Computing Interior Eigenvalues of Domains from Far Fields

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An example

Scattering problem

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D},$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial D} + \tau\, u|_{\partial D} = 0 \qquad \text{on } \partial D,$$

Interior Robin eigenvalue problem $v := u^i$

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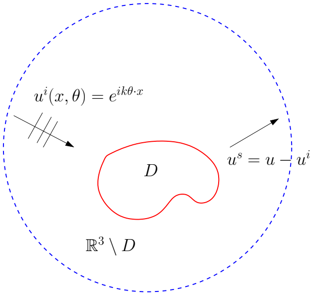

Scattering from a Robin obstacle.

An example

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$$\begin{aligned} \Delta u + k^2 u &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \left.\frac{\partial u}{\partial \nu}\right|_{\partial D} + \tau\, u|_{\partial D} &= 0 && \text{on } \partial D, \end{aligned}$$

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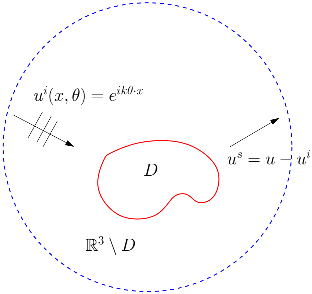

Scattering from a Robin obstacle.

Radiating scattered field

$$u^s(x,\theta) = \frac{\exp(\mathrm{i}k|x|)}{4\pi|x|}\left(u^\infty(\hat{x},\theta) + \mathcal{O}\left(\frac{1}{|x|}\right)\right) \text{ as } |x| \to \infty.$$

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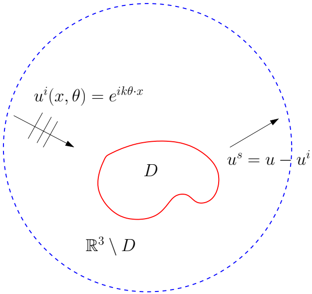

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Far field operator

$$F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2), \quad Fg(\hat{x}) := \int_{\mathbb{S}^2} u^\infty(\hat{x},\theta) g(\theta)\, \mathrm{d}S(\theta), \qquad \hat{x} \in \mathbb{S}^2.$$

Inside-outside duality

- Spectral decomposition: $Fg = \sum_{j\in\mathbb{N}} \lambda_j (g, g_j) g_j \quad \forall g \in L^2(\mathbb{S}^2)$ with $\lambda_j \to 0$ as $j \to \infty$.
- λ_j are on the circle $|z - \mathrm{i}8\pi^2/k| = 8\pi^2/k$.
- $\lambda_j = r_j \exp(\mathrm{i}\vartheta_j)$ with $r_j \geq 0$ and $\vartheta_j \in [0, \pi)$. $\mathrm{Re}\,(\lambda_j) > 0$ for all $j \in \mathbb{N}$ large enough. Since $\lambda_j \to 0$ as $j \to \infty$, $\vartheta_j \to 0$ as $j \to \infty$. Hence $\vartheta^* = \max_{j\in\mathbb{N}} \vartheta_j$ is well defined and attained by some $\lambda^* \neq 0$.

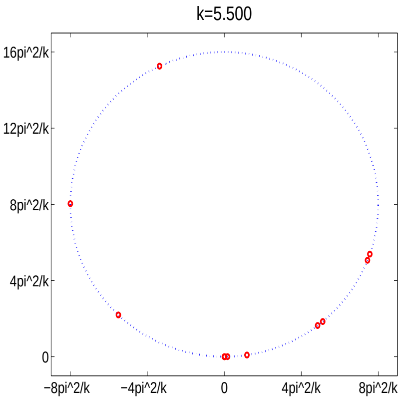
Theorem (Lechleiter-Peters 2014)

If $\lim_{i\to\infty} \vartheta^*(k_i) = \pi$ for some sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i > k$ and $k_i \to k > 0$ as $i \to \infty$, then k^2 is an interior Robin eigenvalue of $-\Delta$ in D. If $k^2 > 0$ is a Robin eigenvalue of $-\Delta$ in D then $\lim_{i\to\infty} \vartheta^*(k_i) = \pi$ for any sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i > k$ that tends to k from above.

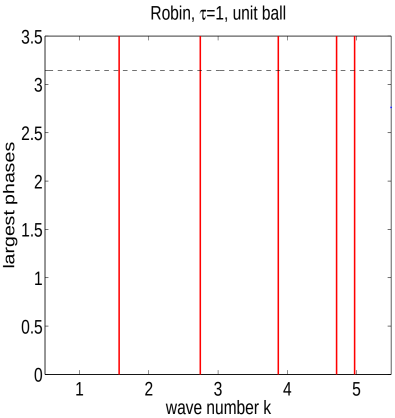
- Inverse scattering: Colton-Kress, Kirsch-Grinberg
- Interior eigenvalues: Cakoni-Colton-Monk 07, Cakoni-Gintides-Haddar 10.
- Inside-outside duality:
  - Obstacles with Dirichlet, Neumann or Robin boundary condition: Eckmann-Pillet 95, Lechleiter-Peters 14.
  - Isotropic or anisotropic penetrable acoustic media: Kirsch-Lechleiter 13, Lechleiter-Peters 15.
  - Electromagnetic scattering form penetrable anisotropic dielectric media: Lechleiter-Rennoch 15.

Straightforward approximation of interior eigenvalues

Eigenvalues.



Largest phases.



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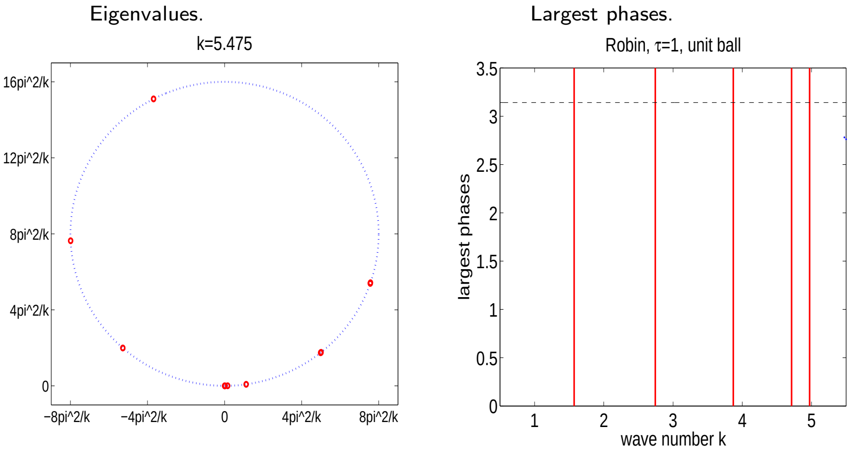

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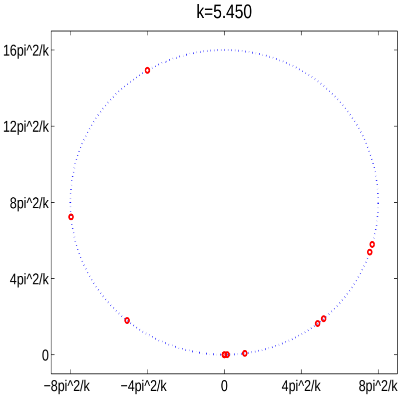

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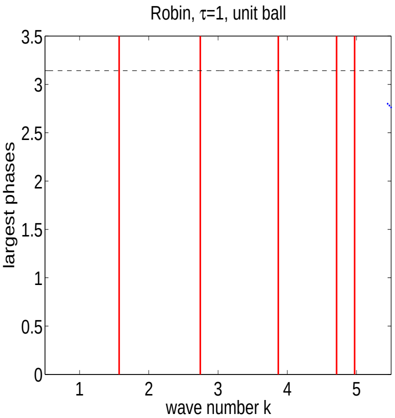

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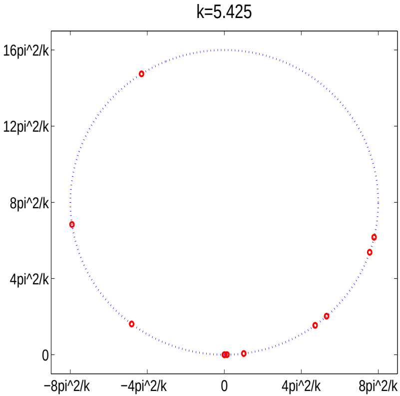

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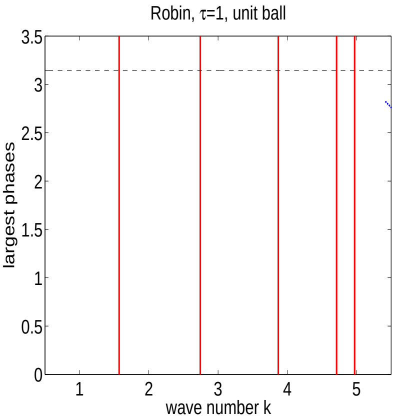

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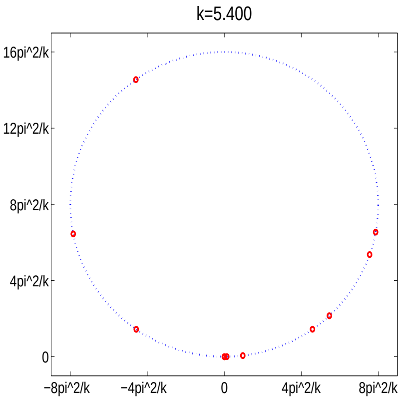

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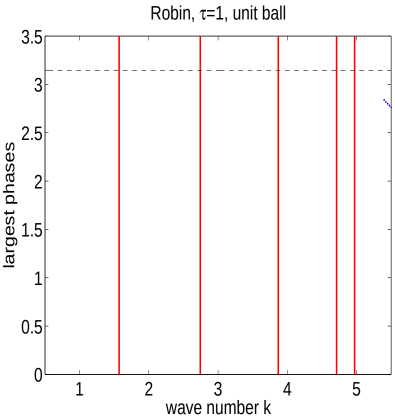

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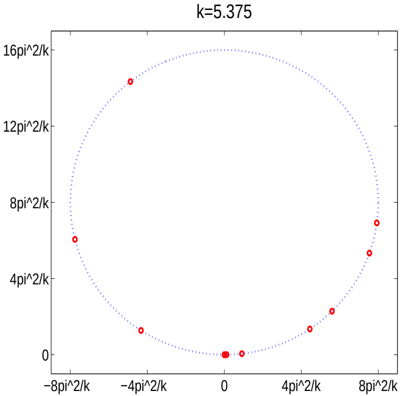

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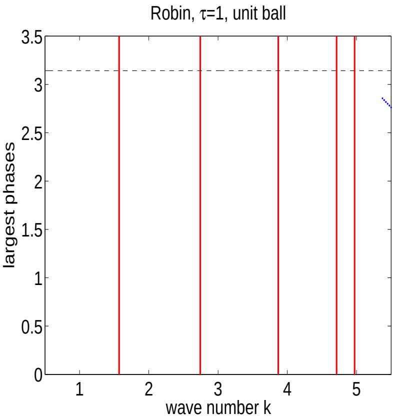

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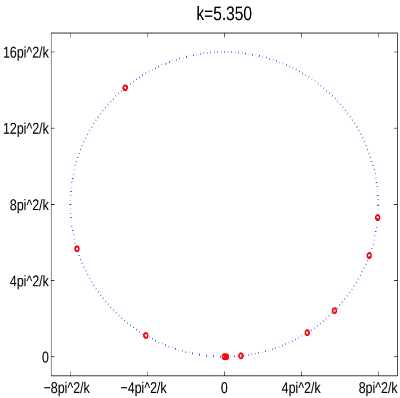

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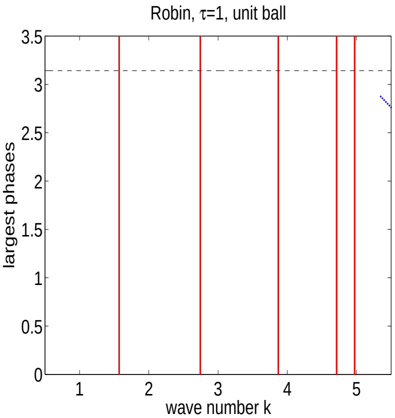

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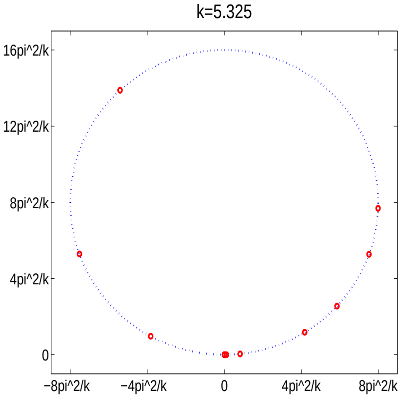

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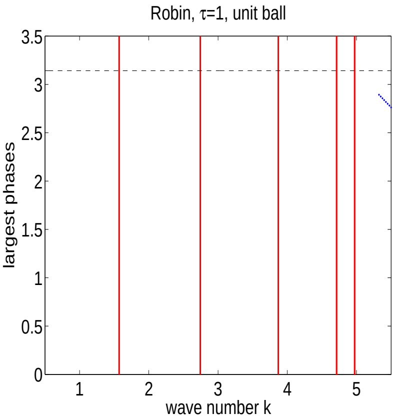

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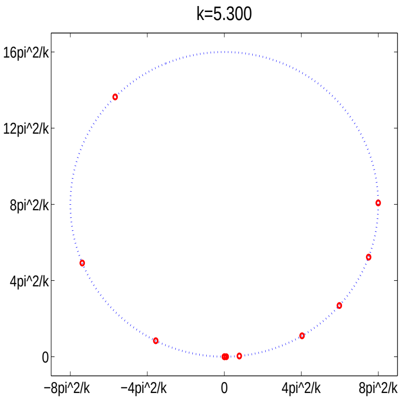

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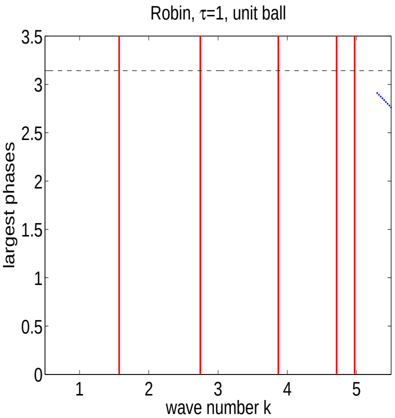

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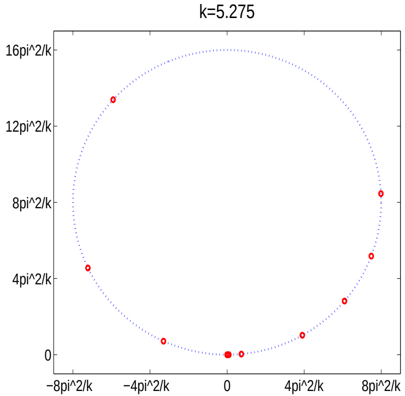

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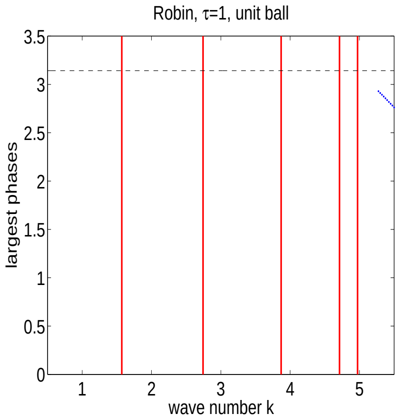

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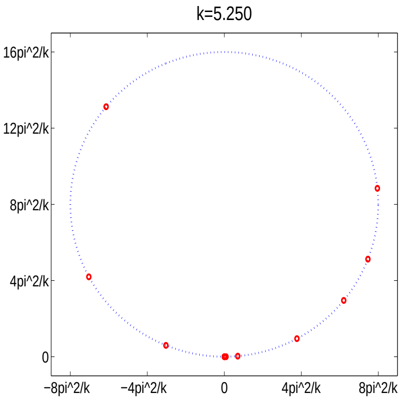

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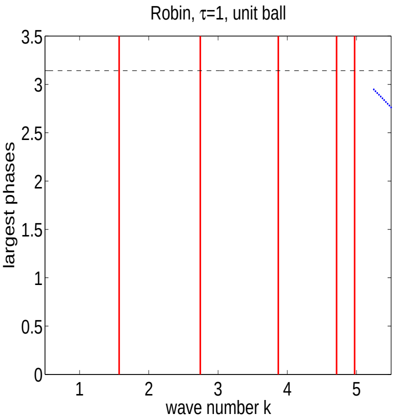

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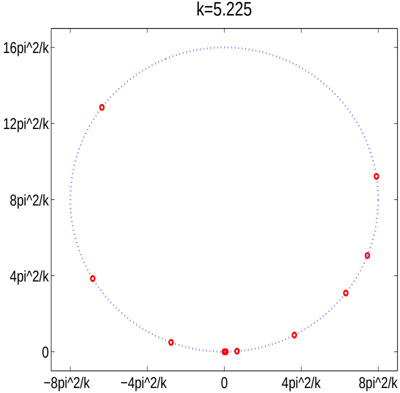

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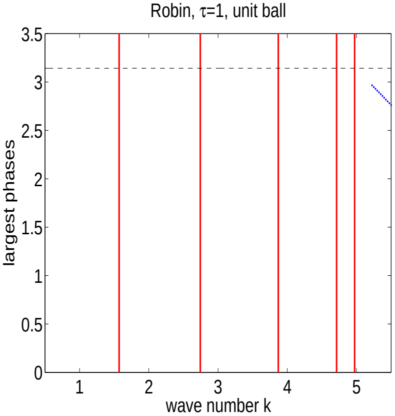

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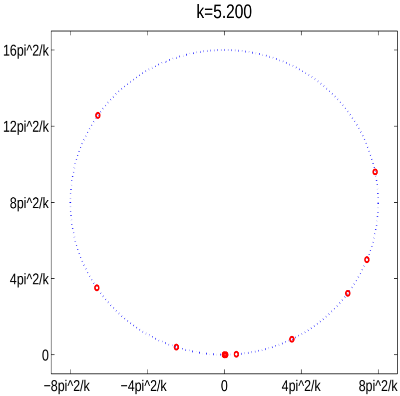

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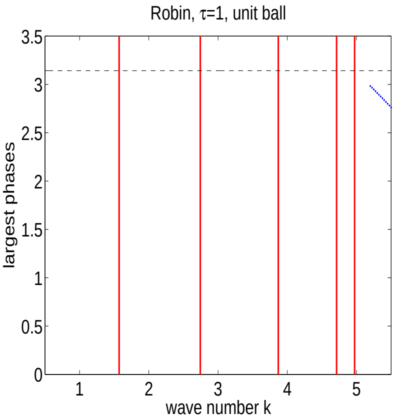

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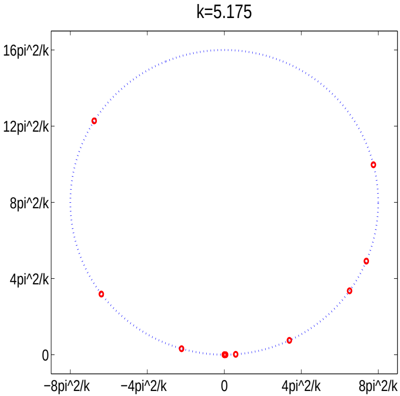

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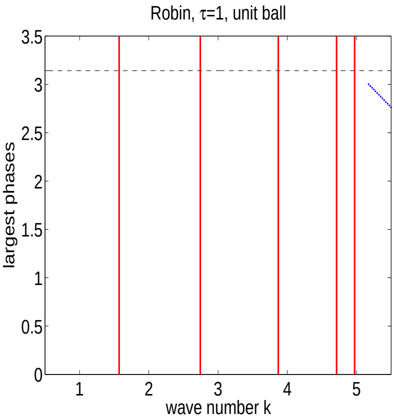

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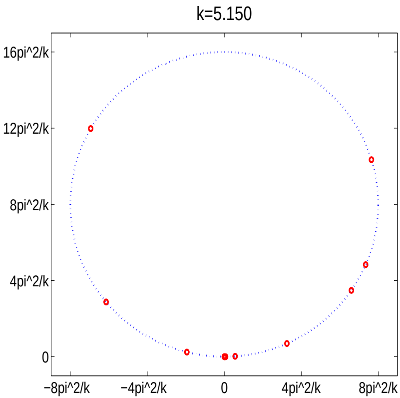

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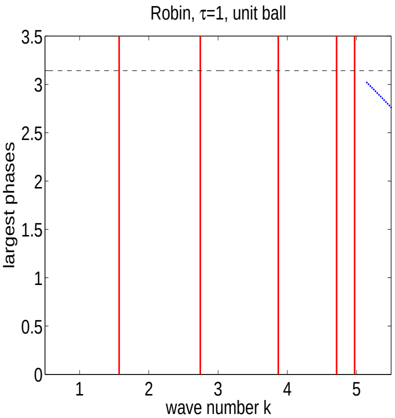

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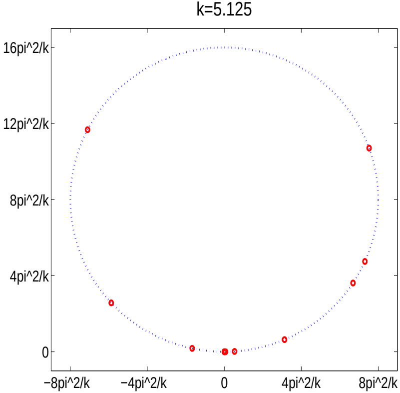

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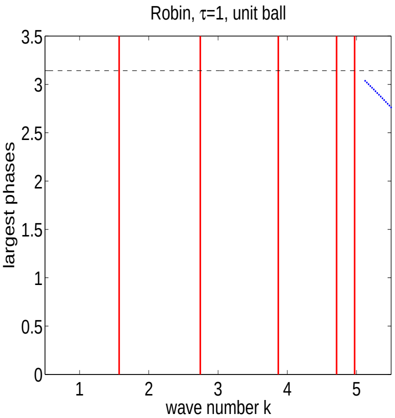

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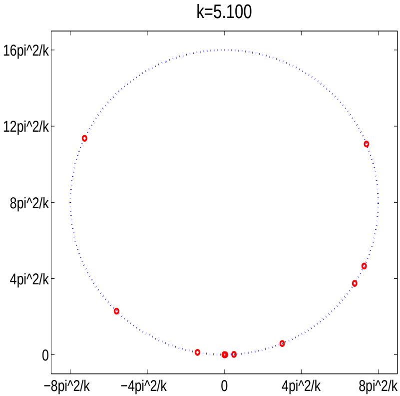

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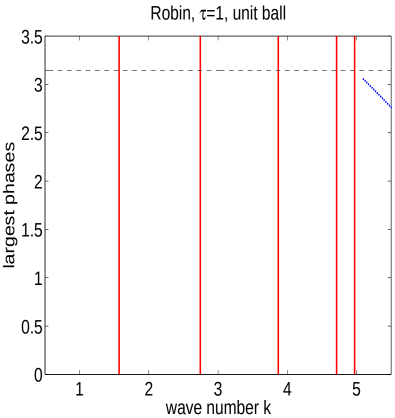

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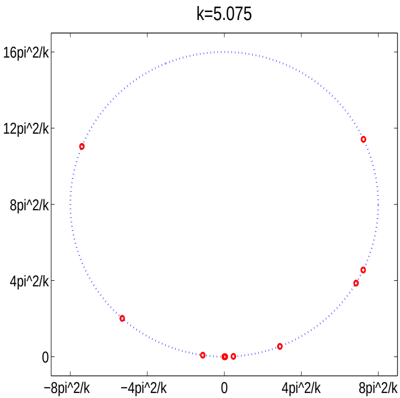

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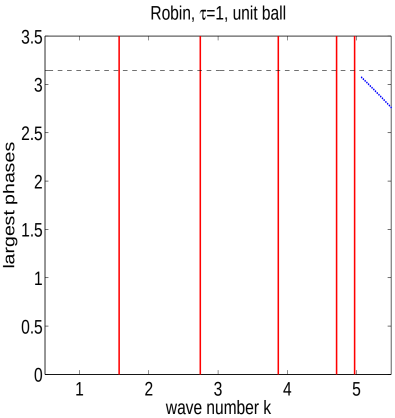

