The inverse scattering problem in quantitative multi-modal tomography

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Outline

- The Full-field OCT system
- The inverse scattering problem
  - isotropic dispersive medium
  - anisotropic dispersive medium
- The multi-modal PAT-OCT system
- The Fredholm integral equation
- Numerical examples

Low (time) coherence interferometer (LCI)

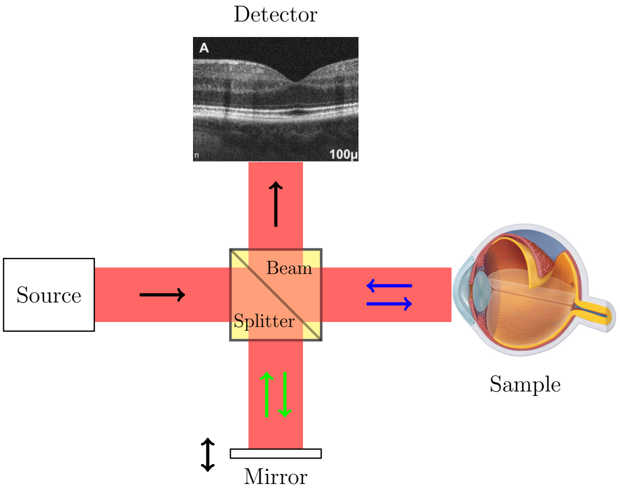


Figure : Standard OCT system based on a Michelson interferometer [Huang *et al* 91].

The FF-OCT system

We consider Maxwell's equations

$$
\begin{aligned}
\operatorname{div}_x D(t,x) &= 0,\\
\operatorname{div}_x B(t,x) &= 0,\\
\operatorname{curl}_x E(t,x) &= -\frac{1}{c}\frac{\partial B}{\partial t}(t,x),\\
\operatorname{curl}_x H(t,x) &= \frac{1}{c}\frac{\partial D}{\partial t}(t,x),
\end{aligned}
$$

for $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$, and the material equations

$$
\begin{aligned}
D(t,x) &= E(t,x) + \int_0^\infty \chi(\tau,x)E(t-\tau,x)\mathrm{d}\tau,\\
B(t,x) &= H(t,x),
\end{aligned}
$$

where χ is the electric susceptibility.

The FF-OCT system

Let $\Omega \subset \mathbb{R}^3$ be the domain where the sample is located. We set

- Isotropic medium: χ is scalar, i.e. multiple of the identity matrix $\chi = \chi \mathbb{1}$.
- Anisotropic medium: $\chi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$
- $\chi(t, x) = 0 \quad \text{for} \quad t < 0,\ x \in \mathbb{R}^3$
- $\chi(t, x) = 0 \quad \text{for} \quad t \in \mathbb{R},\ x \in \mathbb{R}^3 \setminus \Omega,$

Then, the Fourier transform $\hat{E}$ of E, given by

$$\hat{E}(\omega, x) = \int_{-\infty}^{\infty} E(t, x) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t,$$

satisfies

$$\operatorname{curl}_x \operatorname{curl}_x \hat{E}(\omega, x) - \frac{\omega^2}{c^2} (\mathbb{1} + \hat{\chi}(\omega, x)) \hat{E}(\omega, x) = 0$$

for $\omega \in \mathbb{R},\ x \in \mathbb{R}^3$

The FF-OCT system

The sample is illuminated by a laser beam described by the electric field $E^{(0)}$ which sstisfies

$$\operatorname{curl}_x \operatorname{curl}_x \hat{E}^{(0)}(\omega, x) - \frac{\omega^2}{c^2} \hat{E}^{(0)}(\omega, x) = 0.$$

Moreover, we assume that

$$\operatorname{supp} E^{(0)}(t, \cdot) \cap \Omega = \emptyset, \quad \text{for} \quad t \leq 0.$$

Then, the electric field E (generated by this incoming light beam in the presence of the sample) satisfies the initial condition

$$E(t, x) = E^{(0)}(t, x) \quad \text{for} \quad t \leq 0, \; x \in \mathbb{R}^3.$$

The FF-OCT system

The measurements are obtained by the combination of E (backscattered field from the sample) and E_r (back-reflected field from the mirror).
The mirror is placed orthogonal to the unit vector $e_3 = (0,0,1)$ through the point $r\,e_3$ and we assume that

$$E_r(t,x) = E^{(0)}(t,x) \quad \text{for} \quad t < 0,\ x \in \mathbb{R}^3.$$

The intensity

$$I_{r,j}(x) = \int_0^\infty |E_j(t,x) + E_{r,j}(t,x)|^2 \mathrm{d}t, \quad j \in \{1,2,3\}.$$

is measured at some detector points, located on the plane

$$\mathcal{D} = \{x \in \mathbb{R}^3 \mid x_3 = d\}$$

parallel to the mirror at a distance $d > 0$ from the origin.

