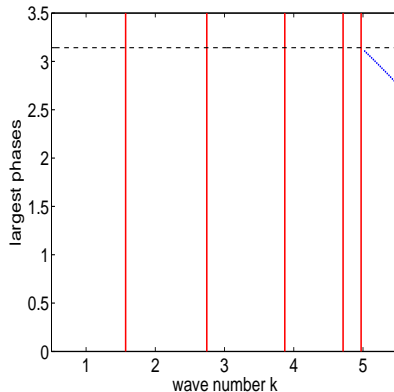
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.025



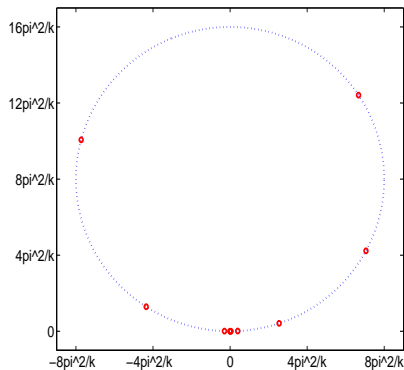
Largest phases.

Robin, $\tau=1$, unit ball

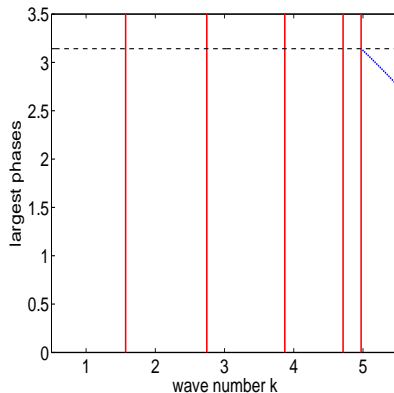
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=5.000



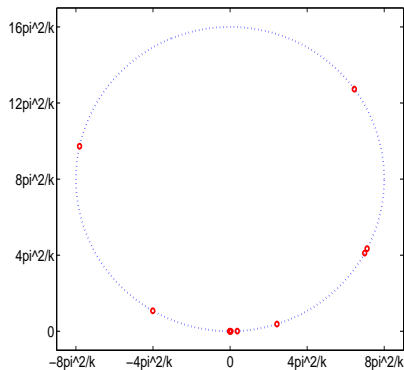
Largest phases.

Robin, $\tau=1$, unit ball

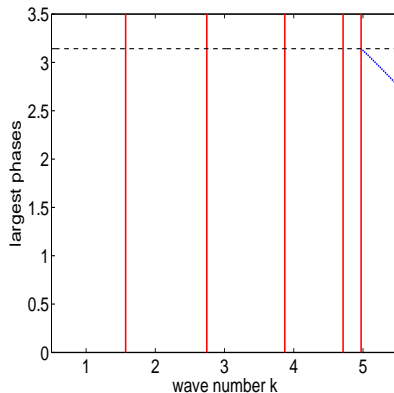
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.975



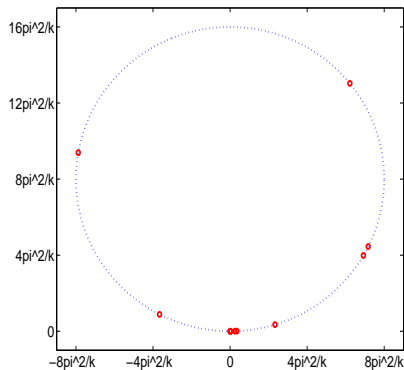
Largest phases.

Robin, $\tau=1$, unit ball

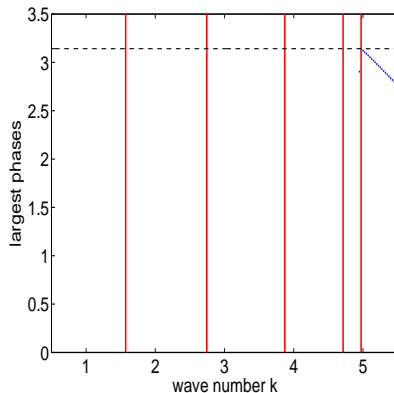
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.950



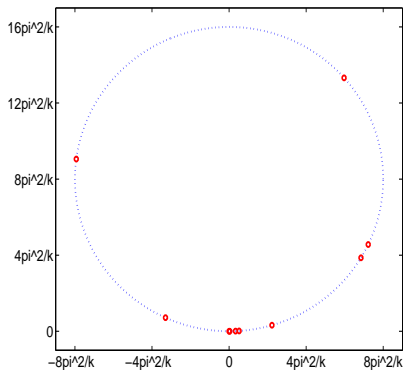
Largest phases.

Robin, $\tau=1$, unit ball

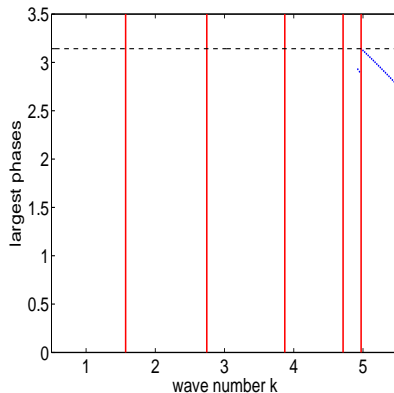
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.925



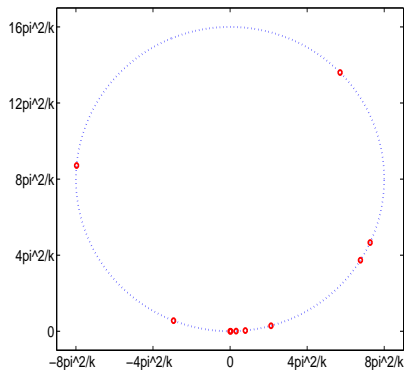
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Robin, $\tau=1$, unit ball

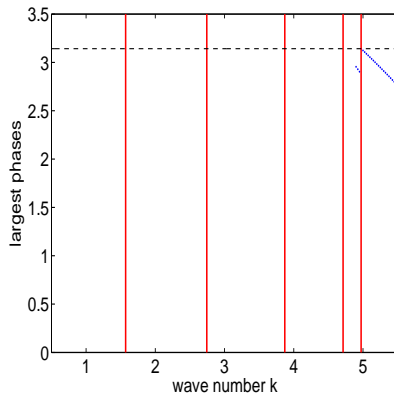
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.900



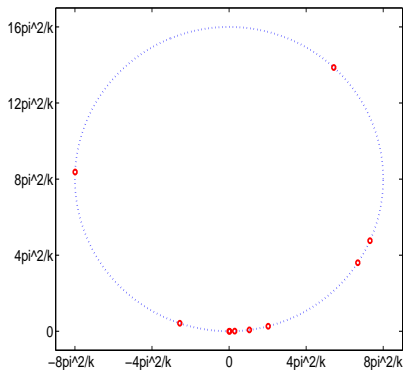
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Robin, $\tau=1$, unit ball

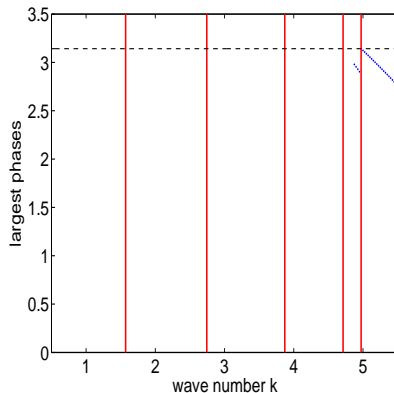
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.875



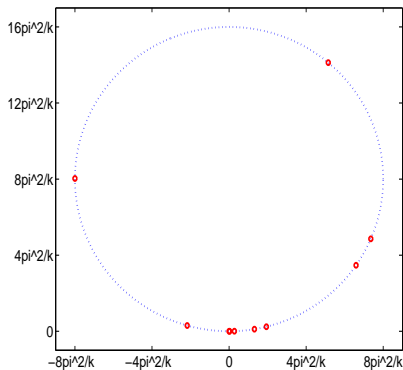
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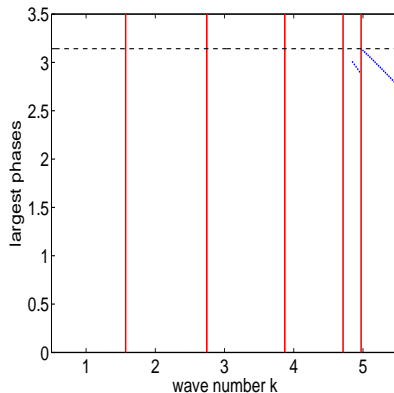
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.850



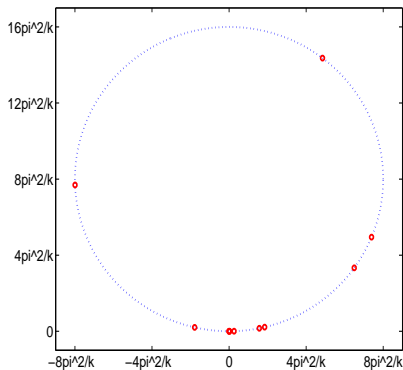
Largest phases.

Robin, $\tau=1$, unit ball

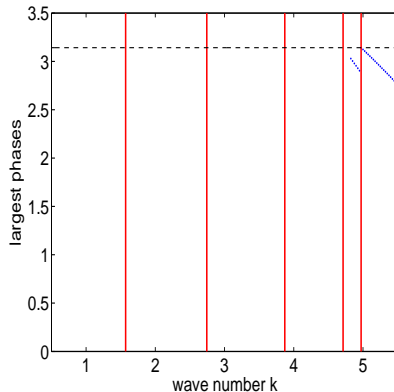
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.825



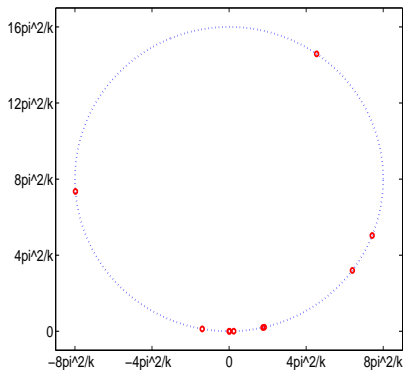
Largest phases.

Robin, $\tau=1$, unit ball

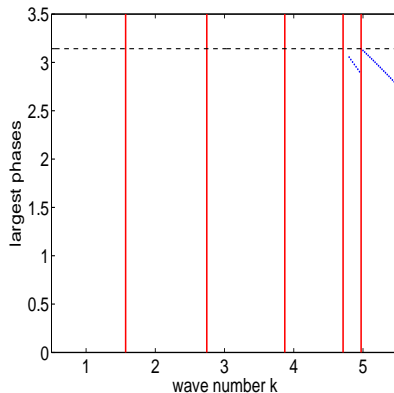
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.800



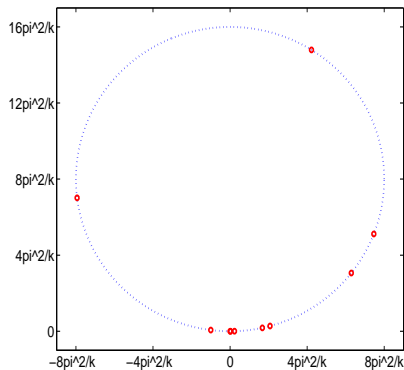
Largest phases.

Robin, $\tau=1$, unit ball

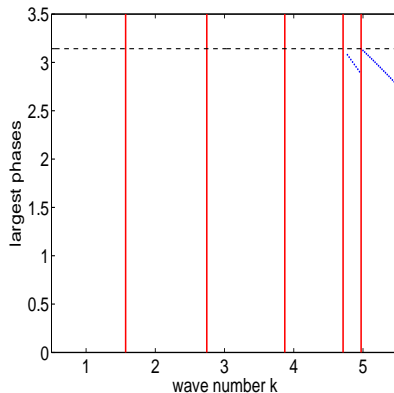
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.775



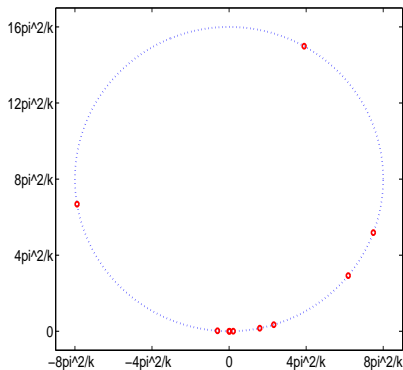
Largest phases.

Robin, $\tau=1$, unit ball

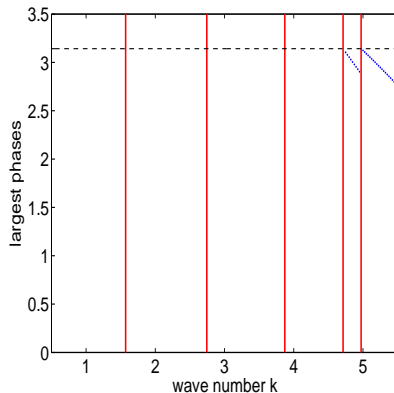
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.750



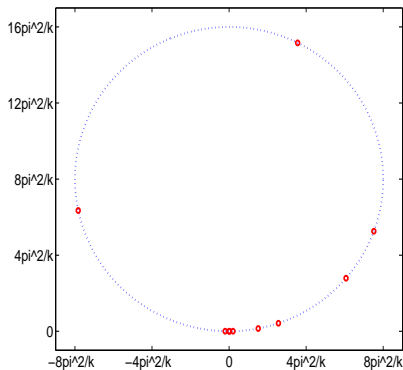
Largest phases.

Robin, $\tau=1$, unit ball

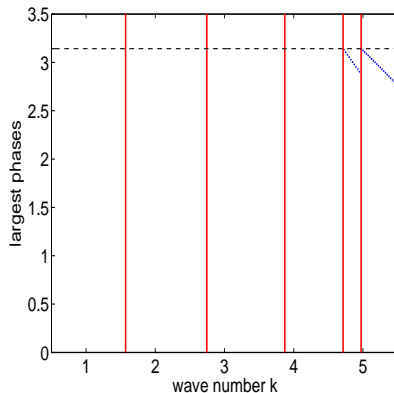
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.725



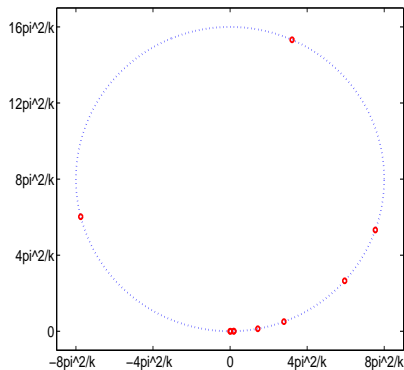
Largest phases.

Robin, $\tau=1$, unit ball

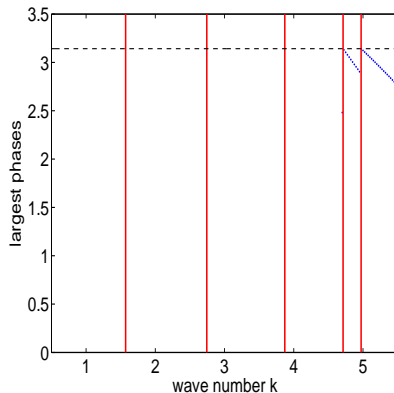
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.700



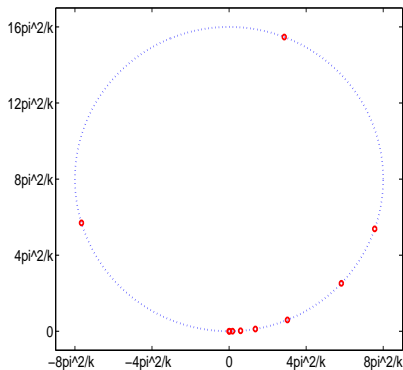
Largest phases.

Robin, $\tau=1$, unit ball

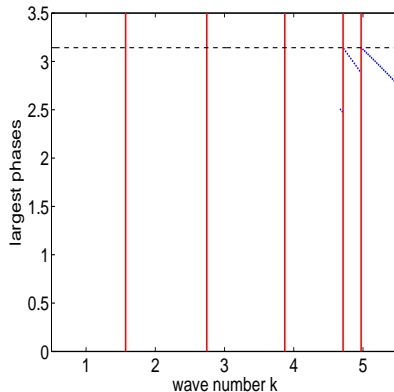
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.675



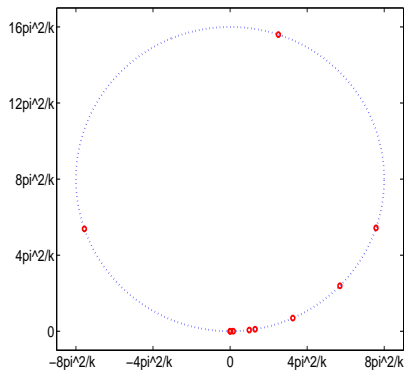
Largest phases.

Robin, $\tau=1$, unit ball

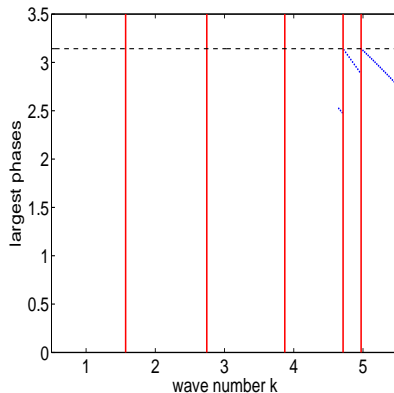
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.650



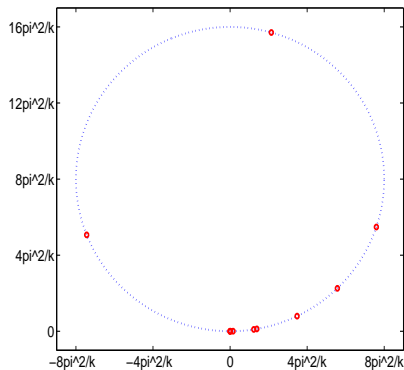
Largest phases.

Robin, $\tau=1$, unit ball

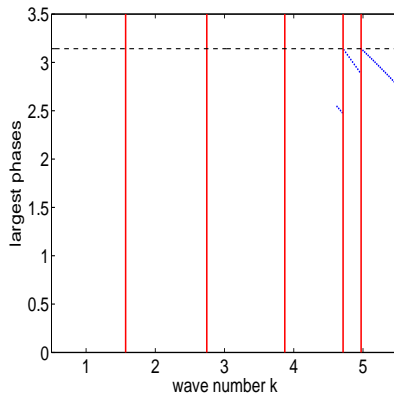
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.625



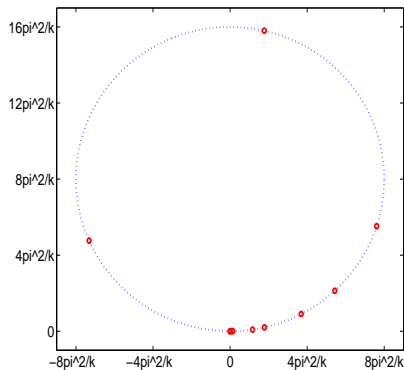
Largest phases.

Robin, $\tau=1$, unit ball

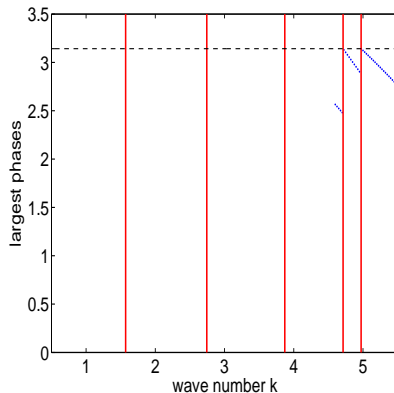
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.600



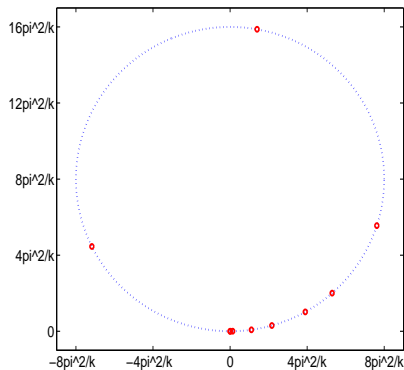
Largest phases.

Robin, $\tau=1$, unit ball

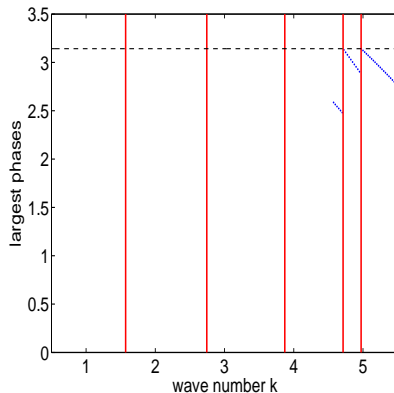
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.575



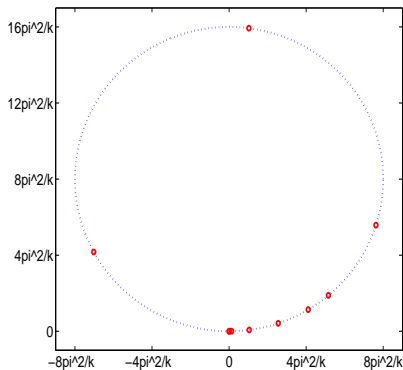
Largest phases.

Robin, $\tau=1$, unit ball

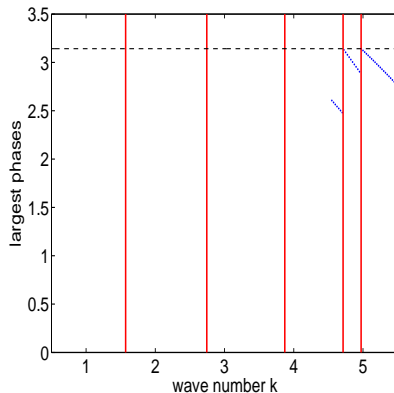
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.550



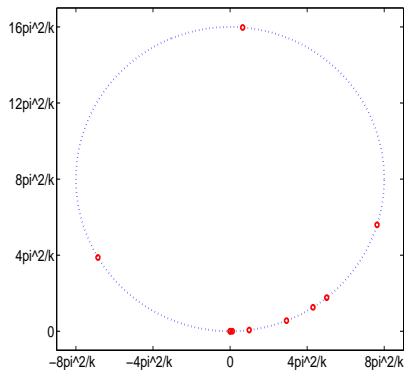
Largest phases.

Robin, $\tau=1$, unit ball

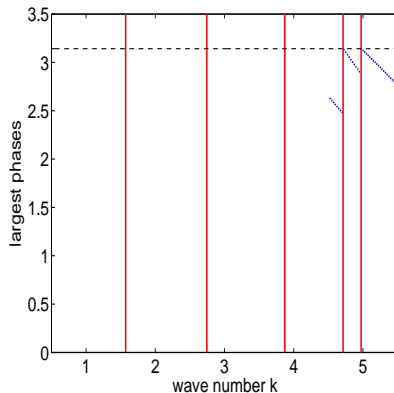
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.525



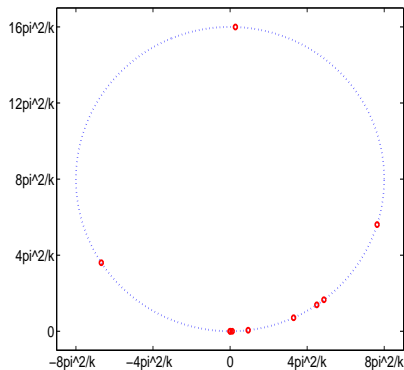
Largest phases.

Robin, $\tau=1$, unit ball

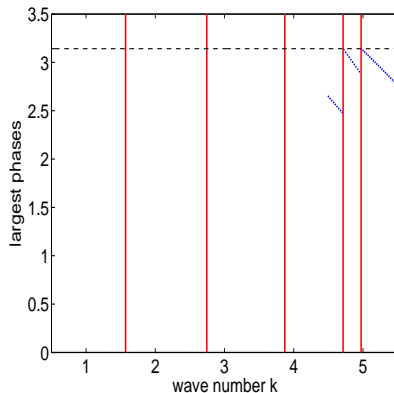
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.500



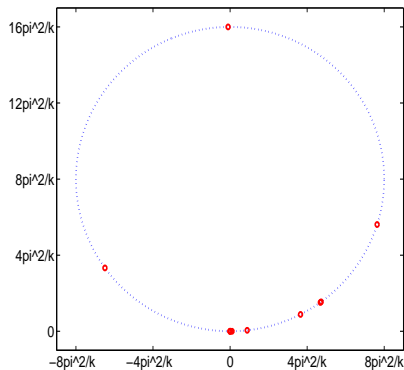
Largest phases.

Robin, $\tau=1$, unit ball

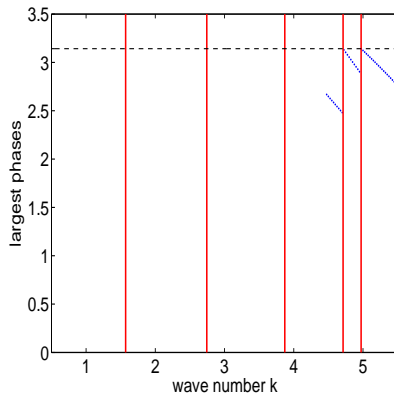
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.475



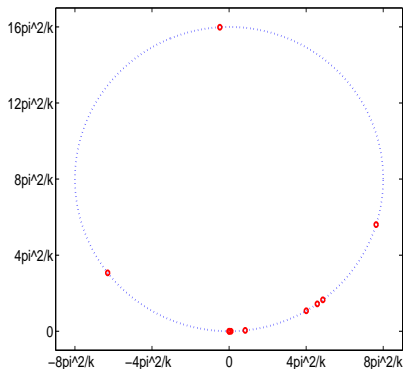
Largest phases.

Robin, $\tau=1$, unit ball

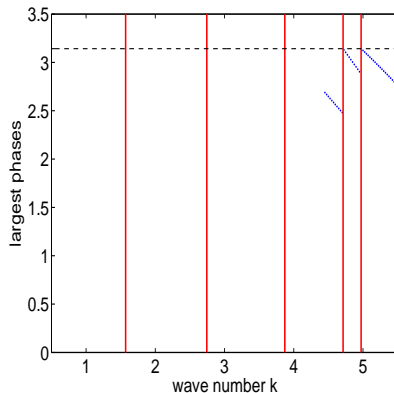
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.450



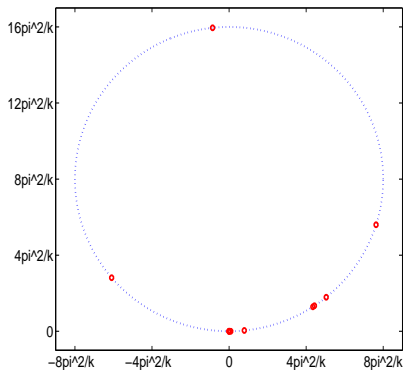
Largest phases.

Robin, $\tau=1$, unit ball

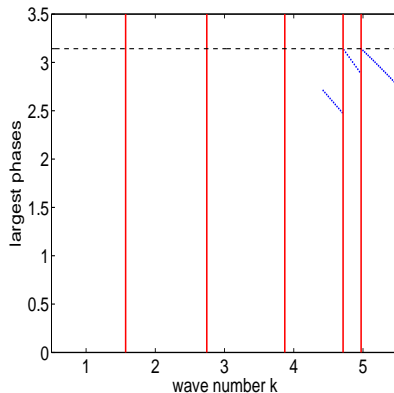
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.425



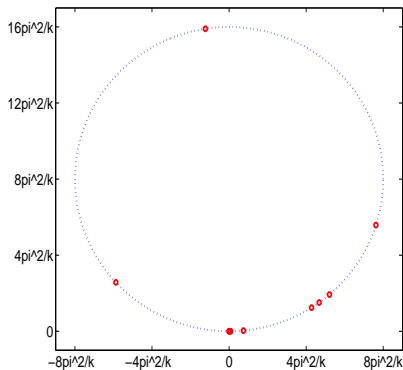
Largest phases.