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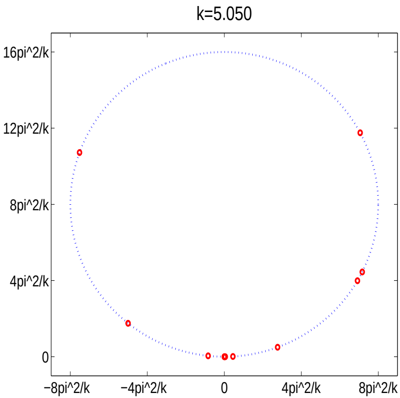

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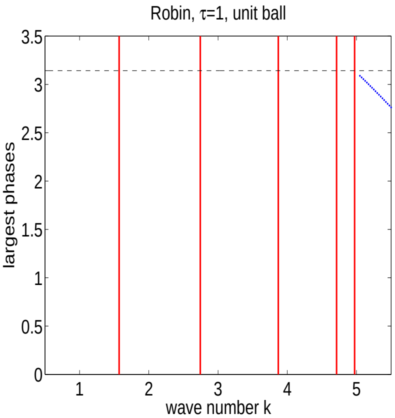

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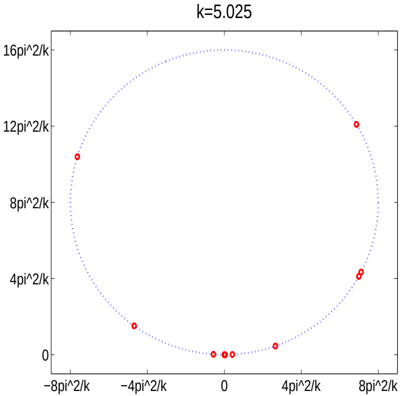

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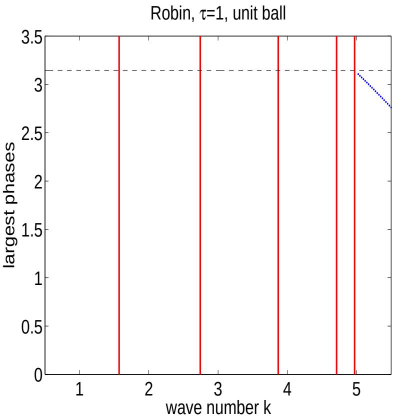

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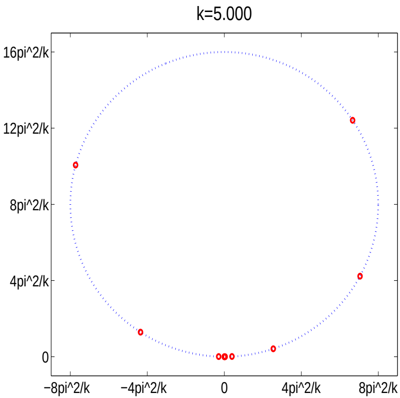

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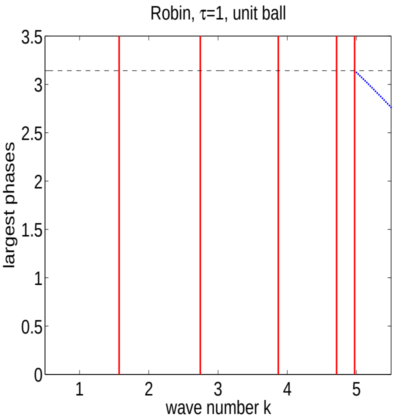

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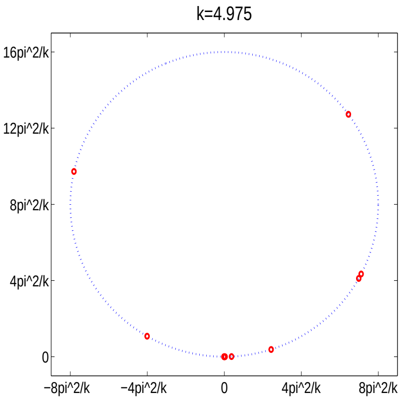

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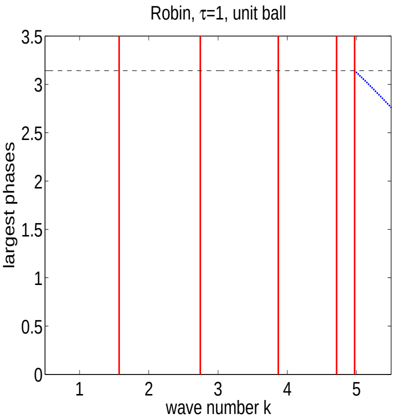

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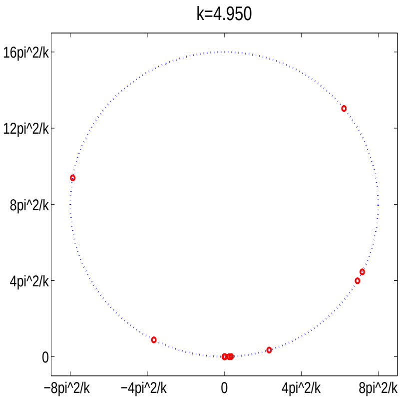

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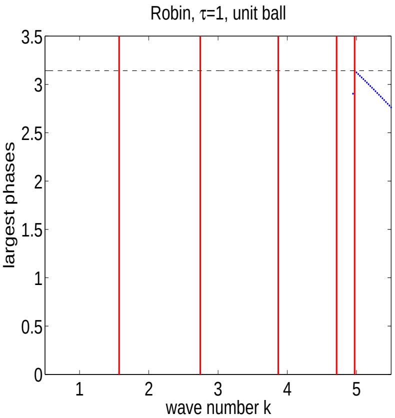

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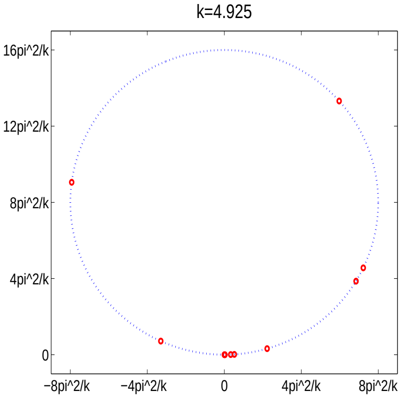

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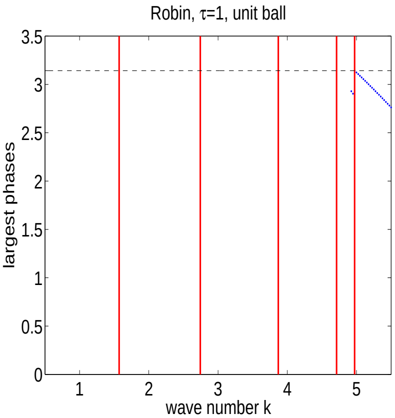

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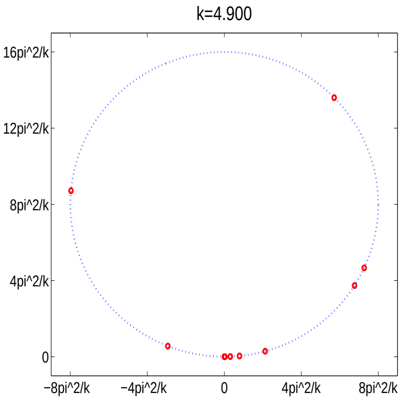

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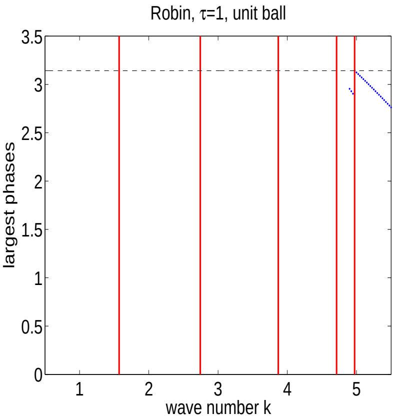

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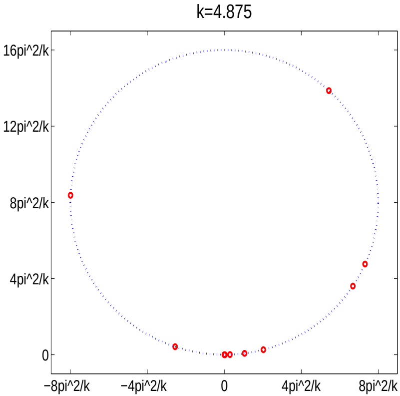

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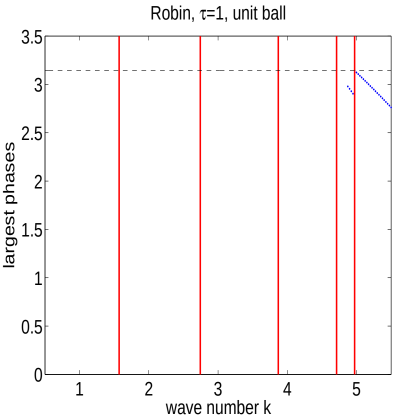

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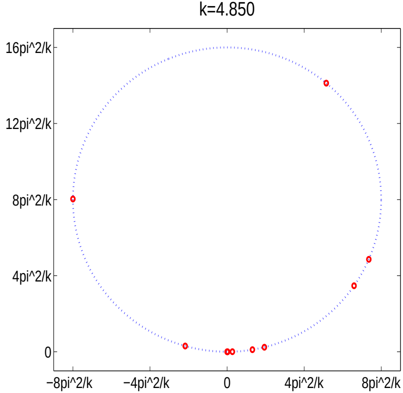

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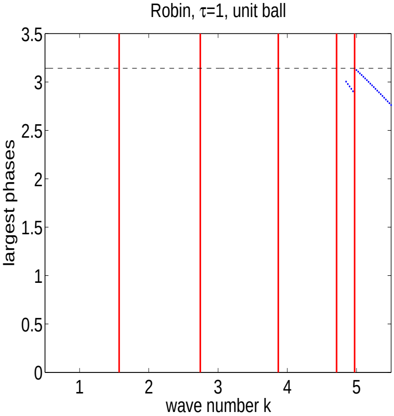

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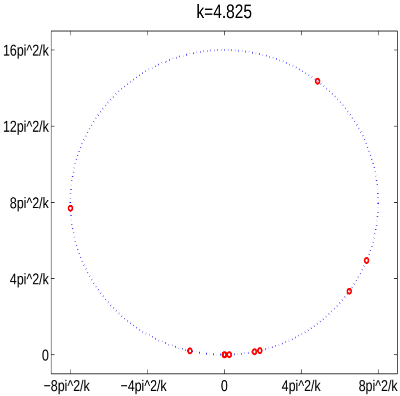

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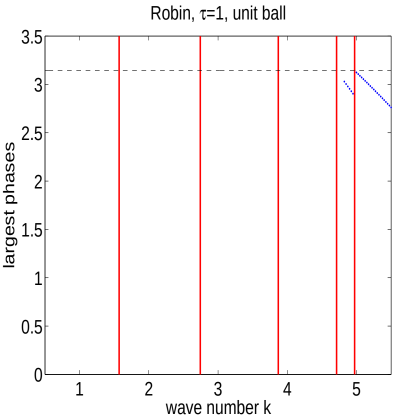

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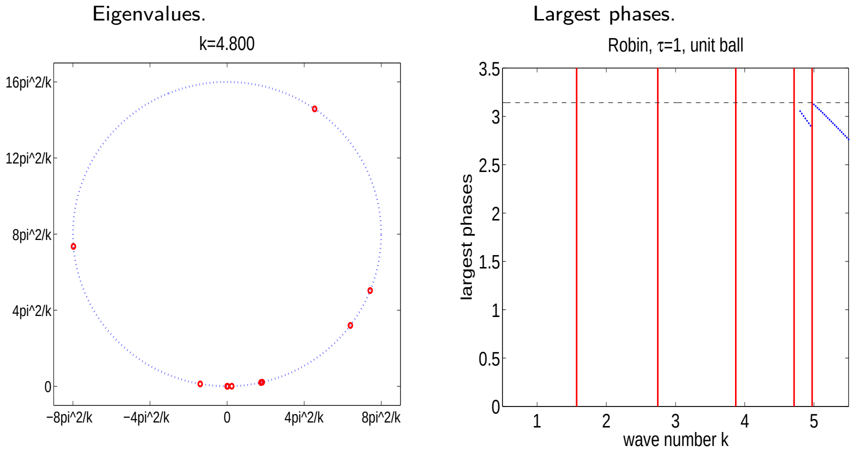

Straightforward approximation of interior eigenvalues

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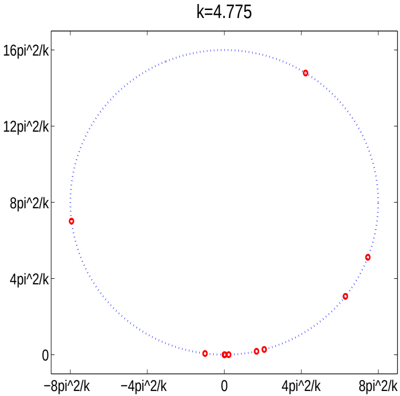
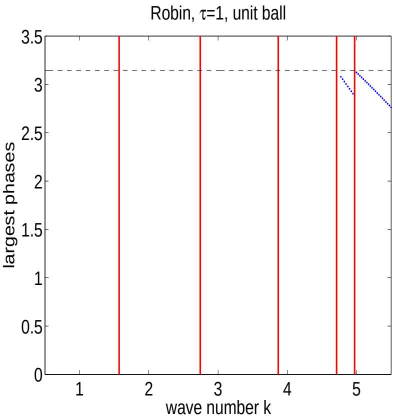
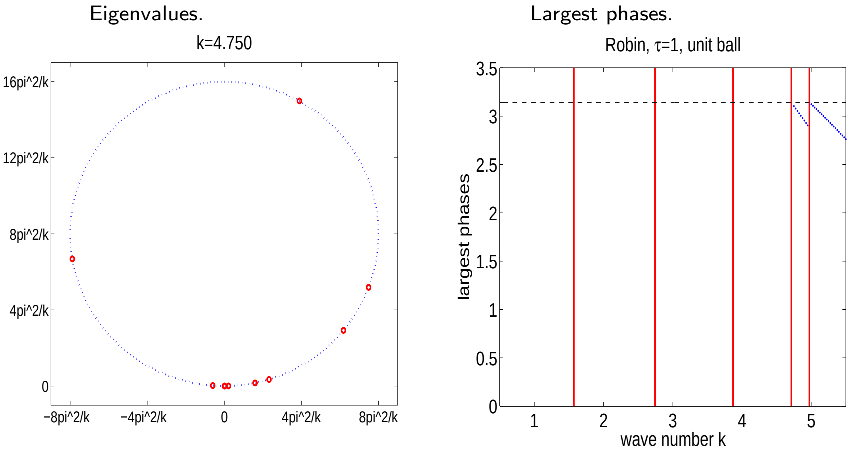


Largest phases.

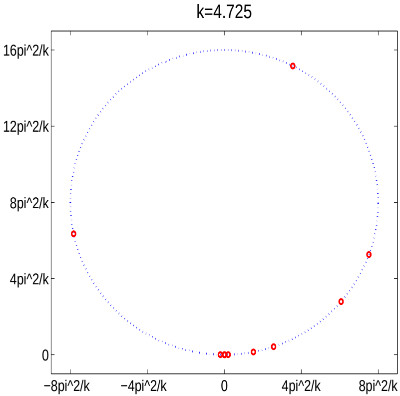


Straightforward approximation of interior eigenvalues



Straightforward approximation of interior eigenvalues

Eigenvalues.



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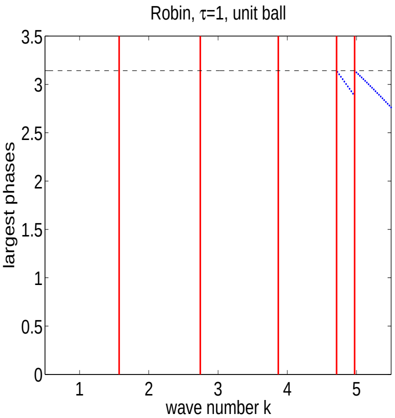
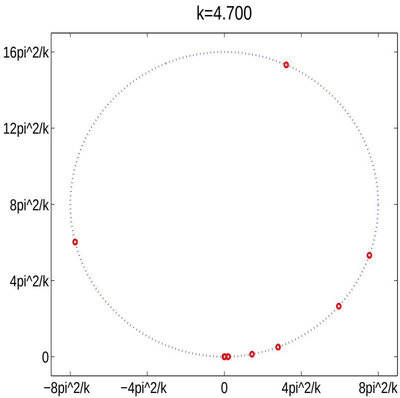


Straightforward approximation of interior eigenvalues

Eigenvalues.



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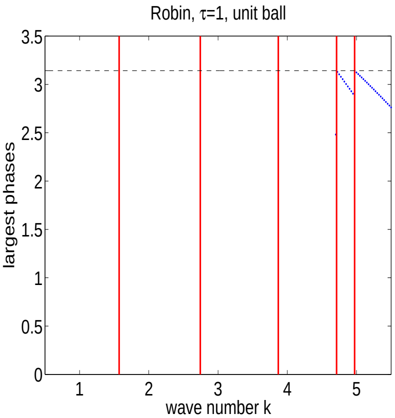
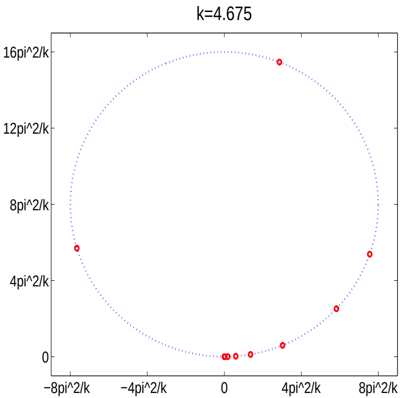


Straightforward approximation of interior eigenvalues

Eigenvalues.



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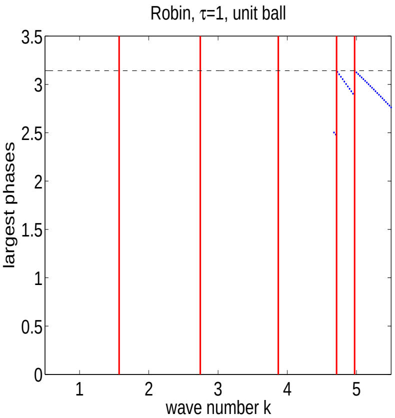
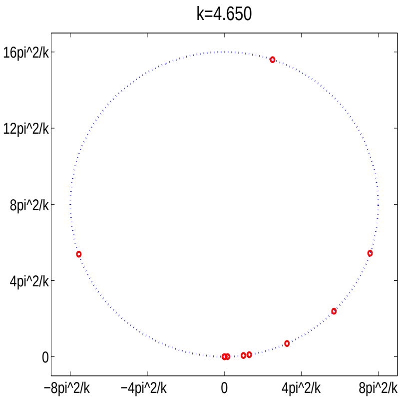


Straightforward approximation of interior eigenvalues

Eigenvalues.



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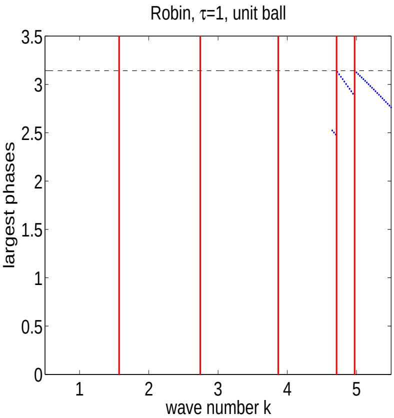
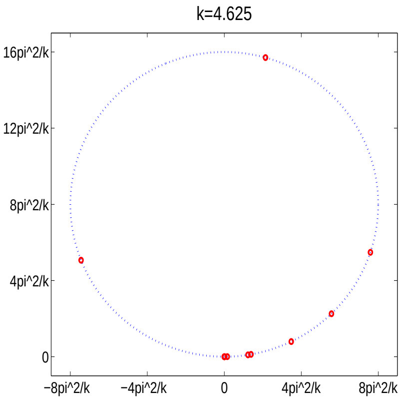


Straightforward approximation of interior eigenvalues

Eigenvalues.



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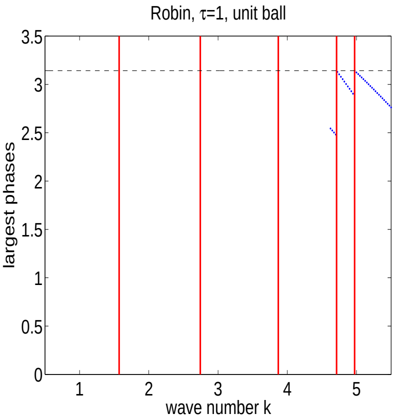
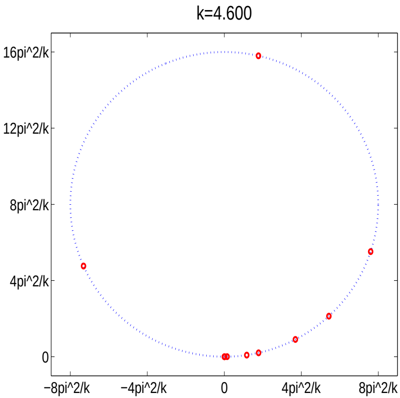


Straightforward approximation of interior eigenvalues

Eigenvalues.



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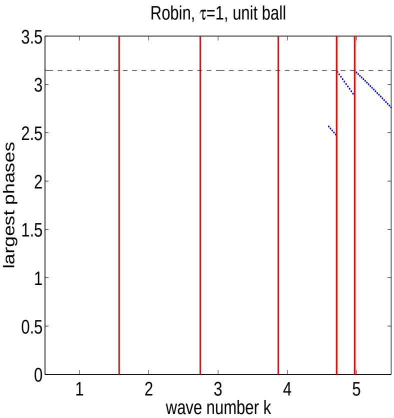
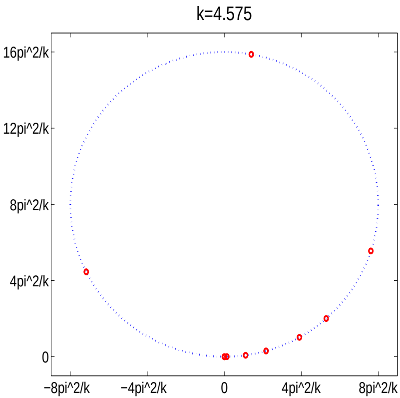


Straightforward approximation of interior eigenvalues

Eigenvalues.



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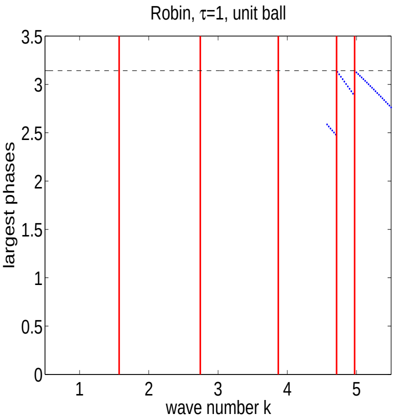
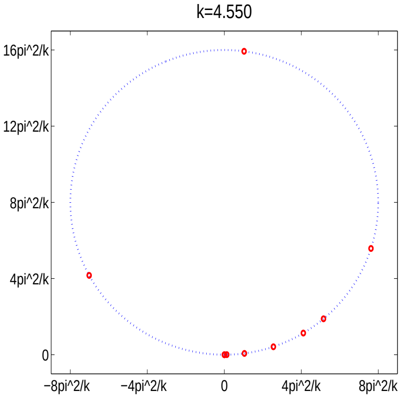


Straightforward approximation of interior eigenvalues

Eigenvalues.