The FF-OCT system

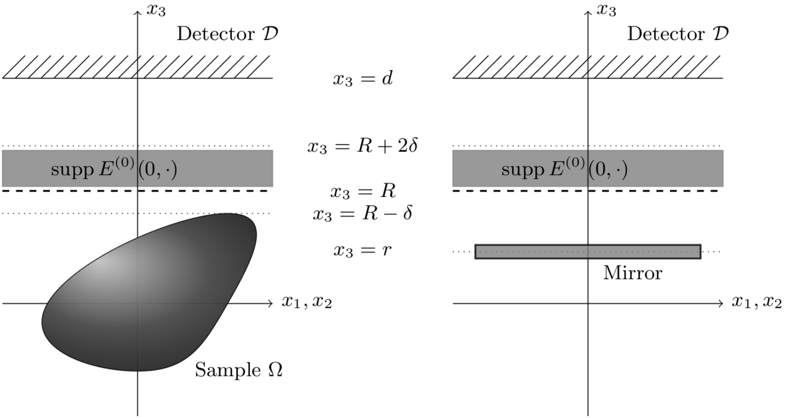


Figure : The two scattering problems involved in OCT.

The FF-OCT system

In this setup, it is easy to acquire besides the intensity I_r also the intensity of the two waves E and E_r separately.
Therefore, we consider instead of I_r the function

$$M_{r,j}(x) = \frac{1}{2}\left(I_{r,j} - \int_0^\infty |E_j(t,x)|^2 \mathrm{d}t - \int_0^\infty |E_{r,j}(t,x)|^2 \mathrm{d}t\right)$$

for $r \in (-\infty, R)$, $j \in \{1,2,3\}$, and $x \in \mathcal{D}$ as our measurement data, or equivalently,

$$\begin{aligned} M_{r,j}(x) &= \int_{-\infty}^{\infty} (E_j - E_j^{(0)})(t,x)(E_{r,j} - E_j^{(0)})(t,x)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} (\hat{E}_j - \hat{E}_j^{(0)})(\omega,x)\overline{(\hat{E}_{r,j} - \hat{E}_j^{(0)})(\omega,x)}\mathrm{d}\omega. \end{aligned}$$

Isotropic case: $M_r(x) = \sum_{j=1}^3 M_{r,j}(x)$.

The FF-OCT system

If E satisfies the (vector) Helmholtz equation and the material equations. Then, $\hat{E}$ solves

$$\hat{E}(\omega, x) = \hat{E}^{(0)}(\omega, x) + \left(\frac{\omega^2}{c^2}\mathbb{1} + \operatorname{grad}_x \operatorname{div}_x\right) \int_{\mathbb{R}^3} G(\omega, x-y)\hat{\chi}(\omega, y)\hat{E}(\omega, y)\mathrm{d}y,$$

where

$$G(\omega, x) = \frac{\mathrm{e}^{\mathrm{i}\frac{\omega}{c}|x|}}{4\pi|x|}, \quad x \neq 0, \, \omega \in \mathbb{R}.$$

Let the medium be weakly scattering and sufficiently far from the detector. Then, the solution is given by

$$\begin{aligned} \hat{E}^{(1)}(\omega, \rho\vartheta) &\simeq \hat{E}^{(0)}(\omega, \rho\vartheta) \\ &\quad - \frac{\omega^2 \mathrm{e}^{\mathrm{i}\frac{\omega}{c}\rho}}{4\pi\rho c^2} \int_{\mathbb{R}^3} \vartheta \times \left(\vartheta \times (\hat{\chi}(\omega, y)\hat{E}^{(0)}(\omega, y))\right)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}\langle\vartheta, y\rangle}\mathrm{d}y, \end{aligned}$$

where $\rho > 0$ and $\vartheta \in S^2$.

