The method of quasi-reversibility for time-dependent wave equation. Application to the inverse obstacle problem

Laurent Bourgeois

(cooperation with Jérémi Dardé,

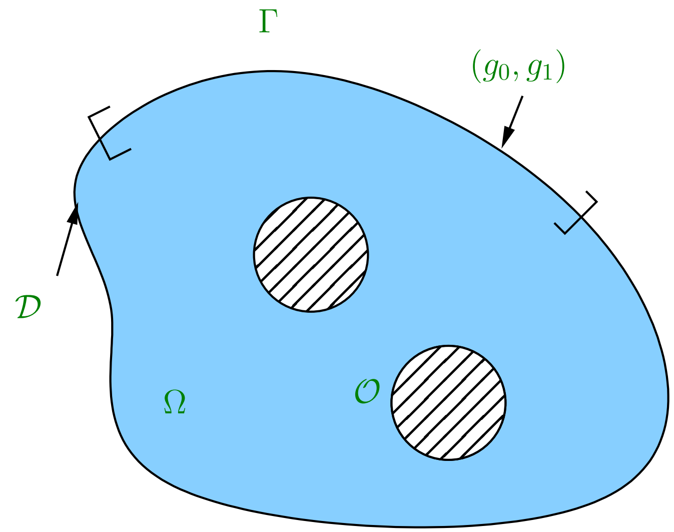
Lucas Franceschini and Eliane Bécache)

Laboratoire POEMS, ENSTA-ParisTech

IWAP, Bremen, 7/4/2015

Inverse obstacle problem (Dirichlet)

- $\mathcal{O} \Subset \mathcal{D}$ (bounded) $\subset \mathbb{R}^d$
- $\Omega(\mathcal{O}) = \mathcal{D} \setminus \overline{\mathcal{O}}$ connected
- $\Gamma \subset \partial\mathcal{D}$ (open Γ s.t. $|\Gamma| > 0$)



For (g_0, g_1), find $\mathcal{O}$ s.t. for some $u \in H^1(Q)$ with $Q = \Omega \times (0,T)$

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times (0,T) \\ u = g_0 & \text{on } \Gamma \times (0,T) \\ \partial_\nu u = g_1 & \text{on } \Gamma \times (0,T) \\ u = 0 & \text{on } \partial\mathcal{O} \times (0,T) \\ (u, \partial_t u) = (0,0) & \text{on } \Omega \times \{0\} \end{cases}$$

Remark :
$H^1(Q) = L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) = H^{1,1}(Q)$

$\rightarrow$ Boundary and Initial Conditions are well-defined

Outline of the talk

State of the art : no identified contribution for one single measurement, partial lateral Cauchy data, finite time domain...

1. Unique continuation for the wave equation
2. The method of quasi-reversibility
3. The "exterior approach" for the inverse obstacle problem
4. Numerical experiments in 1D
5. Conclusion and perspectives

Geodesic distance

We consider a connected open domain $\Omega \subset \mathbb{R}^d$ and $\Gamma \subset \partial\Omega$

Geodesic distance:

$$d_\Omega(x, y) = \inf\{|g|,\ g : [0, 1] \to \Omega,\ g(0) = x,\ g(1) = y\},$$

g: continuous path in Ω of length $|g|$

Remark: for bounded Ω we may have

$$D(x_0, \Omega) := \sup_{x \in \Omega} d_\Omega(x_0, x) = +\infty$$

but $D(x_0, \Omega)$ is finite if Ω is Lipschitz.

For Lipschitz Ω, we define

$$D(\Gamma, \Omega) = \sup_{x \in \Omega} \inf_{x_0 \in \Gamma} d_\Omega(x_0, x) < +\infty$$

The wave equation with lateral Cauchy data

Notation : $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$, $T_0 = D(\Gamma, \Omega)$

For (g_0, g_1), find $u \in H^1(Q)$ such that

$$\begin{cases} \partial_t^2 u - \Delta u & = & 0 & \text{in } Q \\ u & = & g_0 & \text{on } \Sigma \\ \partial_\nu u & = & g_1 & \text{on } \Sigma \end{cases}$$

Uniqueness theorem: assume that $T > 2T_0$ and $(g_0, g_1) = (0,0)$, then $u = 0$ in $\Omega \times (T_0, T - T_0)$.