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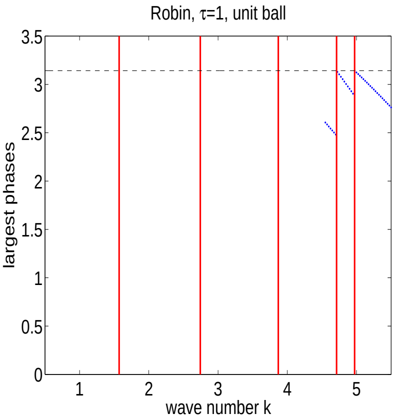
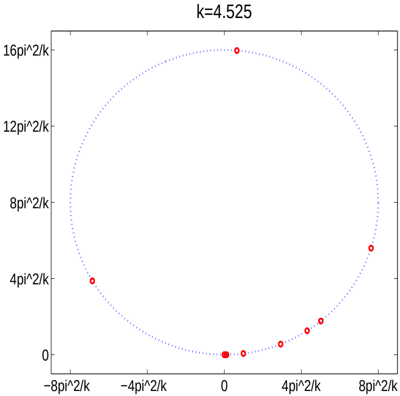


Straightforward approximation of interior eigenvalues

Eigenvalues.



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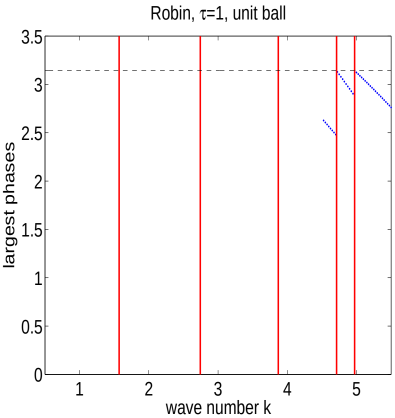

Straightforward approximation of interior eigenvalues

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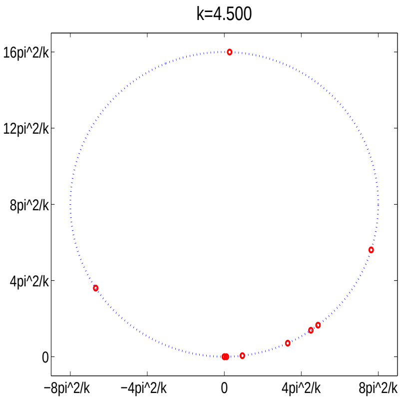

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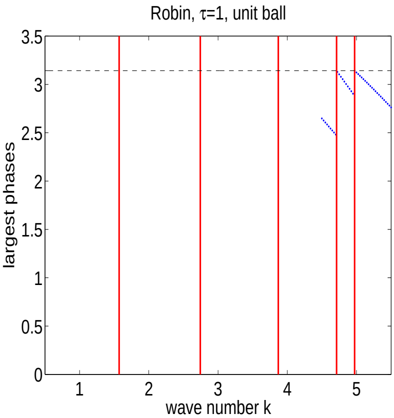
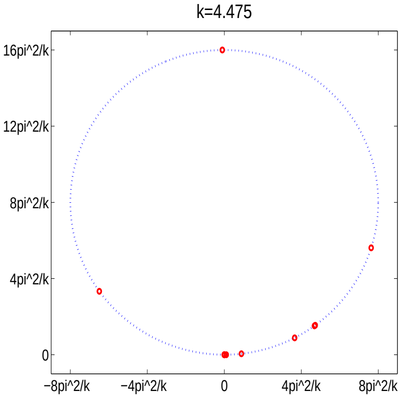


Straightforward approximation of interior eigenvalues

Eigenvalues.



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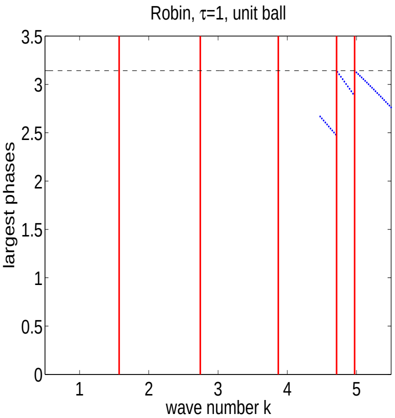
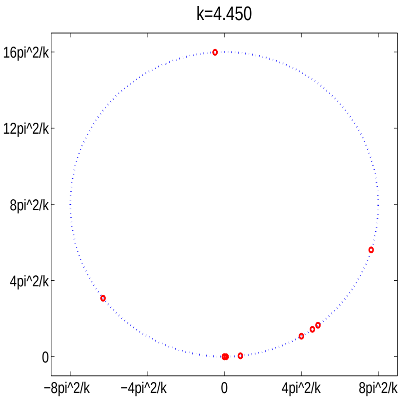


Straightforward approximation of interior eigenvalues

Eigenvalues.



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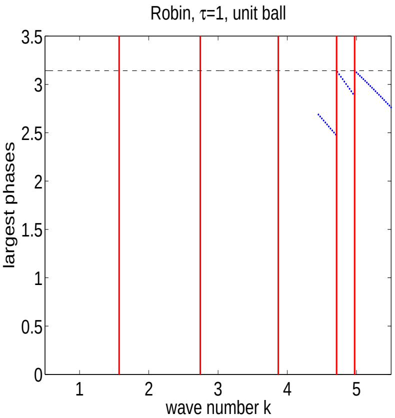
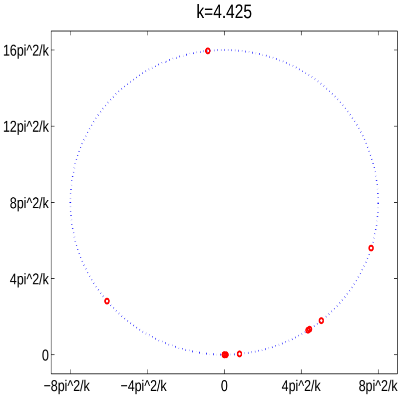


Straightforward approximation of interior eigenvalues

Eigenvalues.



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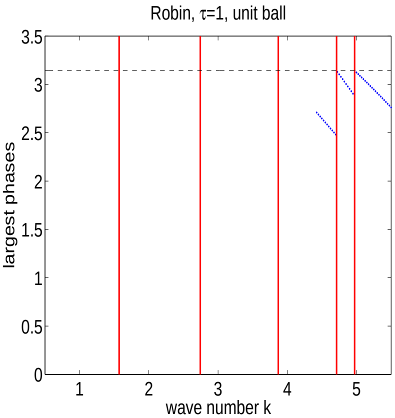
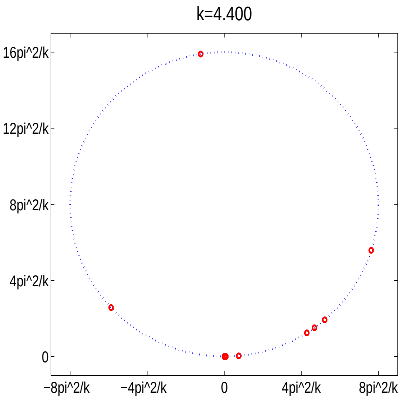


Straightforward approximation of interior eigenvalues

Eigenvalues.



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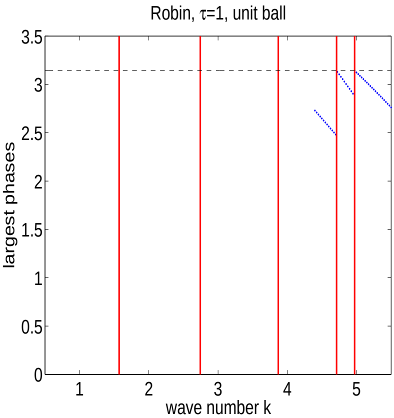
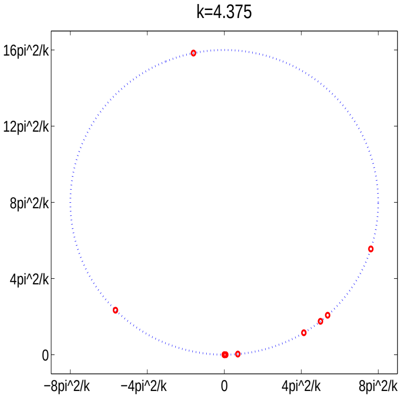


Straightforward approximation of interior eigenvalues

Eigenvalues.



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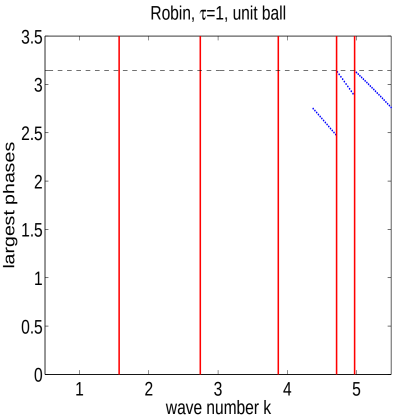
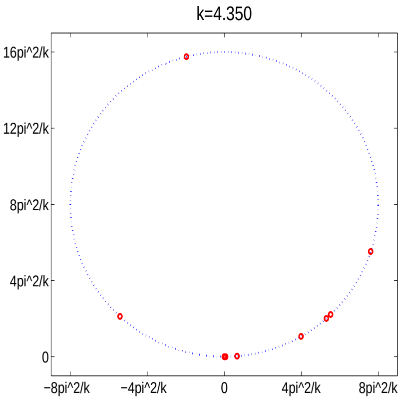


Straightforward approximation of interior eigenvalues

Eigenvalues.



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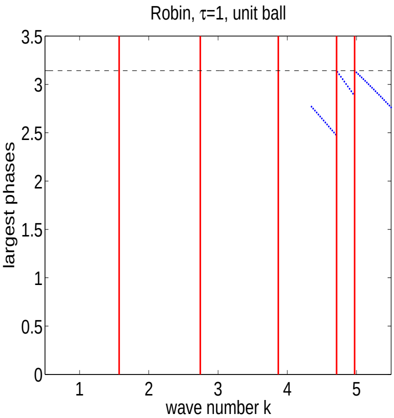
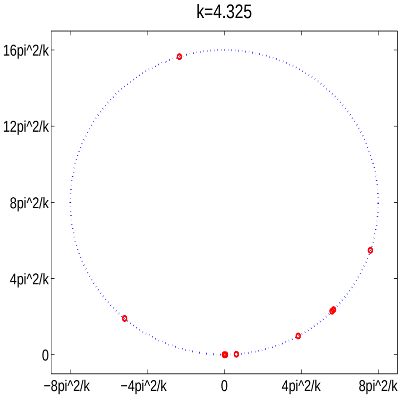


Straightforward approximation of interior eigenvalues

Eigenvalues.



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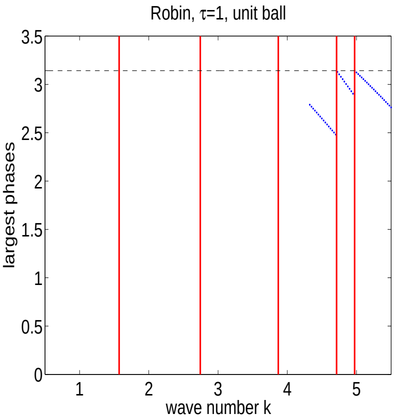
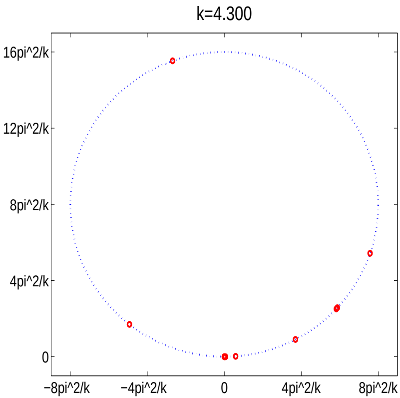


Straightforward approximation of interior eigenvalues

Eigenvalues.



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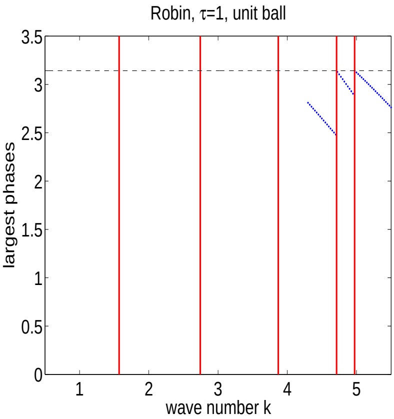
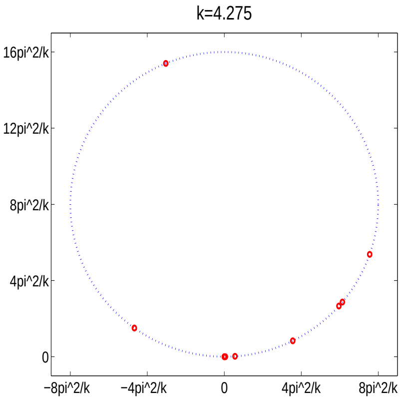


Straightforward approximation of interior eigenvalues

Eigenvalues.



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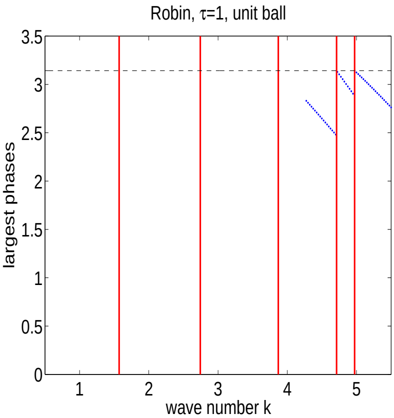

Straightforward approximation of interior eigenvalues

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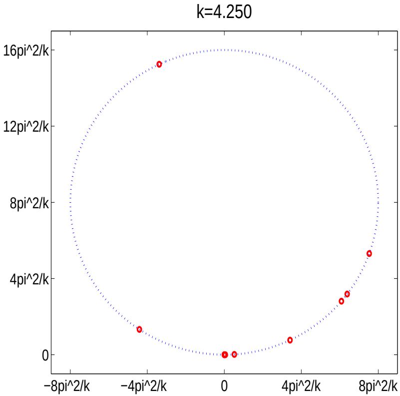

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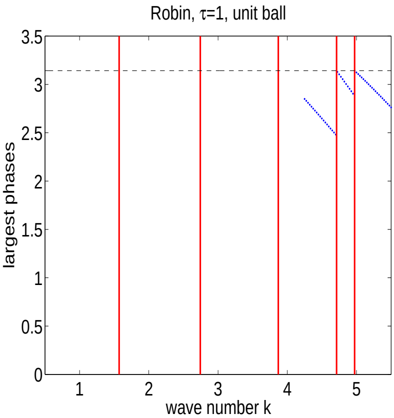
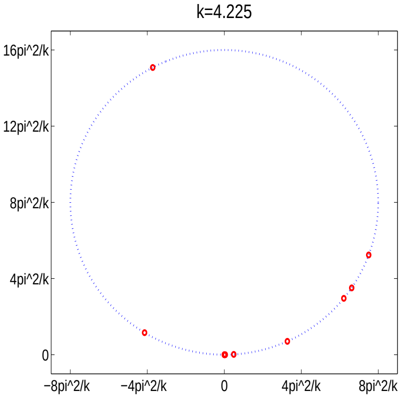


Straightforward approximation of interior eigenvalues

Eigenvalues.



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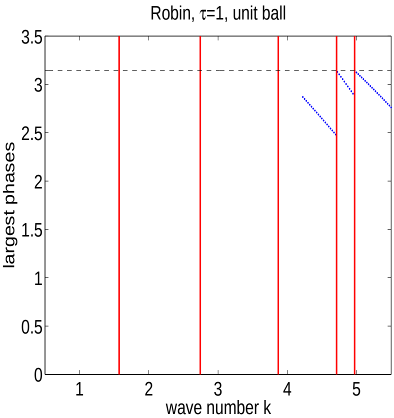

Straightforward approximation of interior eigenvalues

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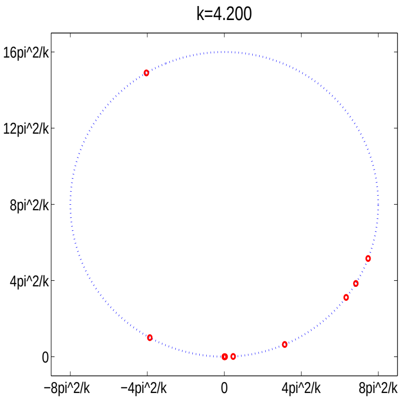

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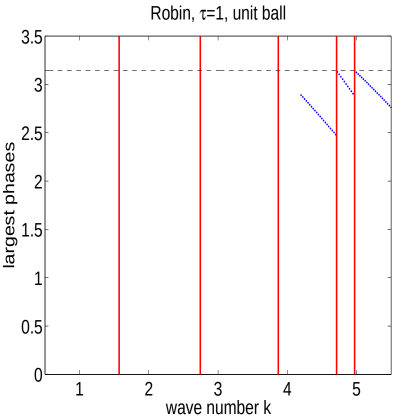

Straightforward approximation of interior eigenvalues

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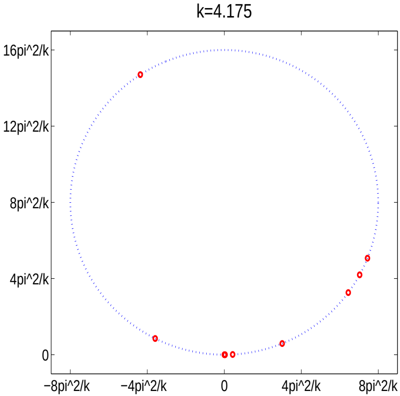

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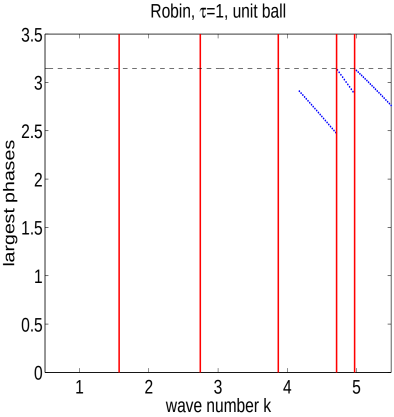

Straightforward approximation of interior eigenvalues

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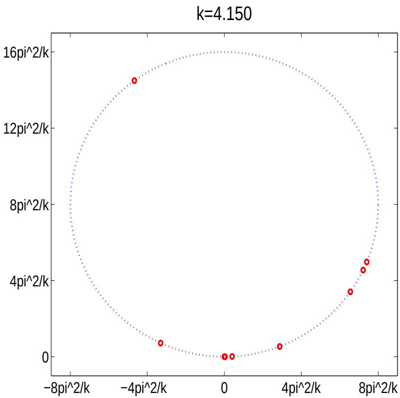

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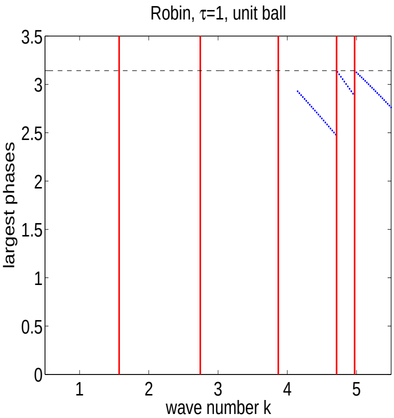
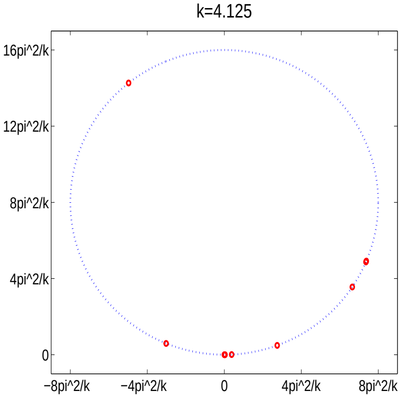


Straightforward approximation of interior eigenvalues

Eigenvalues.



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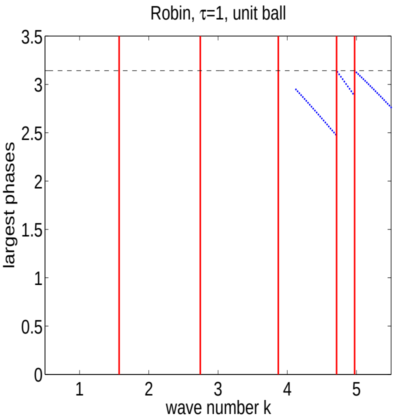
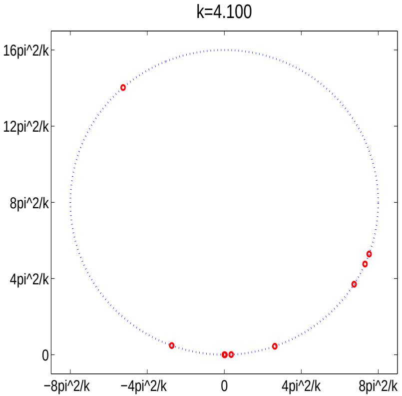


Straightforward approximation of interior eigenvalues

Eigenvalues.



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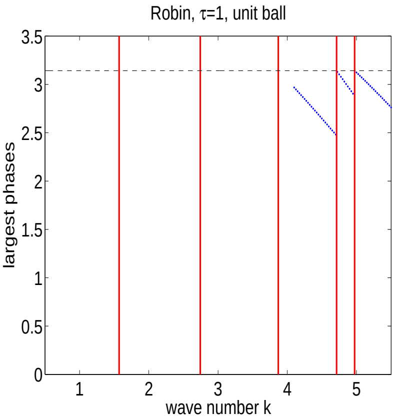

Straightforward approximation of interior eigenvalues

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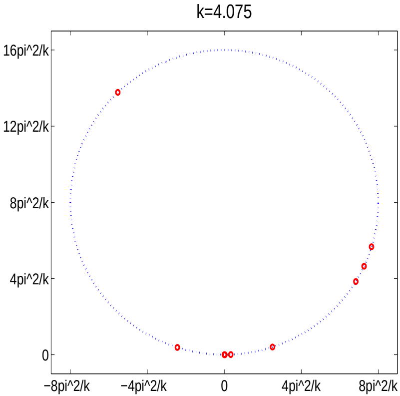

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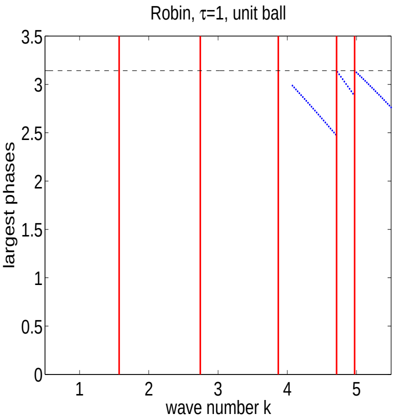

Straightforward approximation of interior eigenvalues

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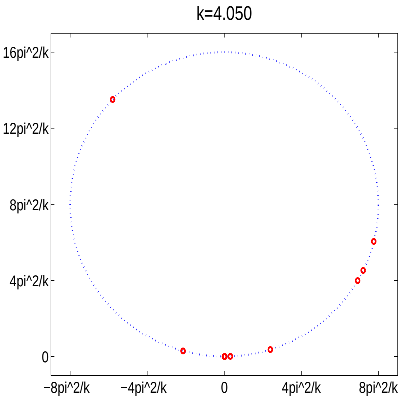

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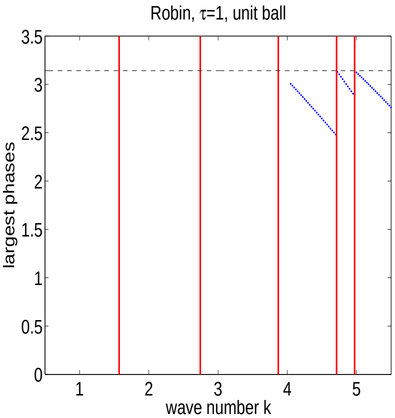

Straightforward approximation of interior eigenvalues

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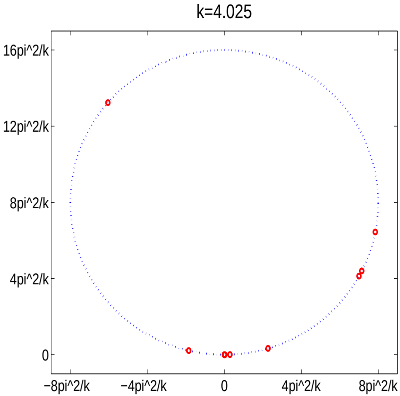

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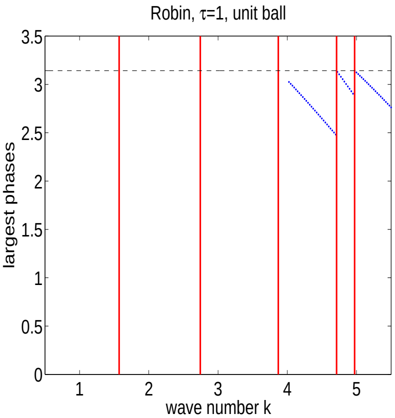

Straightforward approximation of interior eigenvalues

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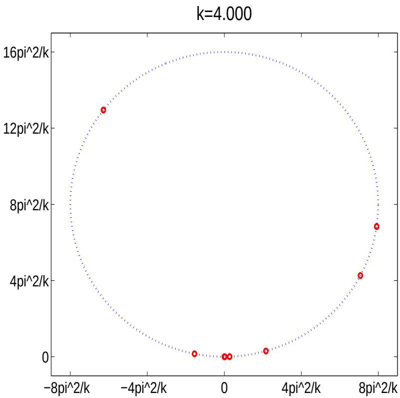

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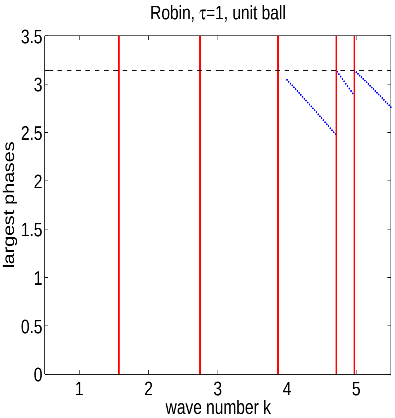

Straightforward approximation of interior eigenvalues

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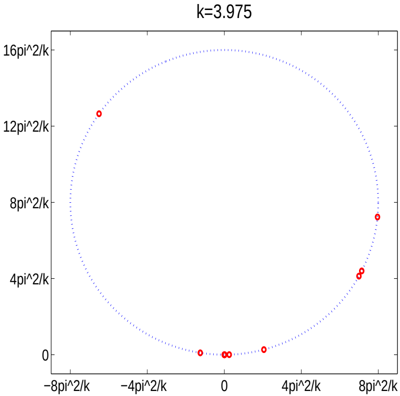

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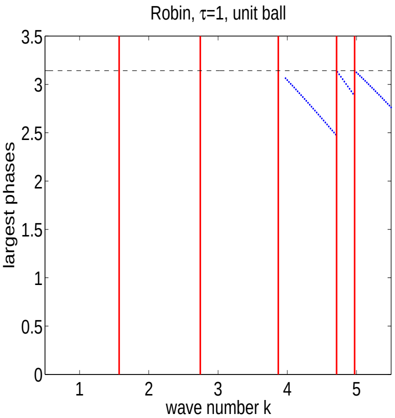
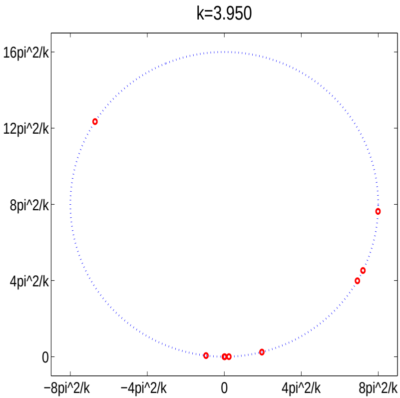


Straightforward approximation of interior eigenvalues

Eigenvalues.