The FF-OCT system

The incident field propagates in the direction $-e_3$, orthogonal to the detector surface $\mathcal{D}$, this means

$$E^{(0)}(t,x) = f(t + \tfrac{x_3}{c})\, p,$$

where $p \in \mathbb{R}^3$, with $p_3 = \langle p, e_3 \rangle = 0$, is the polarisation vector. The vertical distribution $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\operatorname{supp} f \subset (\tfrac{R}{c}, \tfrac{d}{c}).$$

Then, the electric field E_r reflected by the mirror is given by

$$E_r(t,x) = \begin{cases} \big(f(t + \frac{x_3}{c}) - f(t + \frac{x_3}{c} + 2\,\frac{r-x_3}{c})\big)p & \text{if } x_3 > r, \\ 0 & \text{if } x_3 \le r. \end{cases}$$

The FF-OCT system

For $E^{(0)}$ as above, then

$$\hat{E}^{(0)}(\omega, x) = \left(\int_{-\infty}^{\infty} f(t + \tfrac{x_3}{c}) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t \right) p = \hat{f}(\omega) \mathrm{e}^{-\mathrm{i}\frac{\omega}{c} x_3} p,$$

and the measurements are given by

$$\begin{aligned} M_{r,j}(x) &= -p_j \int_{-\infty}^{\infty} (E_j - E_j^{(0)})(t, x) f(t + \tfrac{2r - x_3}{c}) \mathrm{d}t, \\ &= -\frac{p_j}{2\pi} \int_{-\infty}^{\infty} (\hat{E}_j - \hat{E}_j^{(0)})(\omega, x) \hat{f}(-\omega) \mathrm{e}^{\mathrm{i}\frac{\omega}{c}(2r - x_3)} \mathrm{d}\omega \end{aligned}$$

for all $j \in \{1, 2, 3\}$, $r \in (-\infty, R)$, and $x \in \mathcal{D}$.

The FF-OCT system

To summarize,

$$\hat{\chi} \quad \overset{\mathcal{L}}{\longmapsto} \quad \hat{E}^{(1)} - \hat{E}^{(0)} \quad \overset{\mathcal{M}}{\longmapsto} \quad \left(M_{r,j}(x)\right)_{j=1}^{2}$$

where

$$(\mathcal{L}v)(\omega, \rho\vartheta) = -\frac{\omega^2 \mathrm{e}^{\mathrm{i}\frac{\omega}{c}\rho}}{4\pi\rho c^2} \hat{f}(\omega) \int_{\mathbb{R}^3} \vartheta \times \left(\vartheta \times (v(\omega, y)\, p)\right) \mathrm{e}^{-\mathrm{i}\frac{\omega}{c}\langle \vartheta + e_3, y\rangle} \mathrm{d}y,$$

$$(\mathcal{M}v)(r, x) = \left(-\frac{p_j}{2\pi} \int_{-\infty}^{\infty} v_j(\omega, x) \hat{f}(-\omega) \mathrm{e}^{\mathrm{i}\frac{\omega}{c}(2r - x_3)} \mathrm{d}\omega\right)_{j=1}^{2}$$

Thus, combining $\mathcal{L}$, $\mathcal{M}$, the forward operator $\mathcal{F} : \hat{\chi} \mapsto M$, $\mathcal{F} = \mathcal{M}\mathcal{L}$ models the direct problem.
The inverse problem is then formulated as an operator equation

$$\mathcal{F}\hat{\chi} = M.$$

The Inverse Scattering Problem

Let the initial illumination be of the form $\hat{E}^{(0)}(\omega, x) = \hat{f}(\omega)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}x_3}p$, satisfying

$$\operatorname{supp} f \subset \left(\frac{R}{c}, \frac{R}{c} + \frac{2\delta}{c}\right) \subset \left(\frac{R}{c}, \frac{d}{c}\right) \quad \text{for some} \quad \delta > 0.$$

Then,

$$(\hat{E}_j - \hat{E}_j^{(0)})(\omega, x)\overline{\hat{f}(\omega)}p_j = -\frac{2}{c}\int_{-\infty}^{R} M_{r,j}(x)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}(2r - x_3)}\mathrm{d}r$$

for all $j \in \{1, 2, 3\}$, $\omega \in \mathbb{R}$, and $x \in \mathcal{D}$.

The Inverse Scattering Problem

In addition, for every $\omega \in \mathbb{R} \setminus \{0\}$ with $\hat{f}(\omega) \neq 0$, the formula

$$p_j \left[\vartheta \times (\vartheta \times \tilde{\chi}(\omega, \tfrac{\omega}{c}(\vartheta + e_3))p)\right]_j \simeq \frac{8\pi\rho c}{\omega^2 |\hat{f}(\omega)|^2} \int_{-\infty}^{R} M_{r,j}(\rho\vartheta) \mathrm{e}^{-\mathrm{i}\frac{\omega}{c}(2r - \rho(\vartheta_3 - 1))} \mathrm{d}r$$

holds for all $j \in \{1, 2\}$, $\vartheta \in S^2_+ := \{\eta \in S^2 \mid \eta_3 > 0\}$, and $\rho = \frac{d}{\vartheta_3}$ (asymptotically for $\chi \to 0$ and $\rho \to \infty$).