Robin, $\tau=1$, unit ball

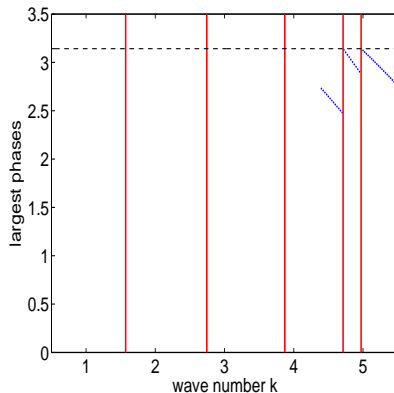
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.400



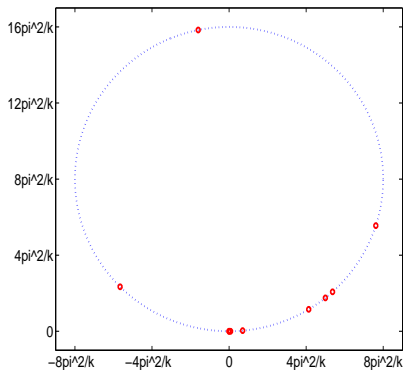
Largest phases.

Robin, $\tau=1$, unit ball

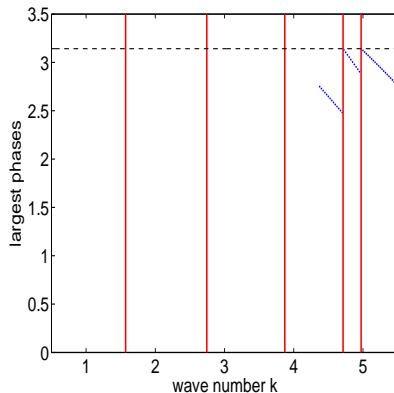
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.375



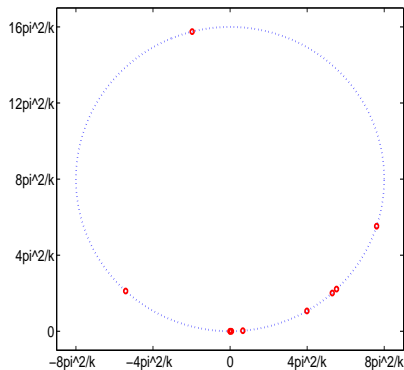
Largest phases.

Robin, $\tau=1$, unit ball

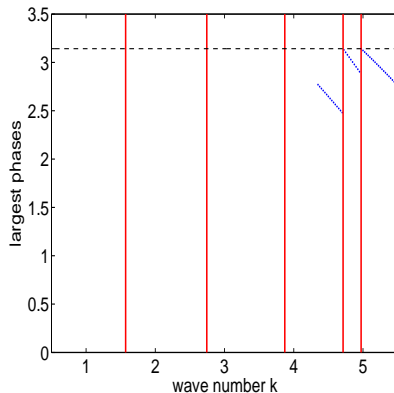
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.350



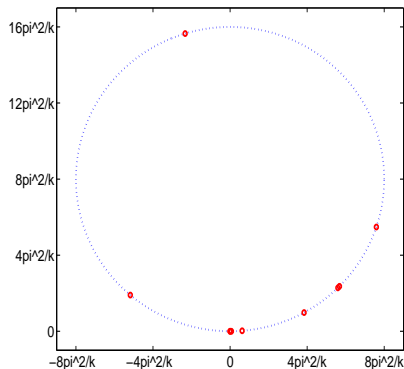
Largest phases.

Robin, $\tau=1$, unit ball

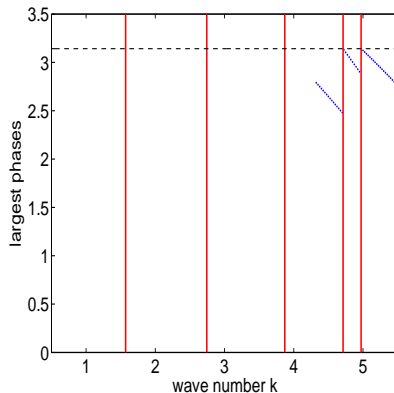
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.325



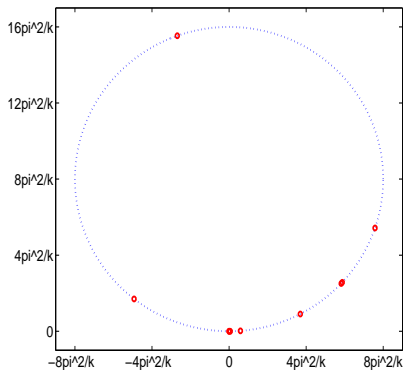
Largest phases.

Robin, $\tau=1$, unit ball

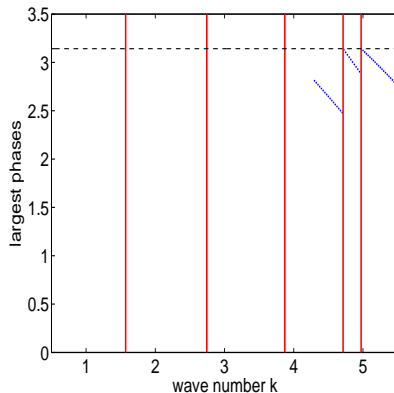
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.300



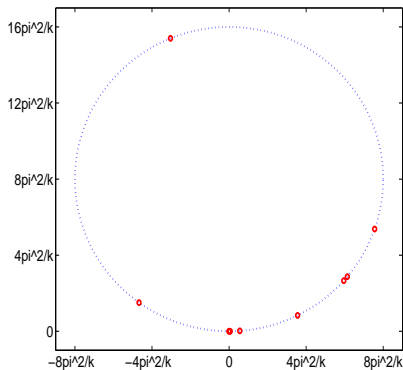
Largest phases.

Robin, $\tau=1$, unit ball

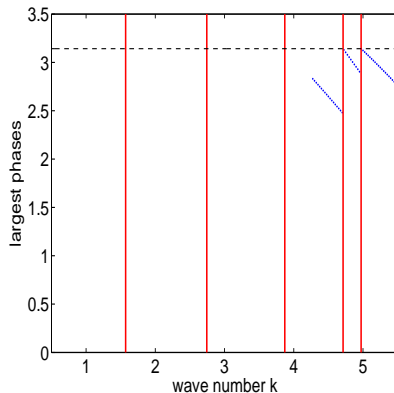
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.275



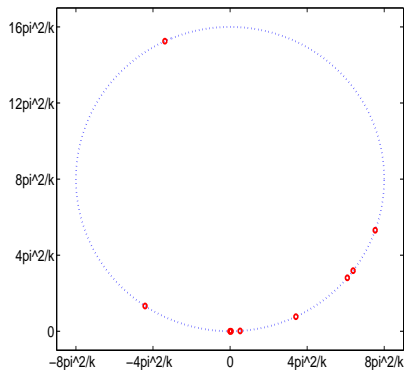
Largest phases.

Robin, $\tau=1$, unit ball

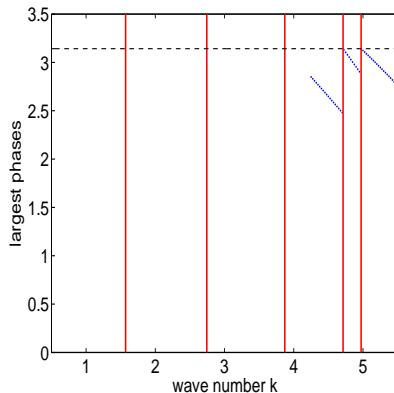
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.250



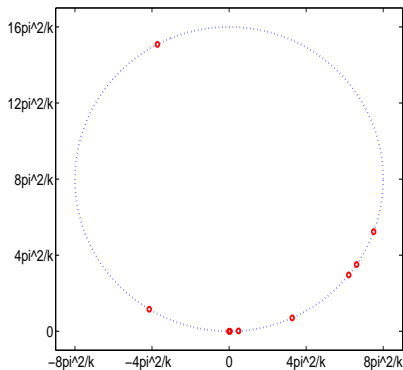
Largest phases.

Robin, $\tau=1$, unit ball

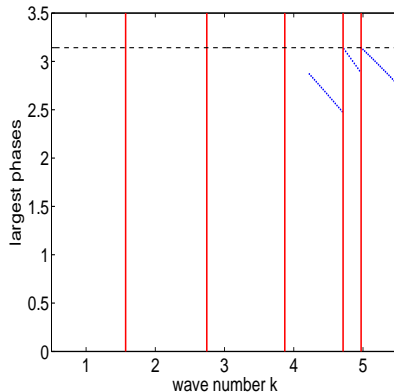
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.225



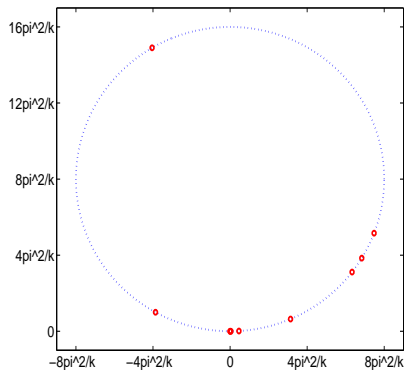
Largest phases.

Robin, $\tau=1$, unit ball

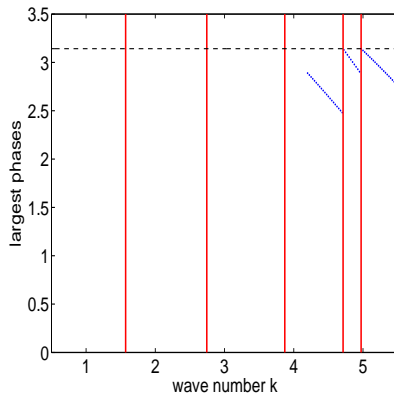
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.200



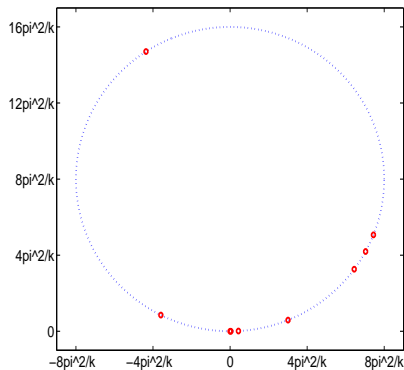
Largest phases.

Robin, $\tau=1$, unit ball

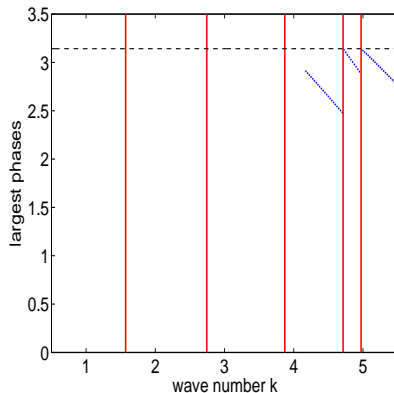
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.175



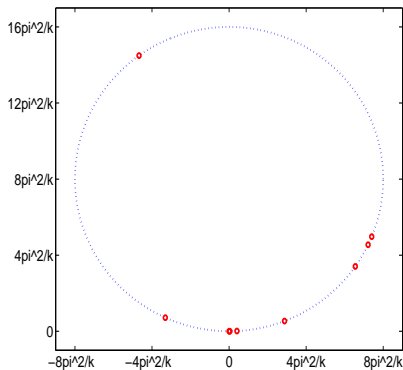
Largest phases.

Robin, $\tau=1$, unit ball

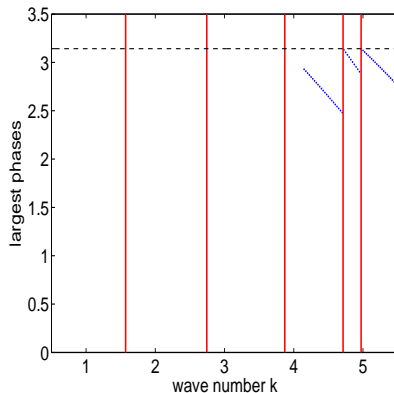
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.150



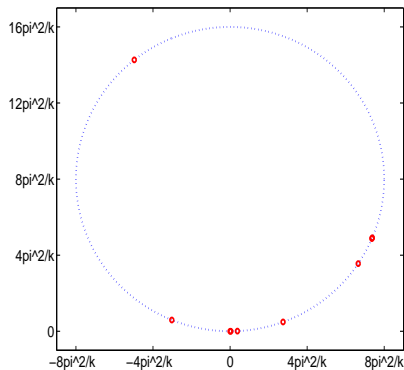
Largest phases.

Robin, $\tau=1$, unit ball

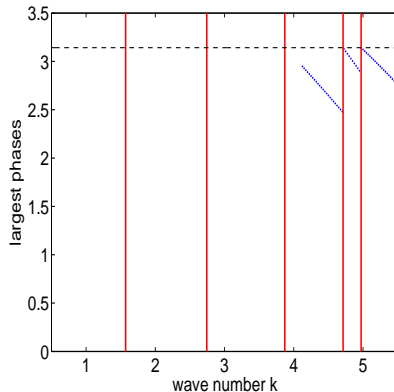
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.125



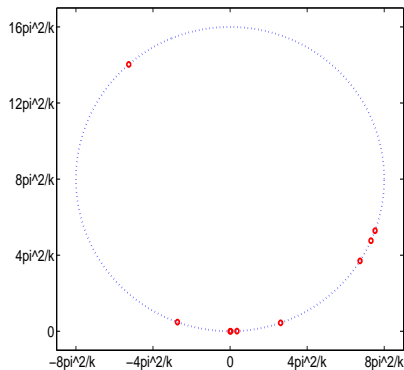
Largest phases.

Robin, $\tau=1$, unit ball

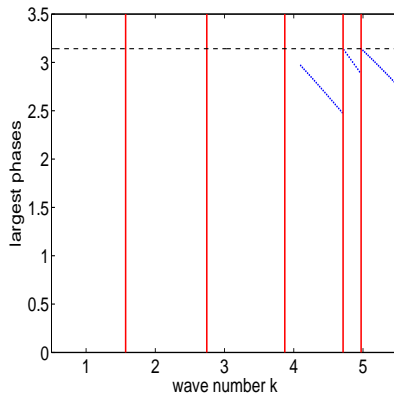
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.100



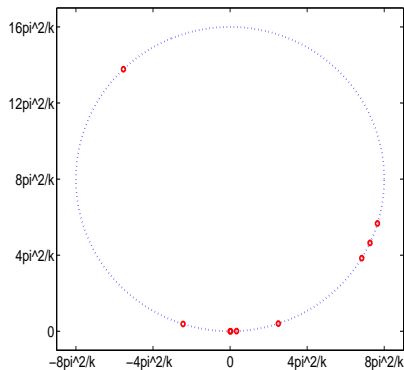
Largest phases.

Robin, $\tau=1$, unit ball

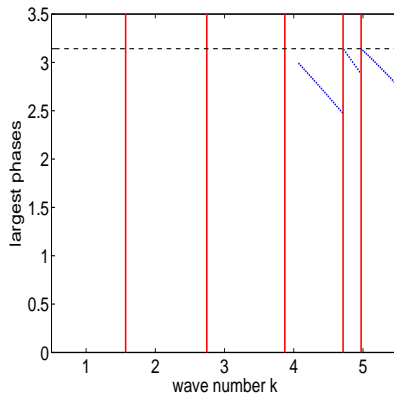
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.075



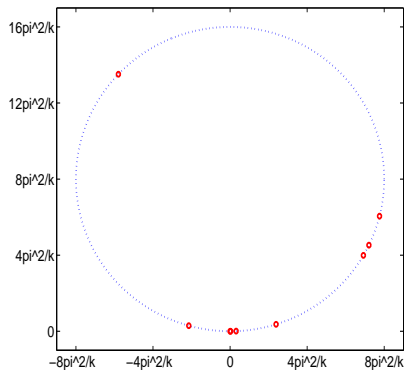
Largest phases.

Robin, $\tau=1$, unit ball

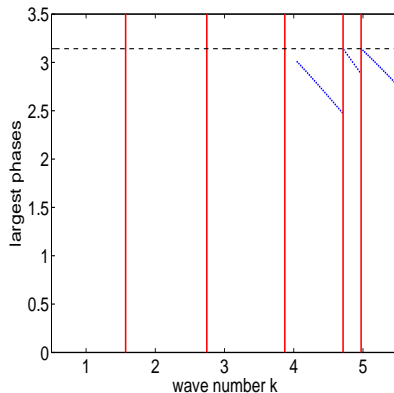
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.050



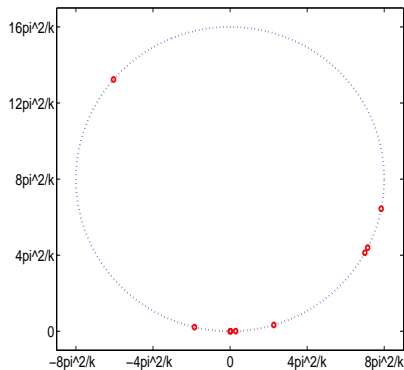
Largest phases.

Robin, $\tau=1$, unit ball

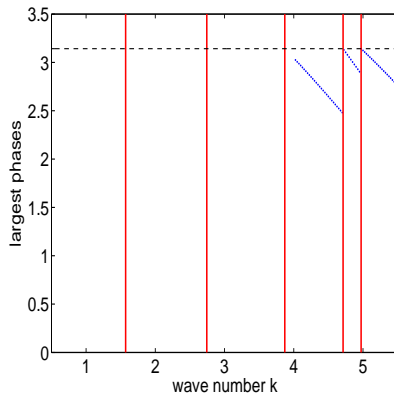
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.025



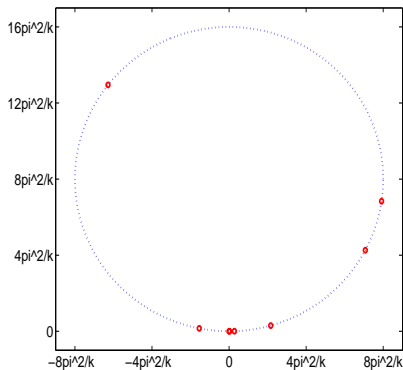
Largest phases.

Robin, $\tau=1$, unit ball

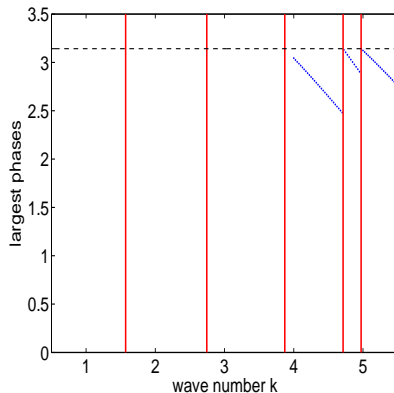
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=4.000



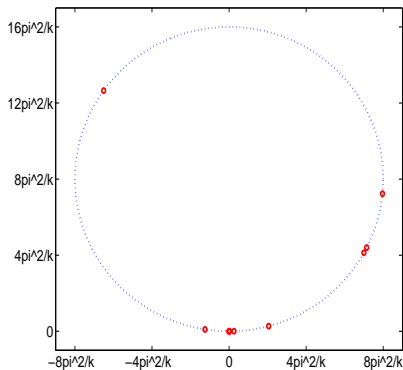
Largest phases.

Robin, $\tau=1$, unit ball

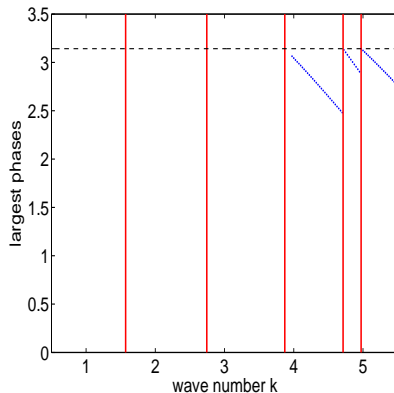
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.975



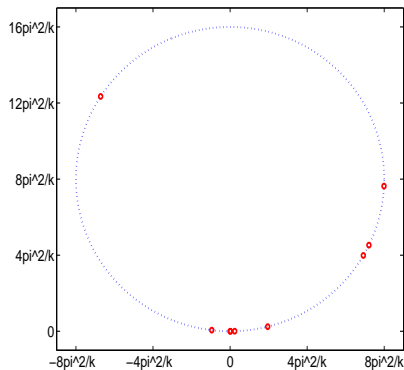
Largest phases.

Robin, $\tau=1$, unit ball

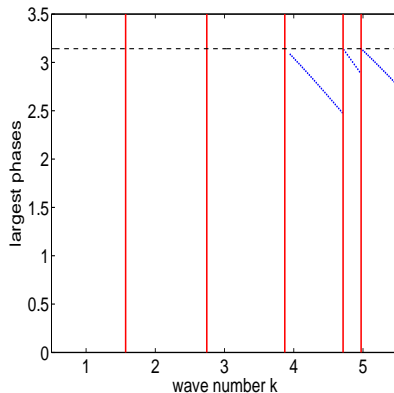
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.950



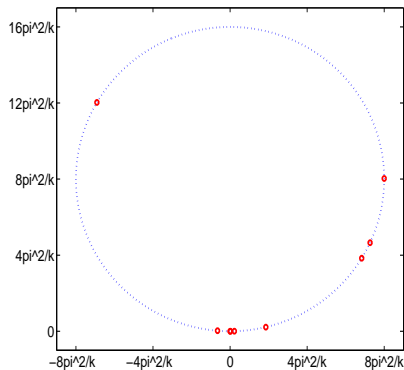
Largest phases.

Robin, $\tau=1$, unit ball

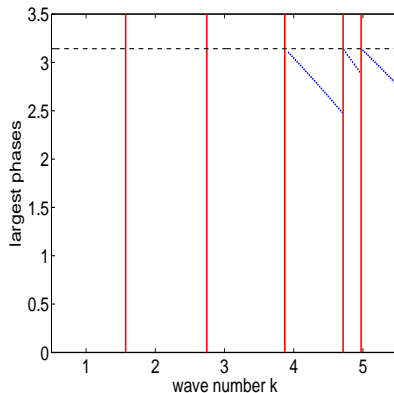
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.925



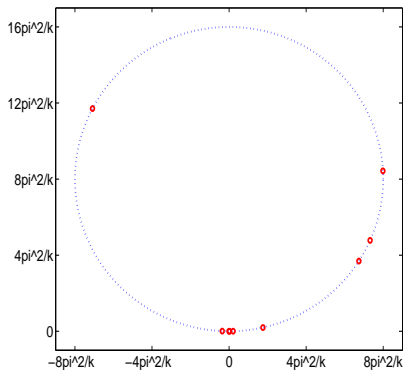
Largest phases.

Robin, $\tau=1$, unit ball

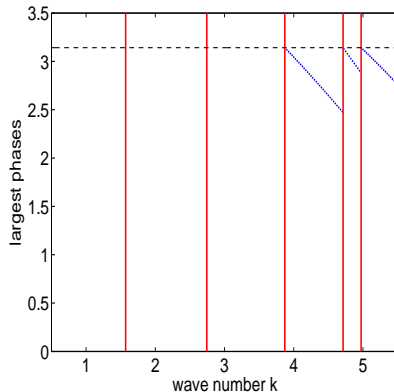
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.900



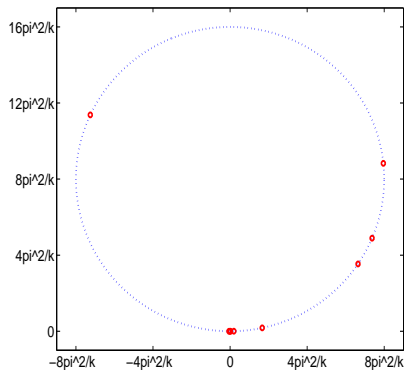
Largest phases.

Robin, $\tau=1$, unit ball

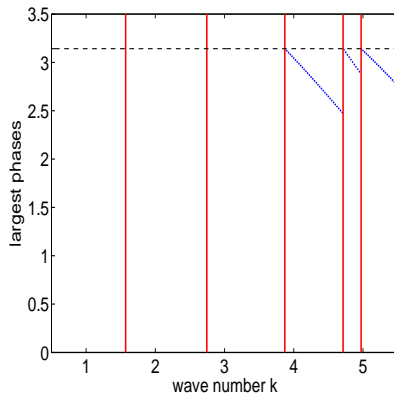
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.875



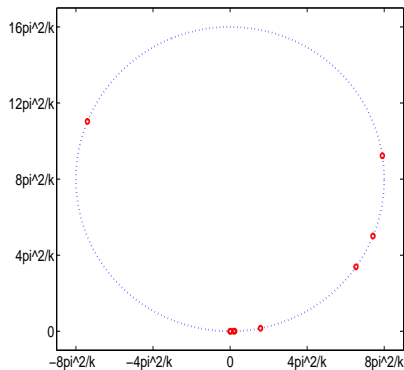
Largest phases.

Robin, $\tau=1$, unit ball

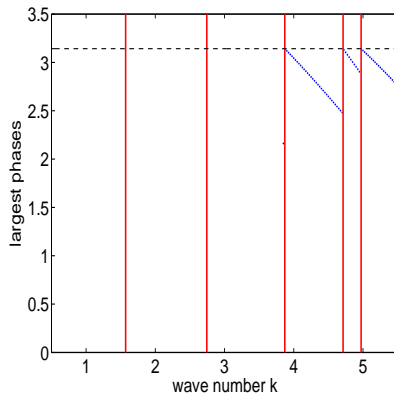
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.850



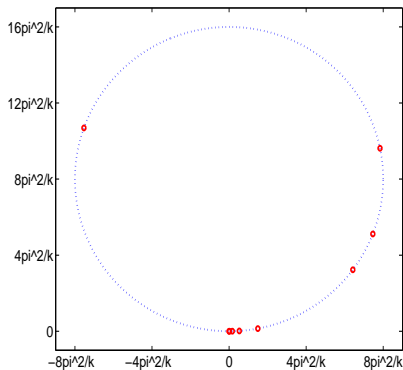
Largest phases.

Robin, $\tau=1$, unit ball

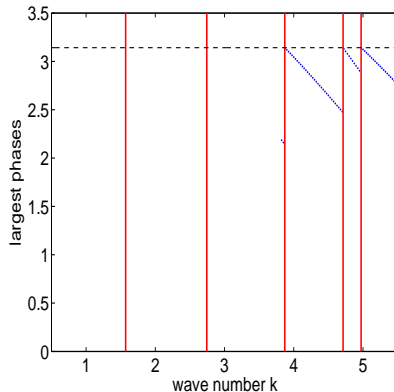
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.825



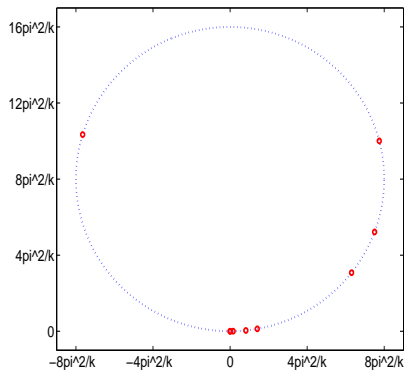
Largest phases.

Robin, $\tau=1$, unit ball

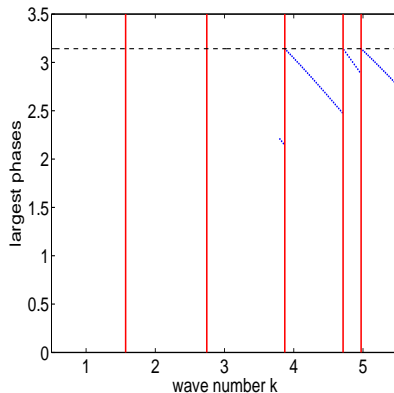
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.800



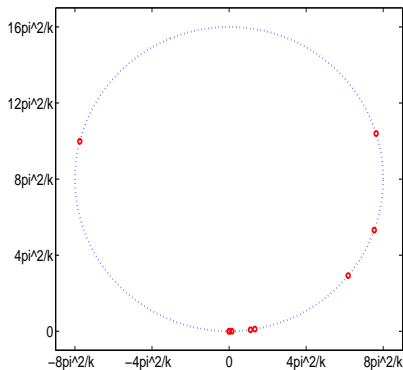
Largest phases.

Robin, $\tau=1$, unit ball

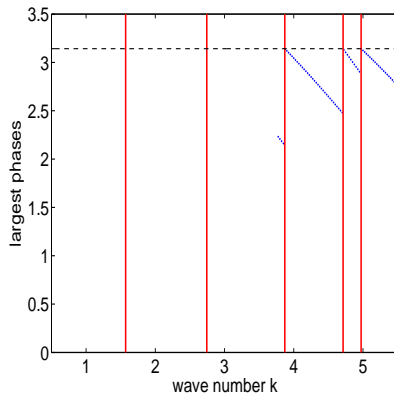
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.775



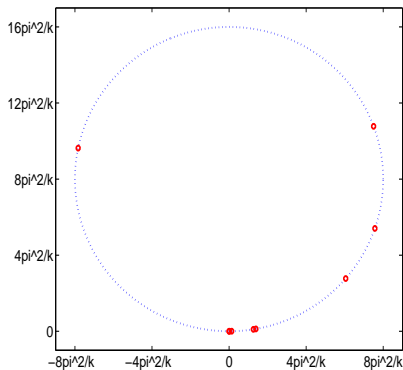
Largest phases.