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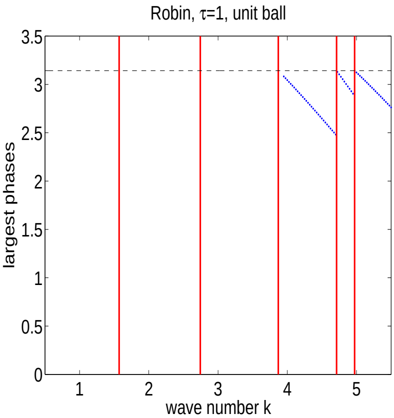
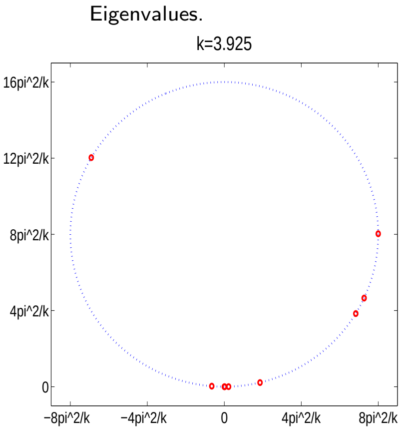
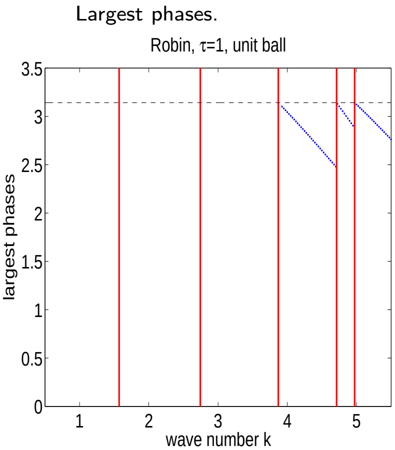


Straightforward approximation of interior eigenvalues





Straightforward approximation of interior eigenvalues

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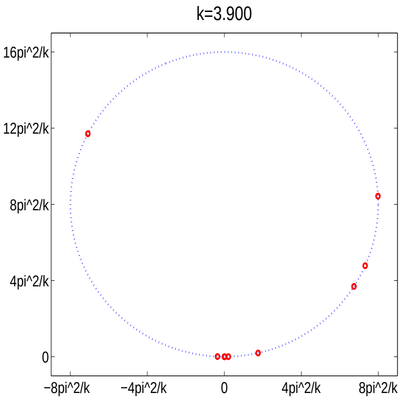

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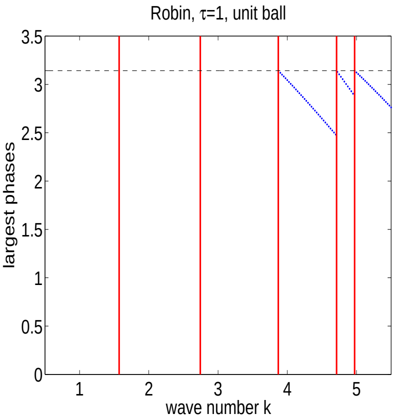
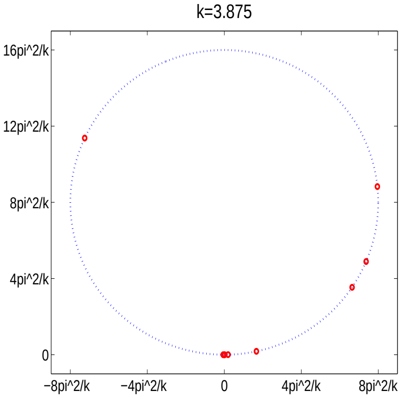
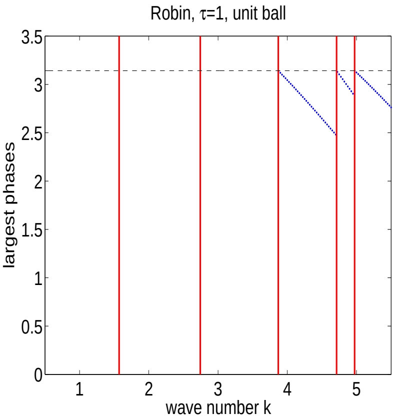


Straightforward approximation of interior eigenvalues

Eigenvalues.

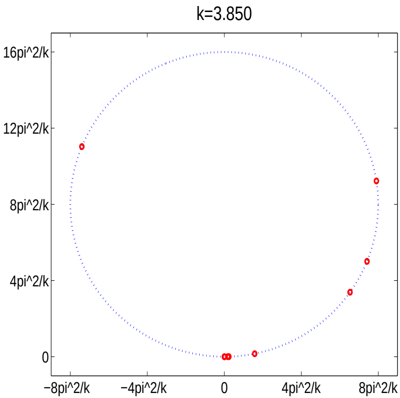


Largest phases.



Straightforward approximation of interior eigenvalues

Eigenvalues.



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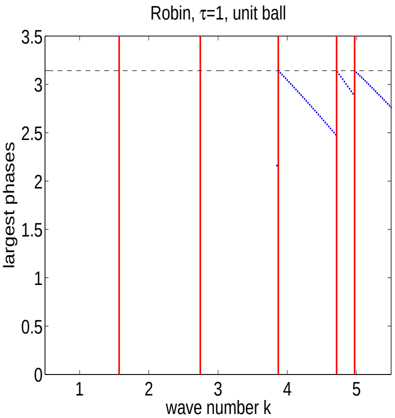
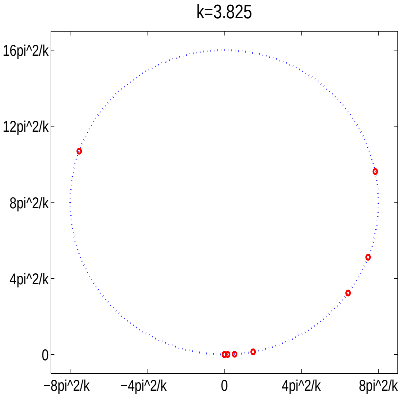


Straightforward approximation of interior eigenvalues

Eigenvalues.



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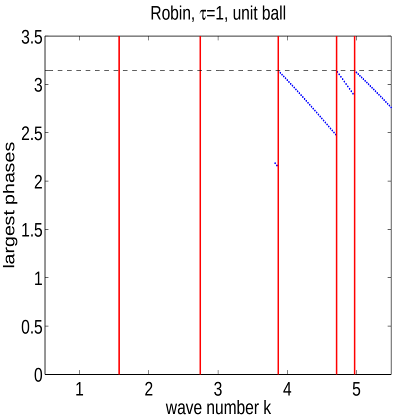
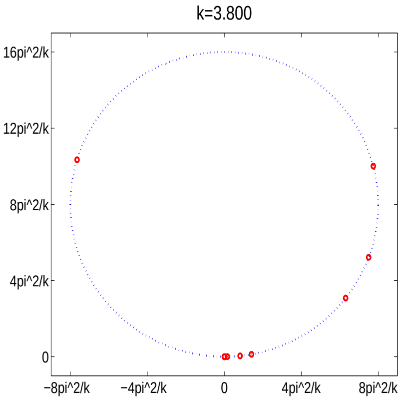


Straightforward approximation of interior eigenvalues

Eigenvalues.



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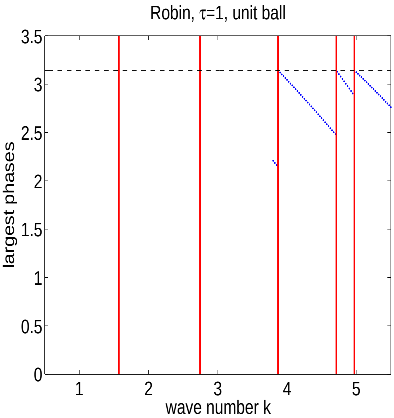
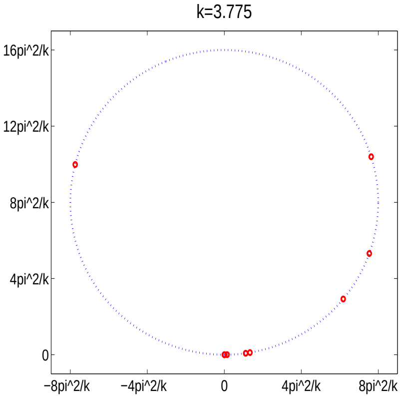


Straightforward approximation of interior eigenvalues

Eigenvalues.



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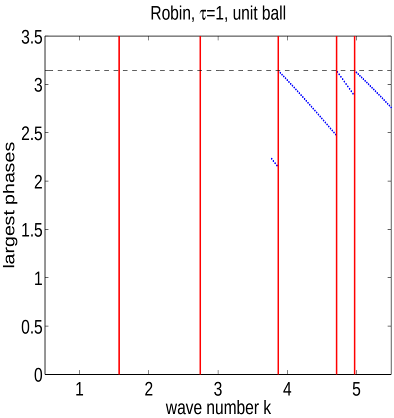
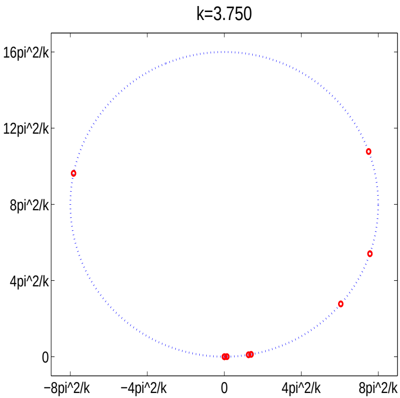


Straightforward approximation of interior eigenvalues

Eigenvalues.



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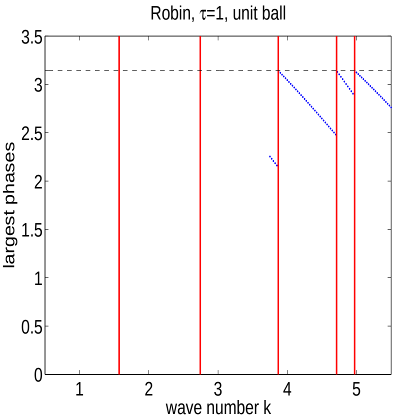
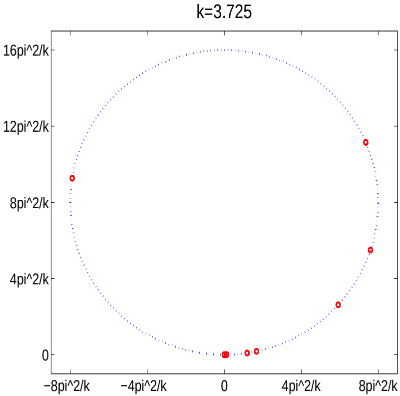


Straightforward approximation of interior eigenvalues

Eigenvalues.



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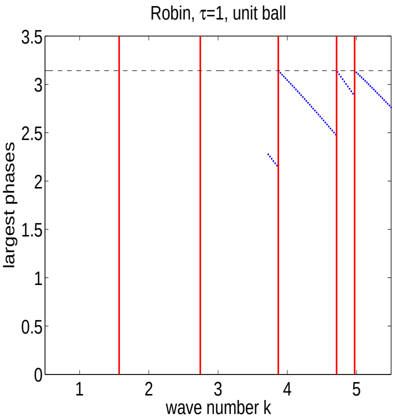
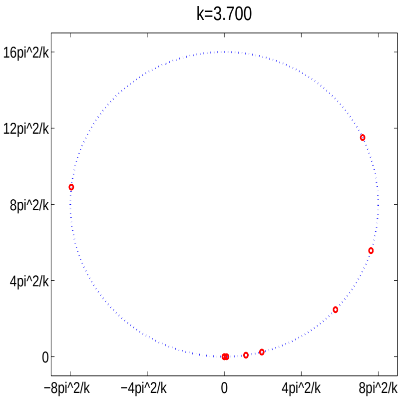


Straightforward approximation of interior eigenvalues

Eigenvalues.



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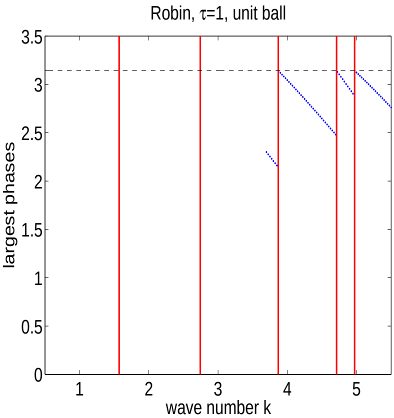
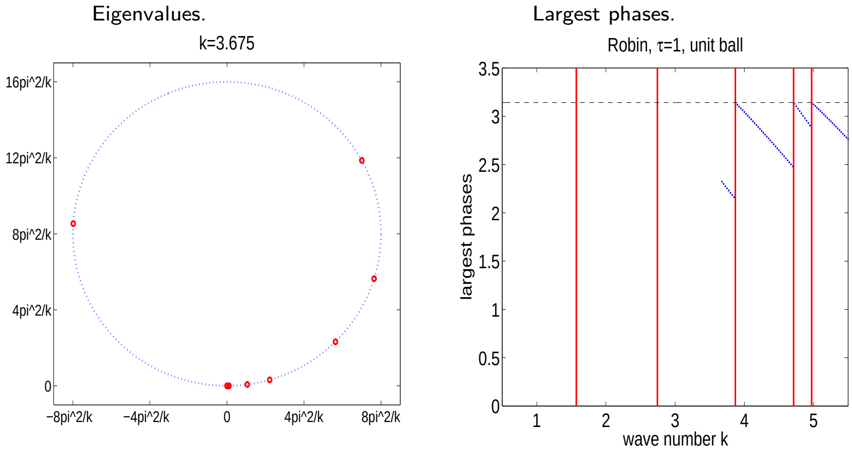
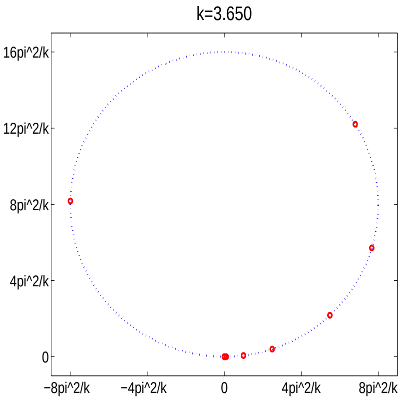


Straightforward approximation of interior eigenvalues



Straightforward approximation of interior eigenvalues

Eigenvalues.



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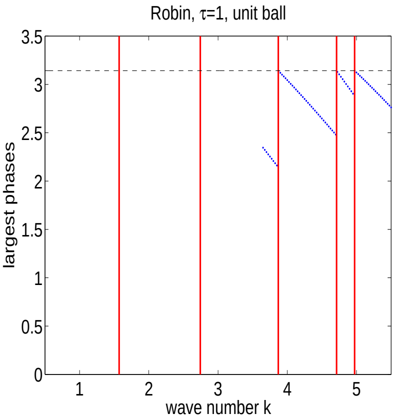
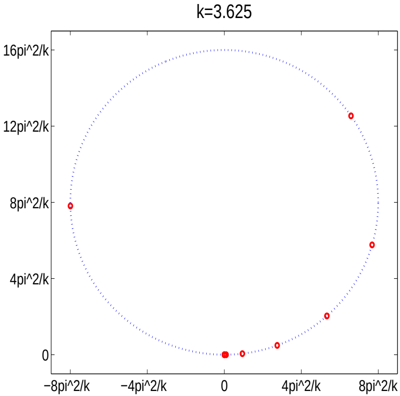


Straightforward approximation of interior eigenvalues

Eigenvalues.



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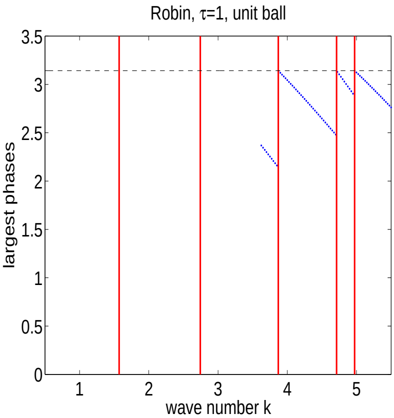
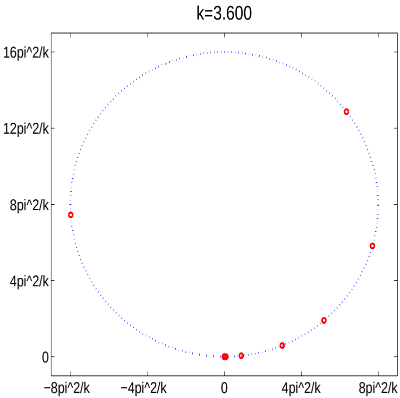


Straightforward approximation of interior eigenvalues

Eigenvalues.



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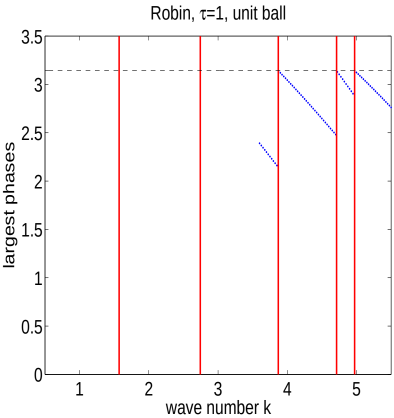
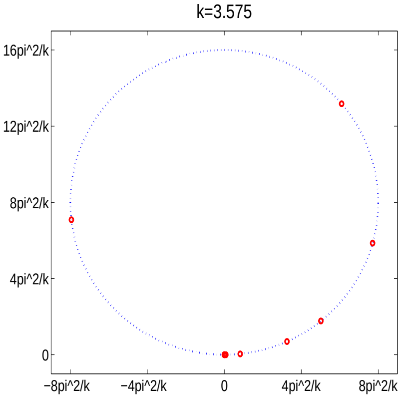


Straightforward approximation of interior eigenvalues

Eigenvalues.



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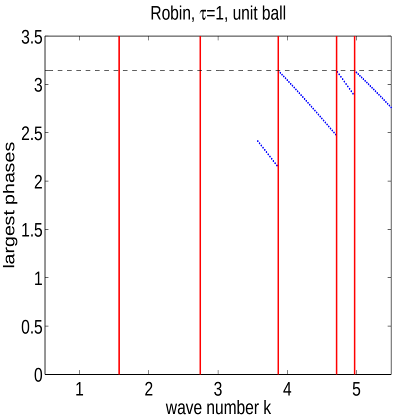
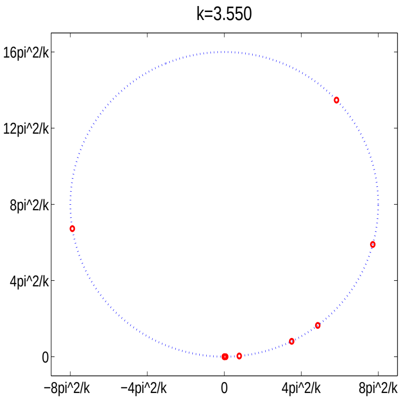


Straightforward approximation of interior eigenvalues

Eigenvalues.



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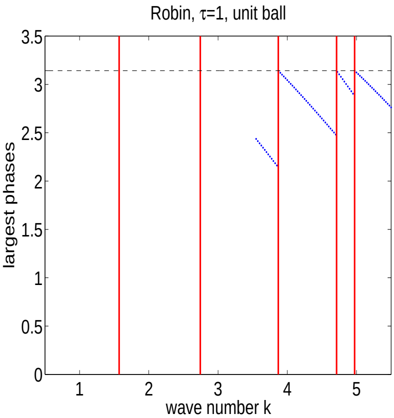
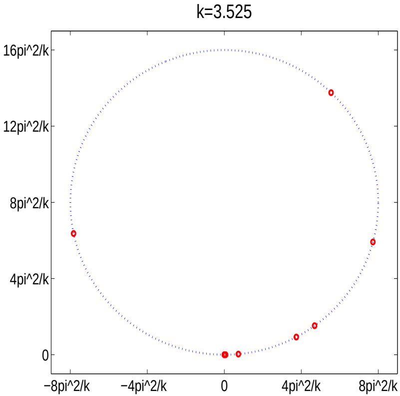


Straightforward approximation of interior eigenvalues

Eigenvalues.



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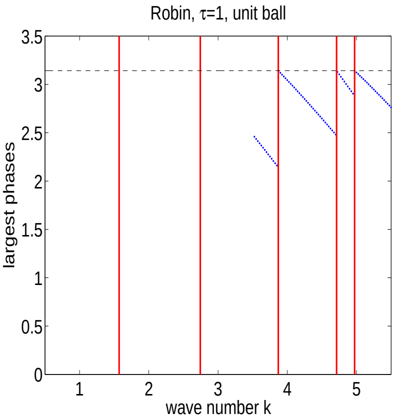
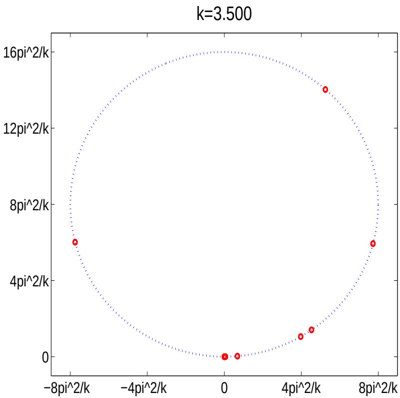


Straightforward approximation of interior eigenvalues

Eigenvalues.