Here $\tilde{\chi}$ denotes the Fourier transform of χ with respect to time and space,

$$\tilde{\chi}(\omega, k) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \chi(t, x) \mathrm{e}^{-\mathrm{i}\langle k, x\rangle} \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}x \mathrm{d}t = \int_{\mathbb{R}^3} \hat{\chi}(\omega, x) \mathrm{e}^{-\mathrm{i}\langle k, x\rangle} \mathrm{d}x.$$

The Inverse Scattering Problem (Isotropic case)

Then, from the sum of $M_{r,1}$ and $M_{r,2}$ we obtain

$$\tilde{\chi}(\omega, \tfrac{\omega}{c}(\vartheta + e_3)) \langle p, \vartheta \times (\vartheta \times p)\rangle = \tilde{\chi}(\omega, \tfrac{\omega}{c}(\vartheta + e_3))(\langle \vartheta, p\rangle^2 - |p|^2).$$

Since $\langle \vartheta, p\rangle^2 < |p|^2$ for every combination of $p \in \mathbb{R}^2 \times \{0\}$ and $\vartheta \in S^2_+$, we have direct access to

$$\tilde{\chi}(\omega, \tfrac{\omega}{c}(\vartheta + e_3)), \quad \omega \in \mathbb{R} \setminus \{0\},\ \vartheta \in S^2_+,$$

of χ in a subset of $\mathbb{R} \times \mathbb{R}^3$.

Remark

Here, the problem is to reconstruct the 4D susceptibility from the 3D measurement data.

The Inverse Scattering Problem (Isotropic case)

Recall that

$$\hat{m} : \mathbb{R} \times S^2_+ \to \mathbb{C}, \quad \hat{m}(\omega, \vartheta) = \tilde{\chi}(\omega, \tfrac{\omega}{c}(\vartheta + e_3)).$$

Lemma

The inverse Fourier transform $m : \mathbb{R} \times S^2_+ \to \mathbb{C}$ of $\hat{m}$ with respect to ω is given by

$$m(t, \vartheta) = \frac{c}{\sqrt{2(1+\vartheta_3)}} \int_{-\infty}^{\infty} \int_{E_{\tau - t, \vartheta}} \chi(\tau, y) \mathrm{d}s(y) \mathrm{d}\tau,$$

for $t \in \mathbb{R}$, $\tau, \sigma \in \mathbb{R}$ and $\vartheta \in S^2_+$, where $E_{\sigma,\vartheta}$ denotes the plane

$$E_{\sigma,\vartheta} = \{y \in \mathbb{R}^3 \mid \langle \vartheta + e_3, y \rangle = c\sigma\}.$$

The Inverse Scattering Problem (Isotropic case)

Thus, the measurements provide the Radon transform of $\chi(\tau, \cdot)$,

$$m(t,\vartheta) = \frac{c}{\sqrt{2(1+\vartheta_3)}} \int_{-\infty}^{\infty} \bar{\chi}(\tau; \tau - t, \vartheta) \mathrm{d}\tau.$$

Discretisation

We assume, for some $T > 0$:

$$\operatorname{supp} \chi(\cdot, x) \subset [0, T] \quad \text{for all} \quad x \in \mathbb{R}^3.$$

Thus, the function $\bar{\chi}(\tau; \cdot, \vartheta)$ is discretised for every $\tau \in \mathbb{R}$ and $\vartheta \in S^2_+$ and the step size depends on the size of the support of $\chi(\cdot, x)$.

The Inverse Scattering Problem (Isotropic case)

Then, we consider the following discretisation

$$\bar{\chi}_n(\tau, \vartheta) = \int_{E_{nT,\vartheta}} \chi(\tau, y) \mathrm{d}s(y), \quad n \in \mathbb{Z}, \ \tau \in (0, T), \ \vartheta \in S^2_+,$$

of the Radon transform of the functions $\chi(\tau, \cdot)$ and we extend it over the planes $E_{nT+\varepsilon,\vartheta}$,

$$\bar{\chi}_n(\tau, \vartheta) \approx \int_{E_{nT+\varepsilon,\vartheta}} \chi(\tau, y) \mathrm{d}s(y), \quad \text{for all} \quad \varepsilon \in [-\tfrac{T}{2}, \tfrac{T}{2}).$$

Then,

$$m(t, \vartheta) \approx \frac{c}{\sqrt{2(1+\vartheta_3)}} \int_0^T \bar{\chi}_{N(\tau-t)}(\tau, \vartheta) \mathrm{d}\tau,$$

where $N(\sigma) = \left\lfloor \frac{\sigma}{T} + \frac{1}{2} \right\rfloor$ denotes the integer closest to $\frac{\sigma}{T}$.