Proof: Holmgren's theorem + sequence of balls following (Robbiano, 91)

Ill-posed problem: neither uniqueness, nor existence
$\longrightarrow$ a regularization is required: method of quasi-reversibility

Classical method of quasi-reversibility

Original idea of Lattès & Lions (67) : for $\varepsilon > 0$, solve

$$(P_\varepsilon) \qquad \inf_{u \in K} J_\varepsilon(u)$$

with

$$J_\varepsilon(u) = \int_Q |\partial_t^2 u - \Delta u|^2 \, \mathrm{d}x\mathrm{d}t + \varepsilon \, ||u||^2_{H^{2,2}(Q)}$$

$$K = \{u \in H^{2,2}(Q), \, u|_\Sigma = g_0, \, \partial_\nu u|_\Sigma = g_1\}$$

$$H^{2,2}(Q) = L^2(0,T; H^2(\Omega)) \cap H^2(0,T; L^2(\Omega))$$

$\longrightarrow$ optimality implies a weak formulation $\longrightarrow$ Finite Element Method

Drawbacks :

- regularity $H^{2,2}(Q)$ is required for the exact solution
- a fourth-order problem: Hermite type finite elements

Mixed formulation

Open domains of ∂Q: $\Sigma = \Gamma \times (0,T)$, $\Sigma' = \partial Q \setminus \overline{\Sigma}$

Functional spaces in Q: $V_{g_0} = \{u \in H^1(Q),\ u|_\Sigma = g_0\}$, $V_0 = V_{g_0}$ for $g_0 = 0$, $V_0' = \{\lambda \in H^1(Q),\ \lambda|_{\Sigma'} = 0\}$

Mixed Q.R. formulation: for $(g_0, g_1) \in H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma)$, find $(u_\varepsilon, \lambda_\varepsilon) \in V_{g_0} \times V_0'$ such that

$$\left\{\begin{array}{c} \displaystyle\int_Q -\partial_t v\, \partial_t \lambda_\varepsilon \,\mathrm{d}x\mathrm{d}t + \varepsilon \int_Q \partial_t u_\varepsilon\, \partial_t v \,\mathrm{d}x\mathrm{d}t \\ \displaystyle + \int_Q \nabla v \cdot \nabla \lambda_\varepsilon \,\mathrm{d}x\mathrm{d}t + \varepsilon \int_Q \nabla u_\varepsilon \cdot \nabla v \,\mathrm{d}x\mathrm{d}t = 0, \quad \forall v \in V_0 \\ \displaystyle - \int_Q \partial_t u_\varepsilon\, \partial_t \mu \,\mathrm{d}x\mathrm{d}t - \int_Q \partial_t \lambda_\varepsilon\, \partial_t \mu \,\mathrm{d}x\mathrm{d}t + \int_Q \nabla u_\varepsilon \cdot \nabla \mu \,\mathrm{d}x\mathrm{d}t \\ \displaystyle - \int_Q \nabla \lambda_\varepsilon \cdot \nabla \mu \,\mathrm{d}x\mathrm{d}t = \int_\Sigma g_1\, \mu \,\mathrm{d}s\mathrm{d}t, \quad \forall \mu \in V_0' \end{array}\right.$$

Mixed formulation (cont.)

Proposition (well-posedness)

Define : $U \in V_{g_0}$ and $\hat{u}_\varepsilon = u_\varepsilon - U \to$ find $(\hat{u}_\varepsilon, \lambda_\varepsilon) \in V_0 \times V_0'$ s.t.