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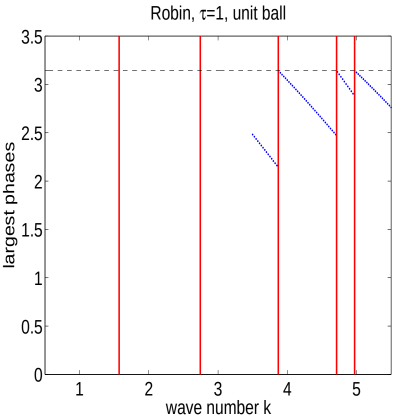
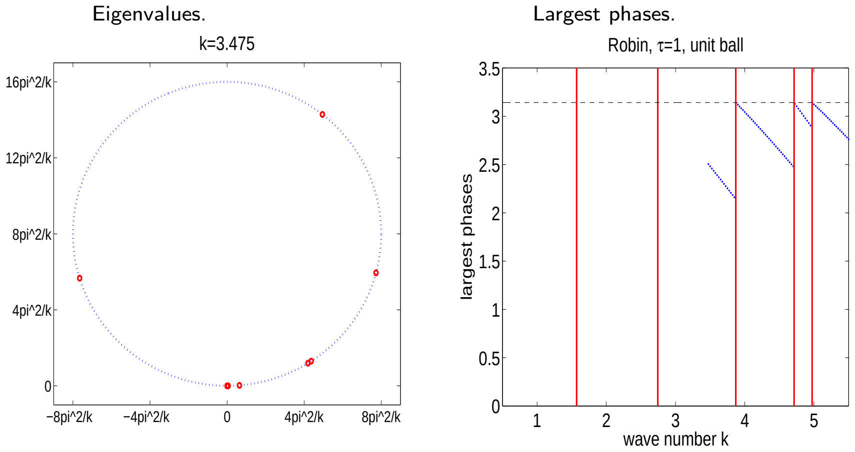
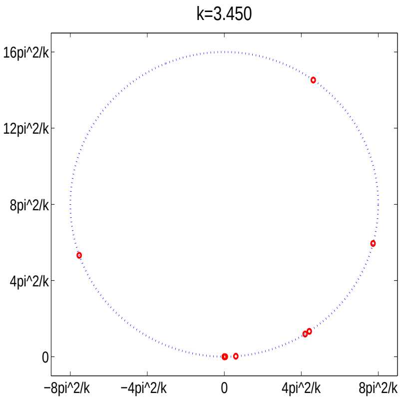


Straightforward approximation of interior eigenvalues



Straightforward approximation of interior eigenvalues

Eigenvalues.



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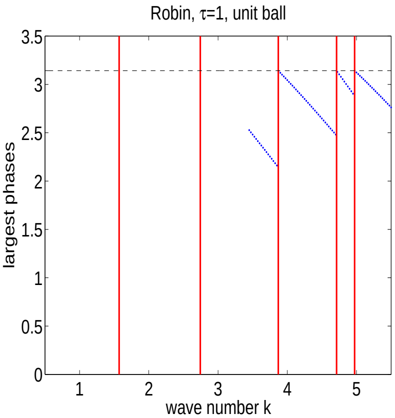
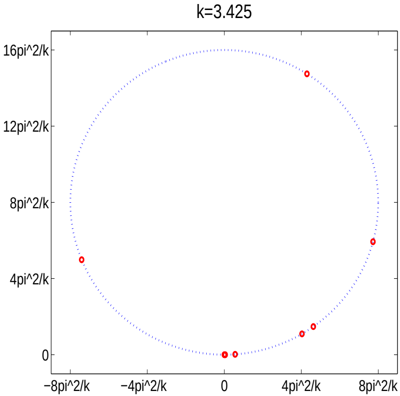


Straightforward approximation of interior eigenvalues

Eigenvalues.



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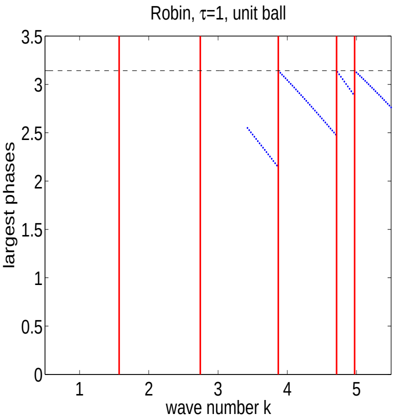
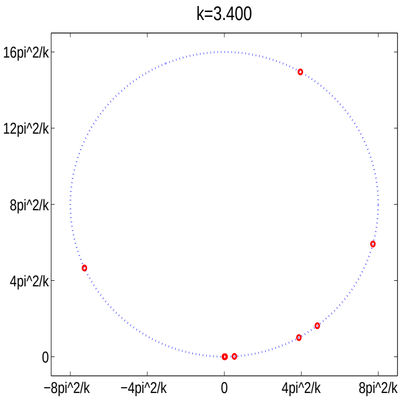


Straightforward approximation of interior eigenvalues

Eigenvalues.



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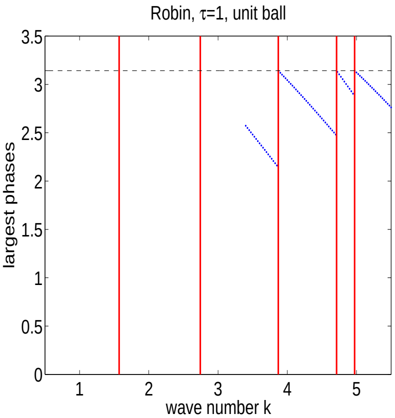
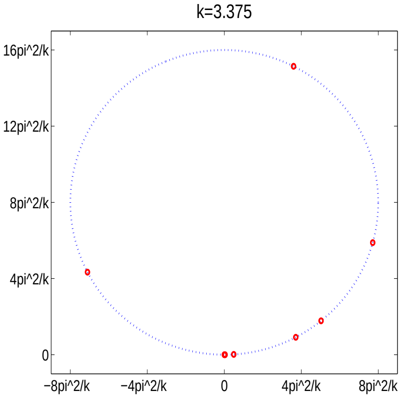


Straightforward approximation of interior eigenvalues

Eigenvalues.



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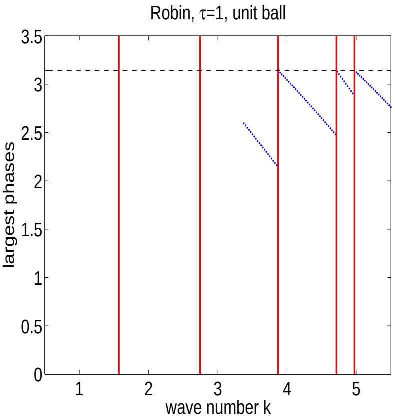
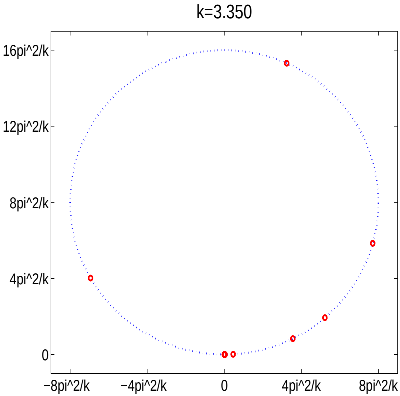


Straightforward approximation of interior eigenvalues

Eigenvalues.



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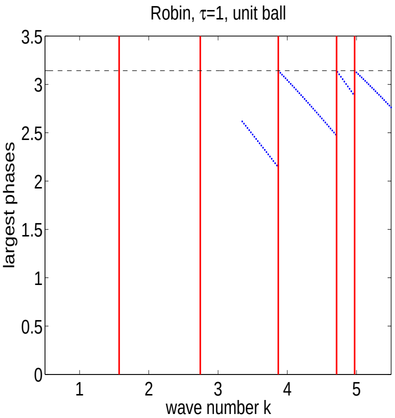
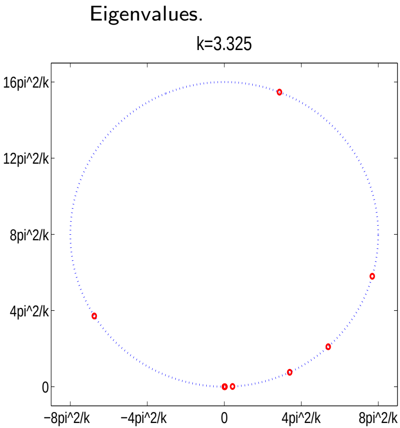
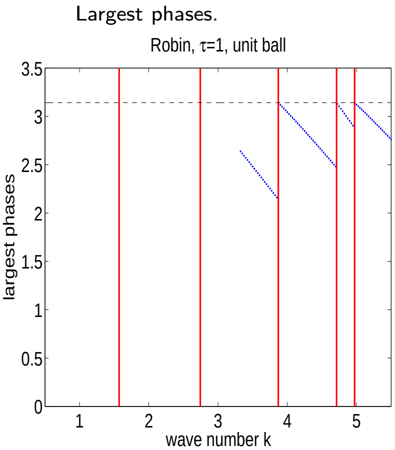


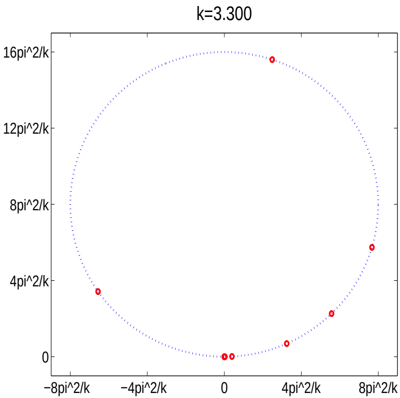
Straightforward approximation of interior eigenvalues





Straightforward approximation of interior eigenvalues

Eigenvalues.



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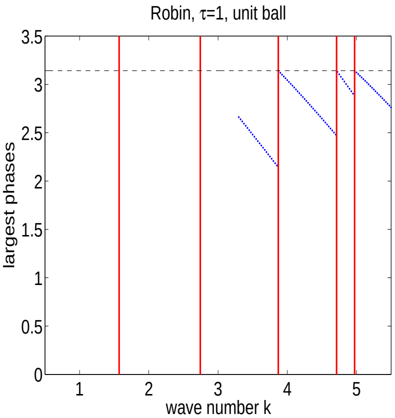
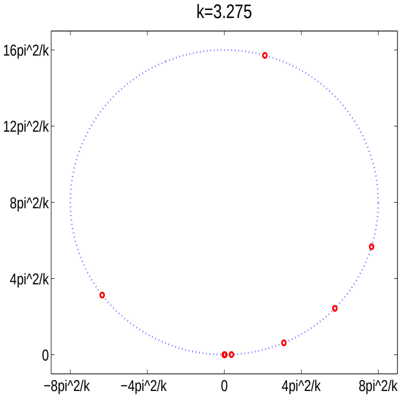


Straightforward approximation of interior eigenvalues

Eigenvalues.



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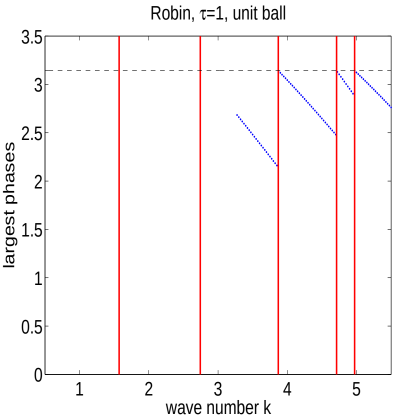
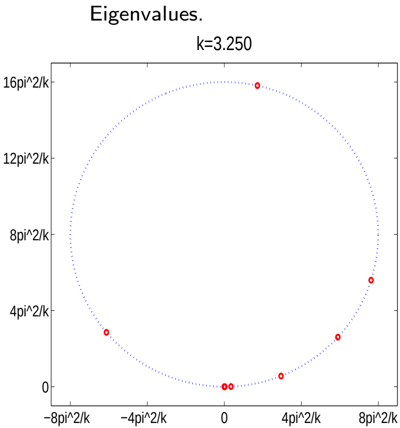
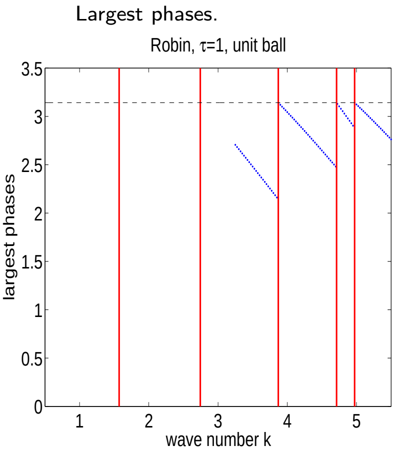


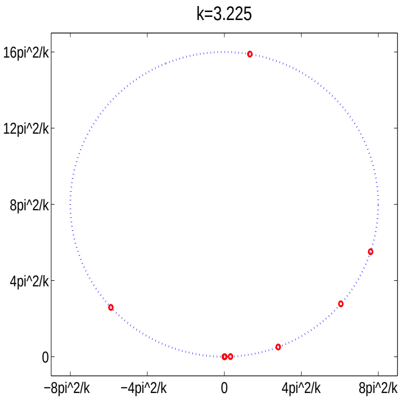
Straightforward approximation of interior eigenvalues





Straightforward approximation of interior eigenvalues

Eigenvalues.



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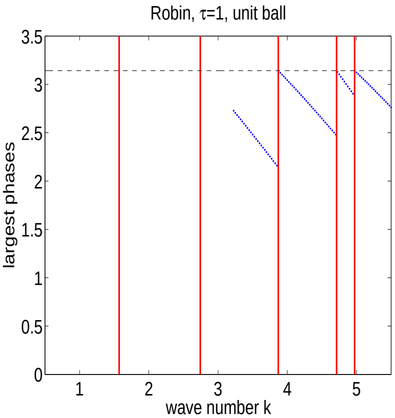
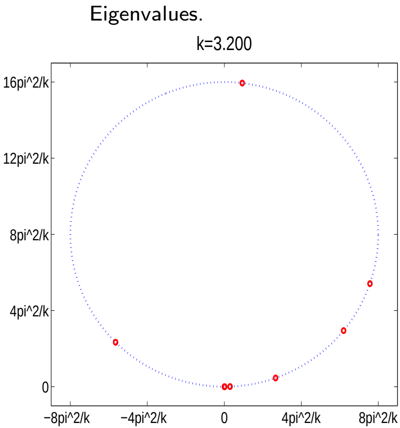
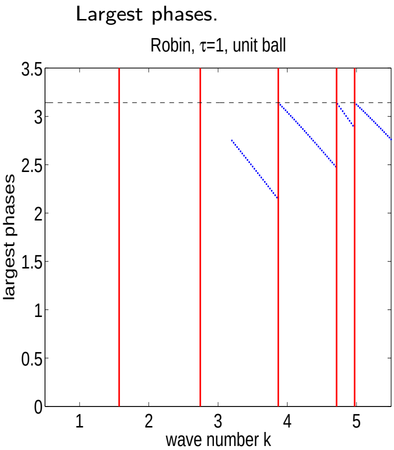


Straightforward approximation of interior eigenvalues





Straightforward approximation of interior eigenvalues

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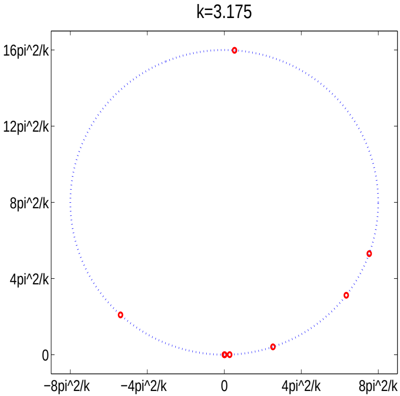

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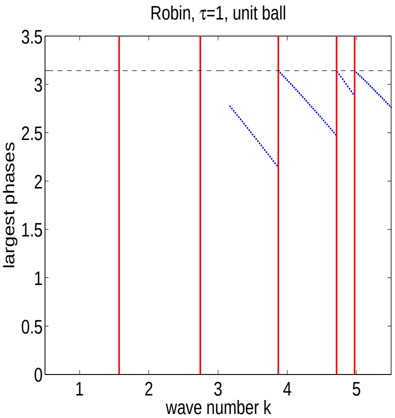

Straightforward approximation of interior eigenvalues

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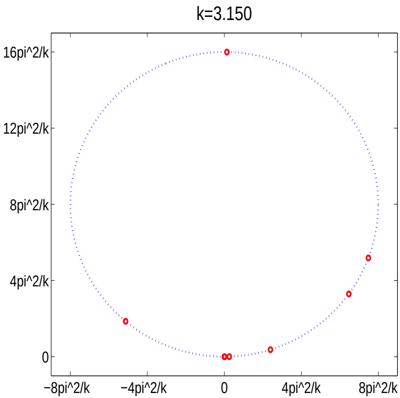

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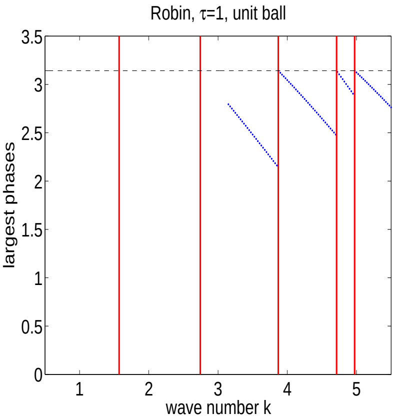

Straightforward approximation of interior eigenvalues

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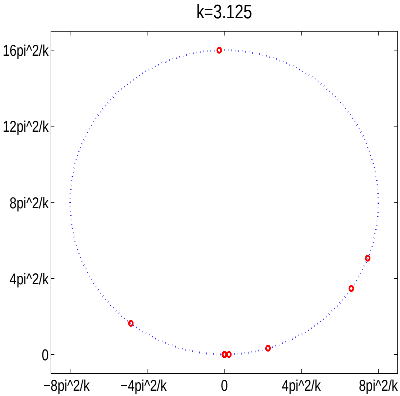

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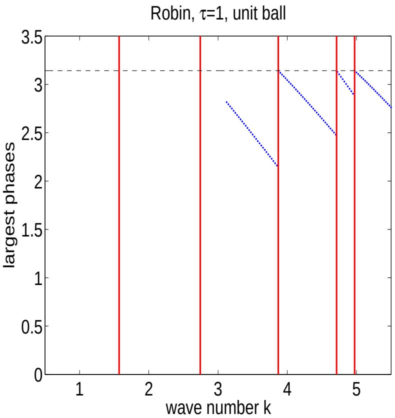

Straightforward approximation of interior eigenvalues

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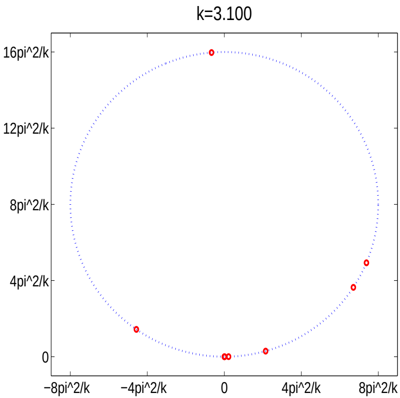

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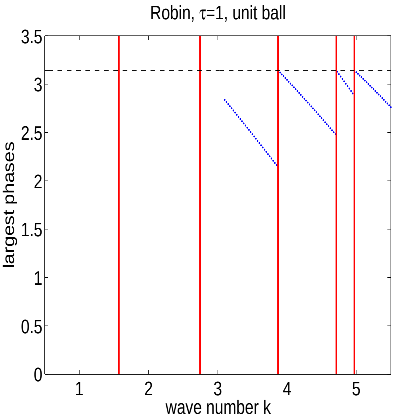

Straightforward approximation of interior eigenvalues

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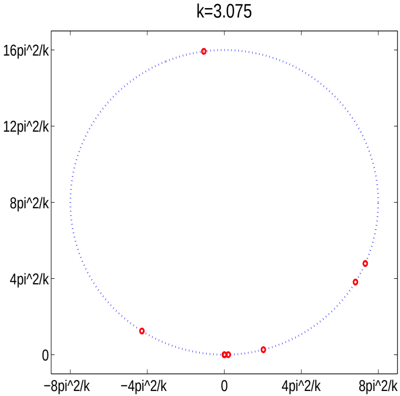

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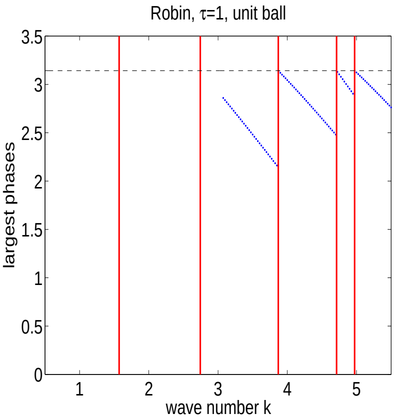

Straightforward approximation of interior eigenvalues

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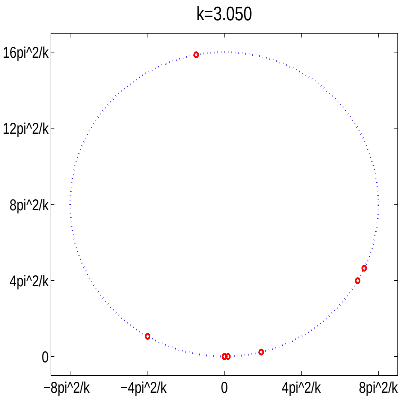

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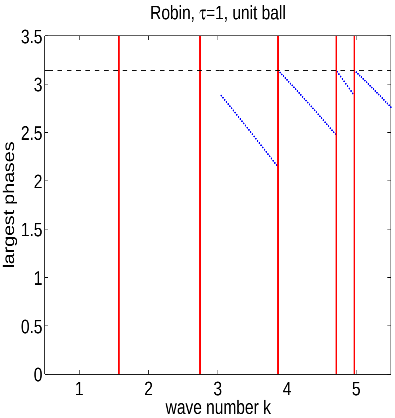

Straightforward approximation of interior eigenvalues

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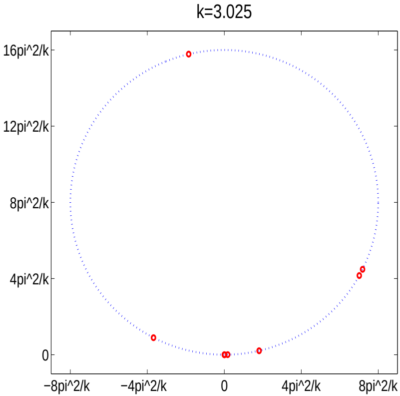

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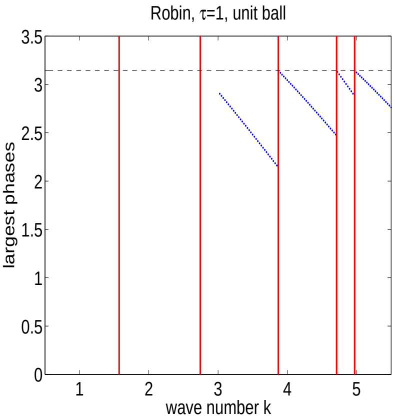

Straightforward approximation of interior eigenvalues

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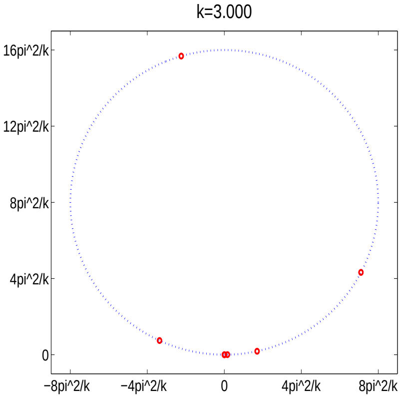

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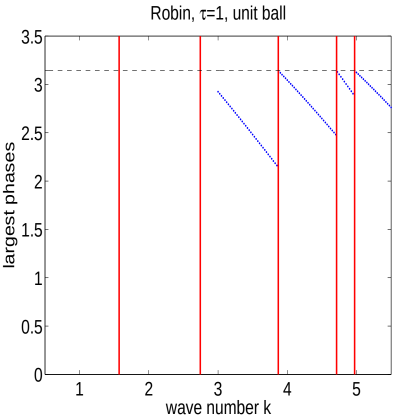

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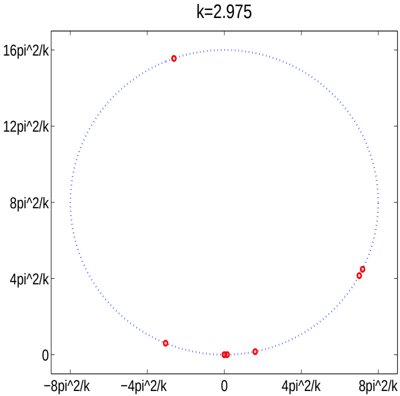
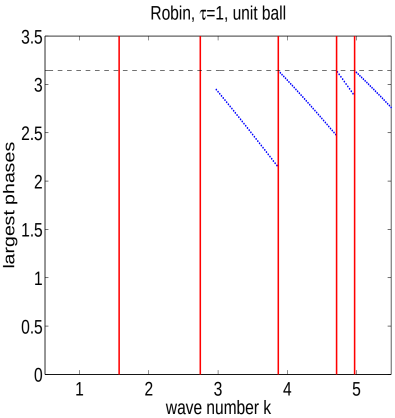
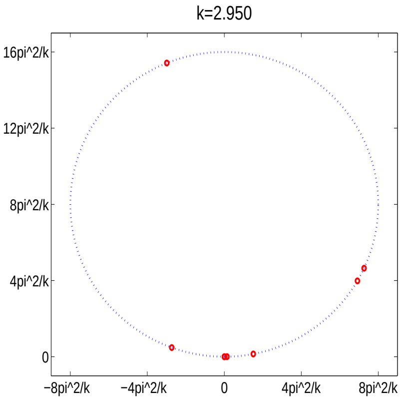


Largest phases.