Robin, $\tau=1$, unit ball

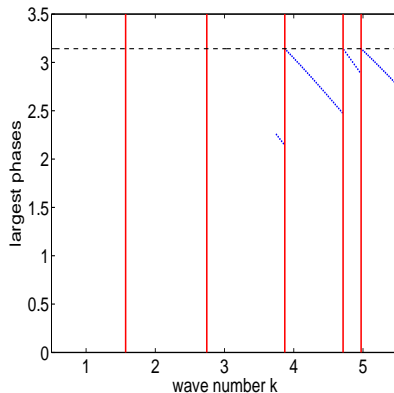
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.750



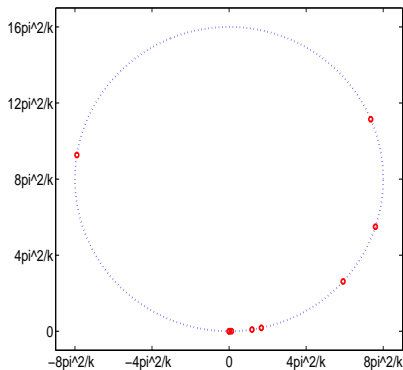
Largest phases.

Robin, $\tau=1$, unit ball

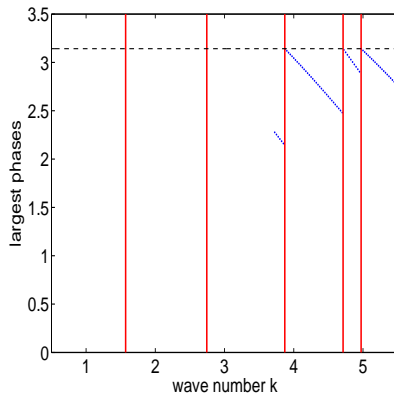
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.725



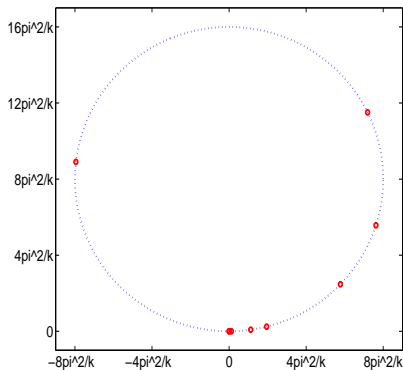
Largest phases.

Robin, $\tau=1$, unit ball

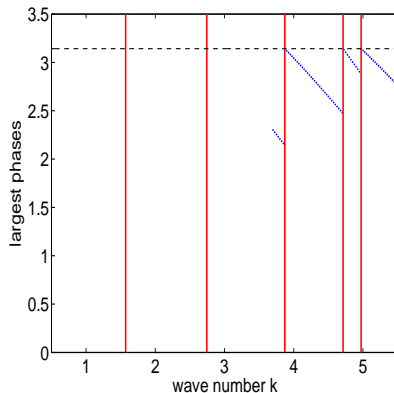
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.700



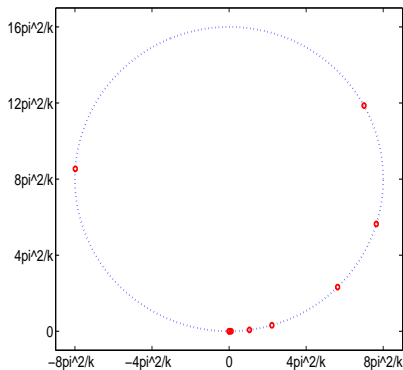
Largest phases.

Robin, $\tau=1$, unit ball

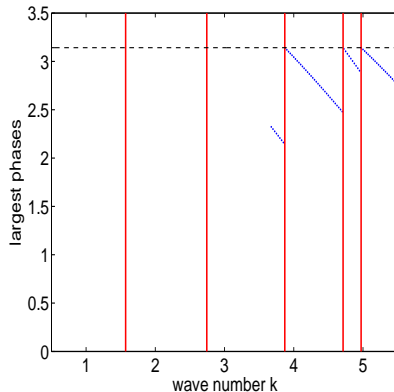
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.675



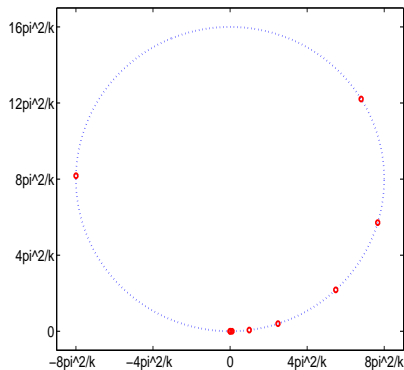
Largest phases.

Robin, $\tau=1$, unit ball

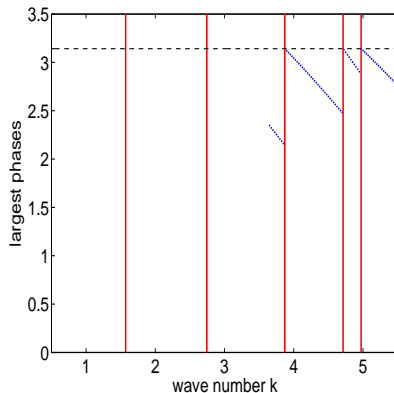
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.650



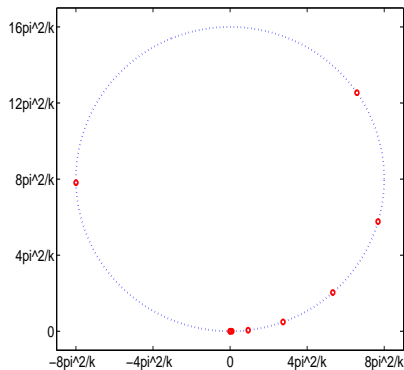
Largest phases.

Robin, $\tau=1$, unit ball

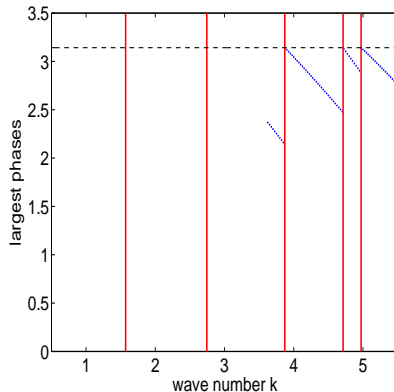
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.625



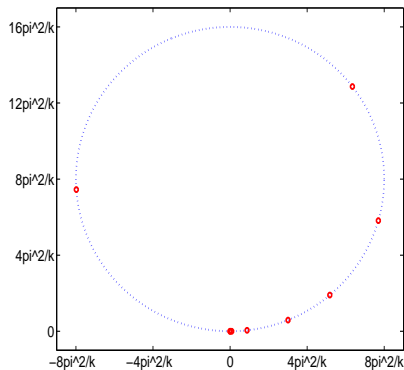
Largest phases.

Robin, $\tau=1$, unit ball

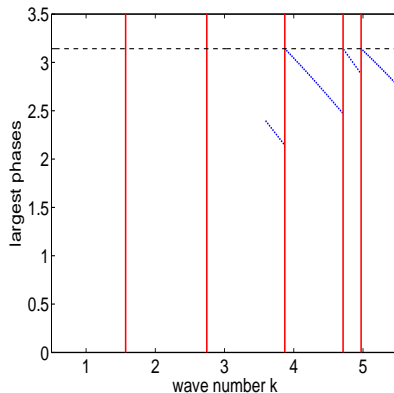
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.600



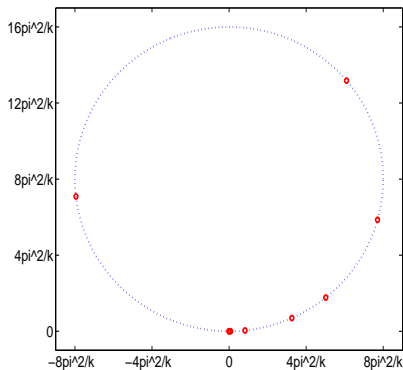
Largest phases.

Robin, $\tau=1$, unit ball

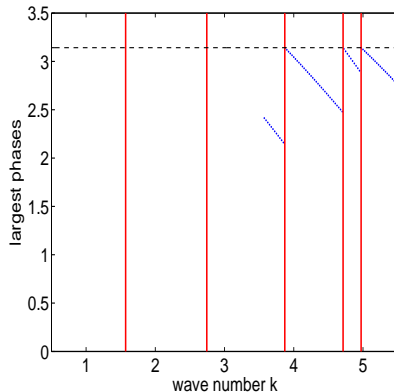
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.575



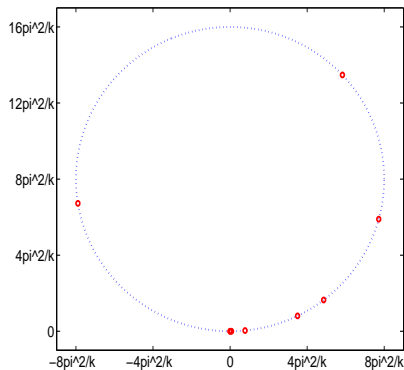
Largest phases.

Robin, $\tau=1$, unit ball

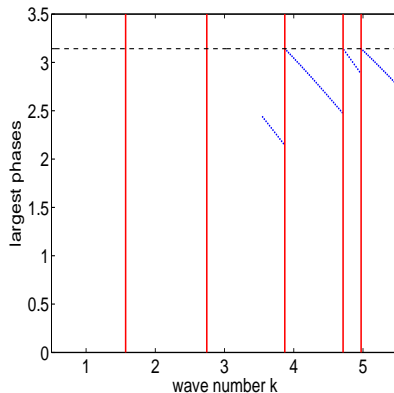
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.550



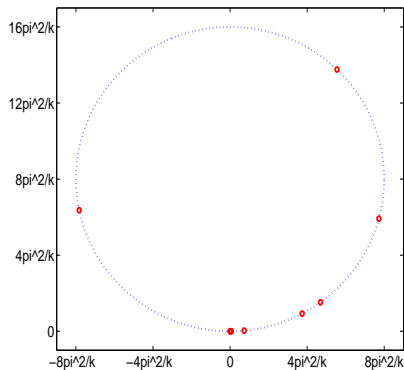
Largest phases.

Robin, $\tau=1$, unit ball

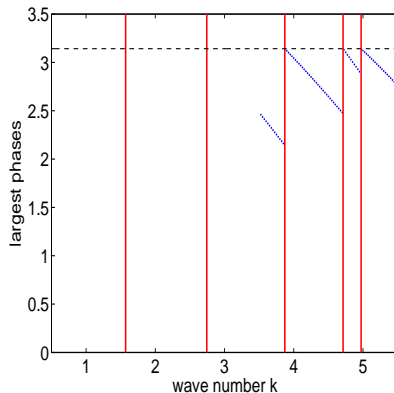
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.525



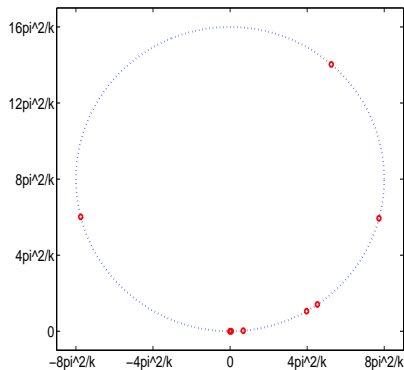
Largest phases.

Robin, $\tau=1$, unit ball

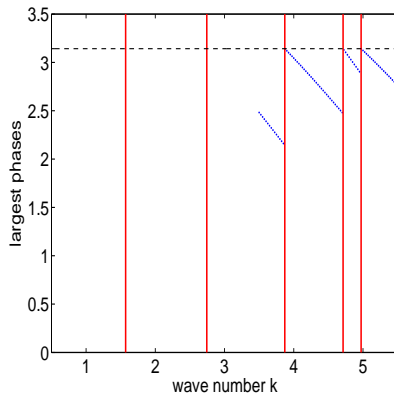
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.500



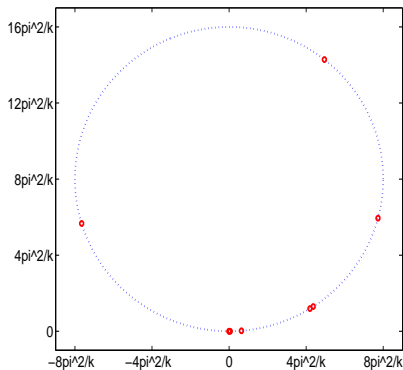
Largest phases.

Robin, $\tau=1$, unit ball

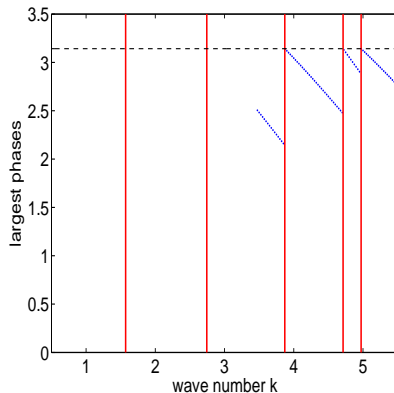
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.475



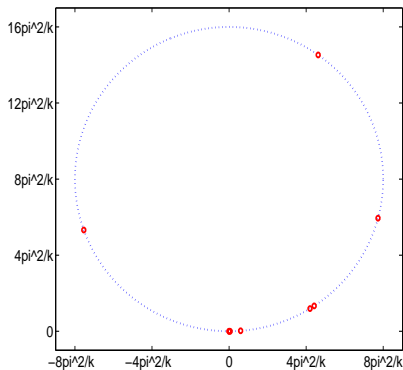
Largest phases.

Robin, $\tau=1$, unit ball

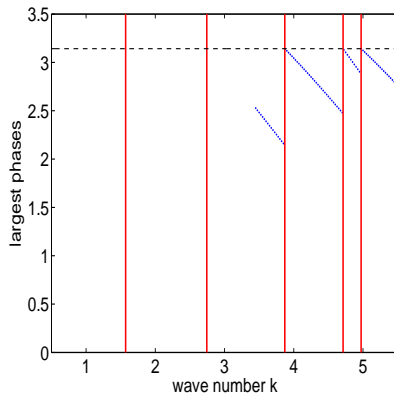
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.450



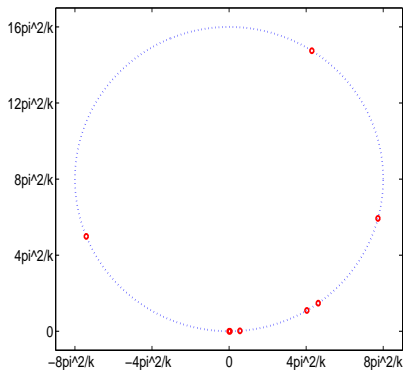
Largest phases.

Robin, $\tau=1$, unit ball

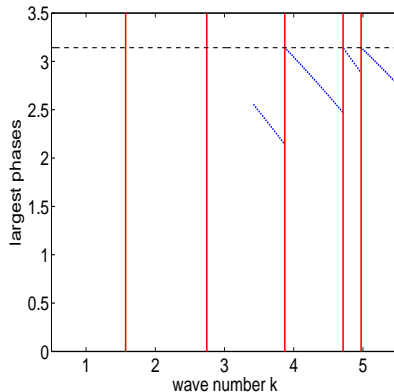
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.425



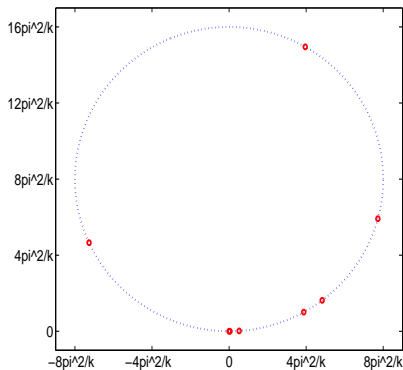
Largest phases.

Robin, $\tau=1$, unit ball

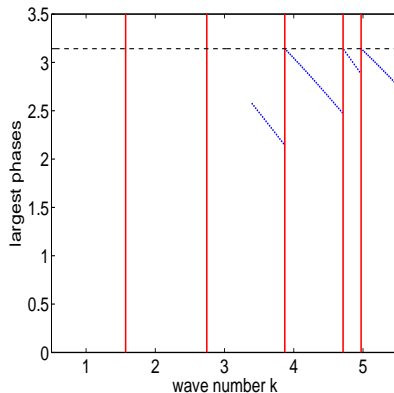
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.400



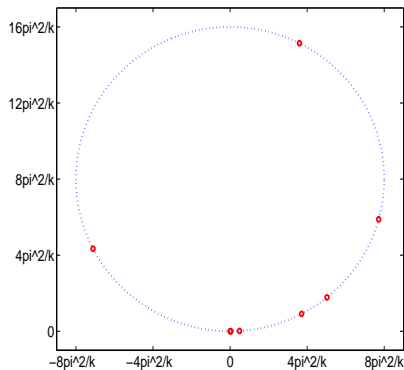
Largest phases.

Robin, $\tau=1$, unit ball

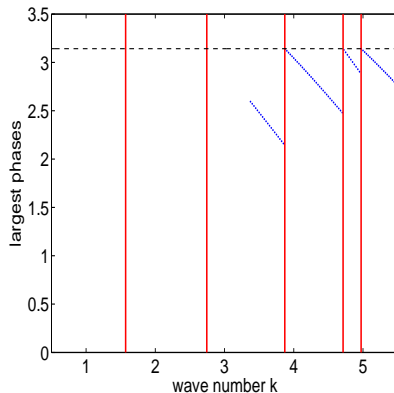
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.375



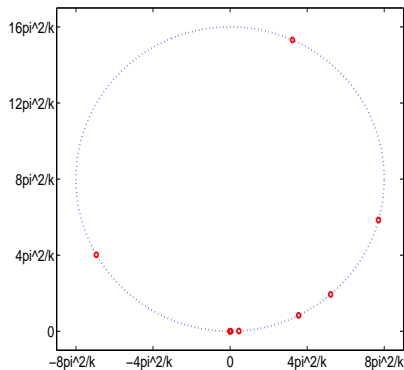
Largest phases.

Robin, $\tau=1$, unit ball

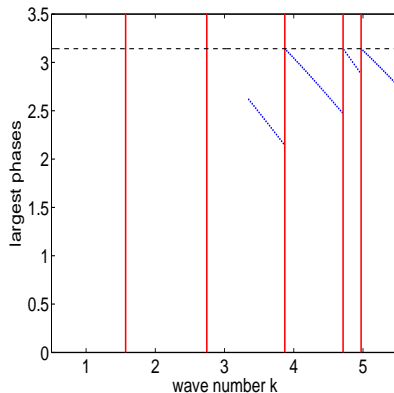
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.350



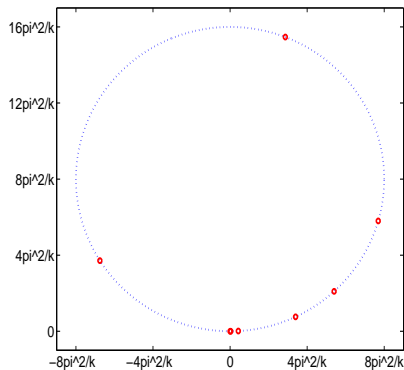
Largest phases.

Robin, $\tau=1$, unit ball

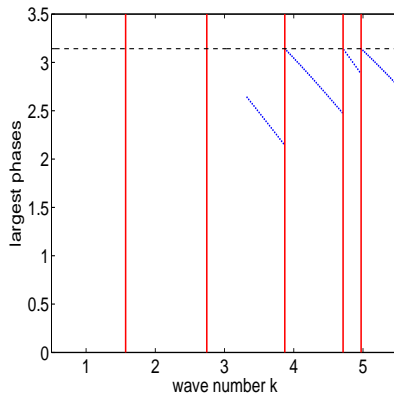
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.325



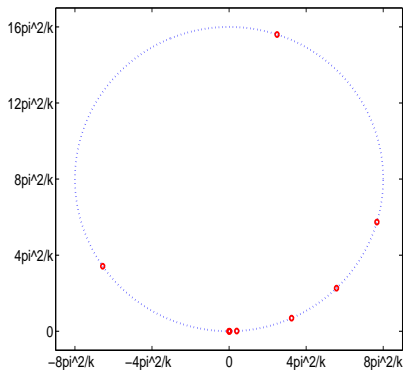
Largest phases.

Robin, $\tau=1$, unit ball

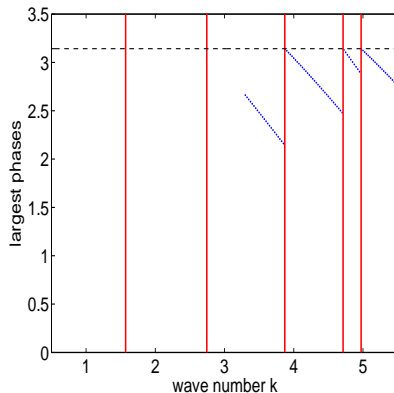
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.300



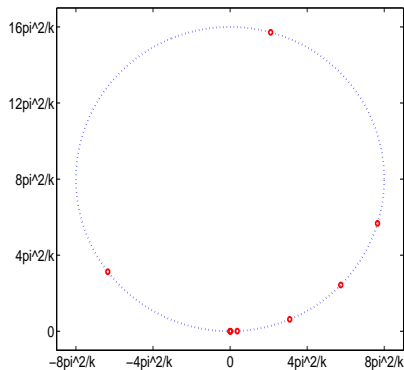
Largest phases.

Robin, $\tau=1$, unit ball

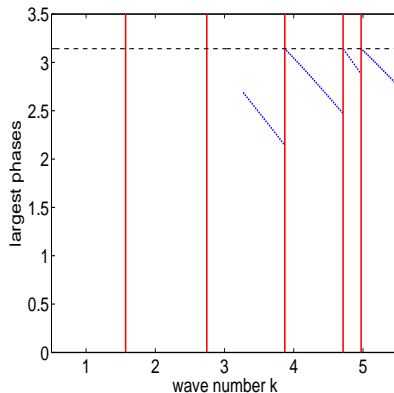
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.275



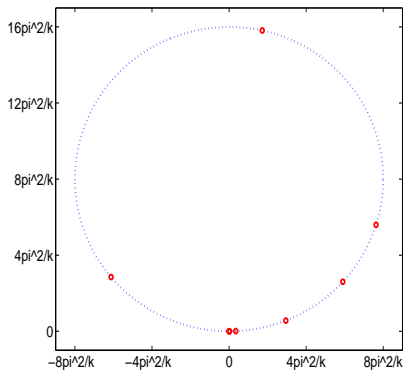
Largest phases.

Robin, $\tau=1$, unit ball

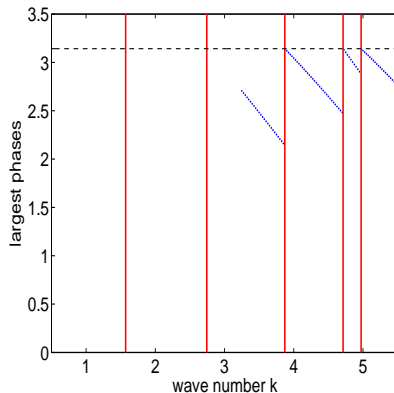
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.250



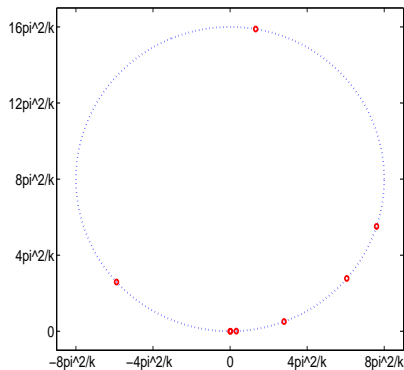
Largest phases.

Robin, $\tau=1$, unit ball

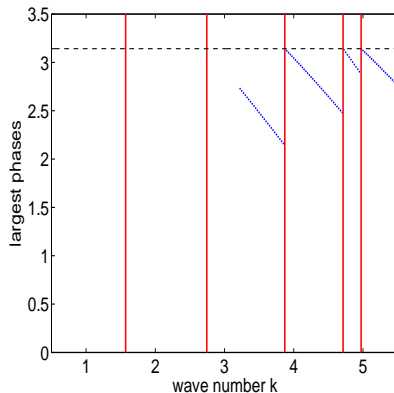
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.225



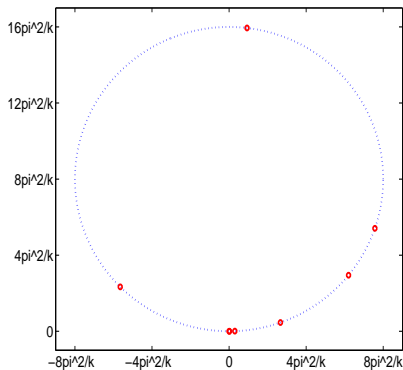
Largest phases.

Robin, $\tau=1$, unit ball

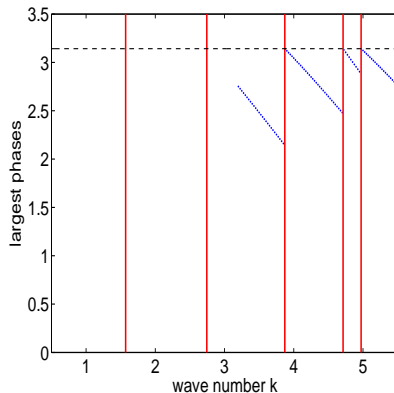
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.200



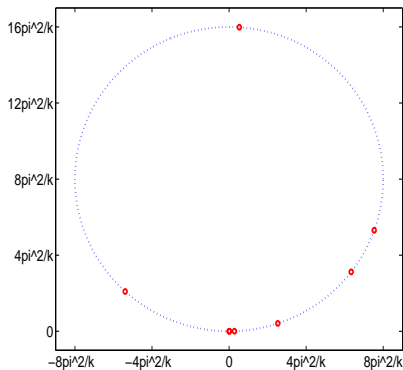
Largest phases.

Robin, $\tau=1$, unit ball

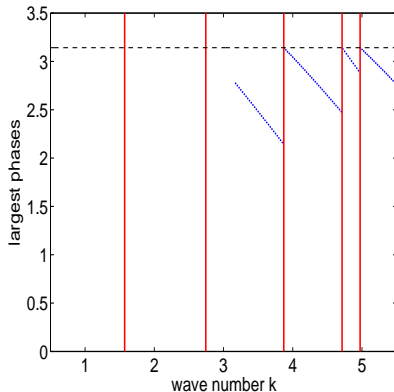
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.175



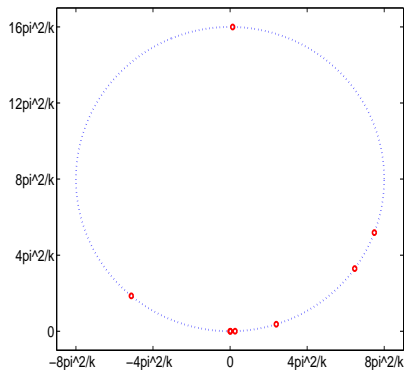
Largest phases.

Robin, $\tau=1$, unit ball

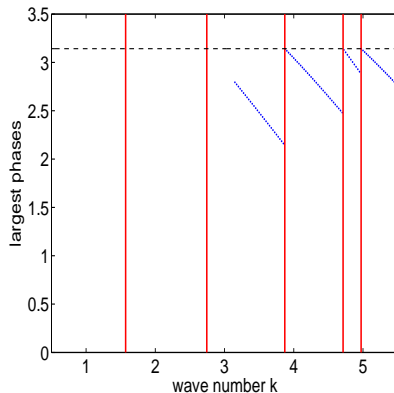
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.150



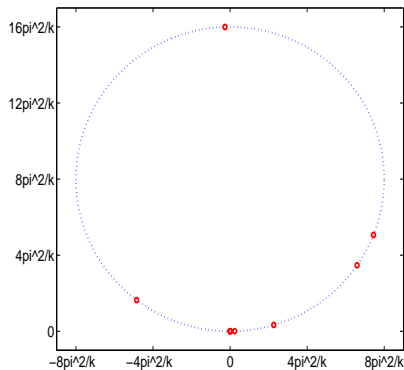
Largest phases.