$$A((\hat{u}_\varepsilon, \lambda_\varepsilon), (v, \mu)) = L(v, \mu), \quad \forall (v, \mu) \in V_0 \times V_0'$$

$$\left\{ \begin{array}{l} A((\hat{u}, \lambda), (v, \mu)) = -\displaystyle\int_Q \partial_t\, v \partial_t \lambda \,\mathrm{d}x\mathrm{d}t + \int_Q \partial_t \hat{u}\, \partial_t \mu \,\mathrm{d}x\mathrm{d}t \\ + \displaystyle\int_Q \nabla v \cdot \nabla \lambda \,\mathrm{d}x\mathrm{d}t - \int_Q \nabla \hat{u} \cdot \nabla \mu \,\mathrm{d}x\mathrm{d}t \\ +\varepsilon \displaystyle\int_Q \partial_t \hat{u}\, \partial_t v \,\mathrm{d}x\mathrm{d}t + \varepsilon \int_Q \nabla \hat{u} \cdot \nabla v \,\mathrm{d}x\mathrm{d}t \\ + \displaystyle\int_Q \partial_t \lambda\, \partial_t \mu \,\mathrm{d}x\mathrm{d}t + \int_Q \nabla \lambda \cdot \nabla \mu \,\mathrm{d}x\mathrm{d}t \end{array} \right.$$

$$A((\hat{u}, \lambda), (\hat{u}, \lambda)) = \varepsilon ||\hat{u}||^2 + ||\lambda||^2, \quad ||\cdot||^2 = ||\partial_t \cdot||^2_{L^2(Q)} + ||\nabla \cdot||^2_{L^2(Q)}$$

Mixed formulation (cont.)

Proposition (convergence): data (g_0, g_1) associated with

- the exact solution u
- the solution $\tilde{u}$ of nimimal norm $||\cdot||$

$$\lim_{\varepsilon\to 0}(u_\varepsilon, \lambda_\varepsilon) = (\tilde{u}, 0) \in H^1(Q) \times H^1(Q)$$

Uniqueness result $\longrightarrow$ for $T > 2T_0$

$$\tilde{u} = u \text{ in } \Omega \times (T_0, T - T_0)$$

Lemma : u satisfies the wave equation with lateral Cauchy data (g_0, g_1) iff $u \in V_{g_0}$ and $\forall \mu \in \tilde{V}_0$

$$-\int_Q \partial_t u \, \partial_t \mu \, \mathrm{d}x\mathrm{d}t + \int_Q \nabla u \cdot \nabla \mu \, \mathrm{d}x\mathrm{d}t = \int_\Sigma g_1 \, \mu \, \mathrm{d}s\mathrm{d}t$$

Mixed formulation (cont.)

Proof of the proposition: one subtracts to the weak formulation $\rightarrow$

$$\left\{\begin{array}{l} -\displaystyle\int_Q \partial_t v\, \partial_t \lambda_\varepsilon \,\mathrm{d}x\mathrm{d}t + \varepsilon \int_Q \partial_t u_\varepsilon\, \partial_t v \,\mathrm{d}x\mathrm{d}t \\ +\displaystyle\int_Q \nabla v \cdot \nabla \lambda_\varepsilon \,\mathrm{d}x\mathrm{d}t + \varepsilon \int_Q \nabla u_\varepsilon \cdot \nabla v \,\mathrm{d}x\mathrm{d}t = 0, \quad \forall v \in V_0 \\ -\displaystyle\int_Q \partial_t (u_\varepsilon - \tilde{u})\, \partial_t \mu \,\mathrm{d}x\mathrm{d}t - \int_Q \partial_t \lambda_\varepsilon\, \partial_t \mu \,\mathrm{d}x\mathrm{d}t \\ +\displaystyle\int_Q \nabla (u_\varepsilon - \tilde{u}) \cdot \nabla \mu \,\mathrm{d}x\mathrm{d}t - \int_Q \nabla \lambda_\varepsilon \cdot \nabla \mu \,\mathrm{d}x\mathrm{d}t = 0, \quad \forall \mu \in V_0' \end{array}\right.$$

We choose $v = u_\varepsilon - \tilde{u} \in V_0$ and $\mu = \lambda_\varepsilon \in \tilde{V}_0$, and subtracting the two equations:

$$\varepsilon((u_\varepsilon - \tilde{u}, u_\varepsilon)) + ||\lambda_\varepsilon||^2 = 0$$

We obtain

$$||u_\varepsilon|| \leq ||\tilde{u}|| \quad \text{et} \quad ||\lambda_\varepsilon|| \leq \sqrt{\varepsilon}||\tilde{u}||$$

Mixed formulation (cont.)