Straightforward approximation of interior eigenvalues

Eigenvalues.



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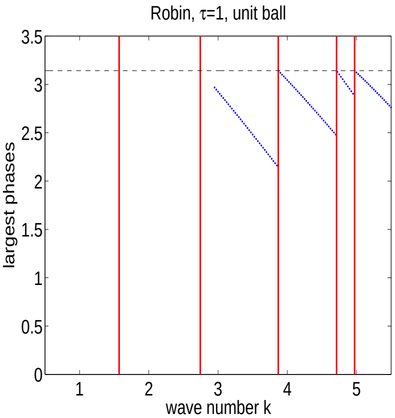
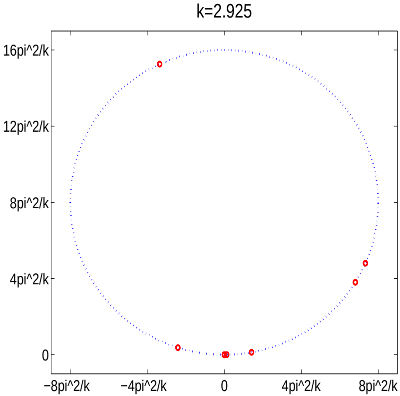


Straightforward approximation of interior eigenvalues

Eigenvalues.



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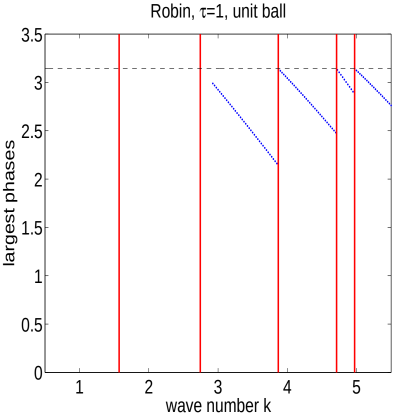
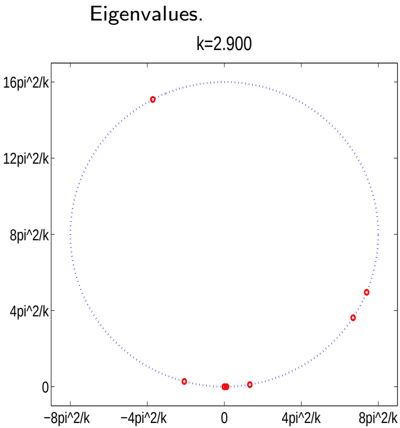
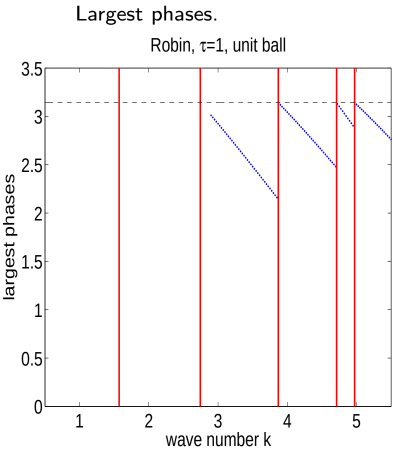


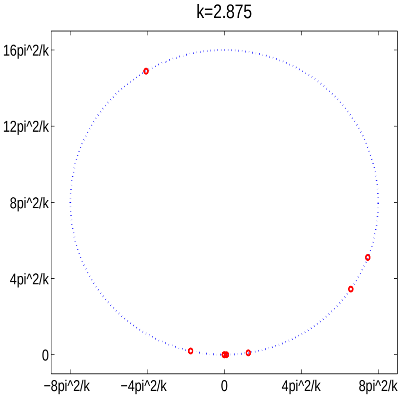
Straightforward approximation of interior eigenvalues





Straightforward approximation of interior eigenvalues

Eigenvalues.



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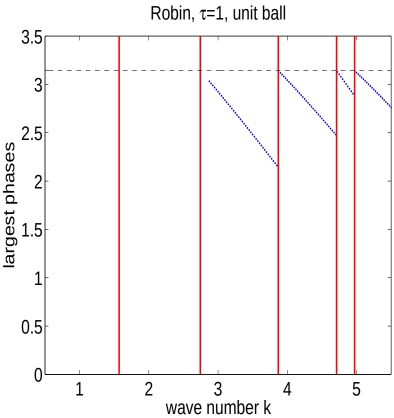
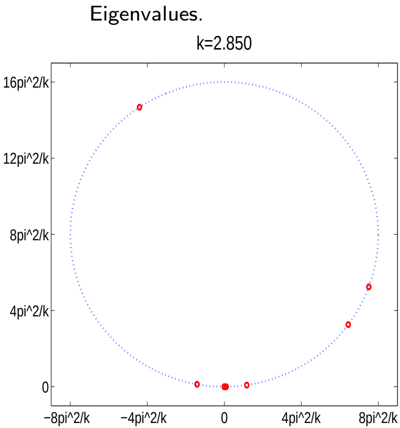
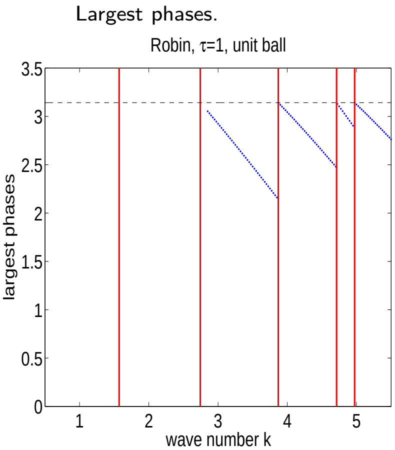


Straightforward approximation of interior eigenvalues

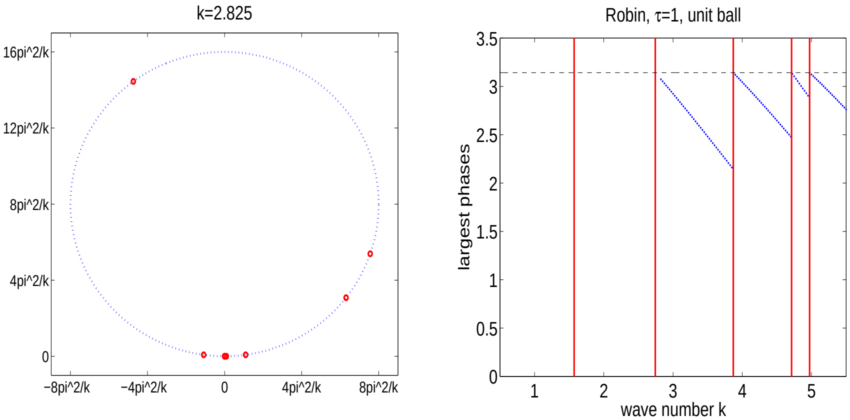




Straightforward approximation of interior eigenvalues

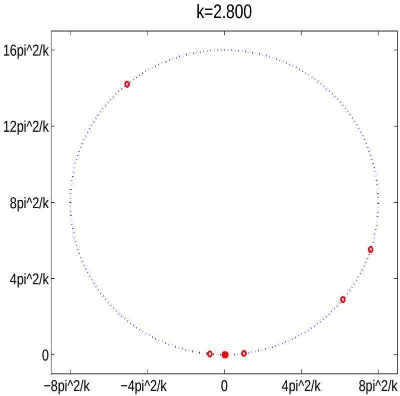
Eigenvalues.

Largest phases.



Straightforward approximation of interior eigenvalues

Eigenvalues.



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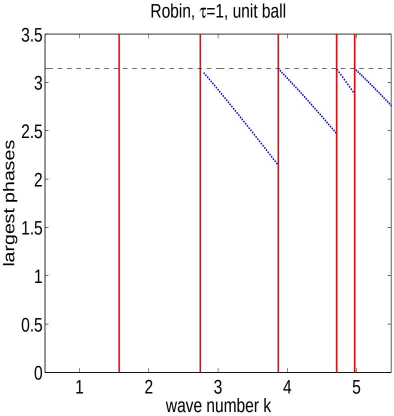
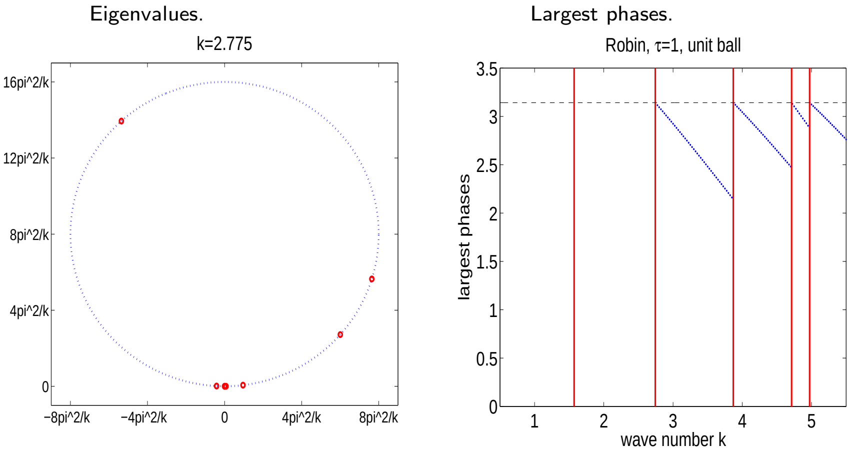
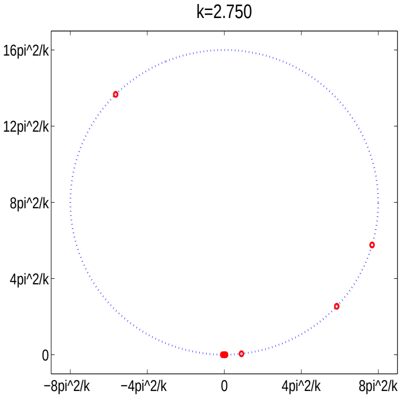


Straightforward approximation of interior eigenvalues



Straightforward approximation of interior eigenvalues

Eigenvalues.



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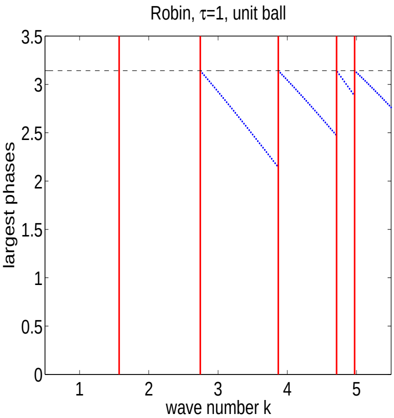
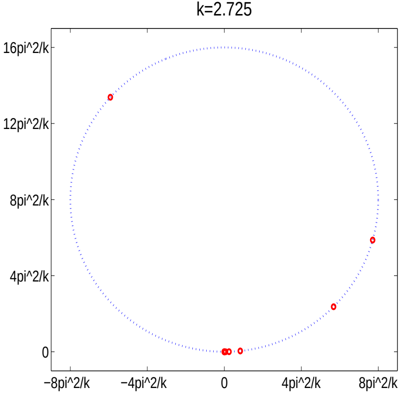


Straightforward approximation of interior eigenvalues

Eigenvalues.



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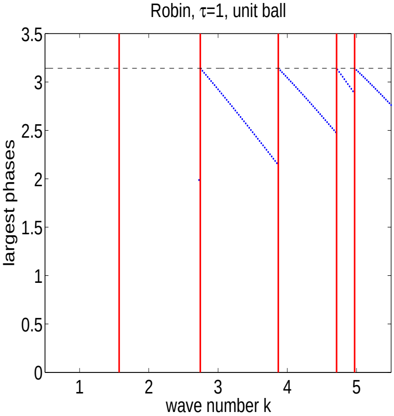
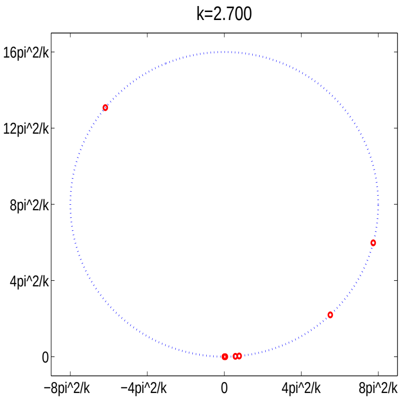


Straightforward approximation of interior eigenvalues

Eigenvalues.



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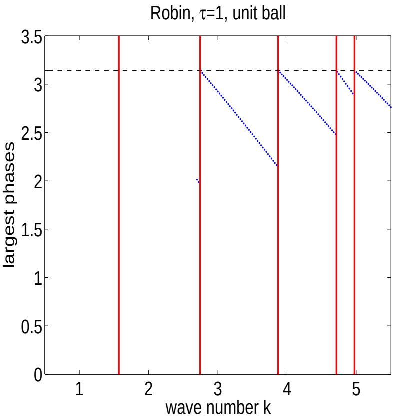
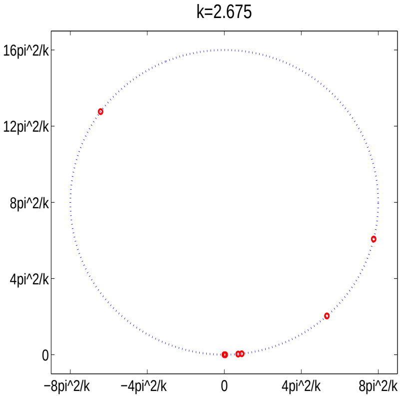


Straightforward approximation of interior eigenvalues

Eigenvalues.



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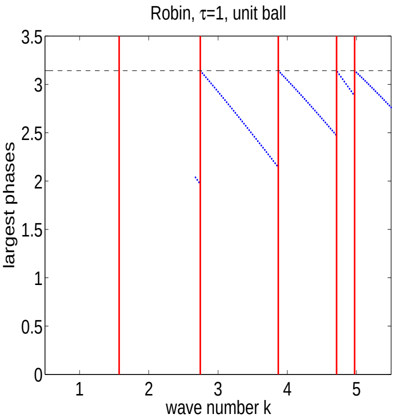
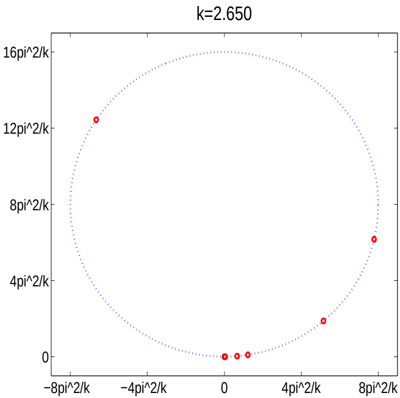


Straightforward approximation of interior eigenvalues

Eigenvalues.



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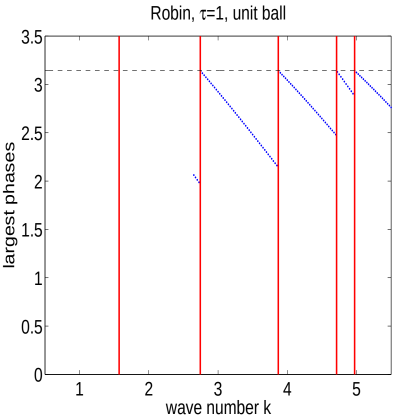
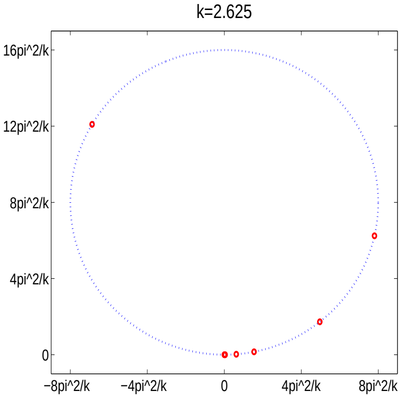


Straightforward approximation of interior eigenvalues

Eigenvalues.



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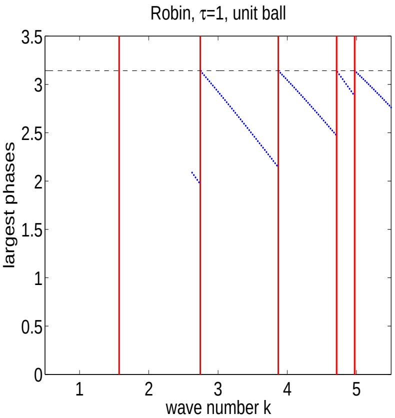
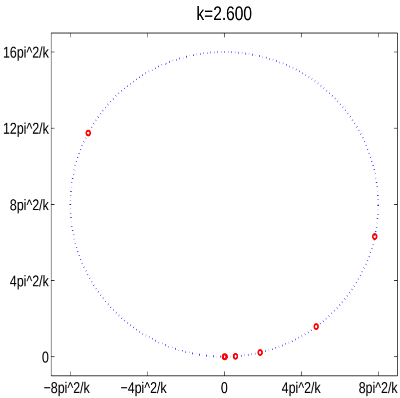


Straightforward approximation of interior eigenvalues

Eigenvalues.



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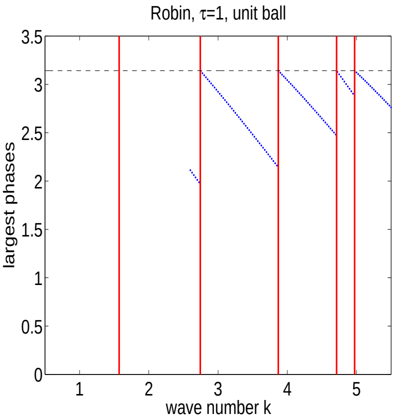
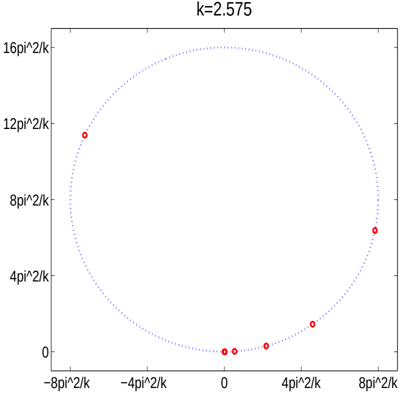


Straightforward approximation of interior eigenvalues

Eigenvalues.



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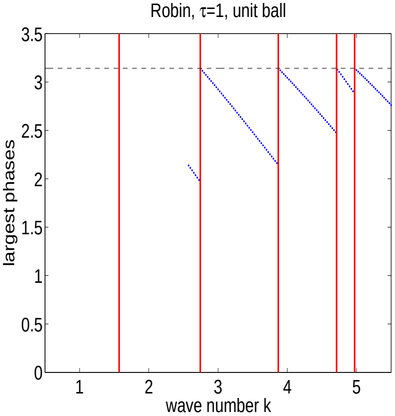
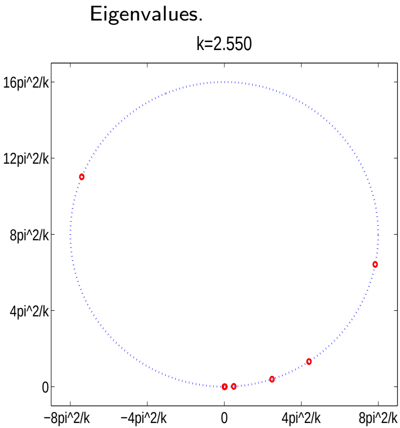
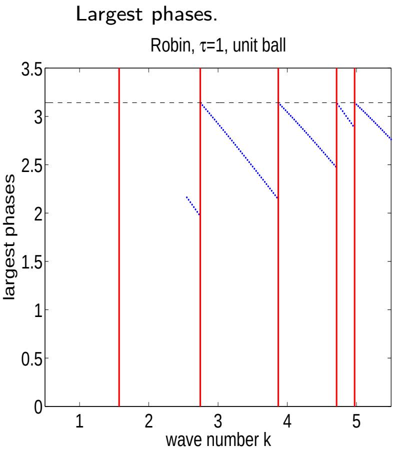


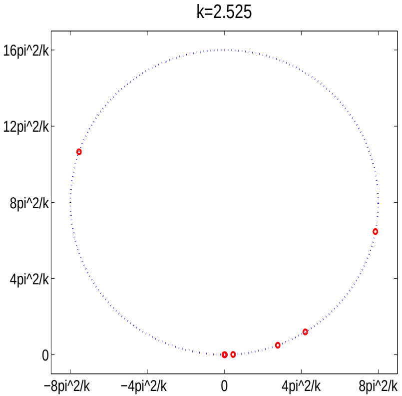
Straightforward approximation of interior eigenvalues





Straightforward approximation of interior eigenvalues

Eigenvalues.



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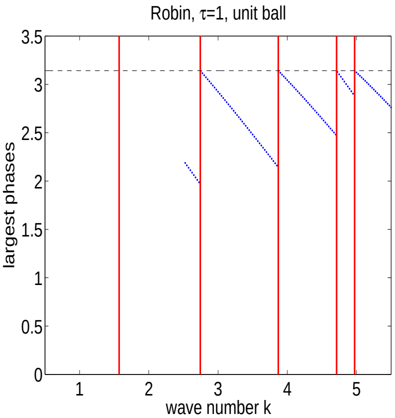
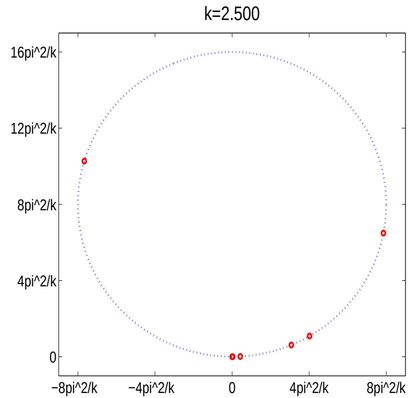


Straightforward approximation of interior eigenvalues

Eigenvalues.