Robin, $\tau=1$, unit ball

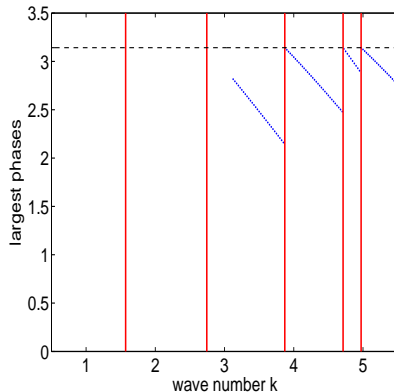
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.125



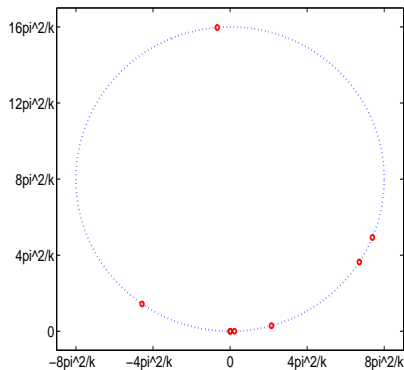
Largest phases.

Robin, $\tau=1$, unit ball

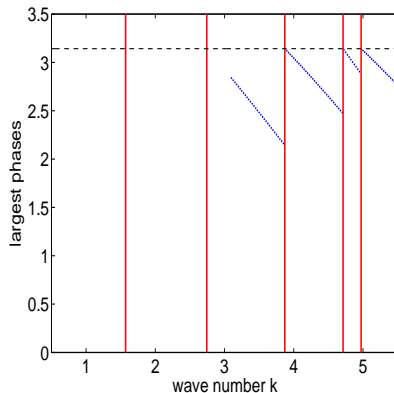
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.100

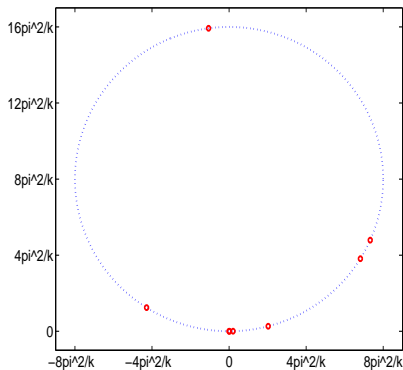


Largest phases.

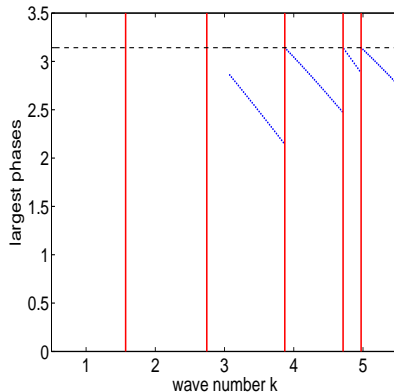
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

 $k=3.075$ 

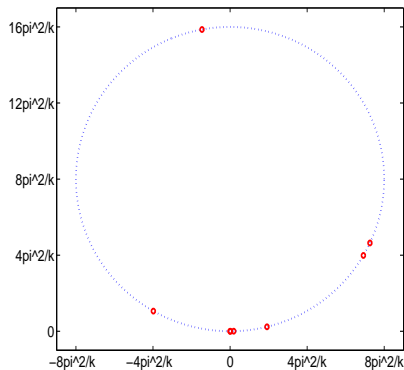
Largest phases.

Robin, $\tau=1$, unit ball

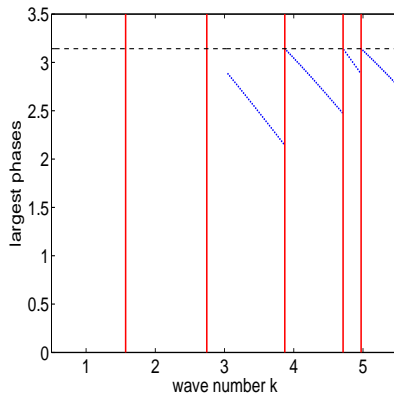
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.050



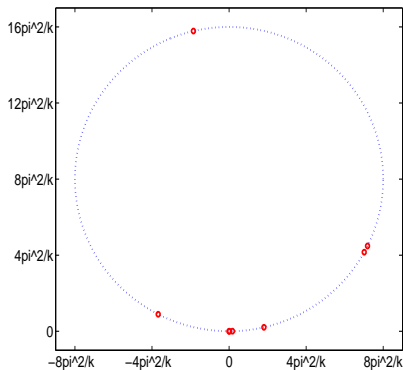
Largest phases.

Robin, $\tau=1$, unit ball

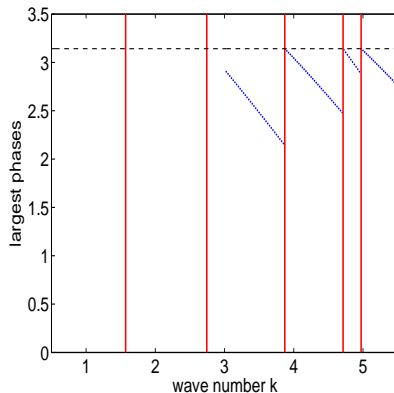
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.025



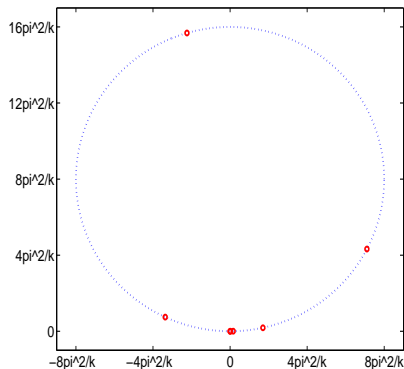
Largest phases.

Robin, $\tau=1$, unit ball

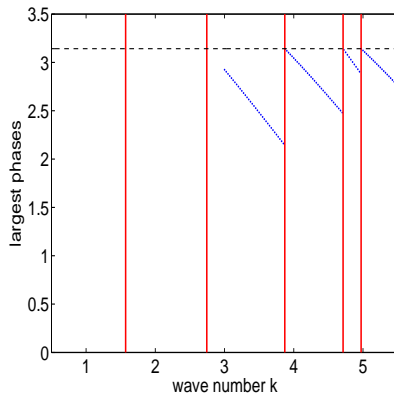
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=3.000



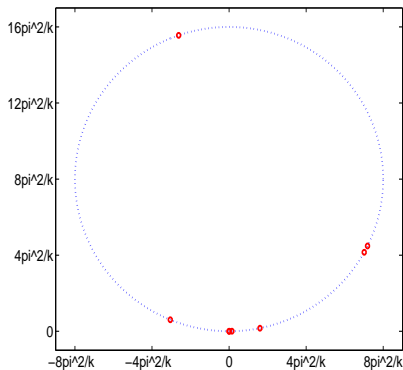
Largest phases.

Robin, $\tau=1$, unit ball

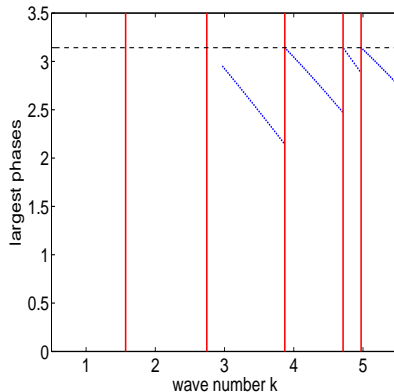
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.975



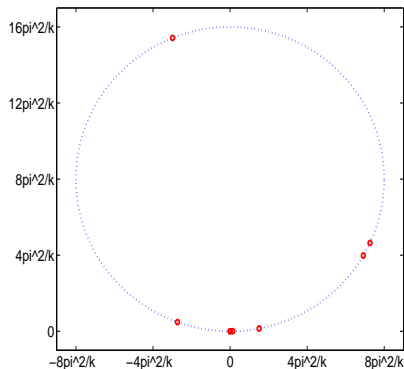
Largest phases.

Robin, $\tau=1$, unit ball

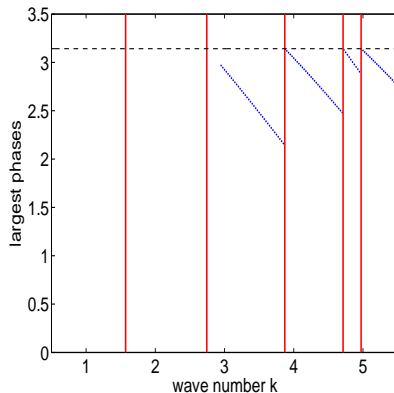
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.950



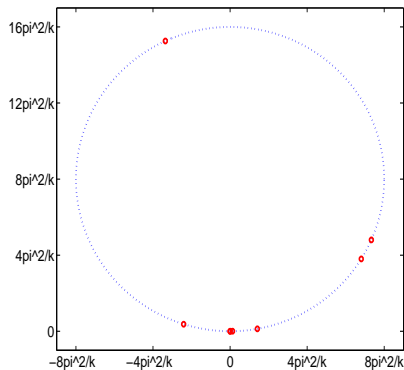
Largest phases.

Robin, $\tau=1$, unit ball

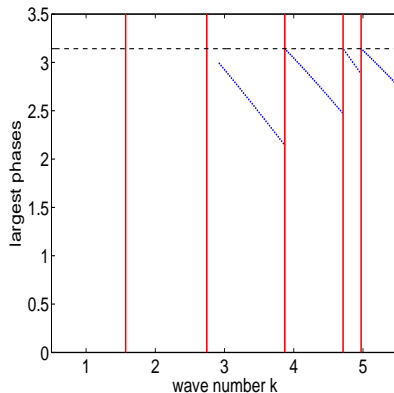
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.925



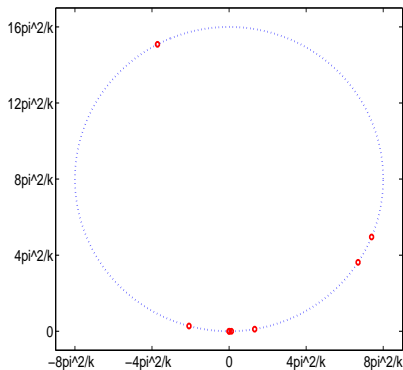
Largest phases.

Robin, $\tau=1$, unit ball

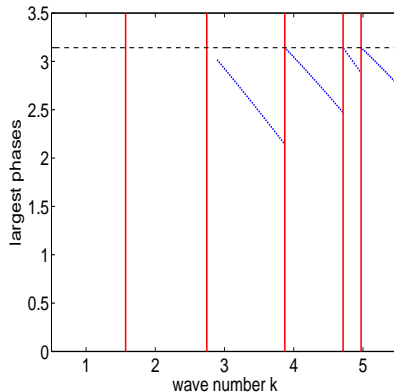
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.900



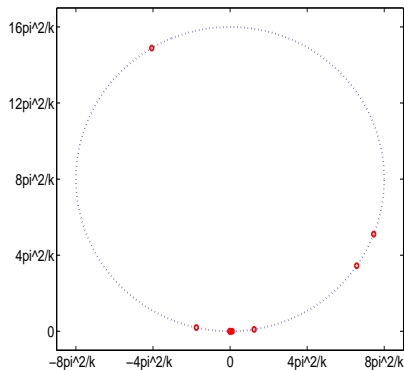
Largest phases.

Robin, $\tau=1$, unit ball

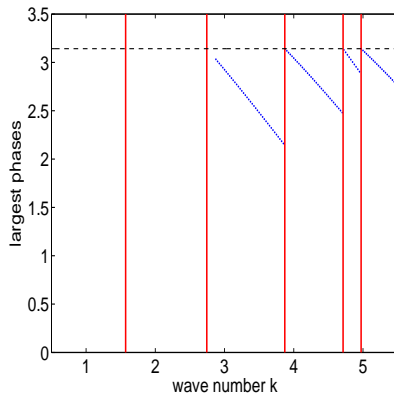
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.875



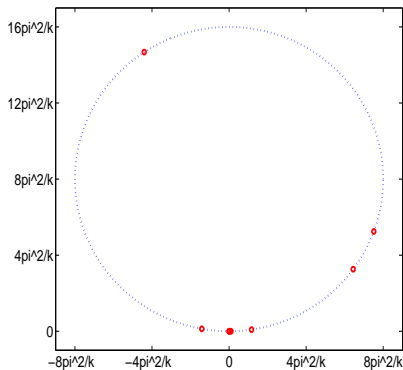
Largest phases.

Robin, $\tau=1$, unit ball

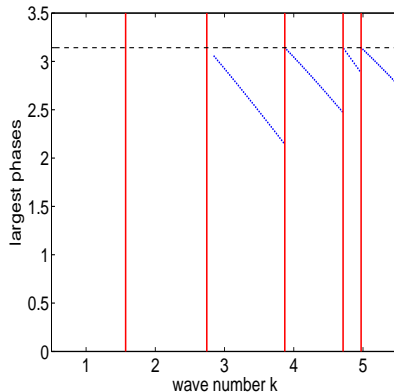
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.850



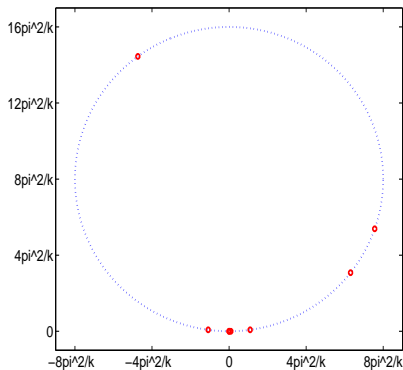
Largest phases.

Robin, $\tau=1$, unit ball

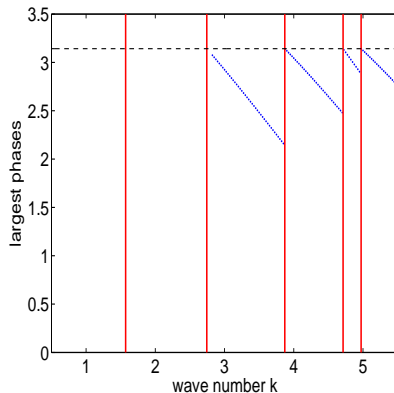
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.825



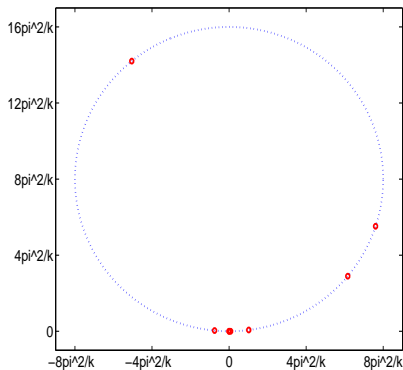
Largest phases.

Robin, $\tau=1$, unit ball

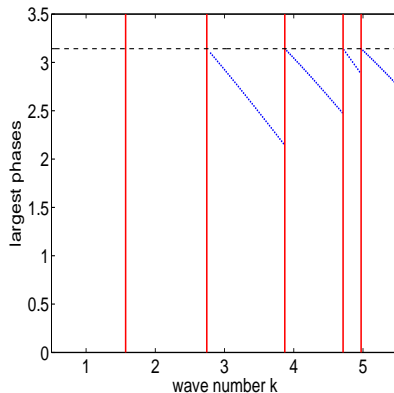
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.800



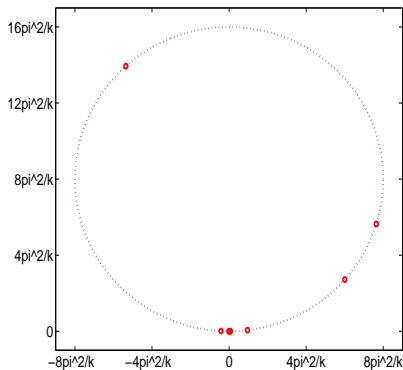
Largest phases.

Robin, $\tau=1$, unit ball

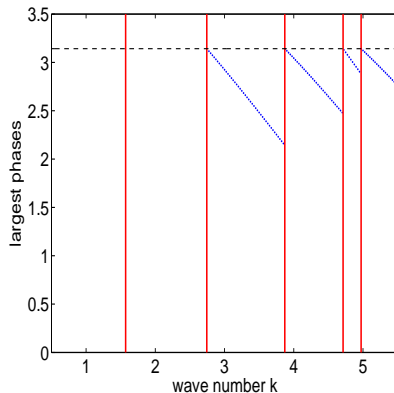
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.775



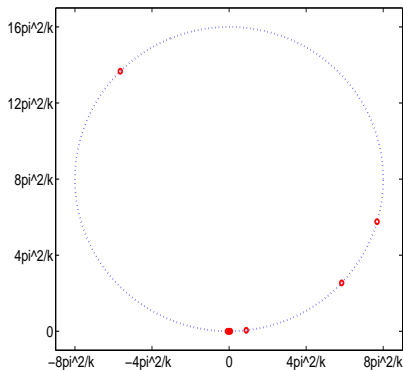
Largest phases.

Robin, $\tau=1$, unit ball

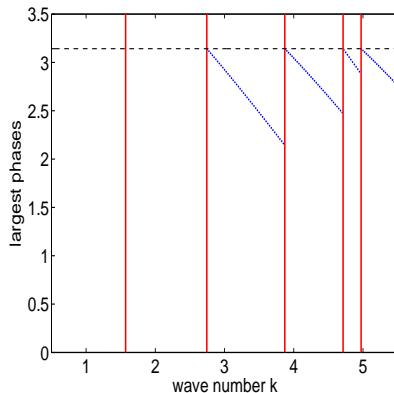
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.750

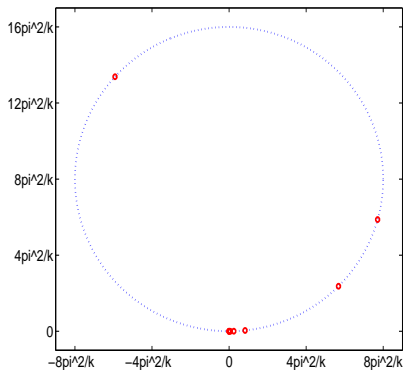


Largest phases.

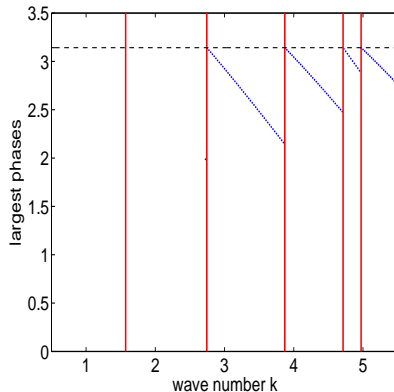
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

 $k=2.725$ 

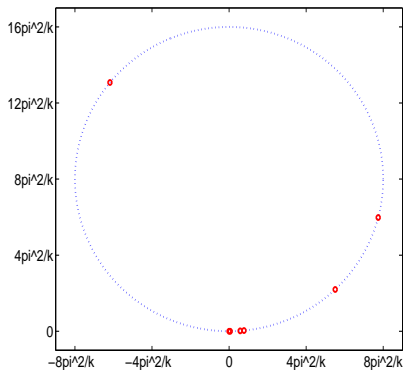
Largest phases.

Robin, $\tau=1$, unit ball

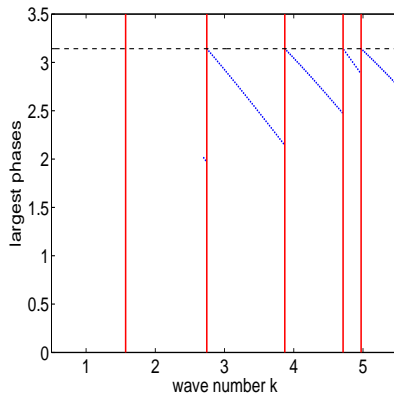
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.700



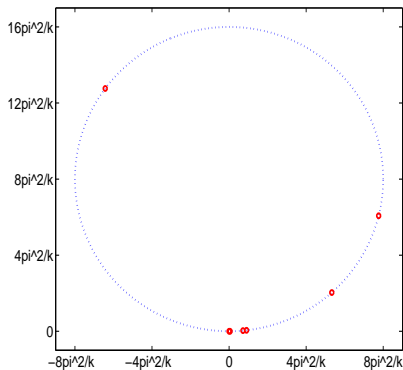
Largest phases.

Robin, $\tau=1$, unit ball

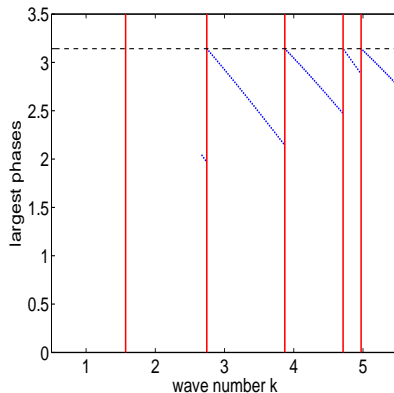
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.675



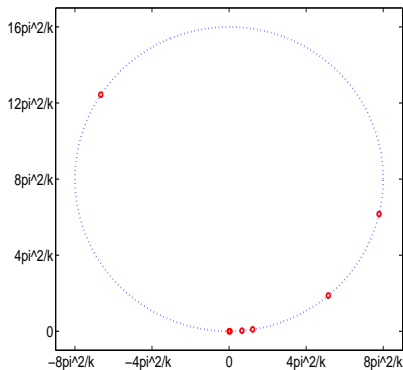
Largest phases.

Robin, $\tau=1$, unit ball

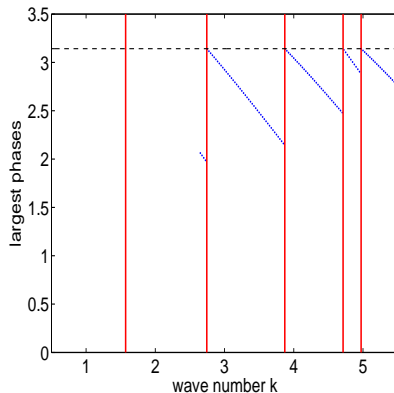
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.650



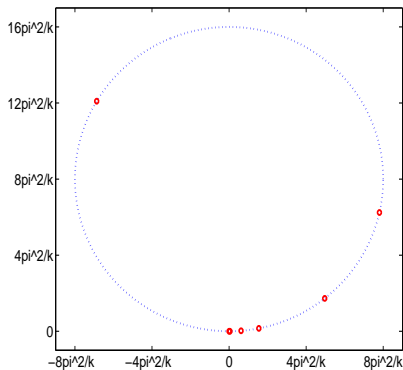
Largest phases.

Robin, $\tau=1$, unit ball

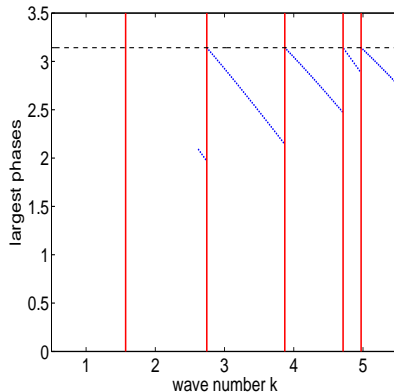
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.625



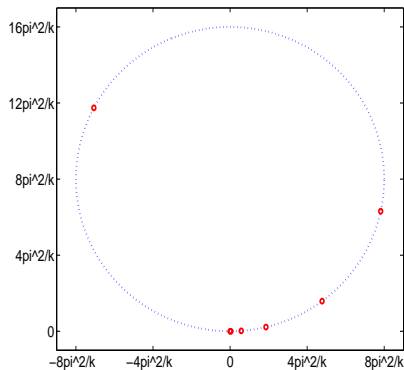
Largest phases.

Robin, $\tau=1$, unit ball

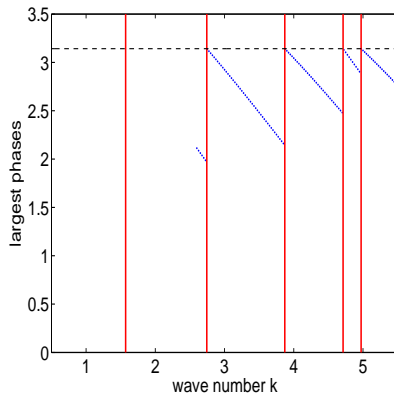
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.600



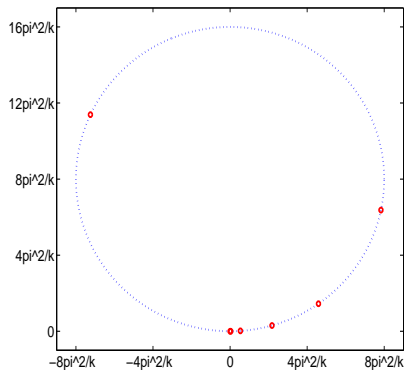
Largest phases.

Robin, $\tau=1$, unit ball

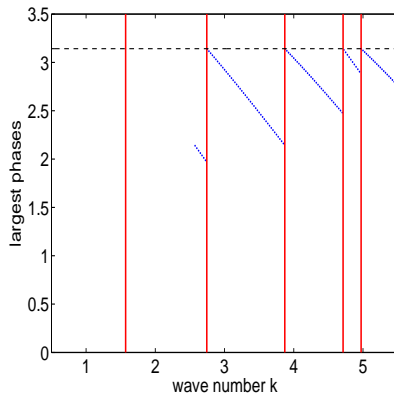
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.575



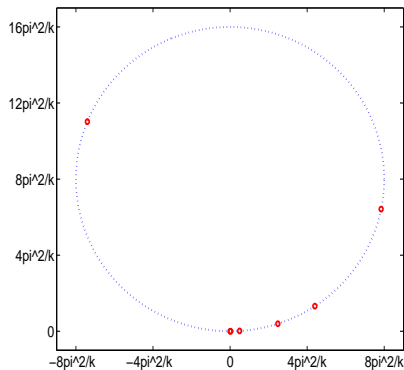
Largest phases.

Robin, $\tau=1$, unit ball

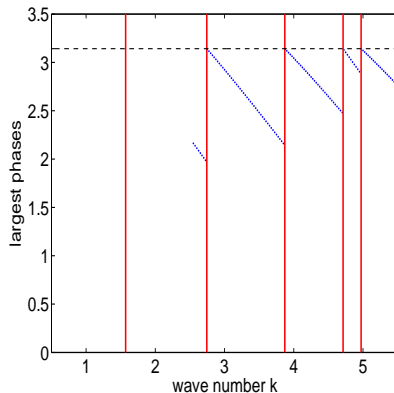
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.550



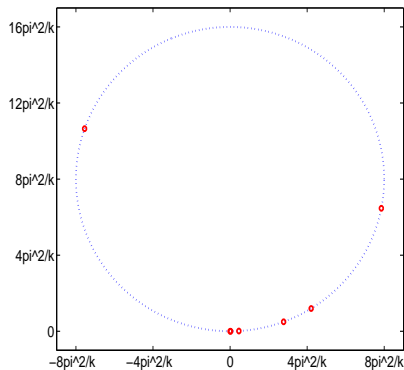
Largest phases.

Robin, $\tau=1$, unit ball

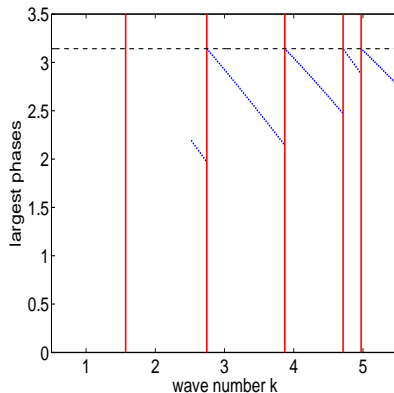
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.525



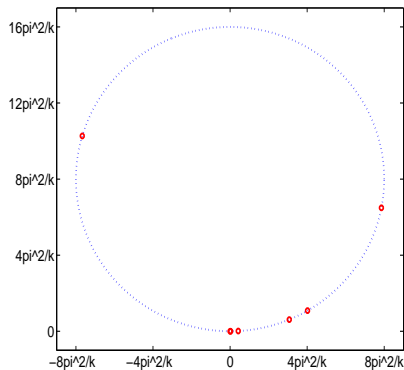
Largest phases.