This approximation can now be iteratively solved for $\bar{\chi}$.

The Inverse Scattering Problem (Isotropic case)

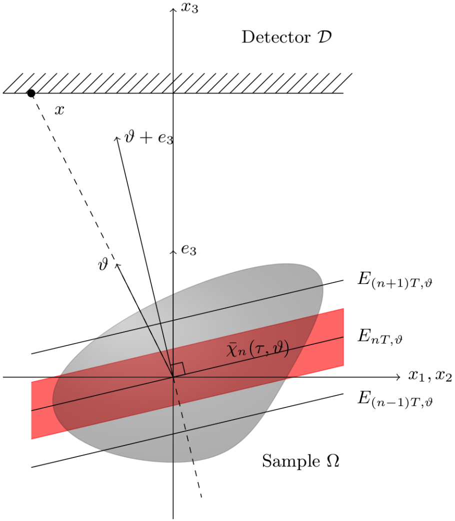


Figure : Discretisation of χ with respect to the detection points.

The Inverse Scattering Problem (Isotropic case)

Theorem

Let

$$\bar{m}(t,\vartheta) = \frac{c}{\sqrt{2(1+\vartheta_3)}} \int_0^T \bar{\chi}_{N(\tau-t)}(\tau,\vartheta)\mathrm{d}\tau, \quad \vartheta \in S^2_+,\ t \in \mathbb{R},$$

for some constant $T > 0$ with the integer valued function $N(\sigma) = \lfloor \frac{\sigma}{T} + \frac{1}{2} \rfloor$.

Then, $\bar{\chi}$ fulfils the recursion relation

$$\bar{\chi}_n(\tau,\vartheta) = \bar{\chi}_{n+1}(\tau,\vartheta) + \frac{\sqrt{2(1+\vartheta_3)}}{c}\frac{\partial \bar{m}}{\partial t}(\tau - (n+\tfrac{1}{2})T,\vartheta),$$

for $\quad n \in \mathbb{Z},\ \tau \in (0,T),\ \vartheta \in S^2_+.$

The Inverse Scattering Problem (Anisotropic case)

The problem is to reconstruct $\chi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ from

$$\chi_{\vartheta,p,j}(\omega,\vartheta) := p_j\big[\vartheta \times (\vartheta \times \tilde{\chi}(\omega, \tfrac{\omega}{c}(\vartheta + e_3))p)\big]_j, \quad j = 1,2,$$

where we assume that measurements for every $p \in \mathbb{R}^2 \times \{0\}$ are available.
As before, a reconstruction formula holds for the functions

$$\bar{\chi}_{p,j}(\tau;\sigma,\vartheta) = \int_{E_{\sigma,\vartheta}} \chi_{\vartheta,p,j}(\tau,y)\mathrm{d}s(y)$$

for all $p \in \mathbb{R}^2 \times \{0\}$, $\tau \in \mathbb{R}$, $\sigma \in \mathbb{R}$, $\vartheta \in S^2_+$, and $j \in \{1,2\}$, where

$$\bar{\chi}(\tau;\sigma,\vartheta) = \int_{E_{\sigma,\vartheta}} \chi(\tau,y)\mathrm{d}s(y)$$

denotes the Radon transform data of $\chi(\tau,\cdot)$.

The Inverse Scattering Problem (Anisotropic case)

Theorem

Let $\tau, \sigma \in \mathbb{R}$ and $\vartheta \in S^2_+$, be fixed, and $P_\vartheta \in \mathbb{R}^{3\times 3}$ denote the orthogonal projection in direction ϑ. Then, using that $\vartheta \times (\vartheta \times Xp) = -P_\vartheta Xp$, the data

$$p_j[\vartheta \times (\vartheta \times \bar{\chi}(\tau;\sigma,\vartheta))]_j$$

for $j = 1, 2$ and the three different polarisation vectors $p = e_1$, e_2 and $p = e_1 + e_2$ uniquely determine the projection

$$(P_\vartheta \bar{\chi}(\tau;\sigma,\vartheta))_{k,\ell} = \int_{E_{\sigma,\vartheta}} (P_\vartheta \chi(\tau, y))_{k,\ell} \, \mathrm{d}s(y) \quad \text{for} \quad k, \ell \in \{1, 2\}.$$

Moreover, measurements for additional polarisations p do not provide further information.