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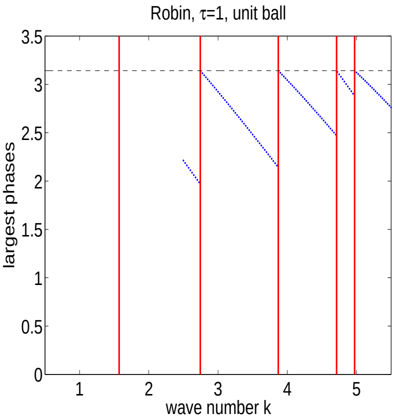
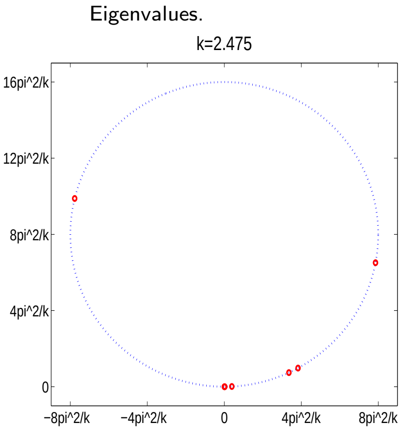
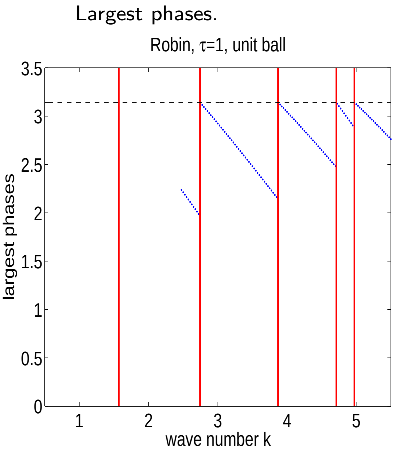


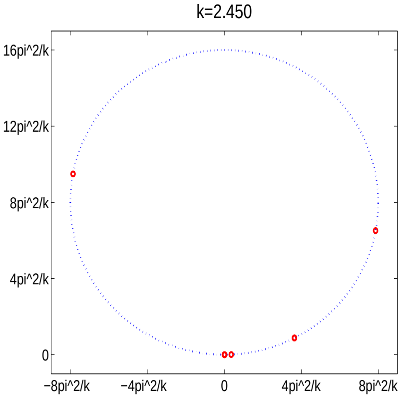
Straightforward approximation of interior eigenvalues





Straightforward approximation of interior eigenvalues

Eigenvalues.



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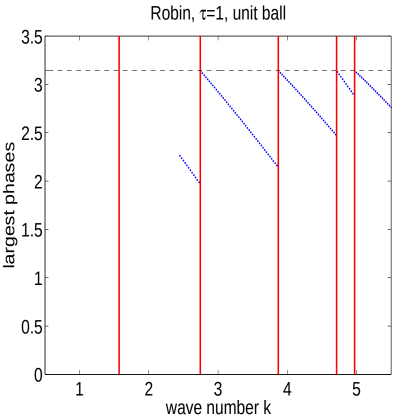
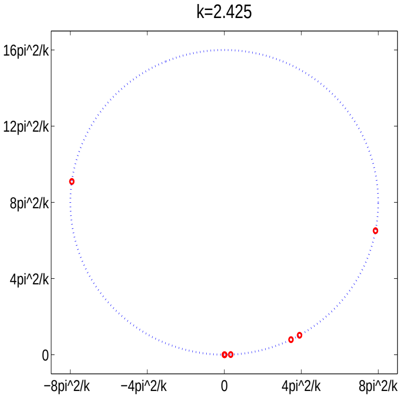


Straightforward approximation of interior eigenvalues

Eigenvalues.



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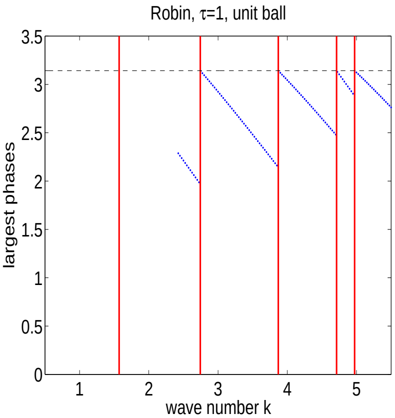
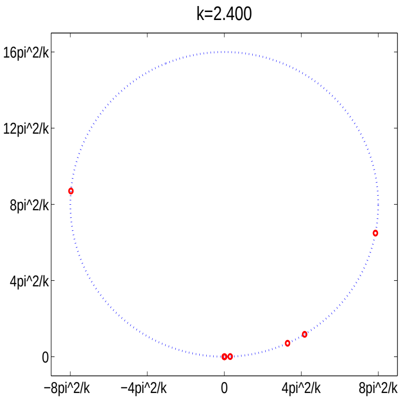


Straightforward approximation of interior eigenvalues

Eigenvalues.



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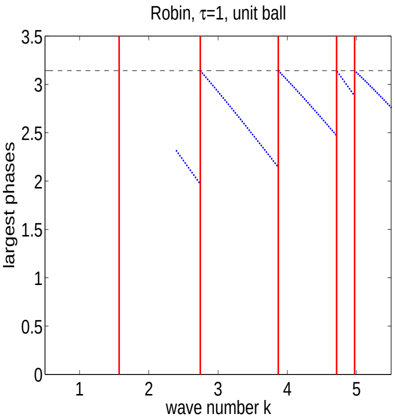
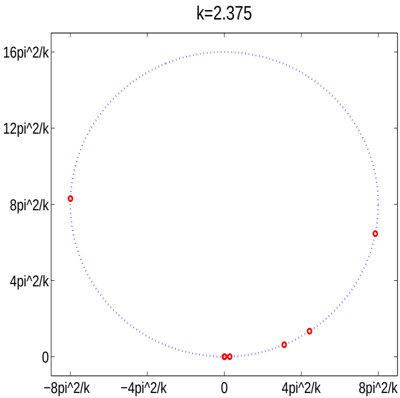


Straightforward approximation of interior eigenvalues

Eigenvalues.



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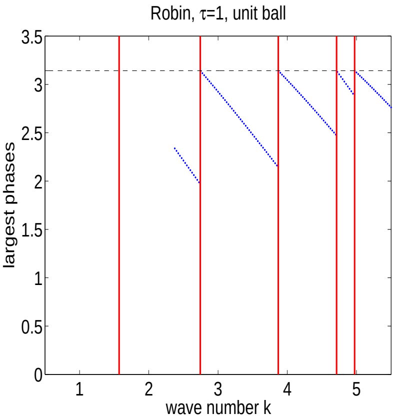
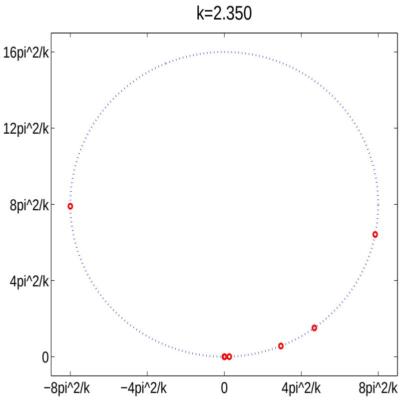


Straightforward approximation of interior eigenvalues

Eigenvalues.



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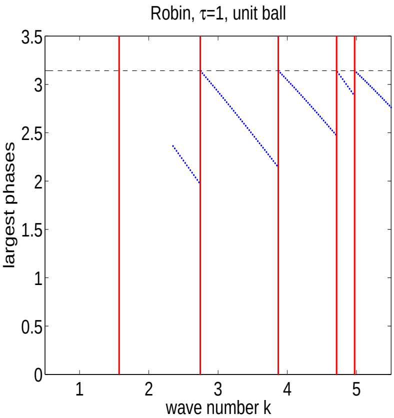
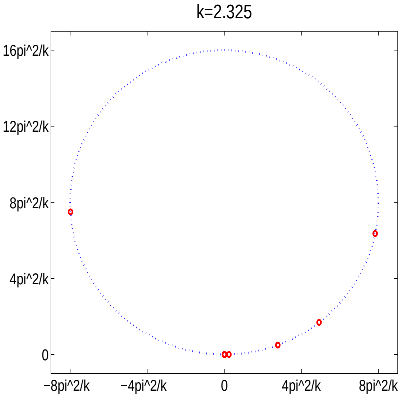


Straightforward approximation of interior eigenvalues

Eigenvalues.



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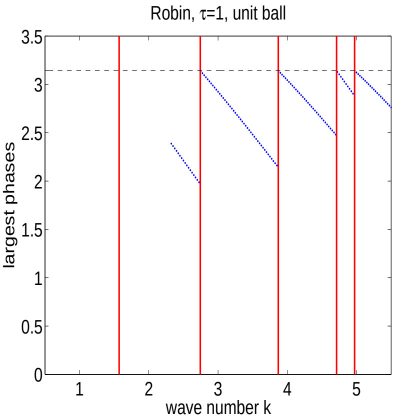
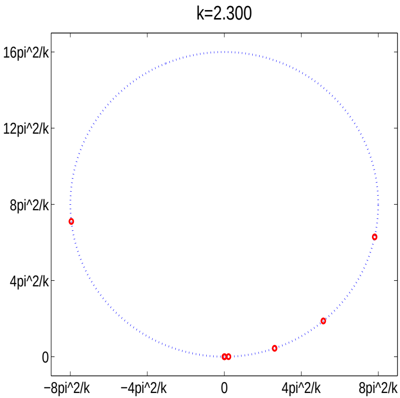


Straightforward approximation of interior eigenvalues

Eigenvalues.



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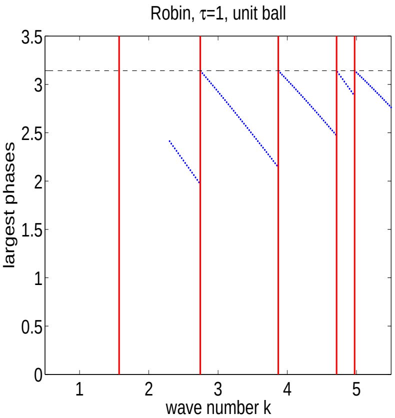
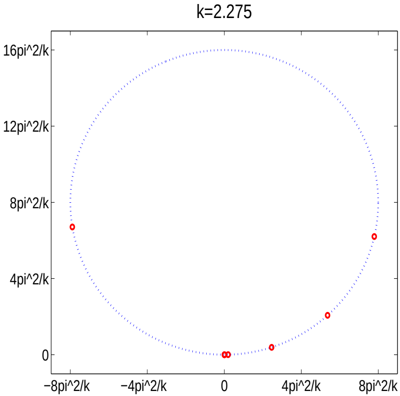


Straightforward approximation of interior eigenvalues

Eigenvalues.



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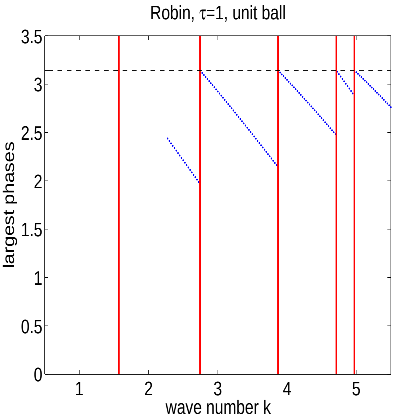
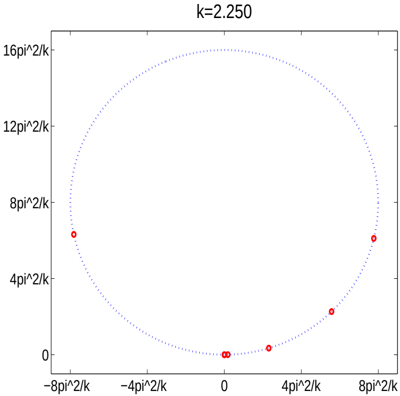


Straightforward approximation of interior eigenvalues

Eigenvalues.



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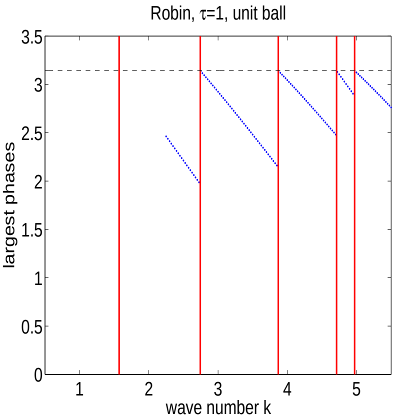
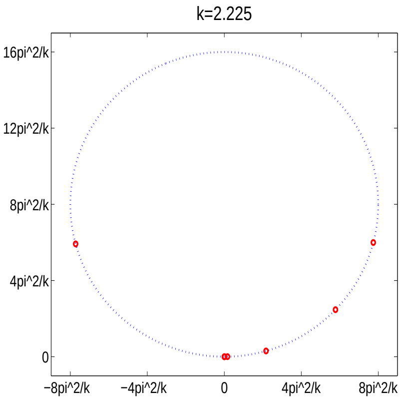


Straightforward approximation of interior eigenvalues

Eigenvalues.



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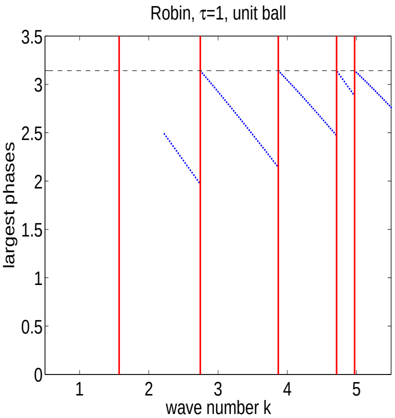
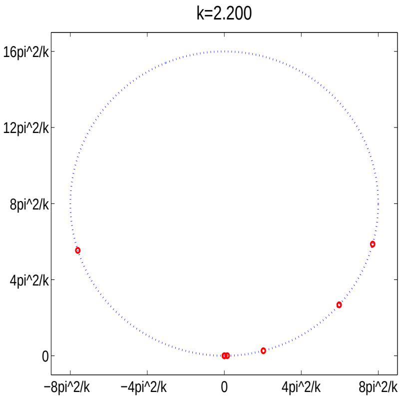


Straightforward approximation of interior eigenvalues

Eigenvalues.



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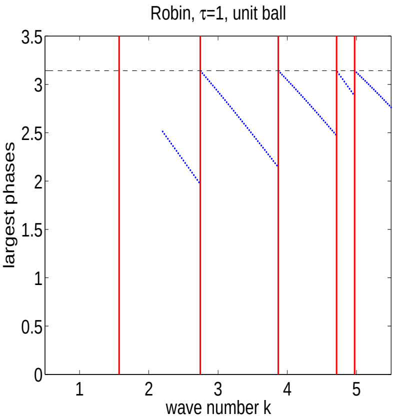
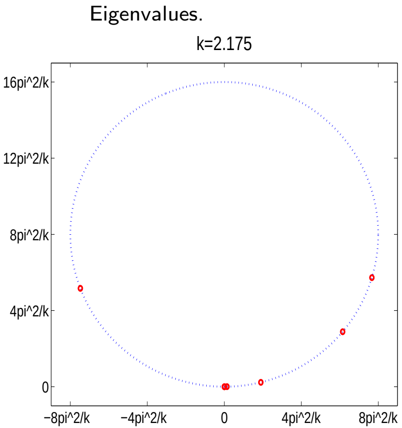
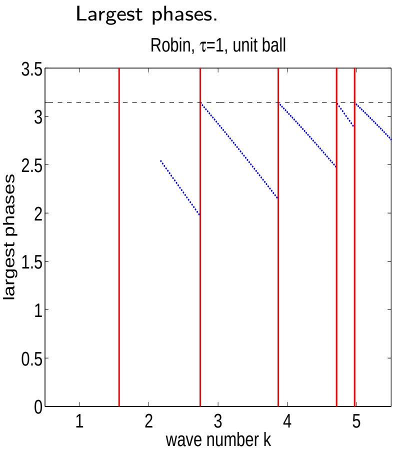


Straightforward approximation of interior eigenvalues





Straightforward approximation of interior eigenvalues

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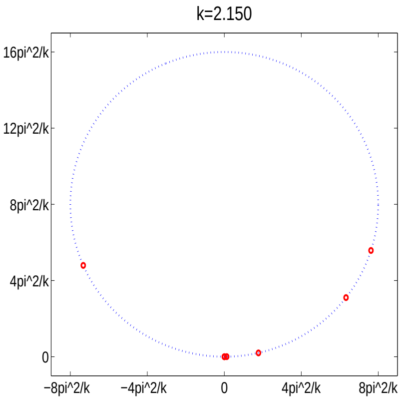

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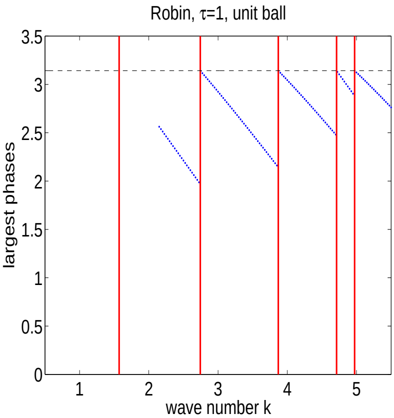
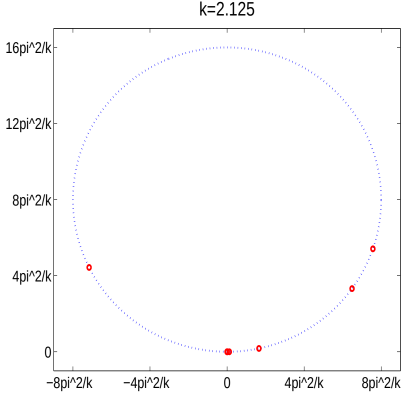


Straightforward approximation of interior eigenvalues

Eigenvalues.



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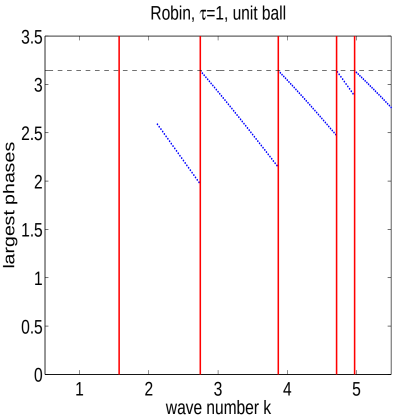

Straightforward approximation of interior eigenvalues

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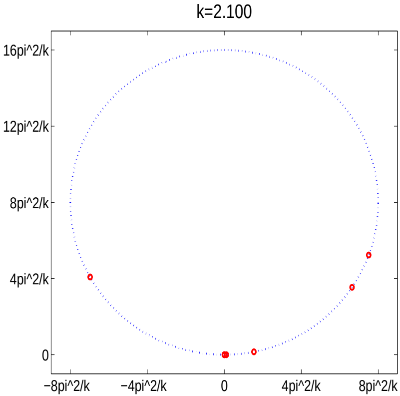

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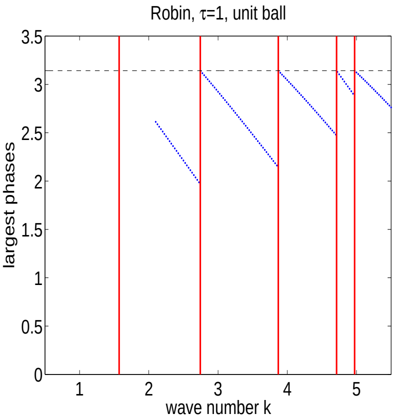

Straightforward approximation of interior eigenvalues

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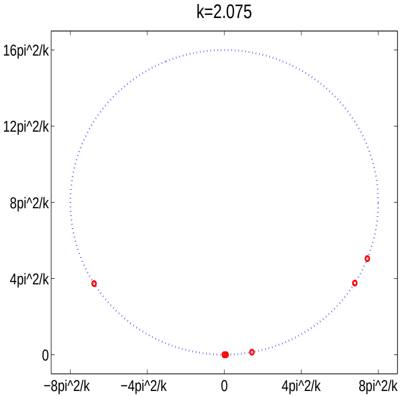

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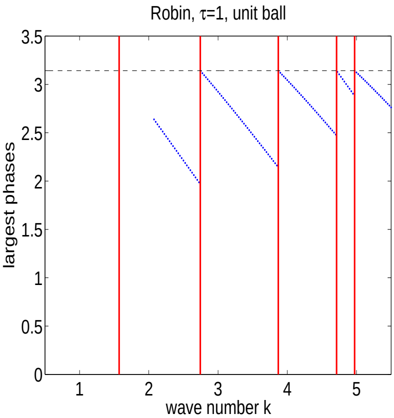
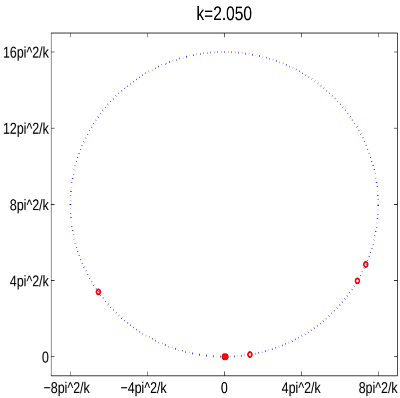
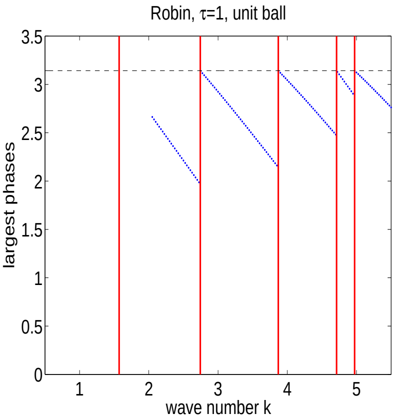


Straightforward approximation of interior eigenvalues

Eigenvalues.

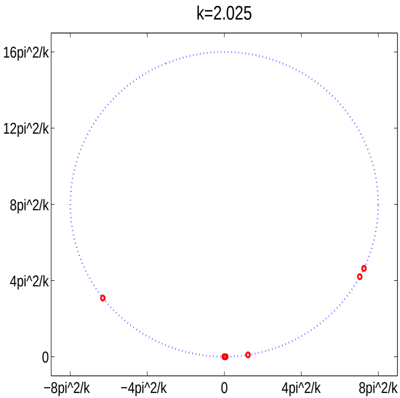


Largest phases.



Straightforward approximation of interior eigenvalues

Eigenvalues.



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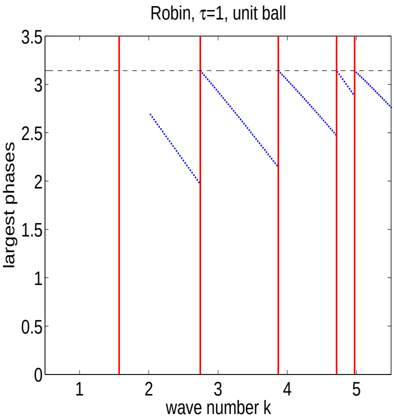
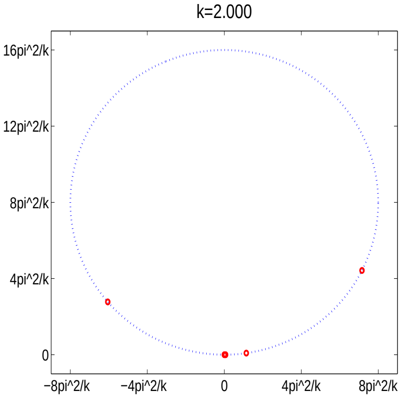


Straightforward approximation of interior eigenvalues

Eigenvalues.



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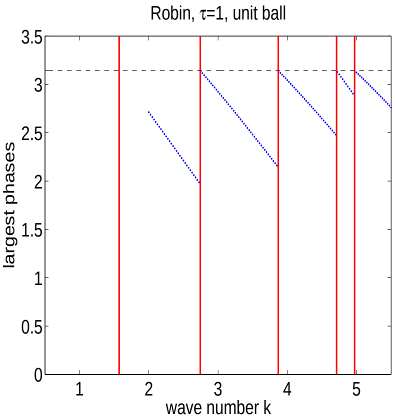
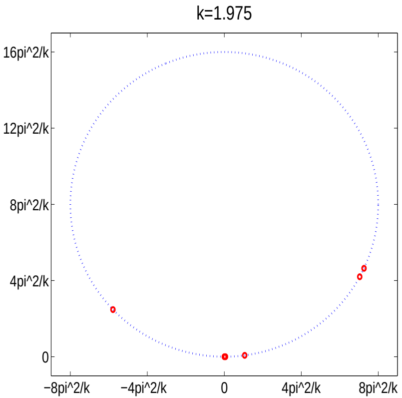


Straightforward approximation of interior eigenvalues

Eigenvalues.