Robin, $\tau=1$, unit ball

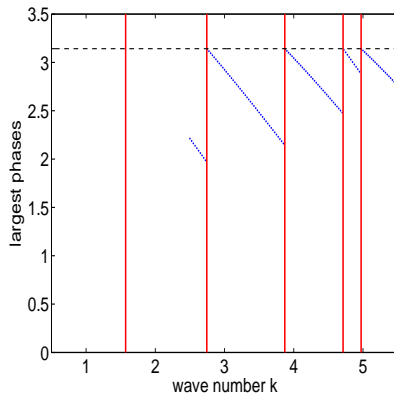
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.500



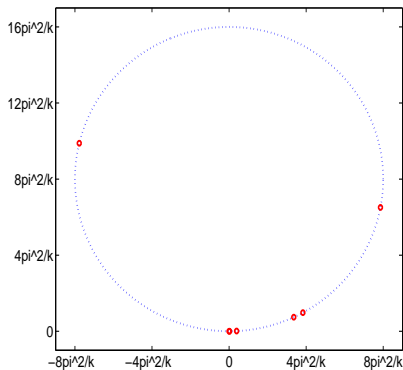
Largest phases.

Robin, $\tau=1$, unit ball

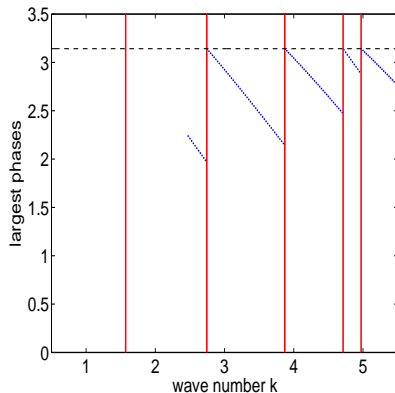
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.475



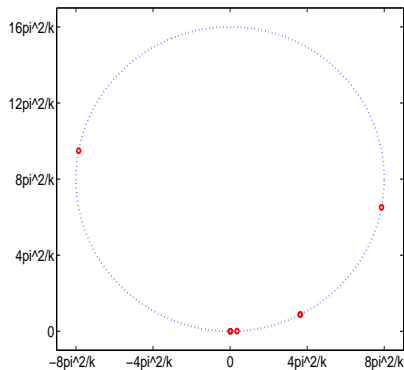
Largest phases.

Robin, $\tau=1$, unit ball

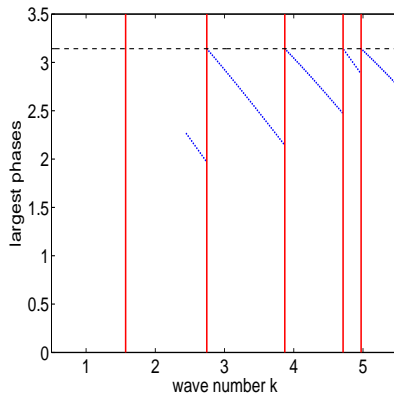
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.450

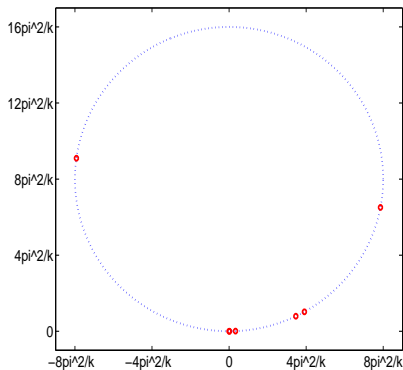


Largest phases.

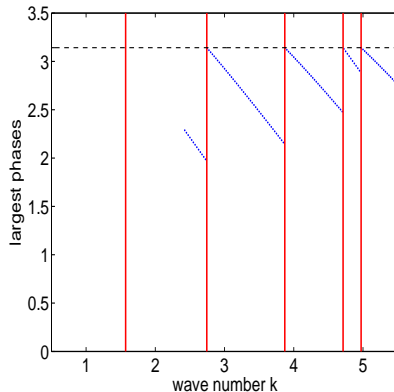
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

 $k=2.425$ 

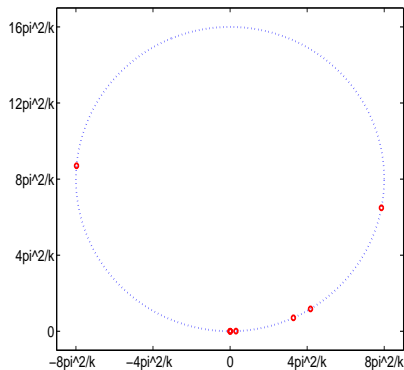
Largest phases.

Robin, $\tau=1$, unit ball

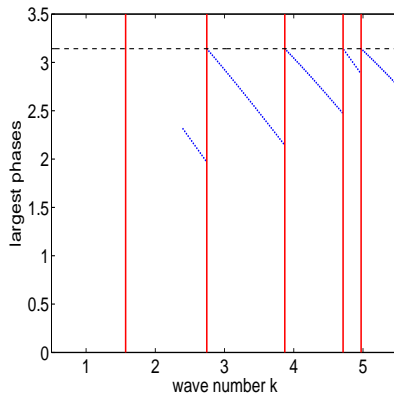
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.400



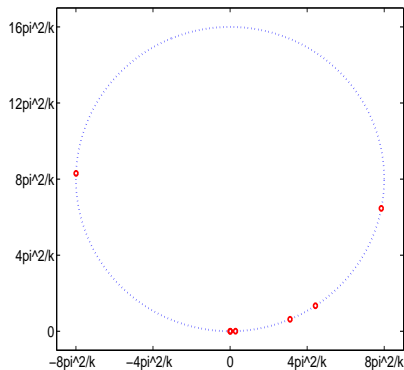
Largest phases.

Robin, $\tau=1$, unit ball

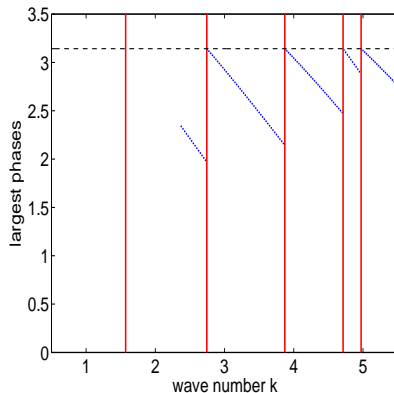
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.375



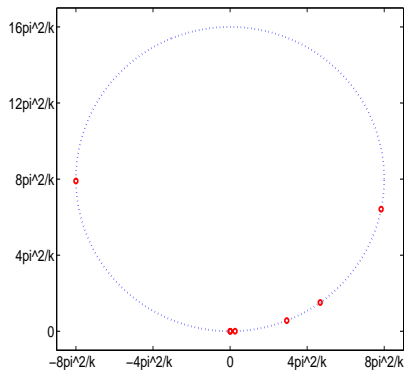
Largest phases.

Robin, $\tau=1$, unit ball

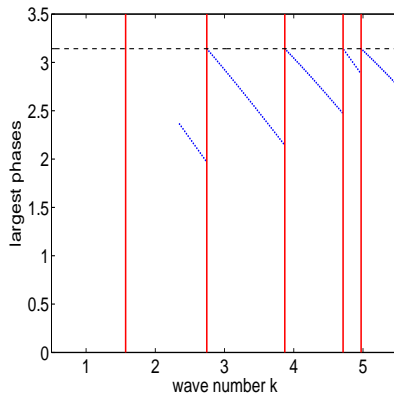
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.350



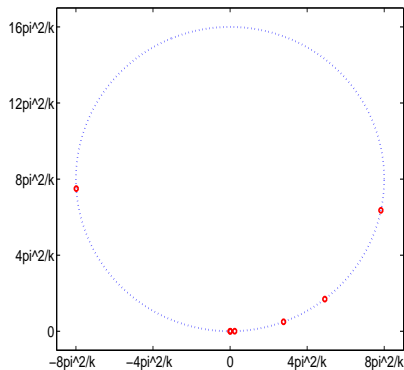
Largest phases.

Robin, $\tau=1$, unit ball

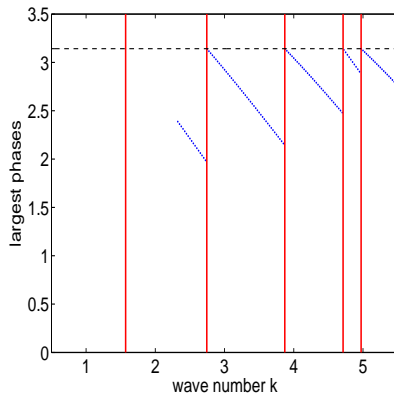
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.325



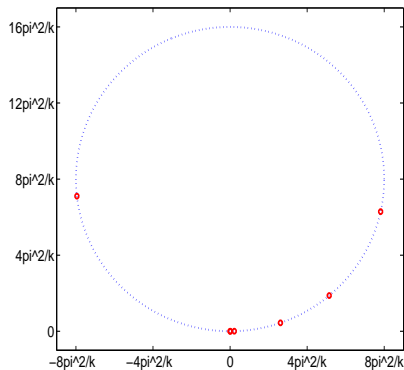
Largest phases.

Robin, $\tau=1$, unit ball

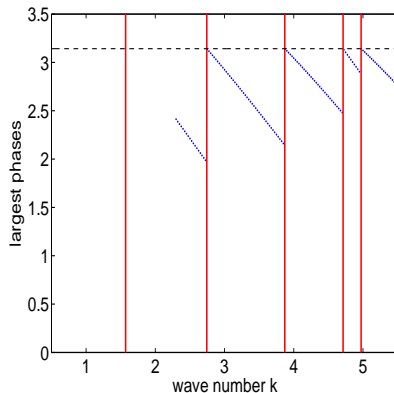
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.300

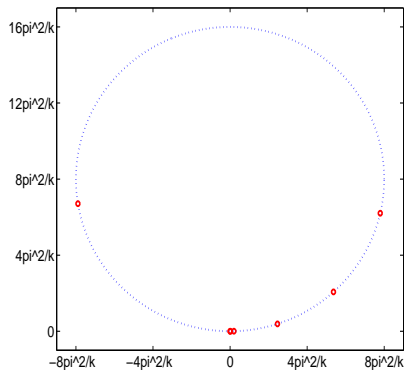


Largest phases.

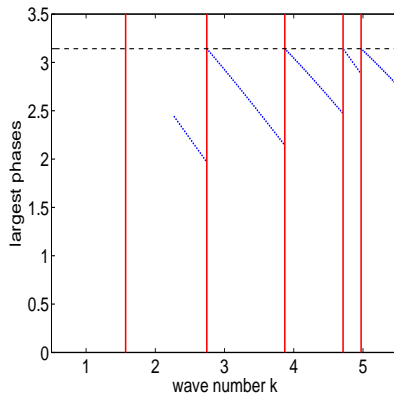
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

 $k=2.275$ 

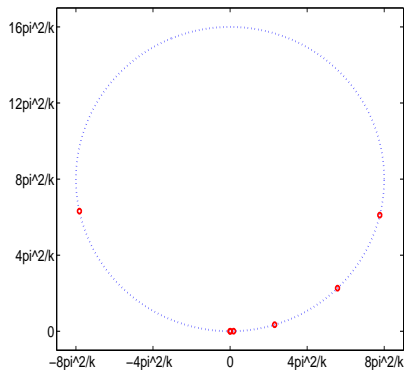
Largest phases.

Robin, $\tau=1$, unit ball

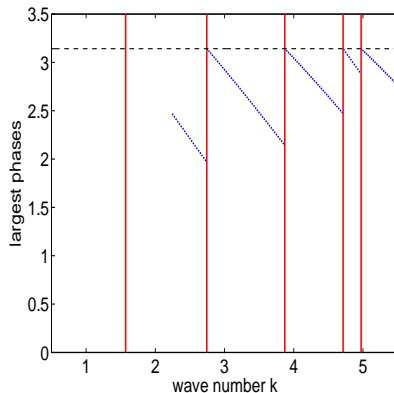
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.250



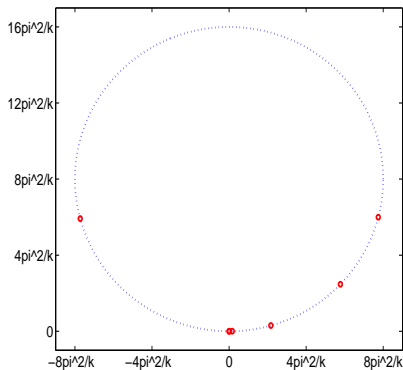
Largest phases.

Robin, $\tau=1$, unit ball

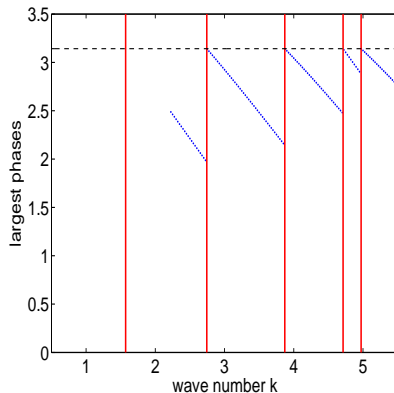
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.225



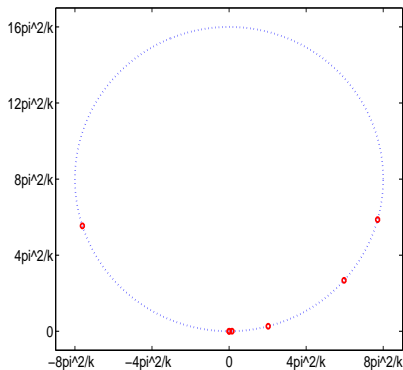
Largest phases.

Robin, $\tau=1$, unit ball

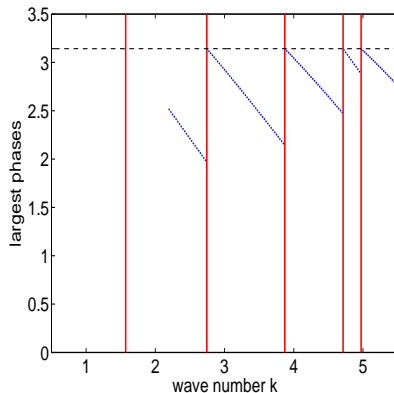
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.200



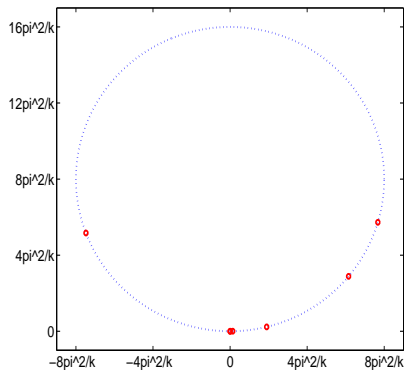
Largest phases.

Robin, $\tau=1$, unit ball

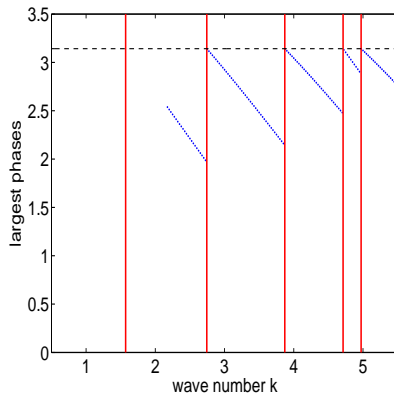
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.175



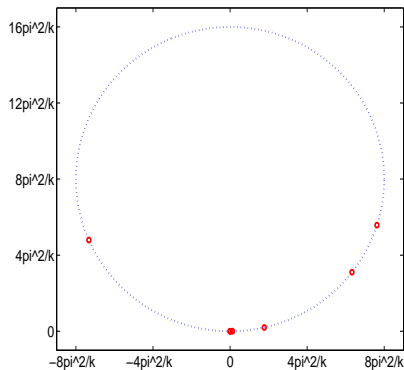
Largest phases.

Robin, $\tau=1$, unit ball

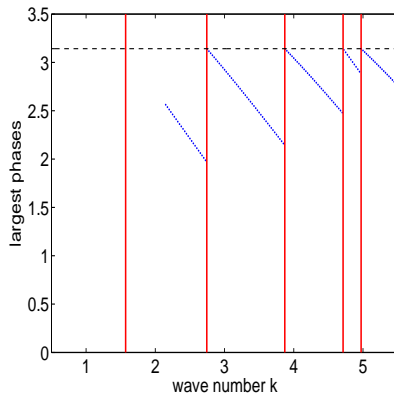
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.150



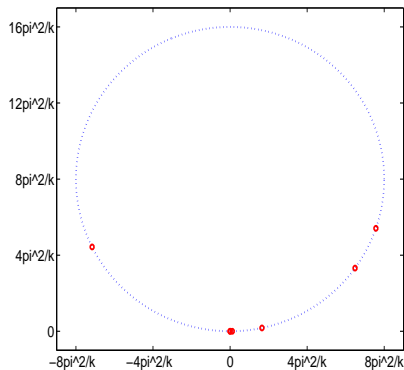
Largest phases.

Robin, $\tau=1$, unit ball

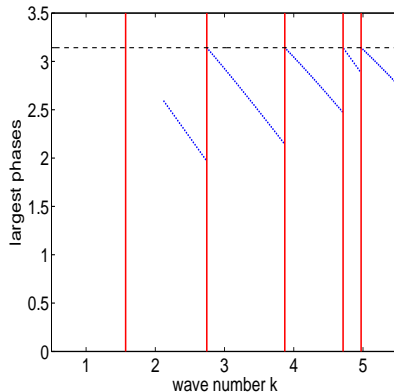
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.125



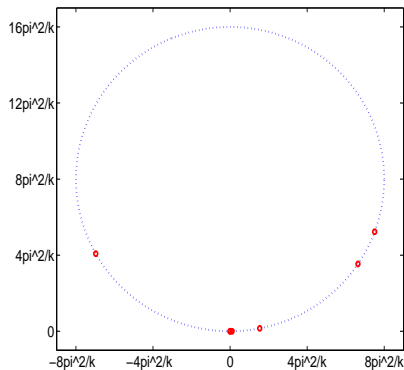
Largest phases.

Robin, $\tau=1$, unit ball

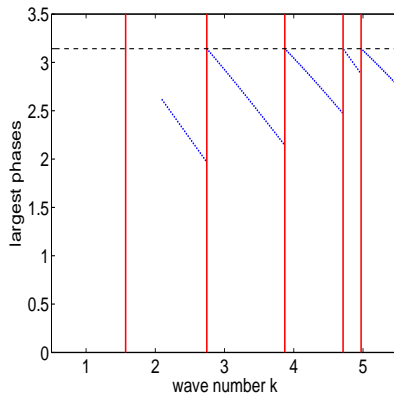
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.100



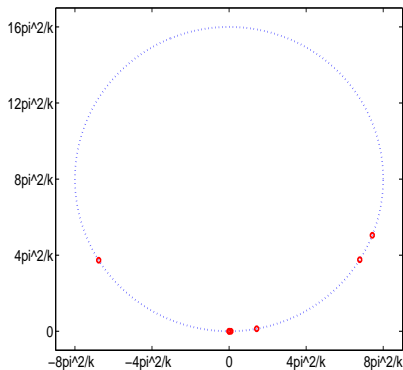
Largest phases.

Robin, $\tau=1$, unit ball

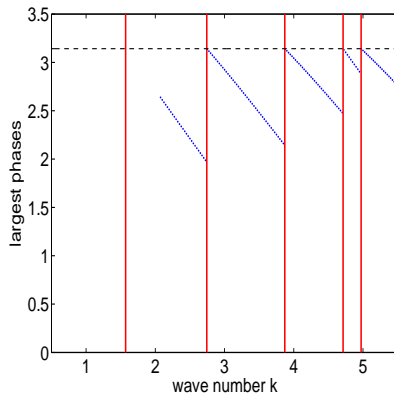
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.075



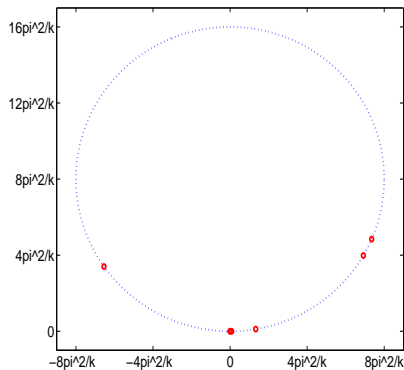
Largest phases.

Robin, $\tau=1$, unit ball

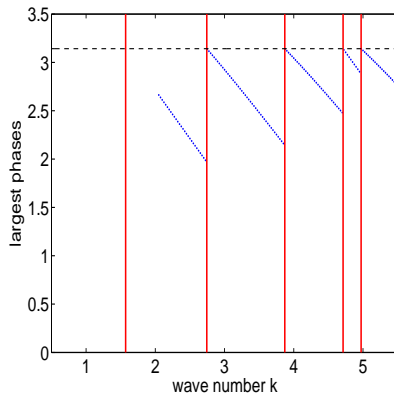
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.050



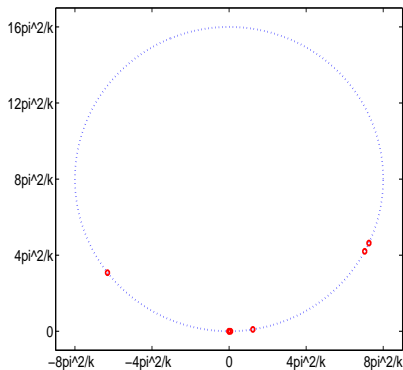
Largest phases.

Robin, $\tau=1$, unit ball

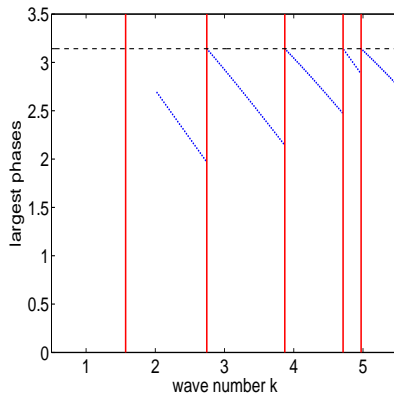
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.025



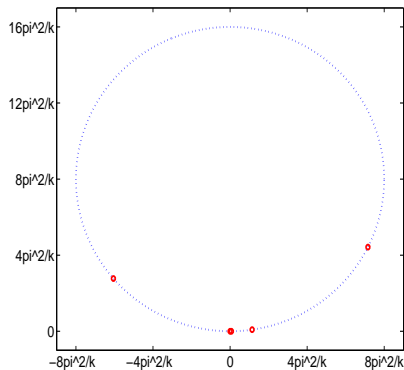
Largest phases.

Robin, $\tau=1$, unit ball

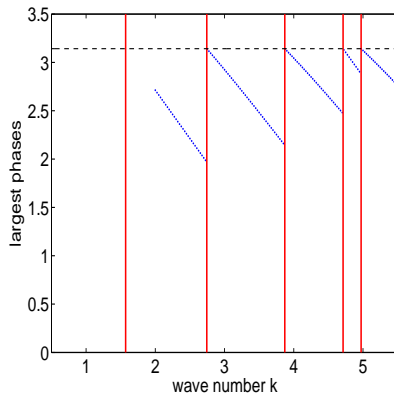
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=2.000



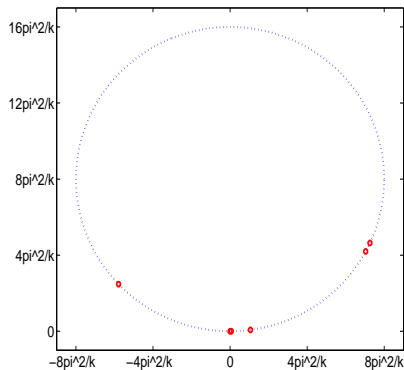
Largest phases.

Robin, $\tau=1$, unit ball

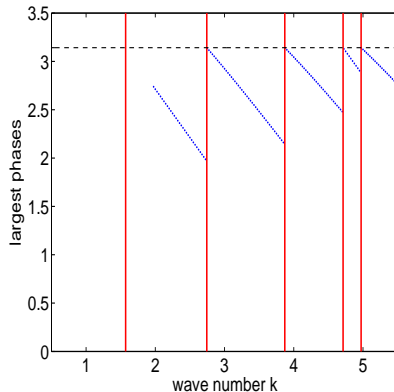
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.975



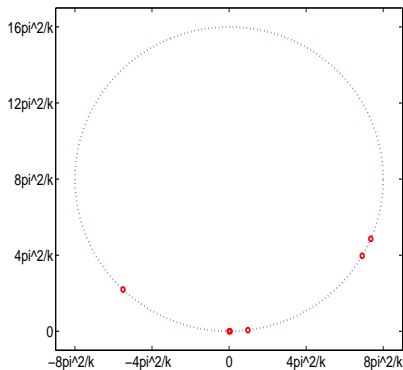
Largest phases.

Robin, $\tau=1$, unit ball

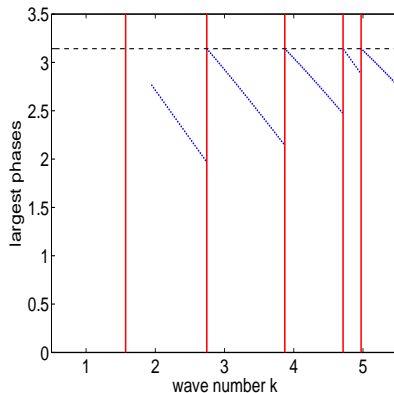
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.950



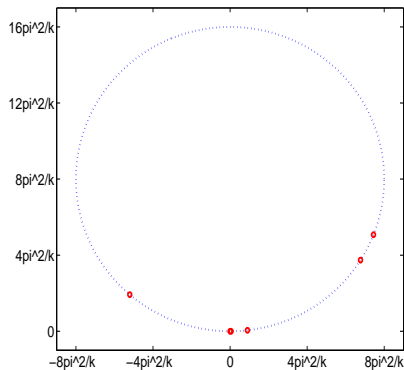
Largest phases.

Robin, $\tau=1$, unit ball

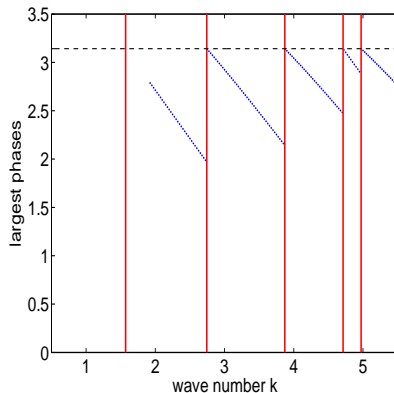
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.925



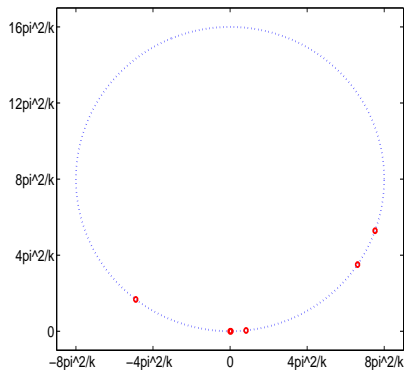
Largest phases.

Robin, $\tau=1$, unit ball

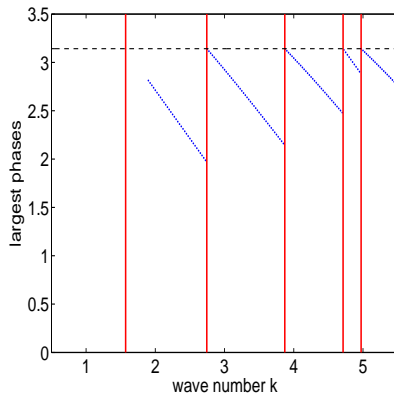
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.900



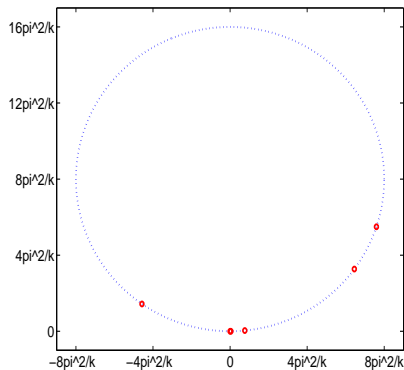
Largest phases.

Robin, $\tau=1$, unit ball

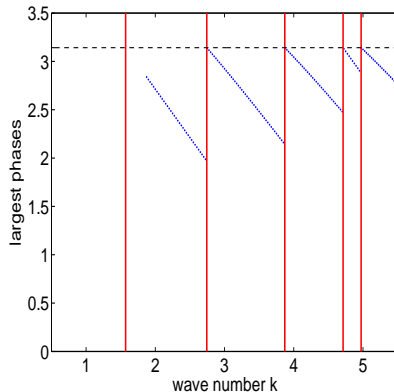
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.875



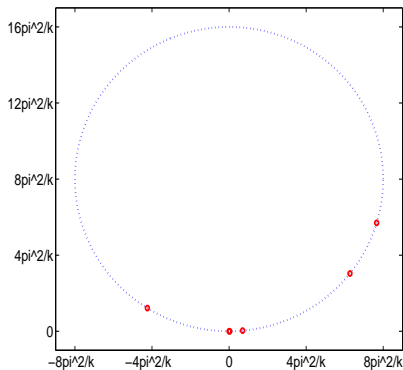
Largest phases.