The Inverse Scattering Problem (Anisotropic case)

Let $R \in \mathrm{SO}(3)$ describe the rotation of the sample. Then, the transformed susceptibility χ_R is given by

$$\chi_R(t, y) = R\,\chi(t, R^{\mathrm{T}} y) R^{\mathrm{T}}.$$

If there exist for every $R \in \{R_0, R_1, R_2\}$ constants $\alpha_R > 0$ and $\vartheta_R \in S^2_+$ satisfying

$$\vartheta_R + e_3 = \alpha_R R(\vartheta + e_3),$$

then, the data corresponding to the measurements of the rotated sample at the detector in direction ϑ_R satisfy

$$\bar{\chi}_{R,p,j}(\tau; \alpha_R \sigma, \vartheta_R) = p_j[\vartheta_R \times (\vartheta_R \times R\bar{\chi}(\tau; \sigma, \vartheta) R^{\mathrm{T}}]_j$$

for all $\tau, \sigma \in \mathbb{R}$, $p \in \mathbb{R}^2 \times \{0\}$, $j = 1, 2$.

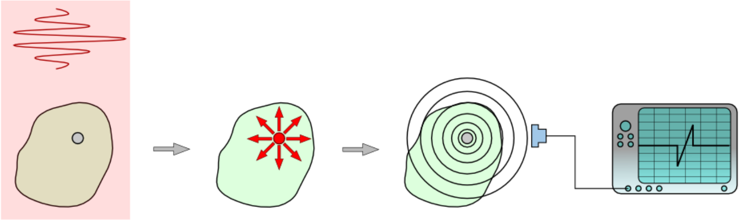
The Inverse Scattering Problem (Anisotropic case)

Theorem

The measurements obtained at the detectors ϑ_R for the polarisations $p = e_1, e_2, e_1 + e_2$ and rotations $R = R_0, R_1, R_2$, so that every proper subset of $\{R_0^{\mathrm{T}} e_3, R_1^{\mathrm{T}} e_3, R_2^{\mathrm{T}} e_3, \vartheta + e_3\}$ is linearly independent and such that $\vartheta_R + e_3 = \alpha_R R(\vartheta + e_3)$, provide sufficient information to reconstruct uniquely the Radon data $\bar{\chi}(\tau; \sigma, \vartheta)$.

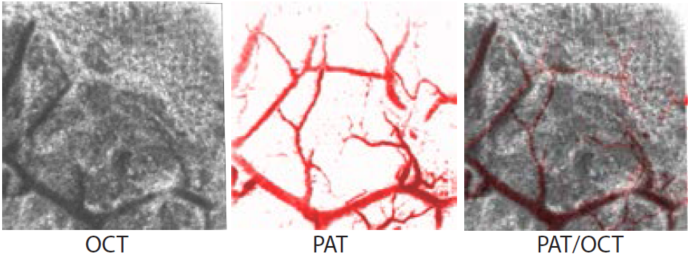
Then, it is possible via an inversion of a limited angle Radon transform to recover the susceptibility χ.

Photoacoustic Tomography (PAT)



- The object is irradiated by a short-pulsed laser beam.
- Some of the light is absorbed and partially converted into heat.
- The heat is converted to a pressure rise via thermoelastic expansion.
- The pressure rise propagates as an ultrasonic wave - the photoacoustic wave.

PAT-OCT system



- The laser beam operates around 1000nm for both systems.
- PAT and OCT use full field illumination.
- PAT detection points and the OCT beam are being co-axially aligned in order the images from each modality to be inherently co-registered.
- PAT and OCT scans are performed consecutively and the data acquisition times differ considerably.

The multi-modal system

Inverse Problem

Recover the optical properties of the sample from internal data (absorbed radiation) and modified far-field data (OCT measurements).

We assume

- The reconstruction of the absorbed energy from the PAT measurements is solved (inverse source problem) [Kuchment and Kunyansky 08].
- The laser beam initialization is short enough such that the pressure is generated instantaneously at $t = 0$.
- The initial pressure $p^{(0)}$ is proportional to the absorbed energy w_A. The proportionality factor is the Grüneisen parameter γ.
- Uniqueness results for non-dispersive medium and boundary data [Bal and Zhou 14].

The multi-modal system (Isotopic case)

Given the Maxwell's equations and the Poynting vector $S = \frac{c}{4\pi}(E \times H)$, considering the conservation of energy for dielectric media, we obtain

$$w_A(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}} \omega \, \Im\mathrm{m} \, \hat{\chi}(\omega, x) |\hat{E}(\omega, x)|^2 \mathrm{d}\omega,$$

for all $x \in \mathbb{R}^3$. Then,

$$p^{(0)}(x) = \gamma(x) w_A(x).$$

Remark

If we assume non-dispersive medium or monochromatic source illumination, then

$$w_A(x) = \sigma(x) \int_{\mathbb{R}} |E(t, x)|^2 \mathrm{d}t,$$

where σ is the conductivity. This formula is commonly used in TAT.