End of the proof:

- $\lambda_\varepsilon \to 0$ in $H^1(Q)$
- There exists a subsequence $u_\varepsilon \rightharpoonup w \in H^1(Q)$, with $w \in V_{g_0}$

Passing to the limit in the second equation of QR form.: $\forall \mu \in \tilde{V}_0$

$$-\int_Q \partial_t w \, \partial_t \mu \, \mathrm{d}x\mathrm{d}t + \int_Q \nabla w \cdot \nabla \mu \, \mathrm{d}x\mathrm{d}t = \int_\Sigma g_1 \, \mu \, \mathrm{d}s\mathrm{d}t$$

From the lemma w is a solution to the problem. In addition

$$||w|| \leq \liminf_{\varepsilon \to 0} ||u_\varepsilon|| \leq ||\tilde{u}||$$

Hence $w = \tilde{u}$, that is $u_\varepsilon \rightharpoonup \tilde{u}$ dans $H^1(Q)$

$$||u_\varepsilon - \tilde{u}||^2 = ((u_\varepsilon, u_\varepsilon - \tilde{u})) - ((\tilde{u}, u_\varepsilon - \tilde{u})) \leq -((\tilde{u}, u_\varepsilon - \tilde{u}))$$

Hence $u_\varepsilon \to \tilde{u}$ in $H^1(Q)$ $\blacksquare$

An alternative problem : lateral Cauchy data + initial condition

For (g_0, g_1), find $u \in H^1(Q)$ such that

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times (0,T) \\ u = g_0 & \text{on } \Gamma \times (0,T) \\ \partial_\nu u = g_1 & \text{on } \Gamma \times (0,T) \\ (u, \partial_t u) = (0,0) & \text{on } \Omega \times \{0\} \end{cases}$$

Uniqueness theorem: assume that $T > T_0$ and $(g_0, g_1) = (0,0)$, then $u = 0$ in $\Omega \times (0, T - T_0)$.

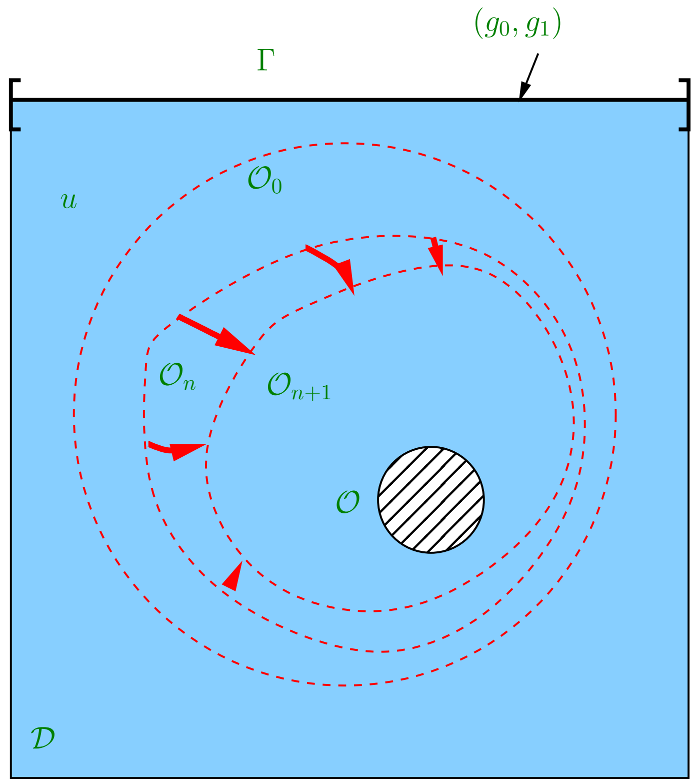
Ill-posed problem again: neither uniqueness, nor existence

$\longrightarrow$ the method of quasi-reversibility can be adapted : change functional spaces !