Robin, $\tau=1$, unit ball

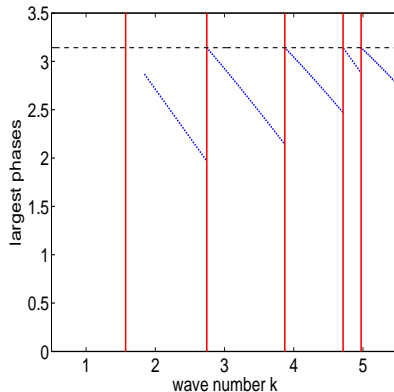
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.850



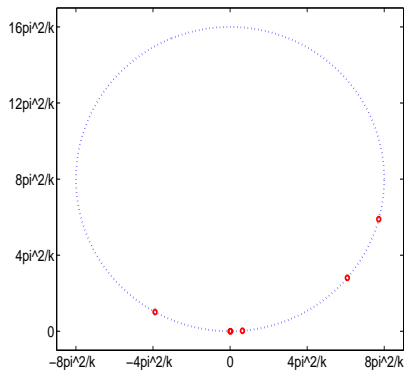
Largest phases.

Robin, $\tau=1$, unit ball

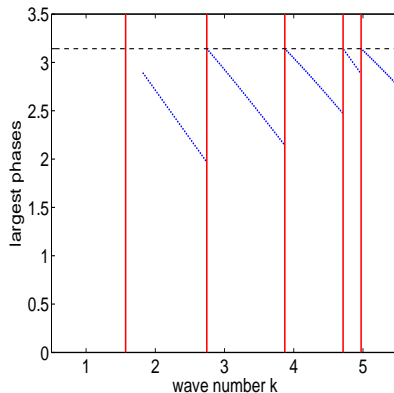
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.825



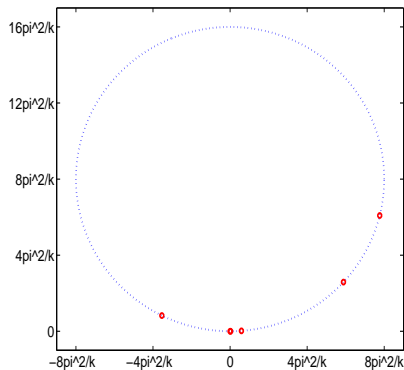
Largest phases.

Robin, $\tau=1$, unit ball

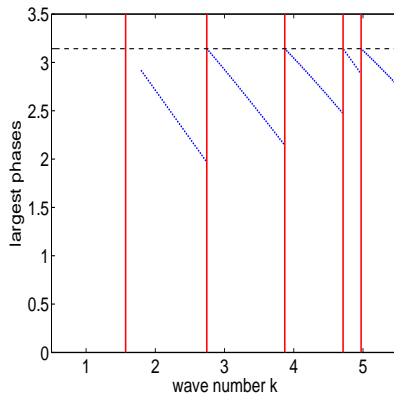
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.800

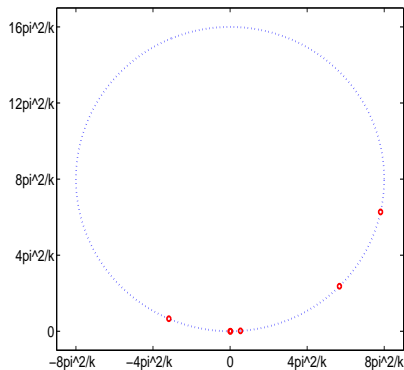


Largest phases.

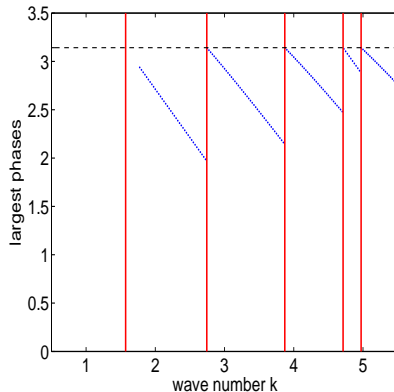
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

 $k=1.775$ 

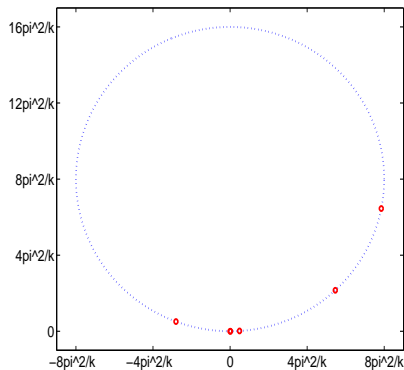
Largest phases.

Robin, $\tau=1$, unit ball

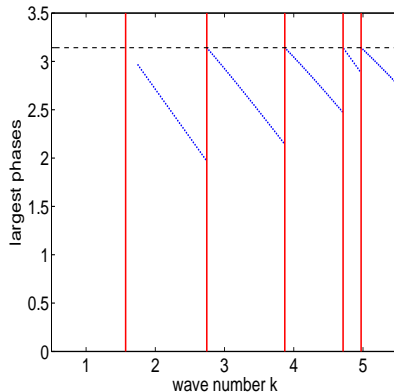
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.750

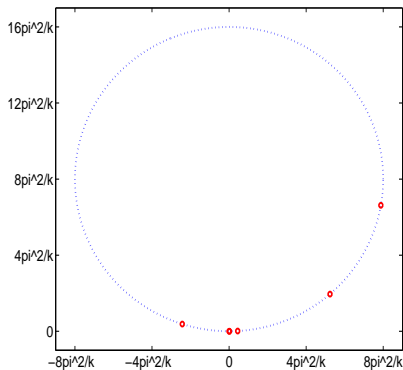


Largest phases.

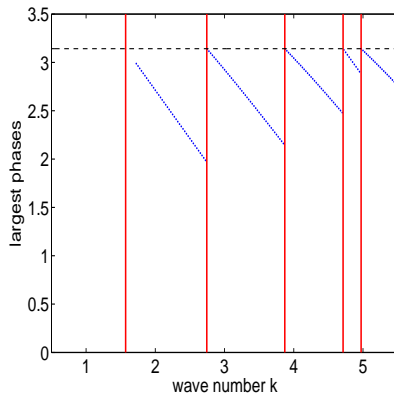
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

 $k=1.725$ 

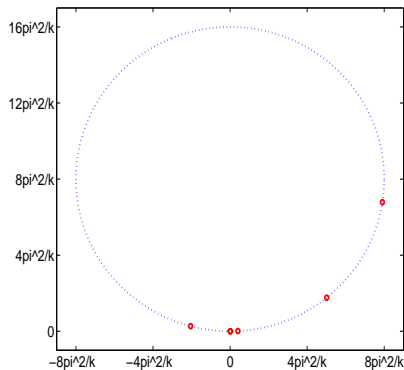
Largest phases.

Robin, $\tau=1$, unit ball

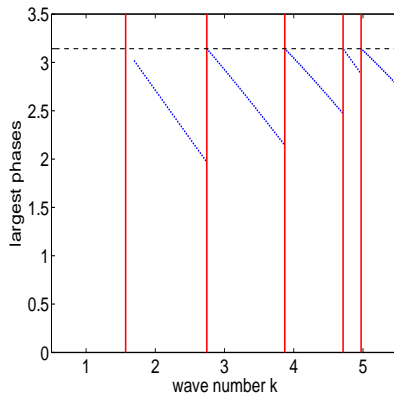
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.700

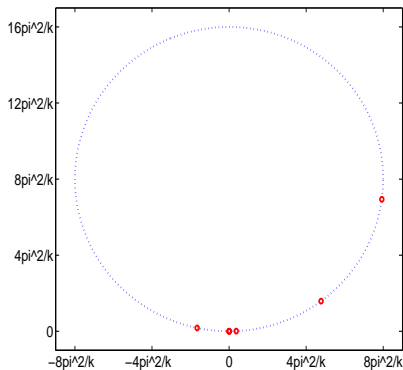


Largest phases.

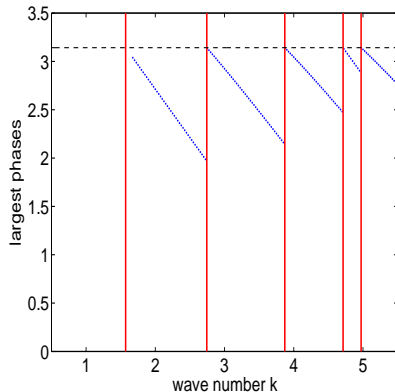
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

 $k=1.675$ 

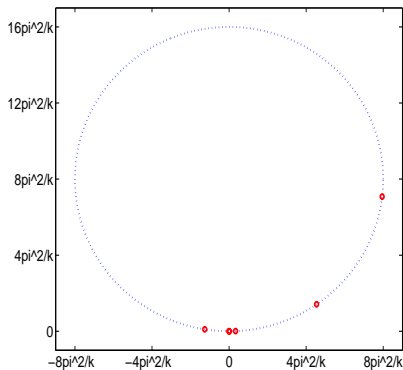
Largest phases.

Robin, $\tau=1$, unit ball

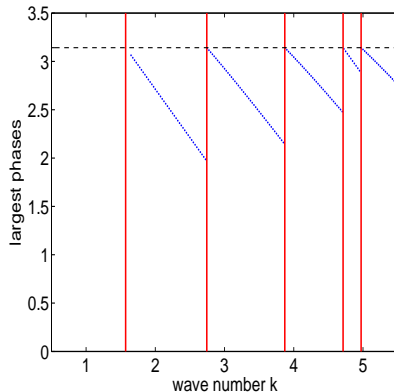
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.650

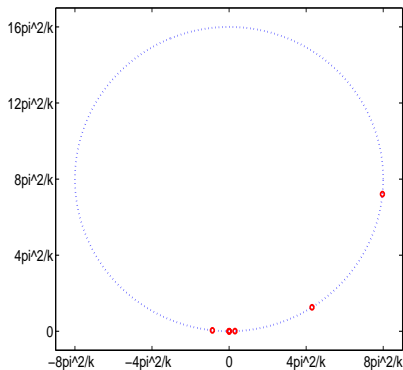


Largest phases.

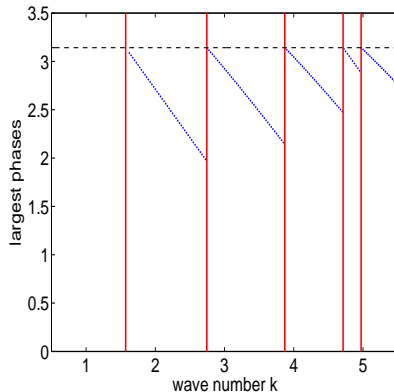
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

 $k=1.625$ 

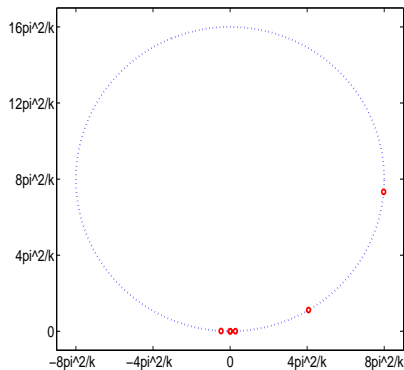
Largest phases.

Robin, $\tau=1$, unit ball

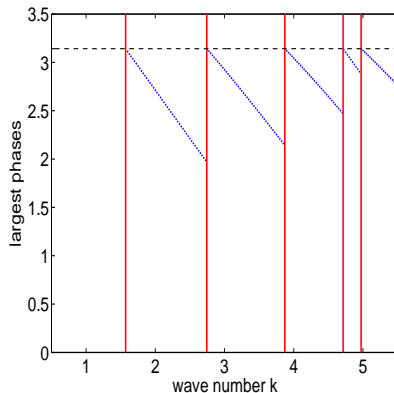
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.600



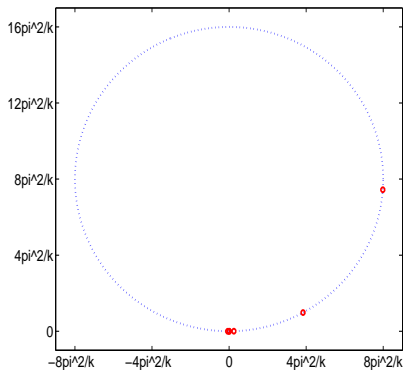
Largest phases.

Robin, $\tau=1$, unit ball

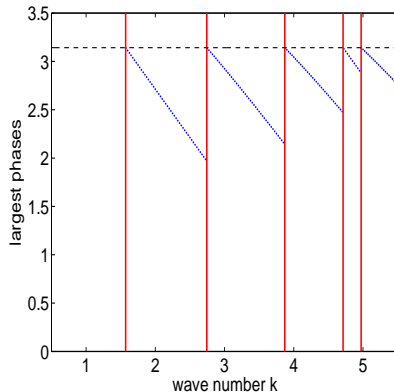
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.575



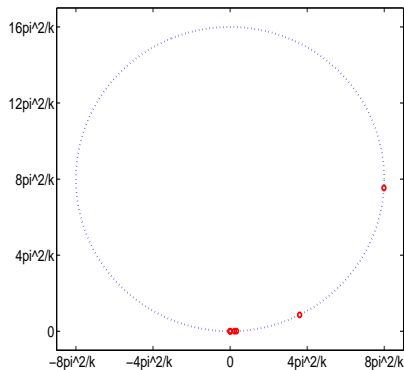
Largest phases.

Robin, $\tau=1$, unit ball

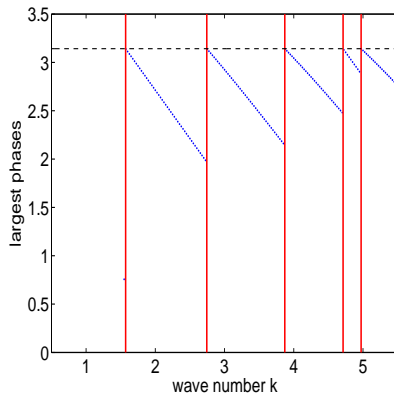
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.550



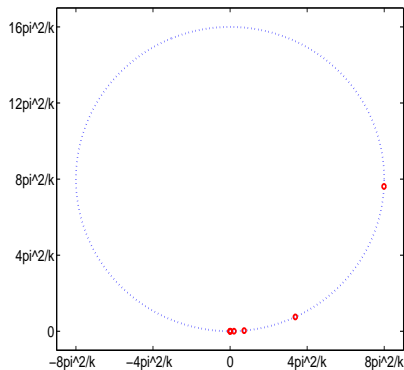
Largest phases.

Robin, $\tau=1$, unit ball

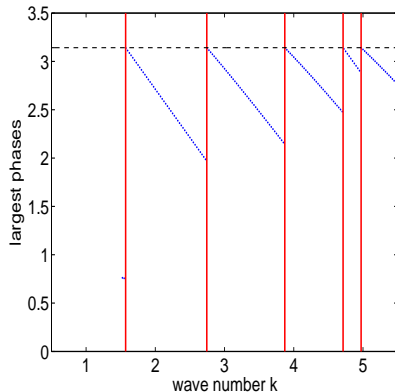
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.525



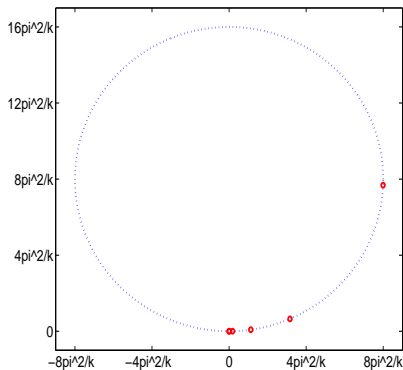
Largest phases.

Robin, $\tau=1$, unit ball

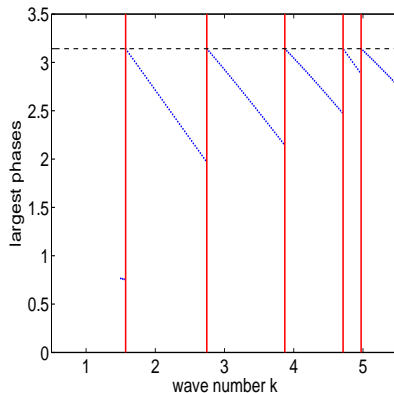
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.500



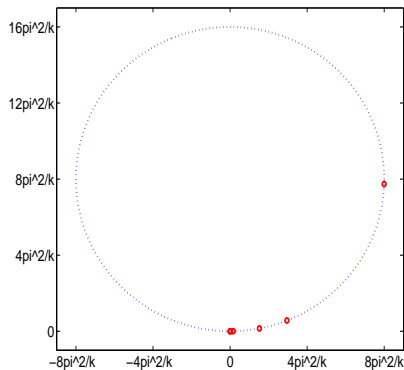
Largest phases.

Robin, $\tau=1$, unit ball

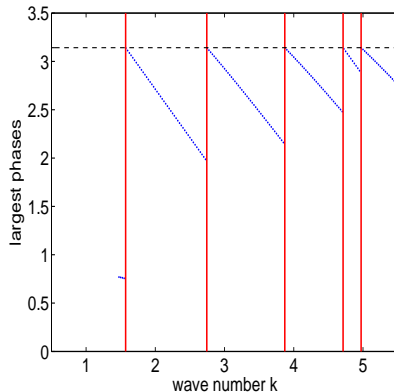
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.475



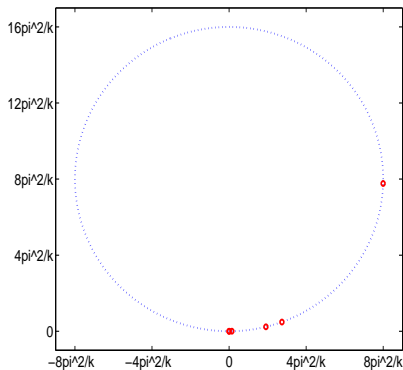
Largest phases.

Robin, $\tau=1$, unit ball

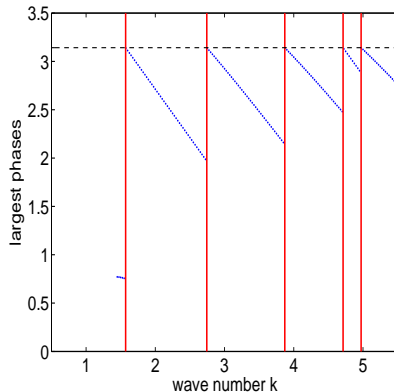
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.450



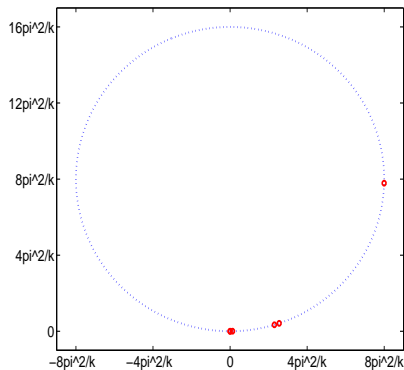
Largest phases.

Robin, $\tau=1$, unit ball

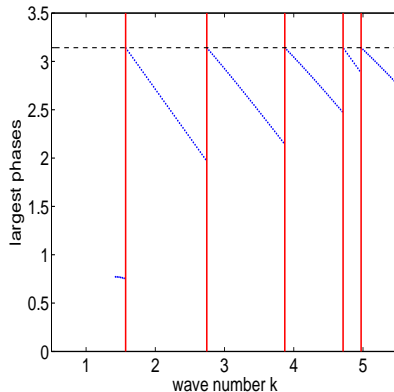
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.425



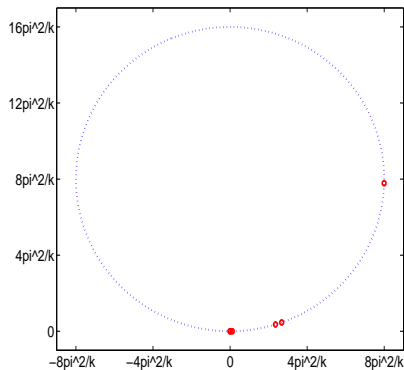
Largest phases.

Robin, $\tau=1$, unit ball

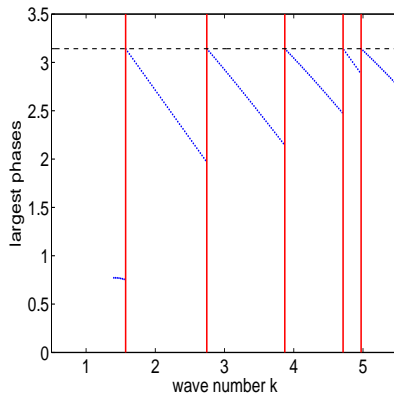
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.400



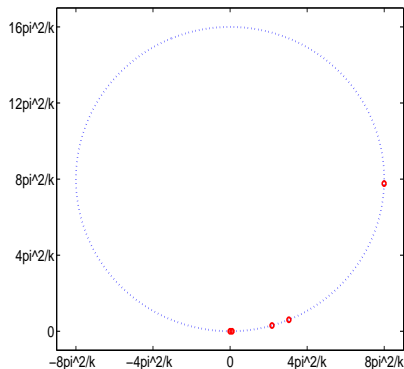
Largest phases.

Robin, $\tau=1$, unit ball

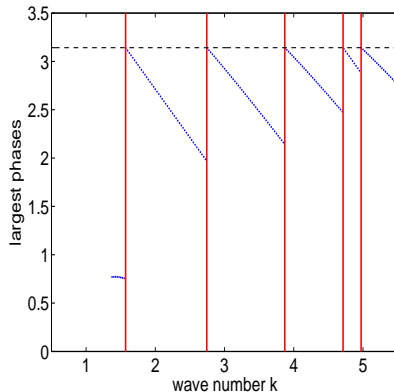
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.375

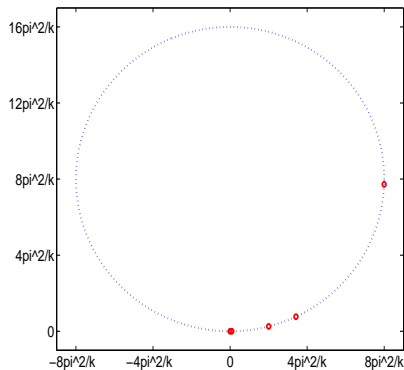


Largest phases.

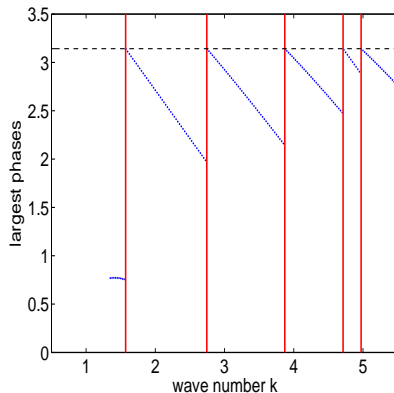
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

 $k=1.350$ 

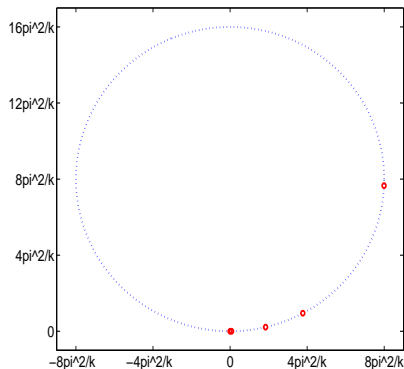
Largest phases.

Robin, $\tau=1$, unit ball

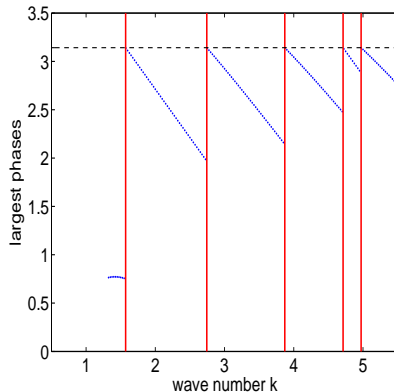
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.325



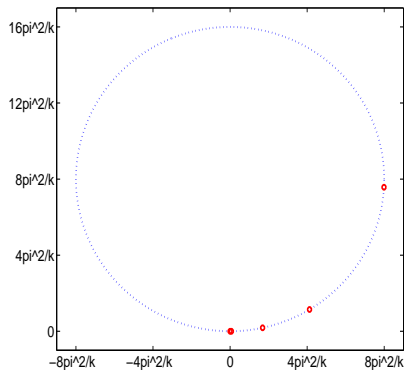
Largest phases.

Robin, $\tau=1$, unit ball

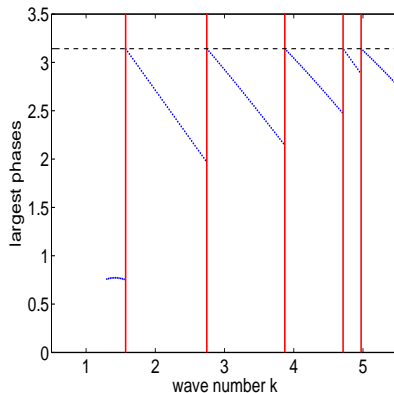
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.300



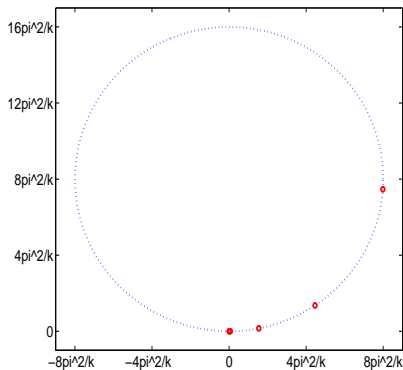
Largest phases.

Robin, $\tau=1$, unit ball

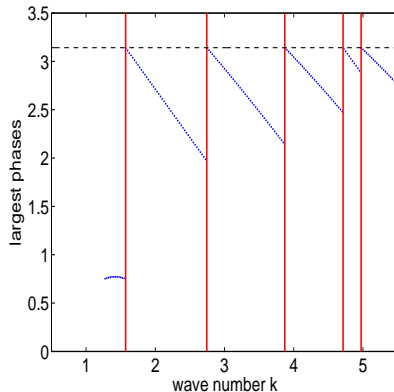
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.275



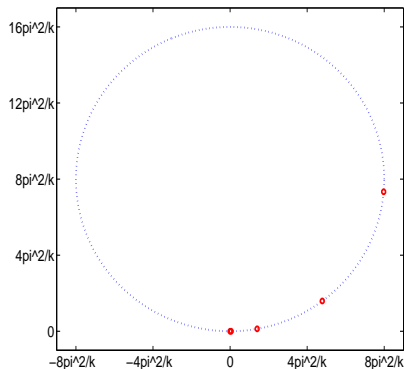
Largest phases.

Robin, $\tau=1$, unit ball

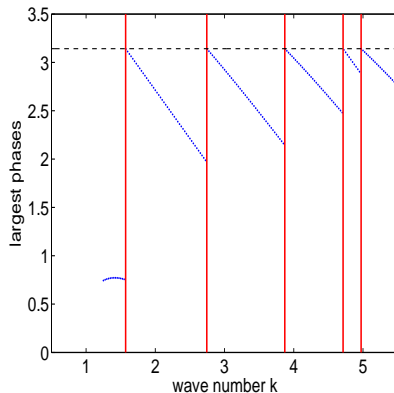
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.250



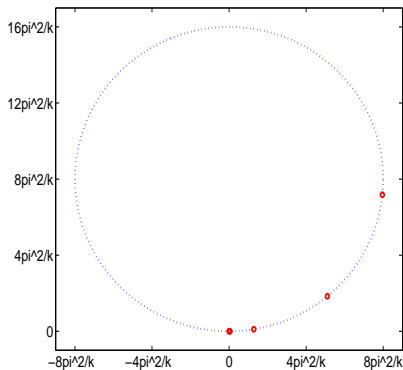
Largest phases.

Robin, $\tau=1$, unit ball

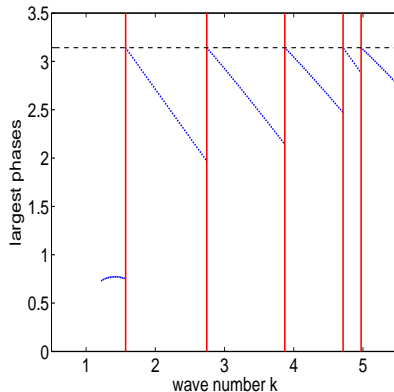
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.225



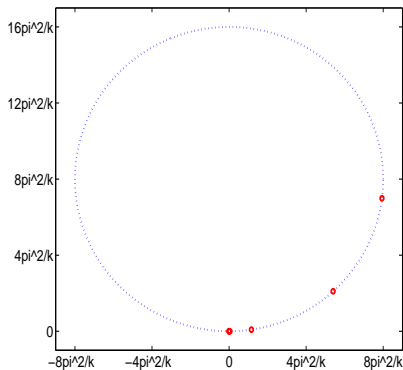
Largest phases.