The multi-modal system

We recall $\hat{E}^{(0)}(\omega, x) = \hat{f}(\omega)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}x_3}p$, and we consider the limiting case $|\hat{f}|^2 = \delta(\cdot - \omega) + \delta(\cdot + \omega)$.
Then, the Inverse Problem is equivalent to the integral equation,

$$\int_{\mathbb{R}^3} \frac{1}{\gamma(y)} p_{\mathcal{H}}(\omega, y) e^{-\mathrm{i}\frac{\omega}{c}\langle\vartheta + e_3, y\rangle} dy = m(\omega, \vartheta),$$

for all $\omega \in \mathbb{R}$ and $\vartheta \in S^2_+$, where

$$p_{\mathcal{H}}(\omega, y) = (\mathcal{H} + \mathrm{i}\,\mathbb{1})\left(\tfrac{8\pi^2}{\omega} p^{(0)}(y)\right),$$

$$m(\omega, \vartheta) = \tfrac{8\pi^2 \rho c^2 \omega^{-2}}{\langle P, \vartheta \times (\vartheta \times P)\rangle} M(r, \vartheta) e^{-\mathrm{i}\frac{\omega}{c}(2r + \rho(1 - \vartheta_3))},$$

and $\mathcal{H}$ stands for the Hilbert transform. Moreover, $\hat{\chi}$ and γ are related through

$$\gamma(x)\hat{\chi}(\omega, x) = p_{\mathcal{H}}(\omega, x).$$

The Fredholm integral equation

Let

$$\Im\mathrm{m}\,\hat{\chi}(\omega, x) = \sum_{j=1}^{N} g_j(\omega) h_j(x),$$

for some smooth functions g_j, h_j. Then, if $\operatorname{supp} G_1 = \mathbb{R}$, and $\operatorname{supp} H_1 \supset \bigcup_{\omega\in\mathbb{R}} \operatorname{supp} \hat{\chi}(\omega,\cdot)$, where $G_j = \mathcal{H} g_j + \mathrm{i} g_j$, $H_j = \gamma h_j$, the function $\Gamma = \frac{H_1}{\gamma}$ fulfils

$$\hat{\Gamma}(v) + \int_{\mathbb{R}^3} K(v,k)\hat{\Gamma}(k)\mathrm{d}k = \bar{m}(v),$$

for $v = \frac{\omega}{c}(\vartheta + e_3)$, where

$$K(v,k) := \frac{1}{(2\pi)^3} \sum_{j=2}^{N} \frac{G_j(\omega)}{G_1(\omega)} \int_{\Omega} \frac{H_j(y)}{H_1(y)} \mathrm{e}^{-\mathrm{i}\langle v-k, y\rangle} \mathrm{d}y$$

and

$$\bar{m}\left(\tfrac{\omega}{c}(\vartheta + e_3)\right) := \frac{m(\omega,\vartheta)}{G_1(\omega)}.$$

Numerical Examples

Focused illumination

We illuminate a small region inside the object. Thus, the unknowns are simplified to,

$$\hat{\chi}(\omega, x) = \delta(x_1)\delta(x_2)\hat{\chi}(\omega, x_3), \quad \gamma(x) = \delta(x_1)\delta(x_2)\gamma(x_3).$$

We set $\vartheta = e_3$ and

$$\mu(x) := \frac{1}{\gamma(x)} \mathbb{1}_{\mathcal{D}} \in L^2(\mathbb{R}).$$

Then, the reduced integral equation reads

$$\int_{\mathbb{R}} \mu(y) p_{\mathcal{H}}(\omega, y) e^{-\mathrm{i}\omega 2y} dy = \bar{m}(\omega).$$

Numerical Examples

To solve the integral equation, we consider the Galerkin method and the hermite functions as an orthonormal basis of $L^2(\mathbb{R})$. We set

$$\mu(x) = \sum_{k=0}^{N} \mu_k h_k(2x)$$

and

$$\bar{p}(\omega, x) = \sum_{k,l=0}^{N} p_{k,l} h_k(\omega) h_l(2x),$$

where the coefficients are given by,

$$\mu_k = \int_{\mathbb{R}} \mu(x) h_k(2x) \mathrm{d}(2x), \quad k = 0, 1, ..., N$$

and

$$p_{k,l} = \int_{\mathbb{R}} \int_{\mathbb{R}} \bar{p}(\omega, x) h_k(\omega) h_l(2x) \mathrm{d}\omega \mathrm{d}(2x), \quad k, l = 0, 1, ..., N$$

Numerical Examples

Then, we end up with the linear system

$$\boldsymbol{A}\boldsymbol{\mu} = \boldsymbol{m},$$

where $\boldsymbol{A} \in \mathbb{C}^{3N \times N}$, $\boldsymbol{\mu} \in \mathbb{R}^{N}$ and $\boldsymbol{m} \in \mathbb{C}^{3N}$.