Back to the inverse obstacle problem: the "exterior approach"

An iterative method coupling the quasi-reversibility method and a level set method $\rightarrow$ no optimization

- **Step 1**: given the obstacle $\mathcal{O}_n$, we find an approximation u_n of u in $\Omega_n \times (0, T - T_0)$ ($\Omega_n = \mathcal{D} \setminus \overline{\mathcal{O}}_n$) with a **method of quasi-reversibility**
- **Step 2**: given the approximate solution u_n in $\Omega_n \times (0, T - T_0)$, we update the obstacle $\mathcal{O}_n$ with a **level set method**



Avantages : partial data ($\Gamma \neq \partial\mathcal{D}$), the number of connected components of $\mathcal{O}$ is unknown

The 1D case

The wave equation with lateral Cauchy data

For (g_0, g_1), find $u \in H^1((0,1)\times(0,T))$ such that

$$\left\{\begin{array}{rcll} \partial_t^2 u - \partial_x^2 u & = & 0 & \text{dans } (0,1)\times(0,T) \\ u & = & g_0 & \text{sur } \{0\}\times(0,T) \\ -\partial_x u & = & g_1 & \text{sur } \{0\}\times(0,T) \end{array}\right.$$

Sharp unique continuation $(T_0 = 1)$: for $T > 2$, u is uniq. defined in

$$LC := \{(x,t),\ 0 < x < 1,\ x < t < T - x\} \supset (0,1)\times(1,T-1)$$

D'alembert formula: u coincides in LC with

$$u_A(x,t) = \frac{1}{2}(g_0(t+x) + g_0(t-x)) - \frac{1}{2}\int_{t-x}^{t+x} g_1(s)\,ds$$

$\to$ The problem is well-posed in the light cone!

The 1D case

Lateral Cauchy data and initial condition

For (g_0, g_1), find $u \in H^1((0,1)\times(0,T))$ such that

$$\begin{cases} \partial_t^2 u - \partial_x^2 u &= 0 & \text{dans } (0,1)\times(0,T) \\ u &= g_0 & \text{sur } \{0\}\times(0,T) \\ -\partial_x u &= g_1 & \text{sur } \{0\}\times(0,T) \\ (u, \partial_t u) &= (0,0) & \text{on } (0,1)\times\{0\} \end{cases}$$

Sharp unique continuation $(T_0 = 1)$: for $T > 1$, u is uniq. defined in

$$LC_0 := \{(x,t),\ 0 < x < 1,\ 0 < t < T - x\} \supset (0,1)\times(0,T-1)$$

The inverse obstacle problem in 1D

The obstacle is an interval $(a,1)$ for $a \in (0,1)$!

For (g_0, g_1), find $a \in (0,1)$ such that for $u \in H^1((0,a) \times (0,T))$:

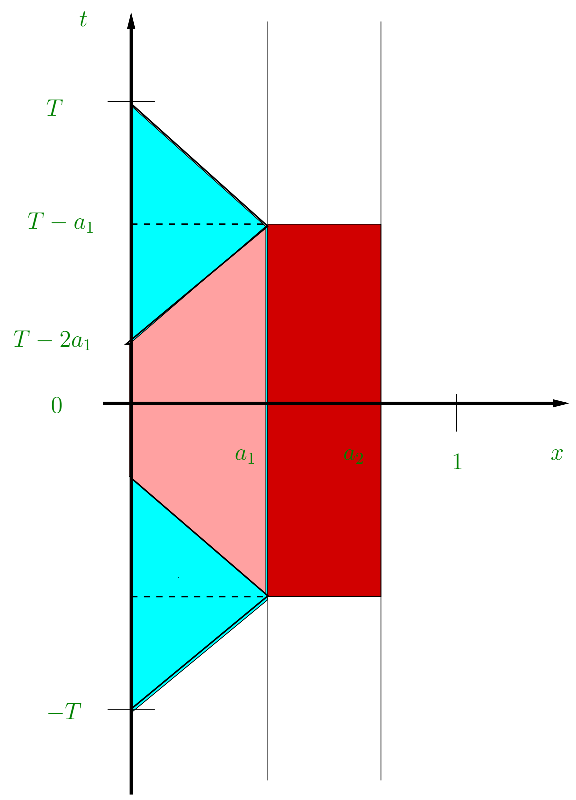
$$\left\{ \begin{array}{ccll} \partial_t^2 u - \partial_x^2 u & = & 0 & \text{dans } (0,a) \times (0,T) \\ u & = & g_0 & \text{sur } \{0\} \times (0,T) \\ -\partial_x u & = & g_1 & \text{sur } \{0\} \times (0,T) \\ u & = & 0 & \text{on } \{a\} \times (0,T) \\ (u, \partial_t u) & = & (0,0) & \text{on } (0,a) \times \{0\} \end{array} \right.$$

The inverse obstacle problem in 1D (cont.)