Robin, $\tau=1$, unit ball

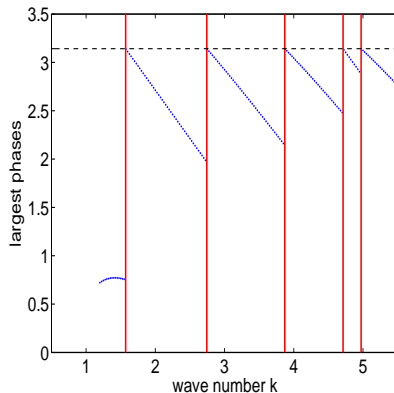
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.200



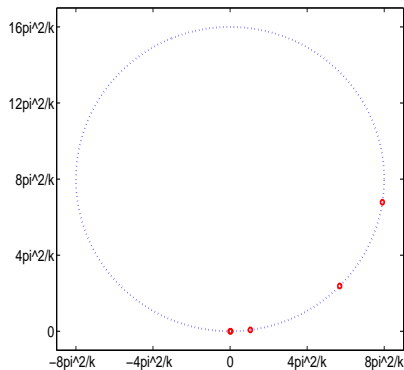
Largest phases.

Robin, $\tau=1$, unit ball

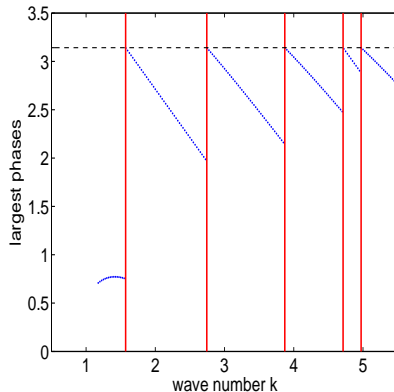
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.175



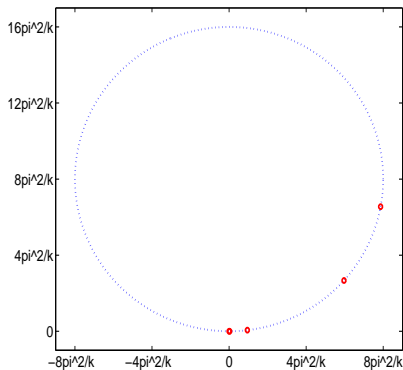
Largest phases.

Robin, $\tau=1$, unit ball

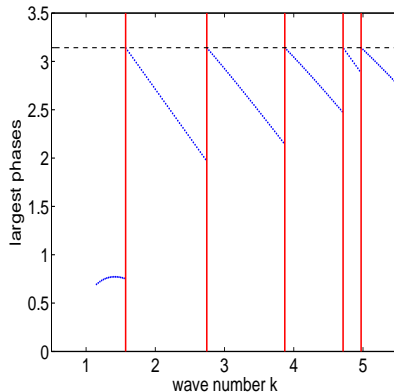
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.150



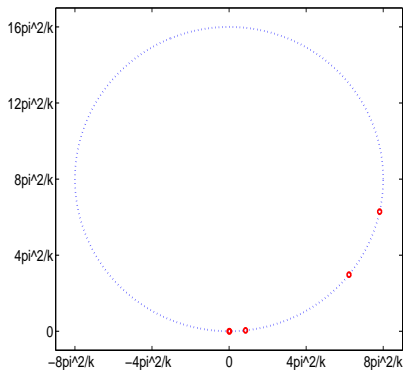
Largest phases.

Robin, $\tau=1$, unit ball

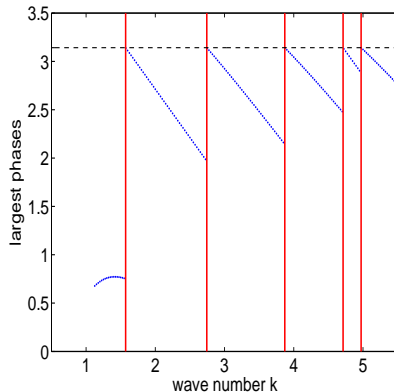
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.125



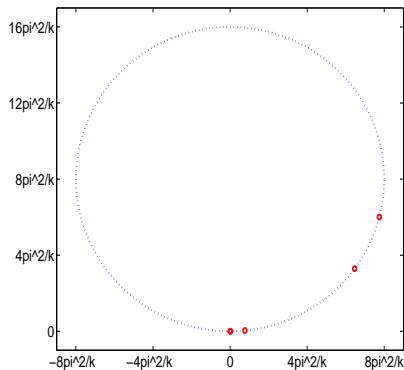
Largest phases.

Robin, $\tau=1$, unit ball

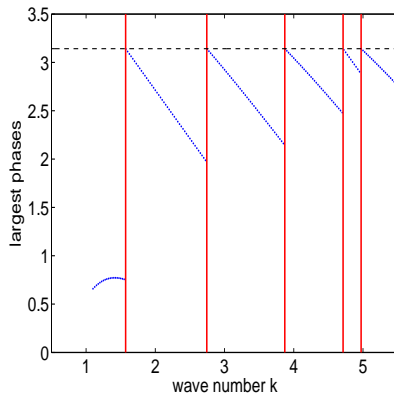
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.100



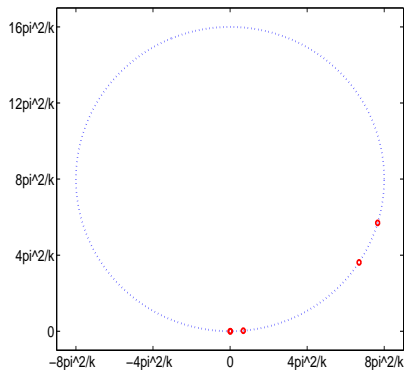
Largest phases.

Robin, $\tau=1$, unit ball

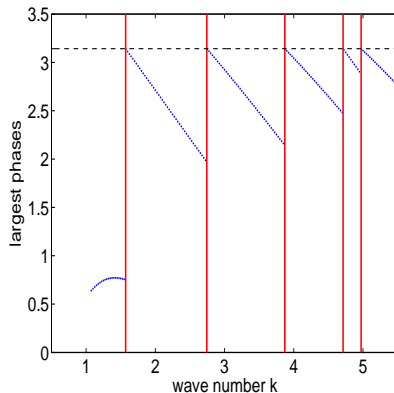
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.075



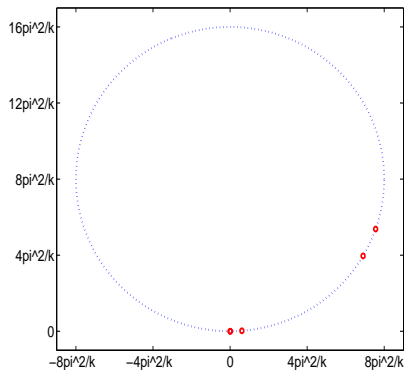
Largest phases.

Robin, $\tau=1$, unit ball

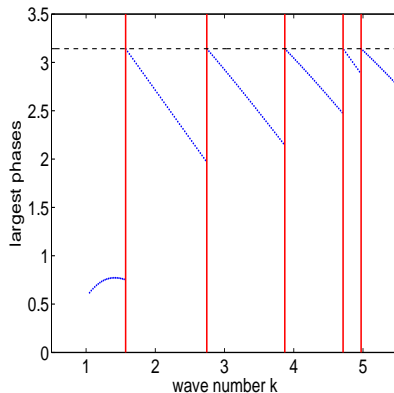
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.050



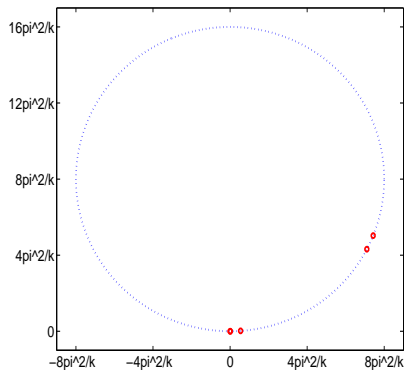
Largest phases.

Robin, $\tau=1$, unit ball

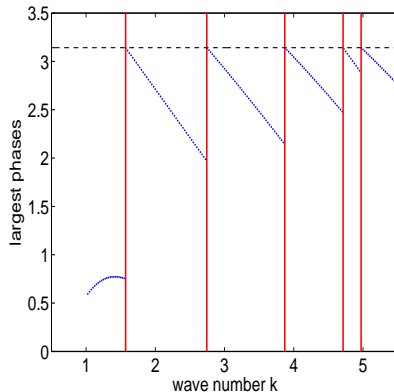
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.025



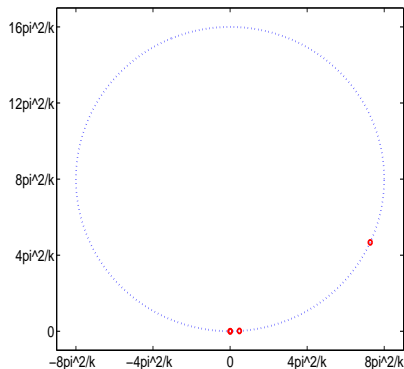
Largest phases.

Robin, $\tau=1$, unit ball

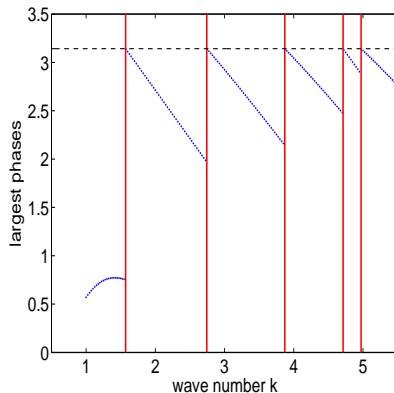
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=1.000

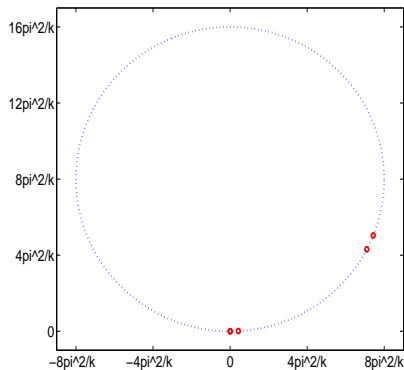


Largest phases.

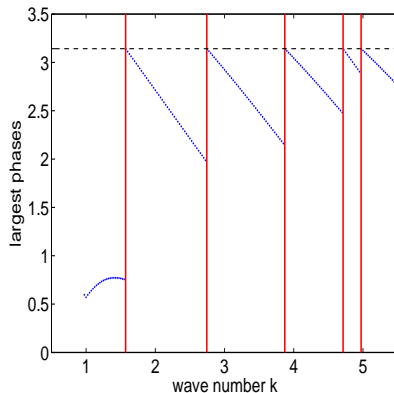
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

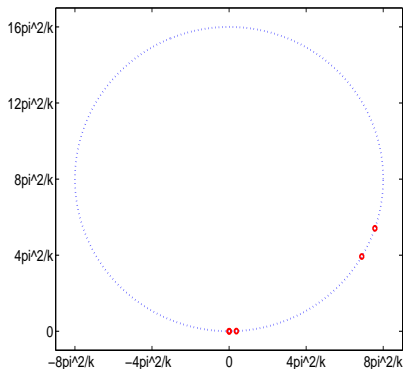
 $k=0.975$ 

Largest phases.

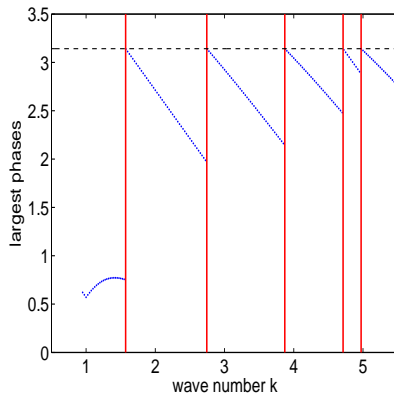
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

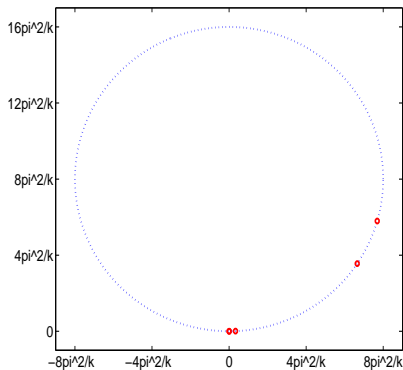
 $k=0.950$ 

Largest phases.

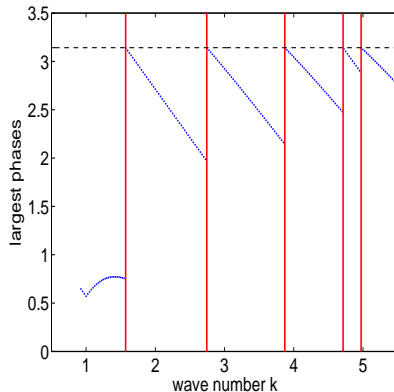
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

 $k=0.925$ 

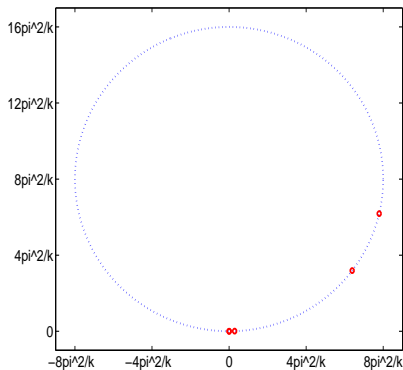
Largest phases.

Robin, $\tau=1$, unit ball

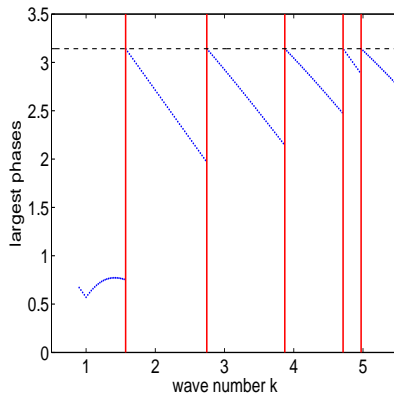
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=0.900

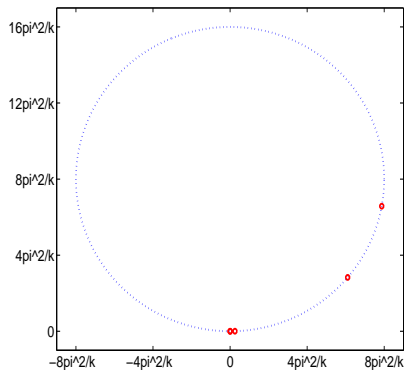


Largest phases.

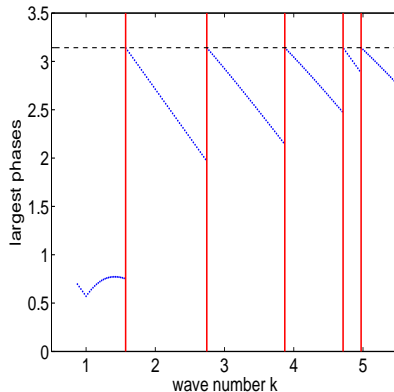
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

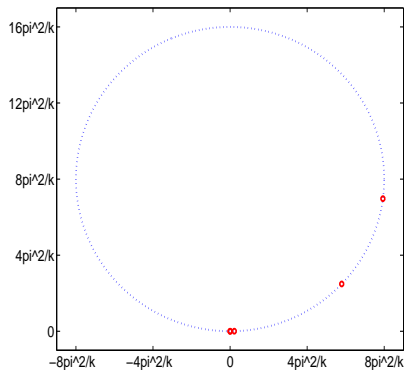
 $k=0.875$ 

Largest phases.

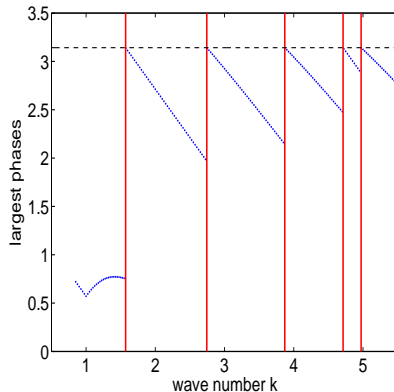
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

 $k=0.850$ 

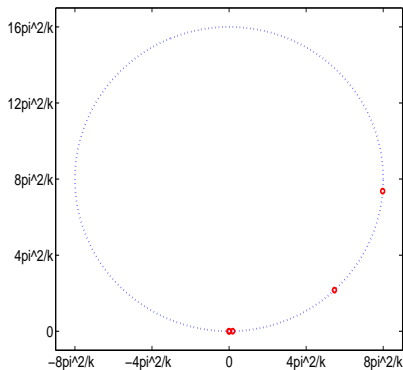
Largest phases.

Robin, $\tau=1$, unit ball

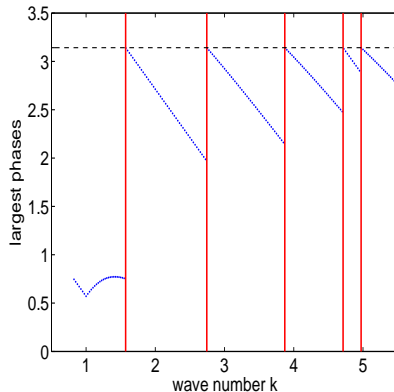
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=0.825



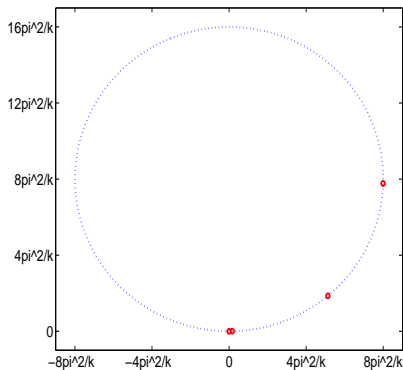
Largest phases.

Robin, $\tau=1$, unit ball

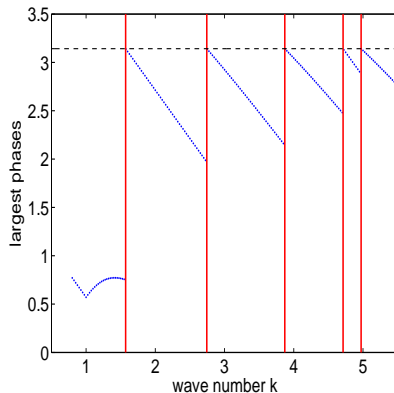
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=0.800

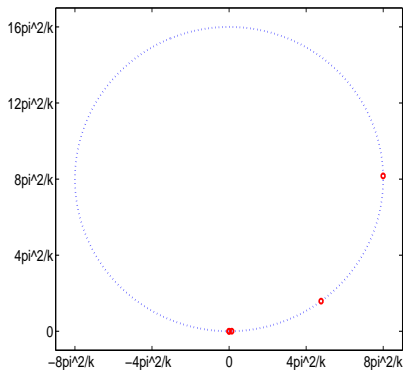


Largest phases.

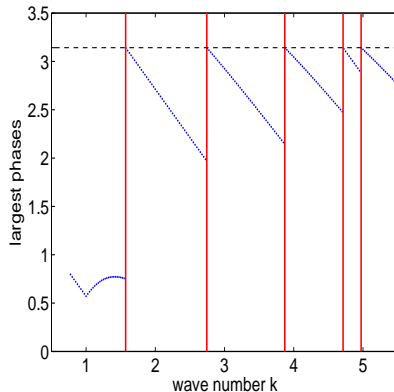
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

 $k=0.775$ 

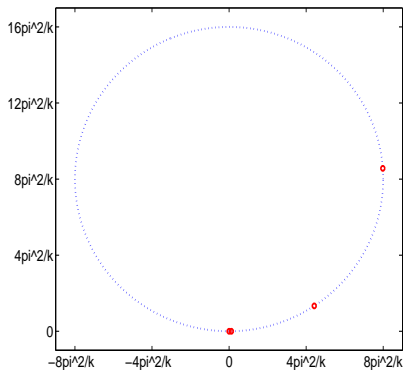
Largest phases.

Robin, $\tau=1$, unit ball

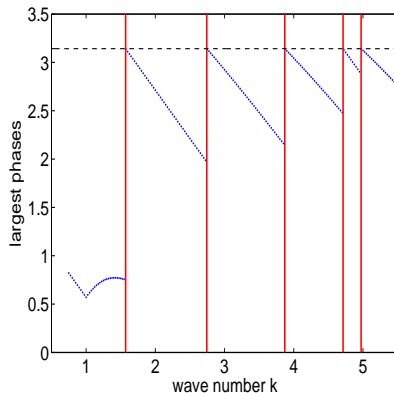
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=0.750



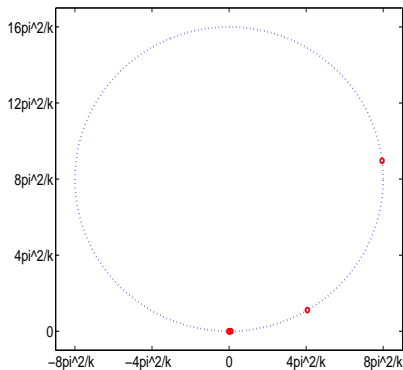
Largest phases.

Robin, $\tau=1$, unit ball

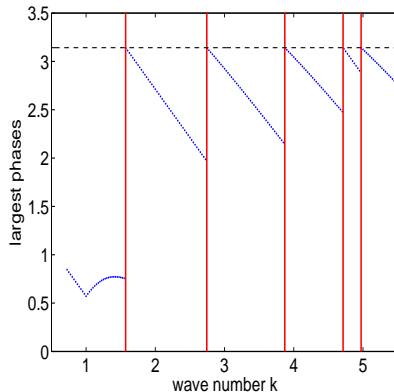
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=0.725



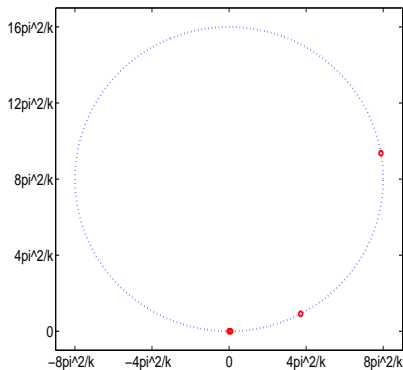
Largest phases.

Robin, $\tau=1$, unit ball

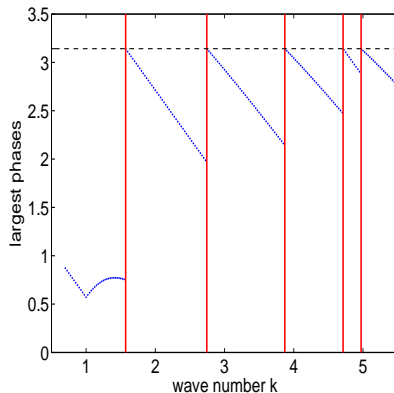
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=0.700



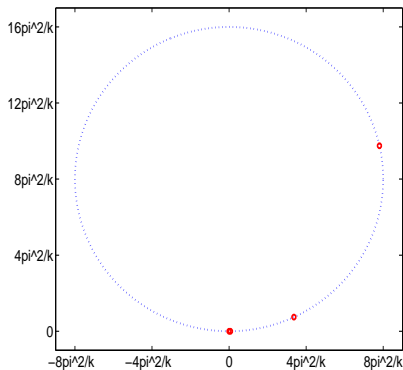
Largest phases.

Robin, $\tau=1$, unit ball

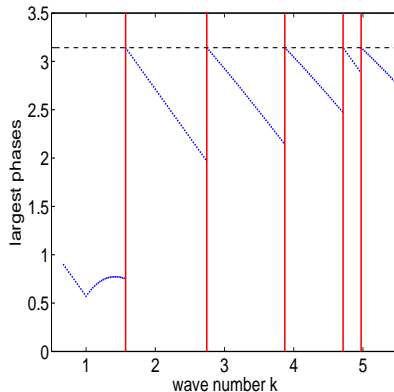
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=0.675



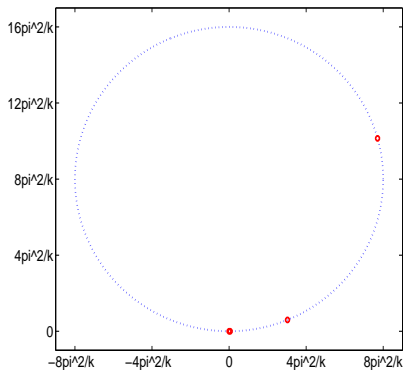
Largest phases.

Robin, $\tau=1$, unit ball

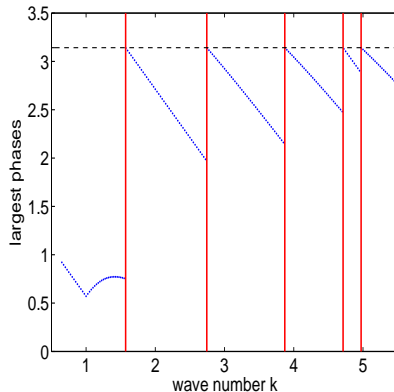
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=0.650



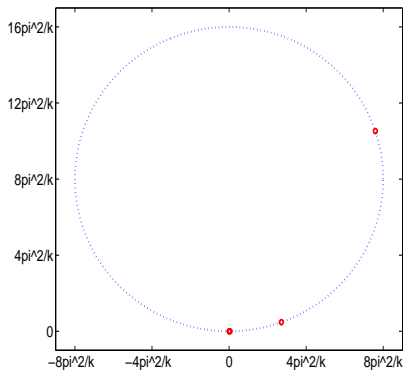
Largest phases.

Robin, $\tau=1$, unit ball

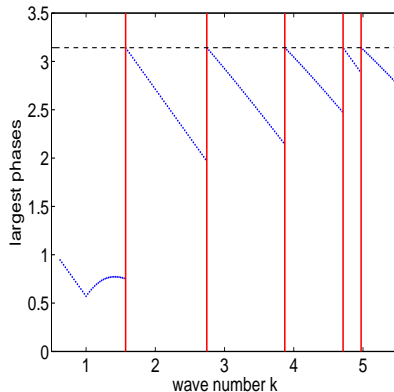
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=0.625



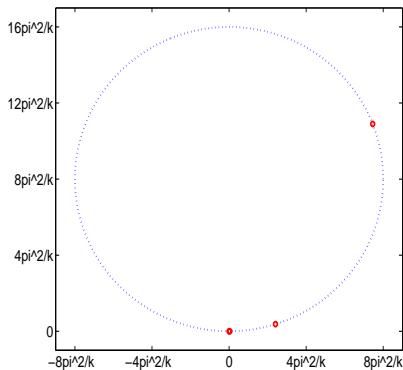
Largest phases.

Robin, $\tau=1$, unit ball

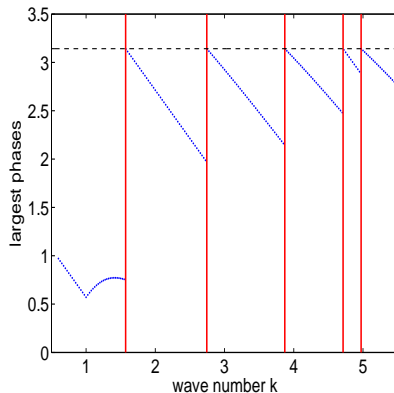
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=0.600



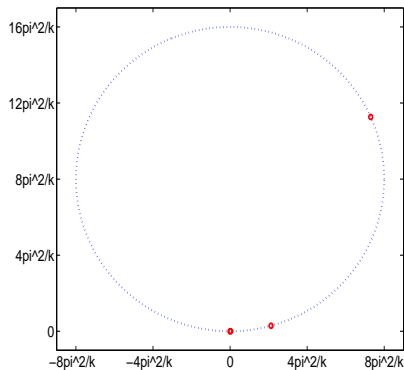
Largest phases.