We solve it, using Tikhonov regularization

$$\min \left\{ \|\boldsymbol{A}\boldsymbol{\mu} - \boldsymbol{m}\|_2^2 + \lambda^2 \|\boldsymbol{L}\boldsymbol{\mu}\|_2^2 \right\},$$

where λ is the regularization parameter and $\boldsymbol{L}$ is the regularization matrix.

Numerical Examples

1^{st} example: Let $\mathcal{D} = [-4, 4]$ and $\mathcal{W} = [-3, 3]$, we consider

$$\mu(x) = \begin{cases} (0.5\, x^5 + x^4 + x^2)\, e^{-x^2}, & x \in \mathcal{D} \\ 0, & x \in \mathbb{R} \setminus \mathcal{D}. \end{cases}$$

and

$$\Re\mathrm{e}\, \hat{\chi}(\omega, x) = \begin{cases} (h_1(\omega) + h_1(2\omega))\mu(x), & (\omega, x) \in \mathcal{W} \times \mathcal{D} \\ 0, & (\omega, x) \in \mathcal{W} \times (\mathbb{R} \setminus \mathcal{D}) \end{cases}$$

Numerical Examples

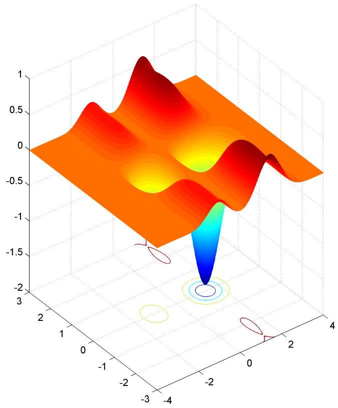
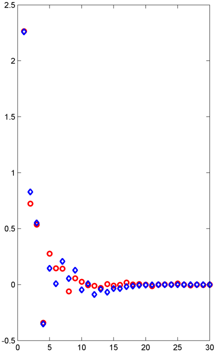


Figure : Exact (left) and reconstructed (right) $\Im m\, \hat{\chi}$. The results are presented for $N = 25$, $\lambda = 0.001$ and 3% noise.

Numerical Examples

2^{nd} example: Let $\mathcal{D} = [-2, 2]$ and $\mathcal{W} = [-3, 3]$, we consider

$$\mu(x) = \begin{cases} h_0(x) + h_0(2x) + h_1(3x), & x \in \mathcal{D} \\ 0, & x \in \mathbb{R} \setminus \mathcal{D}. \end{cases}$$



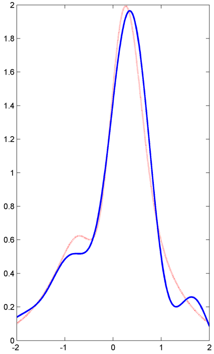


Figure : Reconstruction of the coefficients μ_j (left) and of $\mu(x)$ (right).

Numerical Examples

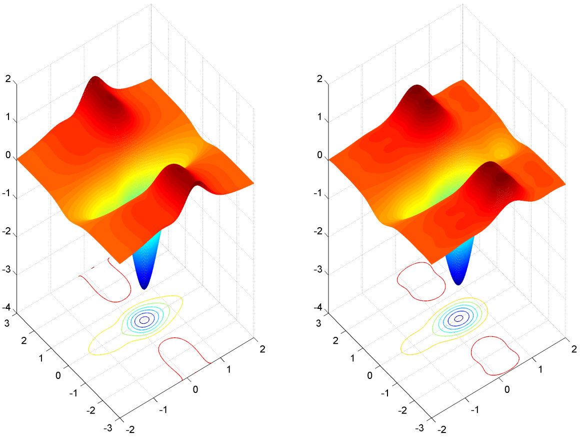


Figure : Exact (left) and reconstructed (right) $\Im\text{m}\,\hat{\chi}$. The results are presented for $N = 30$, $\lambda = 0.001$ and 3% noise.

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- P. Elbau, L. Mindrinos, and O. Scherzer. Reconstruction of optical parameters for the multi-modal PAT-OCT system (in progress).

Thank you for your attention.