Uniqueness theorem: consider $a_1, a_2 \in (0,1)$ such that u_1, u_2 are compatible with data (g_0, g_1) with **assumption** (H):

- for all $t_0 > 0$, $(g_0, g_1) \neq (0,0)$ on $(0, t_0)$
- $T > 2$

$\longrightarrow$ Then $a_1 = a_2$



Remark: initial condition is crucial

Level set method in 1D

Velocity field $V \in H^1(0,1)$ such that

$$\begin{cases} V = \left(\int_0^{T-1} u^2(\cdot, t)\,\mathrm{d}t \right)^{\frac{1}{2}} & \text{on } (0,a) \\ V \leq 0 & \text{sur } (a,1) \end{cases}$$

One defines a sequence (a_n) in $(0,1)$ by $a_0 \in (0,a)$ and

$$a_{n+1} = \inf \mathcal{O}_n, \quad \mathcal{O}_n = \{x \in [a_n, 1), \quad \phi_n(x) \leq 0\}$$

with $\phi_n \in H^1(a_n, 1)$ solution to

$$\begin{cases} -\phi_n' = f & \text{in } (a_n, 1) \\ \phi_n = V & \text{for } x = a_n \end{cases}$$

and $f \in L^2(0,1)$ s.t. $f \geq \max(0, -V')$

Level set method in 1D (cont.)

Theorem (convergence of level sets): assuming (H)

- for all $t_0 > 0$, $(g_0, g_1) \neq (0, 0)$ on $(0, t_0)$
- $T > 2$

then $(0, 1) \setminus \overline{\mathcal{O}_n} = (0, a_{n+1})$ and the sequence (a_n) converges to a.

Remark: V is computed with the help of u, which is unknown ! In practice, with approximate u by the quasi-reversibility method

Back to the "exterior approach"

Algorithm:

1. Initialisation $a_0 \in (0,1)$ such that $a_0 < a$
2. **First step** : for given a_n, compute the Q.R. solution u_n in $(0,a_n) \times (0,T)$ for some ε
3. **Second step** : for given u_n in $(0,a_n) \times (0,T-1)$, solve equation $-\phi_n' = f$ in $(a_n,1)$ with $\phi_n(a_n) = ||u_n(a_n,\cdot)||_{L^2(0,T-1)}$ for some f and define

$$a_{n+1} = \inf\{x \in [a_n,1),\ \phi_n(x) \leq 0\}$$

4. Go back to first step with stopping criteria

Level set method: extension to dimension $d > 1$

Velocity field $V \in H^1(\mathcal{D})$ such that

$$\begin{cases} V = \left(\int_0^{T-T_0} u^2(\cdot,t)\,\mathrm{d}t\right)^{\frac{1}{2}} & \text{in} \quad \Omega \\ V \leq 0 & \text{in} \quad \mathcal{O} \end{cases}$$

For $f \geq 0$ and $f \geq \Delta V$ in $H^{-1}(\mathcal{D})$

$$\begin{cases} \mathcal{O} \subset \mathcal{O}_0 \Subset \mathcal{D} \\ \mathcal{O}_{n+1} = \{x \in \mathcal{O}_n, \quad \phi_n(x) < 0\} \end{cases}$$

with ϕ_n given by

$$\begin{cases} \Delta\phi_n = f & \text{in} \quad \mathcal{O}_n \\ \phi_n = V & \text{on} \quad \partial\mathcal{O}_n \end{cases}$$

Level set method for $d > 1$ (cont.)