Robin, $\tau=1$, unit ball

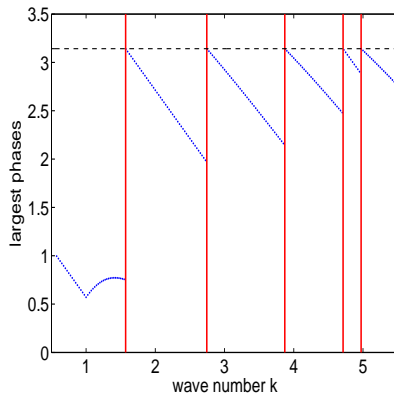
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=0.575



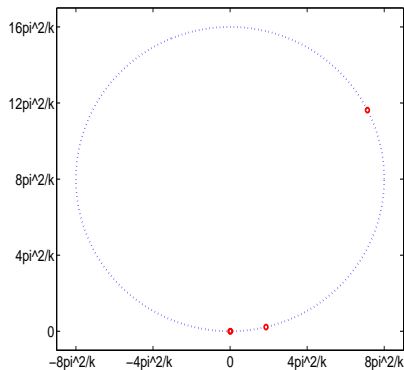
Largest phases.

Robin, $\tau=1$, unit ball

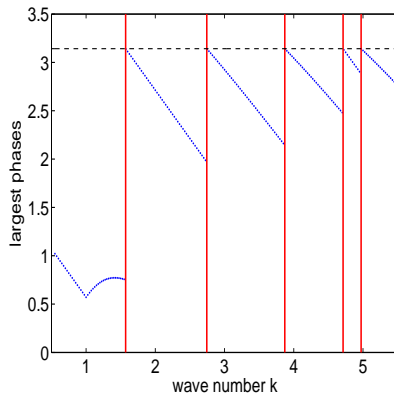
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=0.550

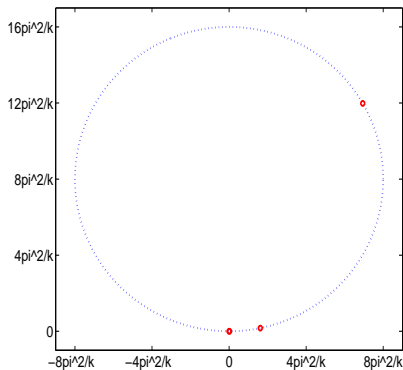


Largest phases.

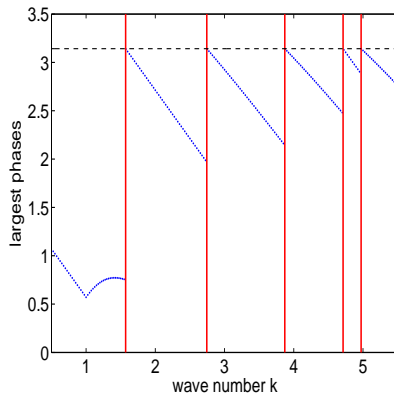
Robin, $\tau=1$, unit ball

Straightforward approximation of interior eigenvalues

Eigenvalues.

 $k=0.525$


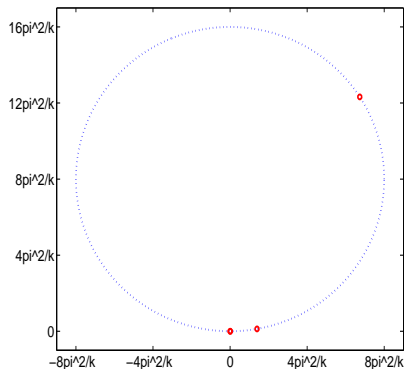
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 Robin, $\tau=1$, unit ball


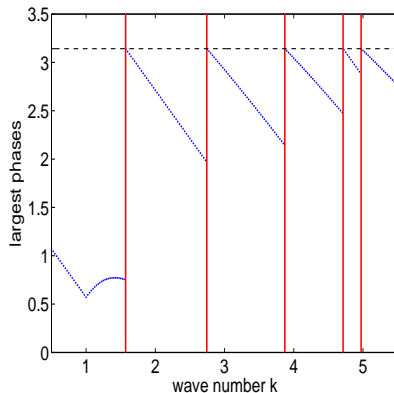
Straightforward approximation of interior eigenvalues

Eigenvalues.

k=0.500



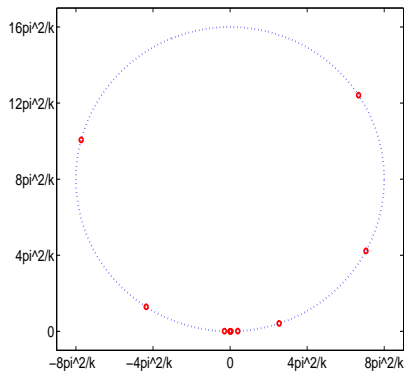
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Robin, $\tau=1$, unit ball

But may be inaccurate

Exact eigenvalues.

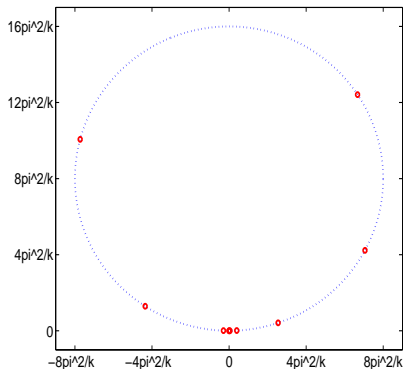
$k=5.000$



But may be inaccurate

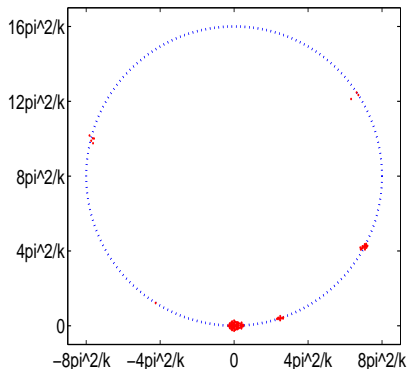
Exact eigenvalues.

k=5.000



Numerical approximations.

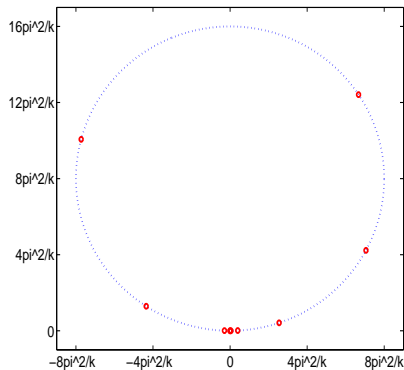
k=5.000, N=120, h=0.1



But may be inaccurate

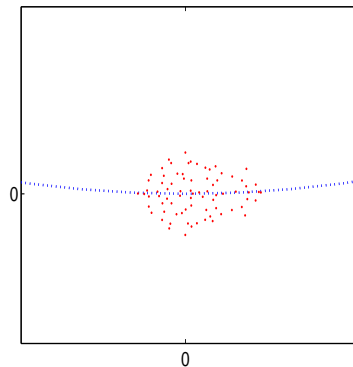
Exact eigenvalues.

k=5.000



Numerical approximations.

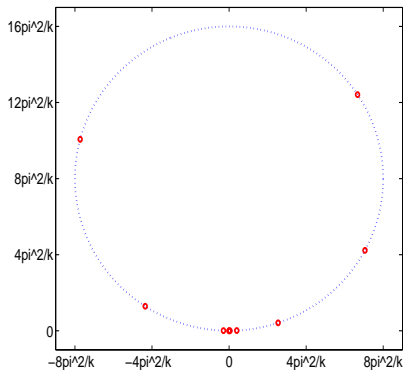
k=5.000, N=120, h=0.1



But may be inaccurate

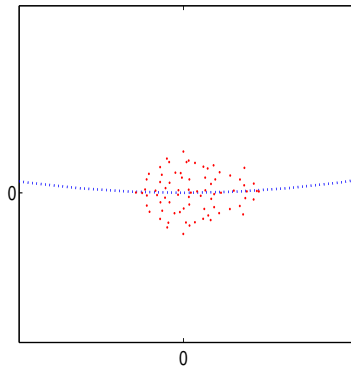
Exact eigenvalues.

k=5.000



Numerical approximations.

k=5.000, N=120, h=0.1



- Error estimate of approximations.
- Regularization.

Outline

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Discrete far field data

Two sources of error

$$Fg(\hat{x}) := \int_{\mathbb{S}^2} u^\infty(\hat{x}, \theta) g(\theta) \, dS(\theta), \quad \hat{x} \in \mathbb{S}^2,$$

Discrete far field data

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Inverse problem: $u_\delta^\infty(\theta_N^{(j)}, \theta_N^{(\ell)}) \rightarrow$ interior eigenvalues.

Regularity assumption on interpolation operators

- Pairwise different directions $\Theta_N := \{\theta_N^{(j)}\}_{j=1}^N \subset \mathbb{S}^2$ such that

$$h_N := \inf_{\theta \in \mathbb{S}^2} \{|\theta - \theta_N^{(j)}|, 1 \leq j \leq N\} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

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$$\mathcal{I}_N[g] = \sum_{j=1}^N [A_{\Phi_N}^{-1} (g(\theta_N^{(j)}))_{j=1}^N](j) \phi_N^{(j)}, \quad \mathcal{I}_N[g](\theta_N^{(j)}) = g(\theta_N^{(j)}) \quad \forall \theta_N^{(j)} \in \Theta_N.$$

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Assumption: for $\sigma = \sigma_{\mathcal{I}} \geq 0$ and $s > 1 - \sigma > 0$, there exists $C_{\mathcal{I}} = C_{\mathcal{I}}(s, \sigma) > 0$ such that

$$\begin{aligned} \|g - \mathcal{I}_N[g]\|_{L^2(\mathbb{S}^2)} &\leq C_{\mathcal{I}} h_N^s \|g\|_{H^{s+\sigma}(\mathbb{S}^2)}, \quad \forall g \in H^{s+\sigma}(\mathbb{S}^2), \quad \forall N \in \mathbb{N}, \\ \|\mathcal{I}_N[g]\|_{L^2(\mathbb{S}^2)} &\leq (1 + C_{\mathcal{I}}) \|g\|_{H^{s+\sigma}(\mathbb{S}^2)}. \end{aligned} \quad (\text{A1})$$

Discrete interpolation operators

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$$Q_N : \mathbb{C}^N \rightarrow L^2(\mathbb{S}^2), \quad Q_N g_N = \sum_{j=1}^N [A_{\Phi_N}^{-1} g_N](j) \phi_N^{(j)} \quad \forall g_N \in \mathbb{C}^N.$$

Property: $\|Q_N\|_{\mathbb{C}^N \rightarrow L^2(\mathbb{S}^2)}^2 \leq C_Q(N)$. $C_Q(N)$ is uniformly bounded in N for interpolations using indicator functions or piecewise polynomial and globally continuous functions.

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Theorem

If $\{\mathcal{I}_N\}_{N \in \mathbb{N}}$ satisfies (A1) for $\sigma = \sigma_{\mathcal{I}} \geq 0$ and $s > 1 - \sigma > 0$ and if $\|\mathbb{F}_N - \mathbb{F}_N^\delta\|_2 \leq \delta$, then

$$\|F - F_N^\delta\|_{L^2(\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2)} \leq Ch_N^s + C_Q(N)\delta, \quad N \in \mathbb{N},$$

with C independent of $N \in \mathbb{N}$.

Idea of proof:

- $\|F - F_N\|_{L^2(\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2)} = \|F - \mathcal{I}_N F \mathcal{I}_N^*\|_{L^2(\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2)} \leq Ch_N^s$ due to (A1).
- $\|F_N - F_N^\delta\|_{L^2(\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2)} = \|Q_N(\mathbb{F}_N - \mathbb{F}_N^\delta)Q_N^*\|_{L^2(\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2)} \leq C_Q(N)\delta$.

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Approximation of eigenvalues

Link condition between directions Θ_N and basis functions Φ_N :

- Vectors $g_{\Theta_N} := (g(\theta_N^{(j)}))_{j=1}^N \in \mathbb{C}^N$, $g_{\Phi_N} = ((g, \phi_N^{(j)})_{L^2(\mathbb{S}^2)})_{j=1}^N$.

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- Weights $w_N(j) := \|\phi_N^{(j)}\|_{L^2(\mathbb{S})}$ and weight matrix $\mathbb{W}_N = \text{diag}(w_N(j))_{j=1}^N \in \mathbb{R}^{N \times N}$.

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Lemma

If (A2) is satisfied, then all eigenvalues of $\mathbb{W}_N \mathbb{F}_N \mathbb{W}_N$ and $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ are eigenvalues of F_N and F_N^δ respectively, and any additional eigenvalue of F_N and F_N^δ must vanish.

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Theorem

F has eigenvalues λ_j , $j \in \mathbb{N}$. For all eigenvalues λ_ℓ^N of F_N^δ it holds that

$$\min_{j \in \mathbb{N}} |\lambda_\ell^N - \lambda_j| \leq \varepsilon := \|F_N^\delta - F\|, \quad \ell = 1, \dots, \mathcal{J}(N).$$

If a finite subset $\sigma_1 \subset \sigma(F)$ satisfies that $\text{dist}_H(\sigma_1, \sigma(F) \setminus \sigma_1) > 2\varepsilon$, then the ε -neighborhood of σ_1 contains at least one eigenvalue of F_N^δ .

Proof: Bauer-Fike theorem.

Approximation of eigenvalues phases

Theorem

Assume that \mathcal{I}_N , Φ_N and Θ_N satisfy (A1), (A2). Consider a sequence of perturbed far field matrices $\mathbb{F}_N^{\delta_N}$ with noise level δ_N and assume in addition that $C_Q(N)\delta_N \rightarrow 0$ as $N \rightarrow \infty$.

- 1 If $\lambda_j \neq 0$ is a non-zero eigenvalue of F , then there exists $\{j'(N)\}_{N \in \mathbb{N}}$ such that

$$|\lambda_{j'(N)}^N - \lambda_j| \leq \|F_N^\delta - F\| =: \varepsilon_N \leq Ch_N^s + C_Q(N)\delta_N \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

The sum of the dimensions of the eigenspaces corresponding to all eigenvalues of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ contained in $B(\lambda_j, \varepsilon_N)$ equals the multiplicity of λ_j .

- 2 If $|\lambda_j| = r_j > \varepsilon_N = \|F_N^\delta - F\|$, then the phase of any eigenvalue $\lambda_{j'}^N \in B(\lambda_j, \varepsilon_N)$ of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ satisfies

$$|\vartheta_{j'}^N - \vartheta_j| < \arcsin(\varepsilon_N / r_j) \leq \frac{\pi \varepsilon_N}{2r_j} \leq \frac{\pi}{2r_j} (Ch_N^s + C_Q(N)\delta_N) \xrightarrow{N \rightarrow \infty} 0.$$

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Phases belong to $(0, \pi)$

- An eigenvalue λ_ℓ^N of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ satisfies $|\lambda_\ell^N| > 4\pi (\varepsilon_N/k)^{1/2} + \varepsilon_N$.

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- $\lambda_j = a + ib$ lies on the circle $\{|z - 8\pi^2 i/k| = 8\pi^2/k\}$

$$a^2 + \left(b - \frac{8\pi^2}{k}\right)^2 = \left(\frac{8\pi^2}{k}\right)^2, \quad \text{that is, } |\lambda_j|^2 = a^2 + b^2 = \frac{16\pi^2}{k}b.$$

So $b = \text{Im}(\lambda_j) > \varepsilon_N$. Hence $\text{Im}(\lambda_\ell^N) > 0$ and $\vartheta_\ell^N \in (0, \pi)$.

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- There exists an eigenvalue λ_j of F such that $|\lambda_\ell^N - \lambda_j| \leq \varepsilon_N$. Then $r_j = |\lambda_j| > 4\pi (\varepsilon_N/k)^{1/2}$.
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$$a^2 + \left(b - \frac{8\pi^2}{k}\right)^2 = \left(\frac{8\pi^2}{k}\right)^2, \quad \text{that is, } |\lambda_j|^2 = a^2 + b^2 = \frac{16\pi^2}{k}b.$$

So $b = \text{Im}(\lambda_j) > \varepsilon_N$. Hence $\text{Im}(\lambda_\ell^N) > 0$ and $\vartheta_\ell^N \in (0, \pi)$.

- Assume that N is so large that $\varepsilon_N^{1/2} < 4\pi/k^{1/2}$, then $r_j > 4\pi (\varepsilon_N/k)^{1/2} > \varepsilon_N$ and

$$|\vartheta_\ell^N - \vartheta_j| \leq \frac{\pi \varepsilon_N}{2 r_j} \leq \frac{1}{8} (k \varepsilon_N)^{1/2}.$$

Phases belong to $(0, \pi)$

- An eigenvalue λ_ℓ^N of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ satisfies $|\lambda_\ell^N| > 4\pi (\varepsilon_N/k)^{1/2} + \varepsilon_N$.
- There exists an eigenvalue λ_j of F such that $|\lambda_\ell^N - \lambda_j| \leq \varepsilon_N$. Then $r_j = |\lambda_j| > 4\pi (\varepsilon_N/k)^{1/2}$.
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Largest regularized discrete phase

$$\vartheta^*(k, N) = \max \left\{ \vartheta_j^N, \lambda_j^N \in \sigma(\mathbb{W}_N \mathbb{F}_N^{\delta N}(k) \mathbb{W}_N), |\lambda_j^N| > 4\pi (\varepsilon_N/k)^{1/2} + \varepsilon_N \right\}.$$

Regularization

Assumptions:

- Link between directions Θ_N and basis functions Φ_N :

$$A_{\Phi_N}^* (\mathbb{W}_N^2 g_{\Theta_N}) = g_{\Phi_N} \text{ for all } g \in \text{span}\{\phi_N^{(1)}, \dots, \phi_N^{(j)}\}.$$

- Density of discrete wave numbers: $K = \{k_i\}_{i \in \mathbb{N}} \subset [k_{\min}, k_{\max}]$ such that $\overline{K} = [k_{\min}, k_{\max}]$.
- Uniform noise level: $\max_{k \in K} \|\mathbb{F}_N^{\delta_N}(k) - \mathbb{F}_N(k)\|_2 \leq \delta_N, N \in \mathbb{N}$.
- $C_Q(N)\delta_N \rightarrow 0$ as $N \rightarrow \infty$.

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Theorem

Under above assumptions, if $\{k_i\}_{i \in \mathbb{N}} \subset K$, $k_i \neq k \in (k_{\min}, k_{\max})$, $k_i \rightarrow k$ as $i \rightarrow \infty$, and denote the eigenvalue of $F(k_i)$ with largest phase by $\lambda^*(k_i)$ and its phase by $\vartheta^*(k_i)$, then

- If $\vartheta^*(k_i, N_i) \rightarrow \pi$ for any sequence $\{N_i\}_{i \geq i_0} \subset \mathbb{N}$ with $N_i \rightarrow \infty$ as $i \rightarrow \infty$, then $\vartheta^*(k_i) \rightarrow \pi$ as $i \rightarrow \infty$ and k^2 is an interior eigenvalue of D .
- If $\vartheta^*(k_i) \rightarrow \pi$ there is $i_0 \in \mathbb{N}$ and $\{N_i\}_{i \geq i_0} \subset \mathbb{N}$ with $N_i \rightarrow \infty$ as $i \rightarrow \infty$ such that

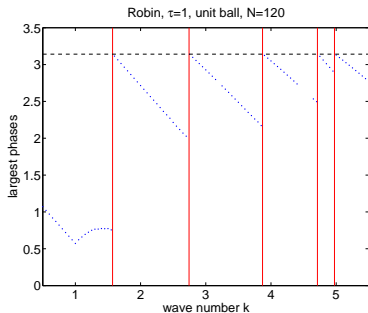
$$4\pi \left(\frac{\varepsilon_{N_i}}{k_i} \right)^{1/2} + 2\varepsilon_{N_i} \leq \min \{ |\lambda^*(k_i)|, \text{dist} [\lambda^*(k_i), \sigma(F(k_i)) \setminus \{\lambda^*(k_i)\}] \}, \quad i \geq i_0,$$

and for any such sequence $\{N_i\}_{i \geq i_0}$ it holds that $\vartheta^*(k_i, N_i) \rightarrow \pi$ as $i \rightarrow \infty$.

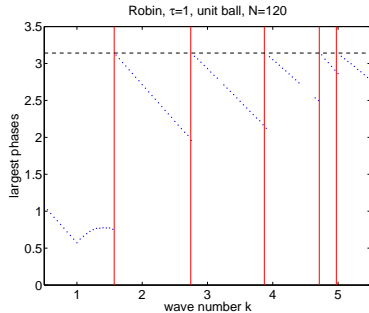
Outline

- 1 Motivation
- 2 Error estimate
 - Approximation of far field operators
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- 3 Regularization
- 4 Numerical Examples**
- 5 Conclusion

Scattering from a unit ball with Robin b.c.

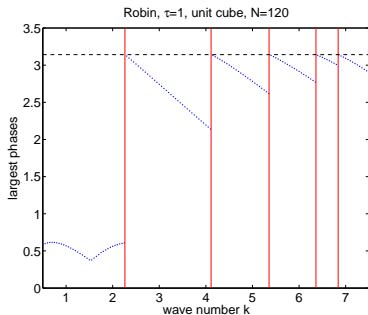


Largest regularised discrete phases, no artificial noise.

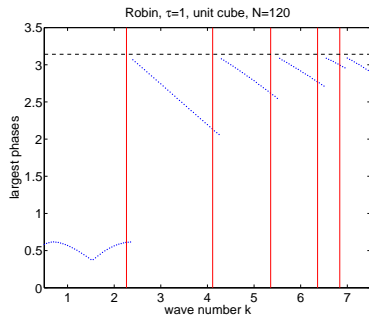


Largest regularised discrete phases, 5% artificial noise.

Scattering from a unit cube with Robin b.c.



Largest regularised discrete phases, no artificial noise.



Largest regularised discrete phases, 5% artificial noise.

Convergence

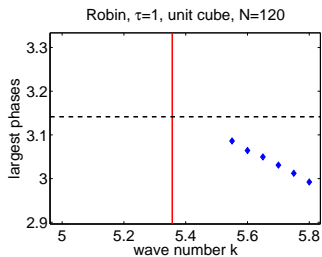
Roots of interior eigenvalues (ball)	1st	2nd	3rd	4th	5th
Exact value (4 digits)	1.571	2.743	3.870	4.712	4.973
$N = 48, \Delta k = 0.2$, no artificial noise	1.70	2.90	3.90	4.90	5.10
$N = 80, \Delta k = 0.1$, no artificial noise	1.60	2.80	3.90	4.80	5.00
$N = 120, \Delta k = 0.05$, no artificial noise	1.60	2.75	3.90	4.75	5.00
$N = 120, \Delta k = 0.05$, 1% artificial noise	1.60	2.80	3.90	4.75	5.00
$N = 120, \Delta k = 0.05$, 5% artificial noise	1.60	2.80	3.95	4.75	5.05

Table: Estimates of the square roots of the first five Robin eigenvalues of $-\Delta$ in the unit ball from far field data for different levels of artificial additive noise.

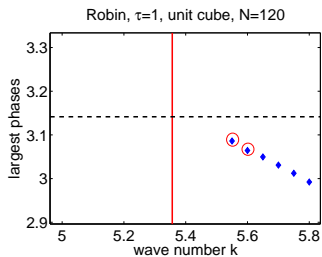
Roots of interior eigenvalues (cube)	1st	2nd	3rd	4th	5th
Exact value (4 digits)	2.263	4.112	5.357	6.362	6.839
$N = 48, \Delta k = 0.2$, no artificial noise	2.30	4.30	5.50	6.50	6.90
$N = 80, \Delta k = 0.1$, no artificial noise	2.30	4.20	5.40	6.40	6.90
$N = 120, \Delta k = 0.05$, no artificial noise	2.30	4.15	5.40	6.40	6.85
$N = 120, \Delta k = 0.05$, 1% artificial noise	2.30	4.15	5.40	6.40	6.90
$N = 120, \Delta k = 0.05$, 5% artificial noise	2.35	4.20	5.45	6.50	6.95

Table: Estimates of the square roots of the first five Robin eigenvalues of $-\Delta$ in the cube $(0, 1)^3$ from far field data for different levels of artificial additive noise.

Better approximation using extrapolation

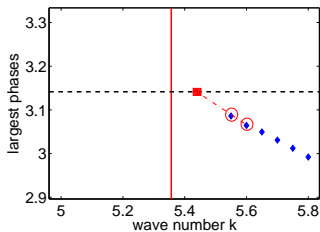


Better approximation using extrapolation



Better approximation using extrapolation

Robin, $\tau=1$, unit cube, $N=120$

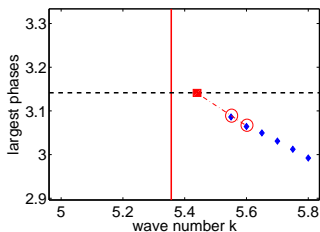


Extrapolation

$$k_{\text{appr}}^{\text{rob}} = k_* + \frac{\pi - \vartheta^*(k_*, N)}{\vartheta^*(k_* + \Delta k, N) - \vartheta^*(k_*, N)} \Delta k$$

Better approximation using extrapolation

Robin, $\tau=1$, unit cube, $N=120$



Extrapolation

$$k_{\text{appr}}^{\text{rob}} = k_* + \frac{\pi - \vartheta^*(k_*, N)}{\vartheta^*(k_* + \Delta k, N) - \vartheta^*(k_*, N)} \Delta k$$

Roots of interior eigenvalues	1st	2nd	3rd	4th	5th
Ball – Exact value (4 digits)	1.571	2.743	3.870	4.712	4.973
Before extrapolation	1.600	2.800	3.950	4.750	5.050
Extrapolation	1.575	2.763	3.884	4.719	4.988
Relative error	0.25%	0.73%	0.36%	0.14%	0.30%
Cube – Exact value (4 digits)	2.263	4.112	5.357	6.362	6.839
Before extrapolation	2.350	4.200	5.450	6.500	6.950
Extrapolation	2.267	4.133	5.307	6.406	6.827
Relative error	0.17%	0.51%	0.93%	0.69%	0.17%

Table: Estimates of the square roots of the first five Robin eigenvalues by the extrapolation procedure. The relative artificial noise level equals 5%. Fixed parameters are $N = 120$, $\Delta k = 0.05$

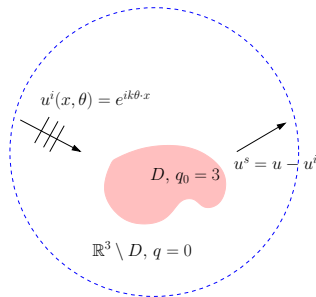
Approximating interior transmission eigenvalues of penetrable media

Scattering problem from an inhomogeneous medium

$$\Delta u + k^2(1+q)u = 0 \quad \text{in } \mathbb{R}^3,$$

Interior transmission eigenvalue problem $(v, w) := (u, u^i)$

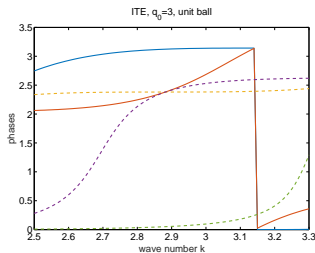
$$\begin{cases} \Delta v + k^2(1+q)v = 0 & \text{in } D, \\ \Delta w + k^2w = 0 & \text{in } D, \\ v = w & \text{on } \partial D, \\ \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} & \text{on } \partial D. \end{cases}$$



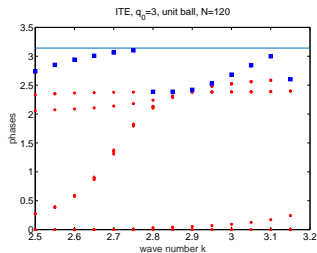
Inhomogeneous medium with contrast q_0 supported in D .

Roots of interior eigenvalues (ball)	1st		2nd	3rd	4th
Exact value (4 digits)	3.142		3.692	4.262	4.832
Estimate, no artificial noise	3.00		3.65	4.25	4.85
Extrapolation, no artificial noise	3.116		3.697	4.268	4.857
Relative error (extrapolation)	0.81%		0.14%	0.14%	0.52%
Estimate, 5% artificial noise	2.75	3.10	3.65	4.25	4.80
Extrapolation, 5% artificial noise	2.803	3.144	3.698	4.268	4.845
Relative error (extrapolation)	10.78%	0.08%	0.16%	0.14%	0.27%

Table: Estimates of the square roots of the first four interior transmission eigenvalues of the unit ball for various noise levels. Fixed parameters are $N = 120$, $\Delta k = 0.05$, $q_0 = 3$.



Exact phases of F .



Computed phases, 5% art. noise.

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Conclusion

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- Suggest a regularized inversion method to approximate interior eigenvalues of domains.
- Illustrate the feasibility and robustness with some examples.

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Thank you for your attention!