Theorem (convergence of level set): assuming (H)

- $u \in L^2(0,T;C^0(\overline{\Omega}))$
- $\exists\, x_0 \in \Gamma$ such that for all $t_0 > 0$, $g_0(x_0,\cdot) \neq 0$ on $(0,t_0)$
- $T > D(\Gamma,\Omega) + D(x_0,\Omega)$

If in addition $\mathcal{D} \setminus \overline{\mathcal{O}_0}$ is connected and $\phi_n \in C^0(\overline{\mathcal{O}_n})\ \forall n$,
$\rightarrow$ then the $\mathcal{D} \setminus \overline{\mathcal{O}_n}$ are connected

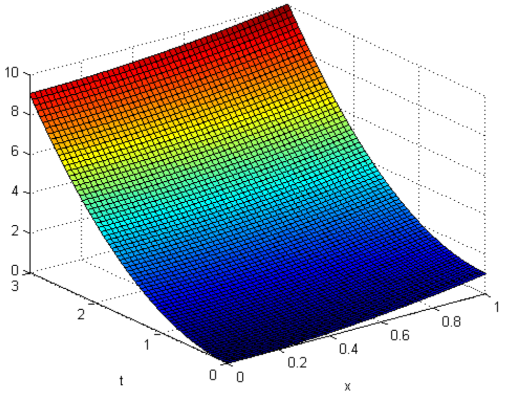
If in addition $d = 2$ or the $\mathcal{O}_n$ are unif. Lipschitz,
$\rightarrow$ then in the sense of Hausdorff distance

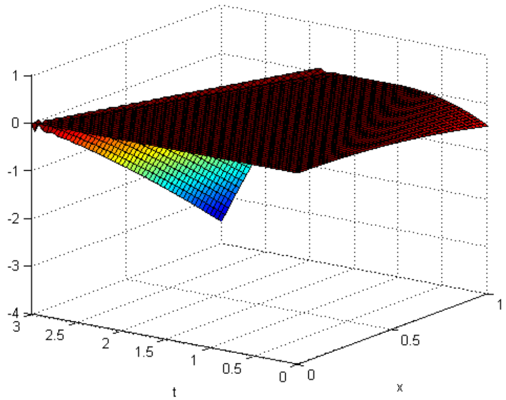
$$\overbrace{\bigcap_n \mathcal{O}_n}^{\circ} = \mathcal{O}$$

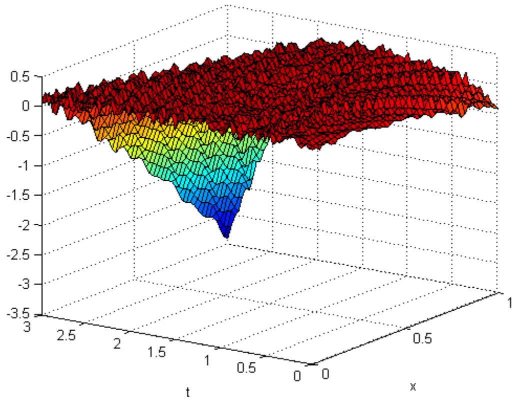
Numerical experiments in 1D

QR method with lateral Cauchy data: $u(x,t) = x^2 + t^2$ and $T = 3$

$\to$ noisy data (0% and 5%) given by $\|g_0^\delta - g_0\|_{L^2} \leq \delta \, \|g_0\|_{L^2}$

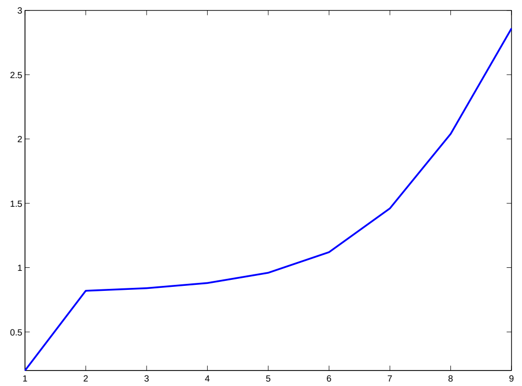


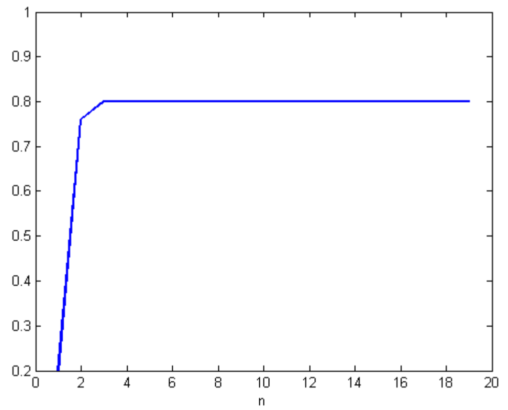


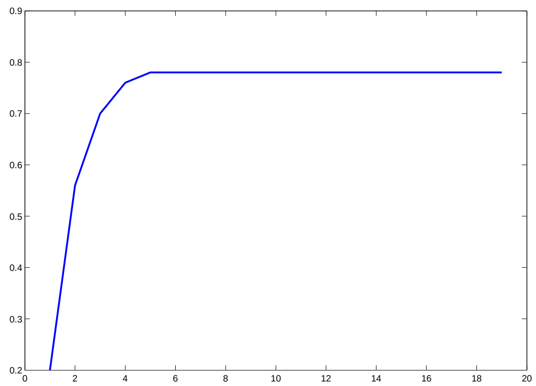


Numerical experiments in 1D (cont.)

Inverse obstacle problem: $g_1(t) = \sin(\pi t/T)$ for $T = 3$ and $a = 0.8$
$\rightarrow$ influence of f ($f = 1.2$, $f = 1.3$ and $f = 2$)



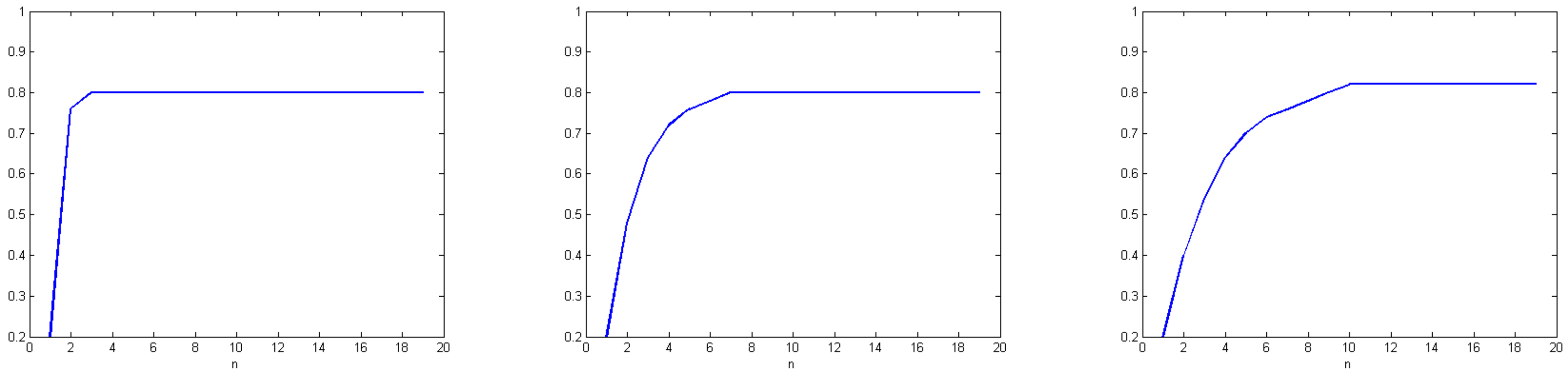




Numerical experiments in 1D (cont.)

Inverse obstacle problem: $g_1(t) = \sin(\pi t/T)$ for $T = 3$ and $a = 0.8$

$\rightarrow$ noisy data (0%, 5% and 10%)



Perspectives

Main features of the "exterior approach" :

- No optimization
- Number of obstacles unknown a priori
- Partial lateral Cauchy data

Extensions :

- $2D$ case : to be done
- Backward heat equation
- Heat equation and inverse obstacle problem with lateral Cauchy data
- Other boundary conditions on the obstacle

THANK'S FOR YOUR ATTENTION !