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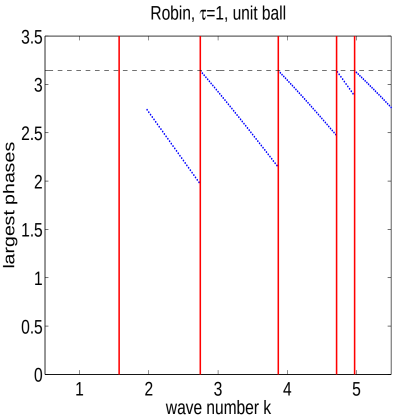
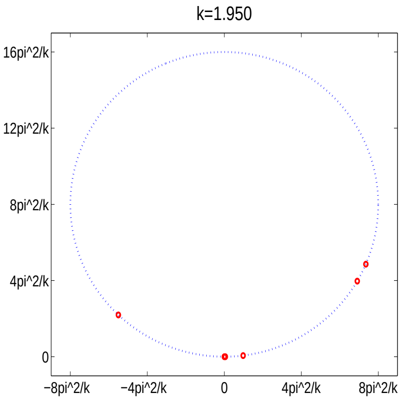


Straightforward approximation of interior eigenvalues

Eigenvalues.



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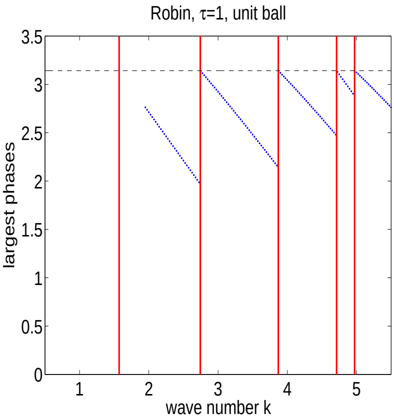
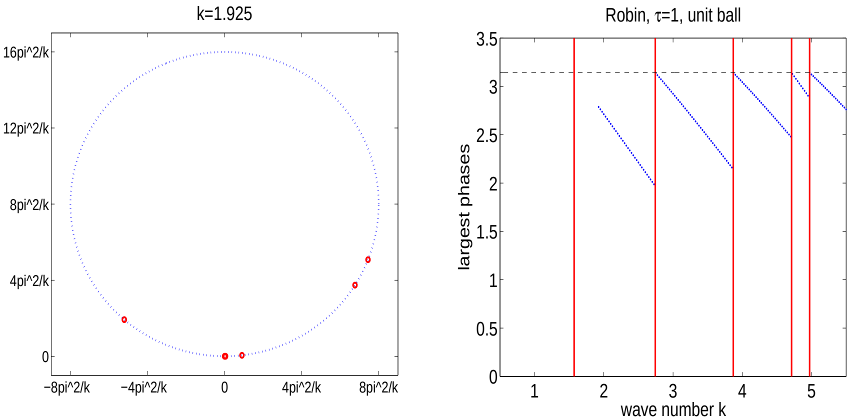


Straightforward approximation of interior eigenvalues

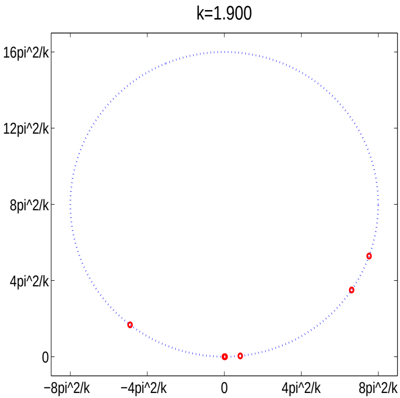
Eigenvalues.

Largest phases.



Straightforward approximation of interior eigenvalues

Eigenvalues.



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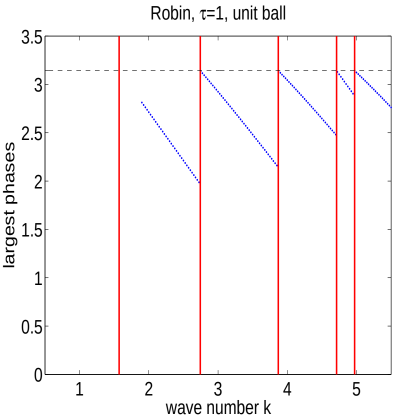
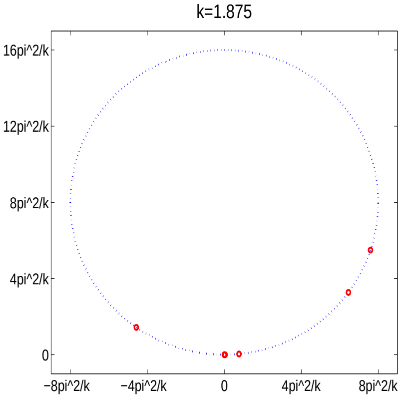


Straightforward approximation of interior eigenvalues

Eigenvalues.



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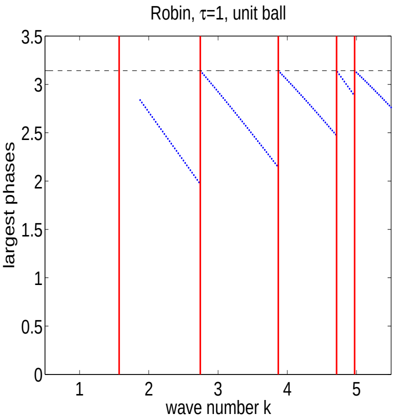
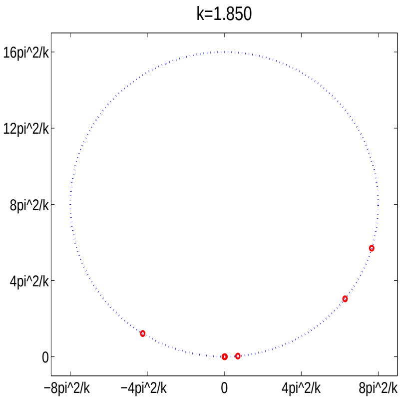


Straightforward approximation of interior eigenvalues

Eigenvalues.



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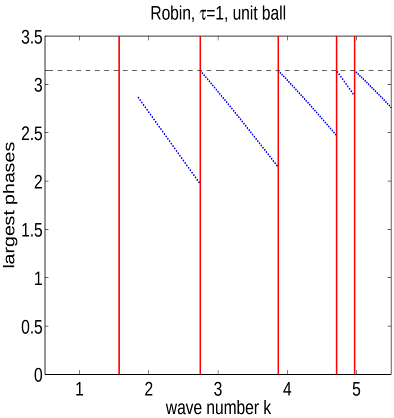
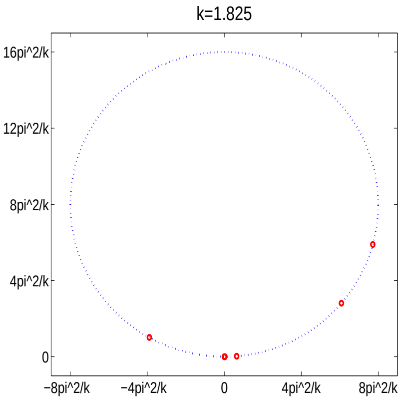


Straightforward approximation of interior eigenvalues

Eigenvalues.



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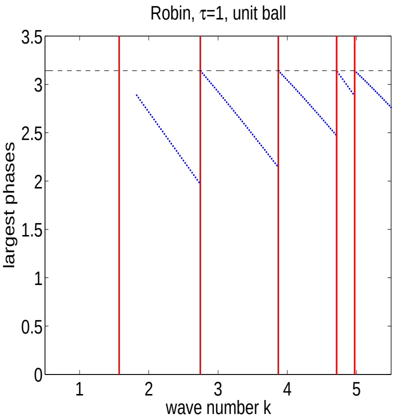
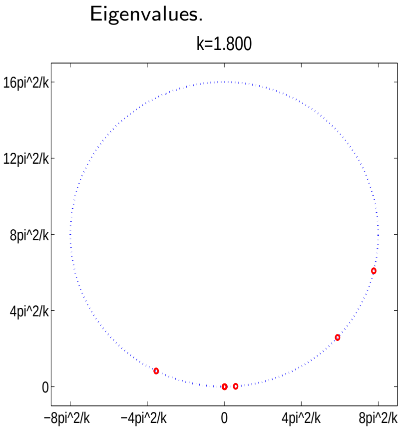
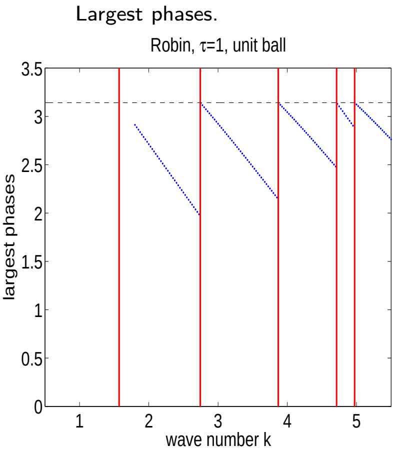


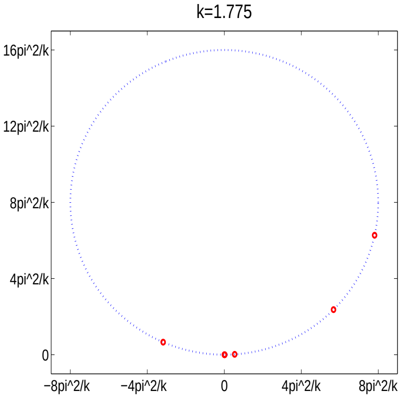
Straightforward approximation of interior eigenvalues





Straightforward approximation of interior eigenvalues

Eigenvalues.



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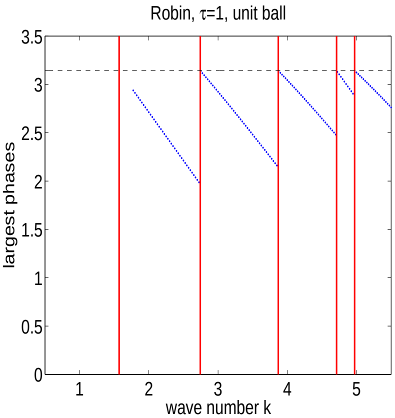
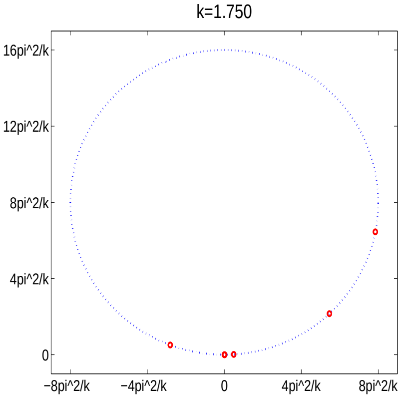


Straightforward approximation of interior eigenvalues

Eigenvalues.



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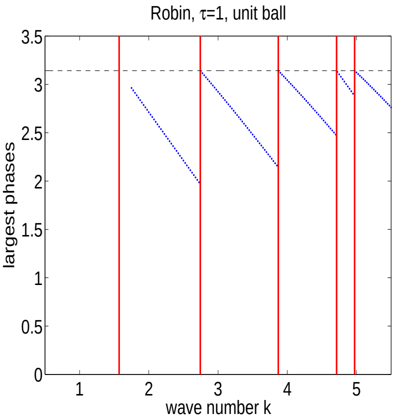
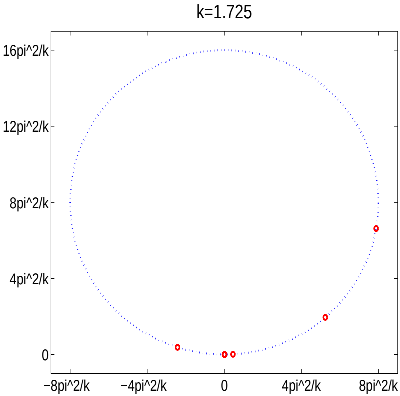


Straightforward approximation of interior eigenvalues

Eigenvalues.



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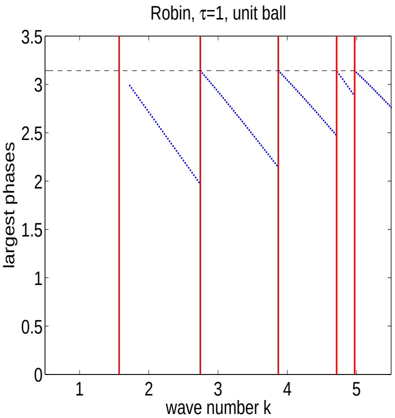
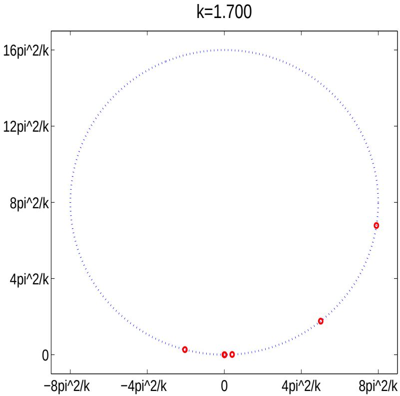


Straightforward approximation of interior eigenvalues

Eigenvalues.



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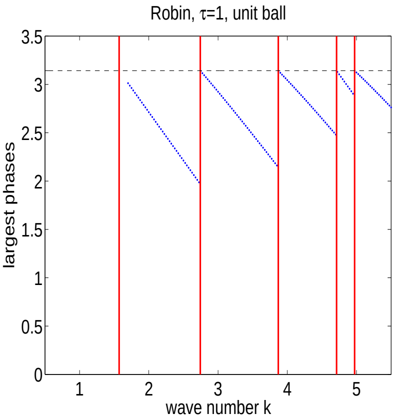
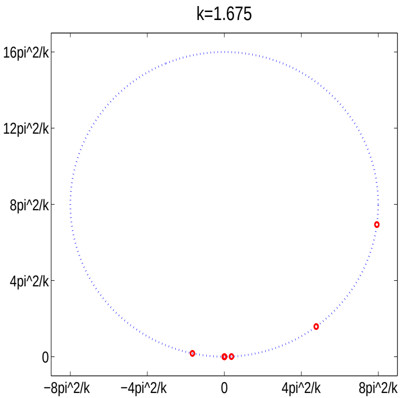


Straightforward approximation of interior eigenvalues

Eigenvalues.



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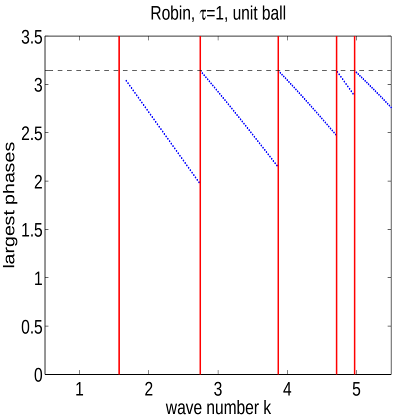
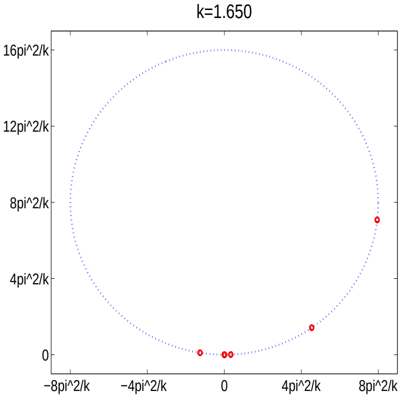


Straightforward approximation of interior eigenvalues

Eigenvalues.



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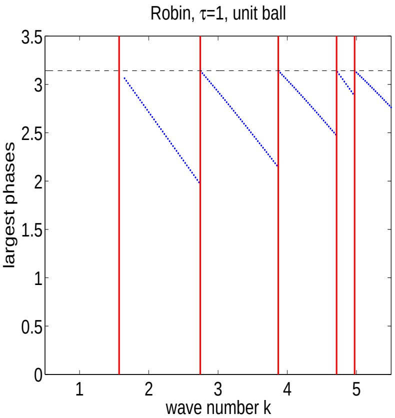
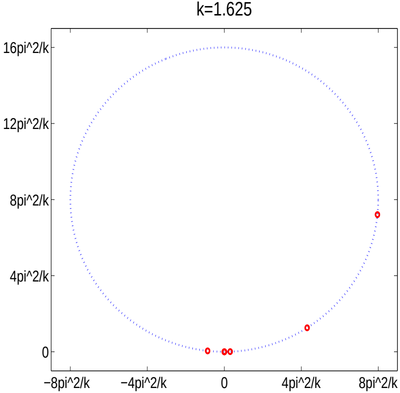


Straightforward approximation of interior eigenvalues

Eigenvalues.



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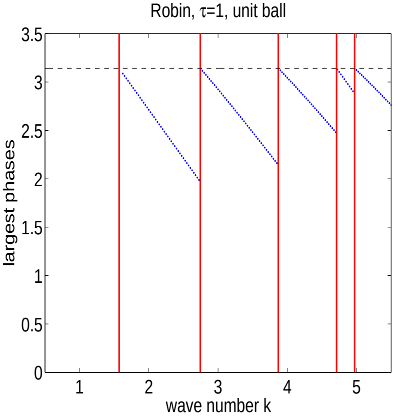

Straightforward approximation of interior eigenvalues

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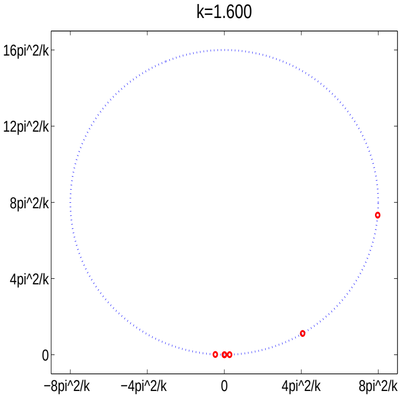

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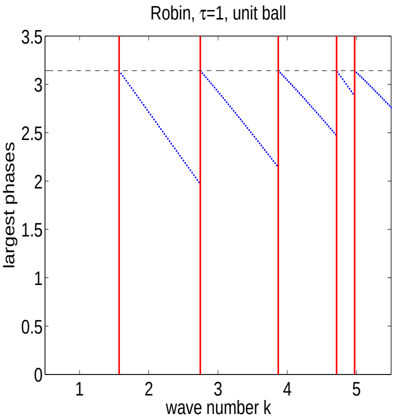
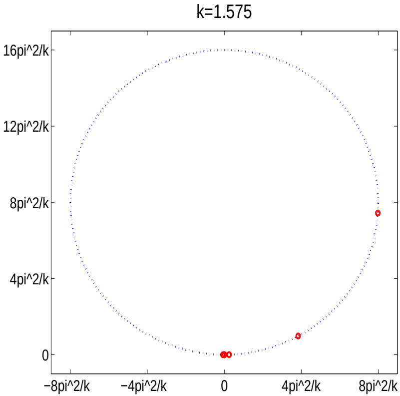


Straightforward approximation of interior eigenvalues

Eigenvalues.



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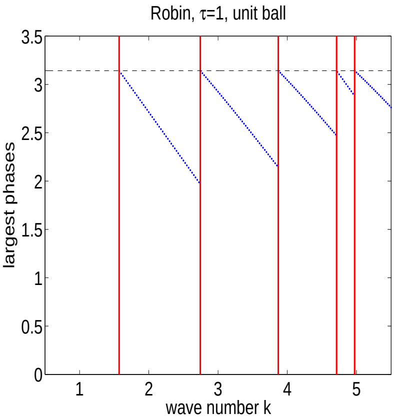
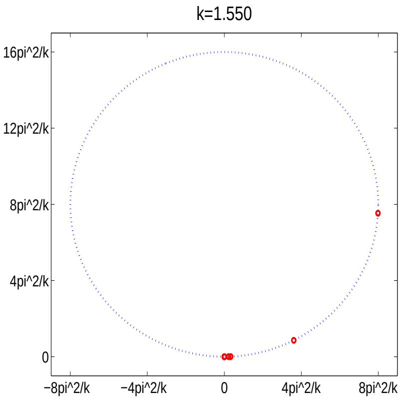


Straightforward approximation of interior eigenvalues

Eigenvalues.



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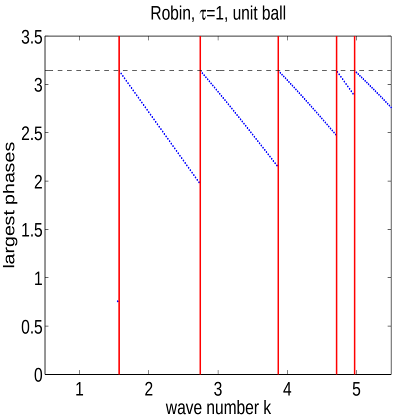
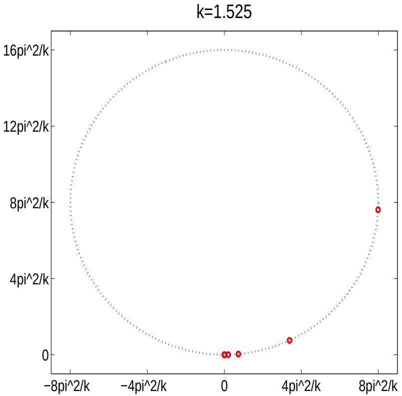


Straightforward approximation of interior eigenvalues

Eigenvalues.



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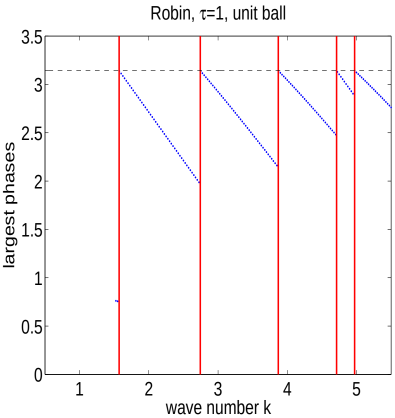
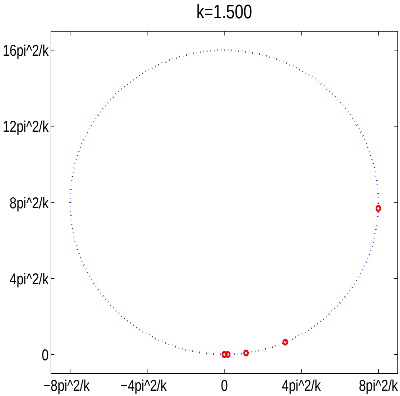


Straightforward approximation of interior eigenvalues

Eigenvalues.



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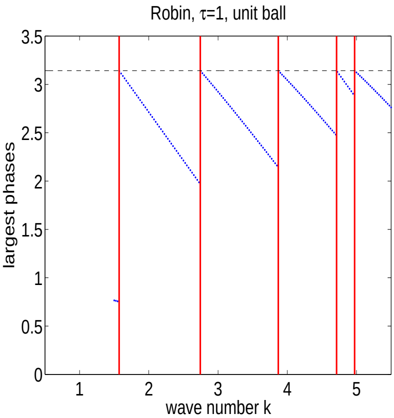

Straightforward approximation of interior eigenvalues

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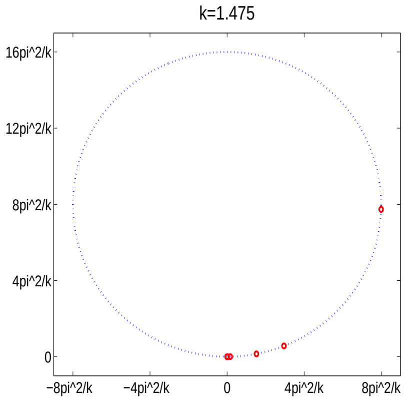

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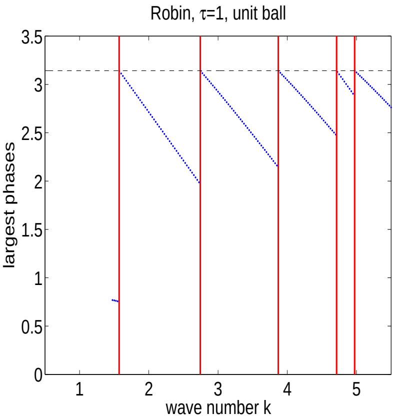

Straightforward approximation of interior eigenvalues

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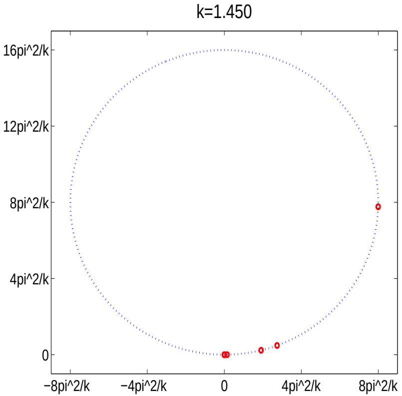

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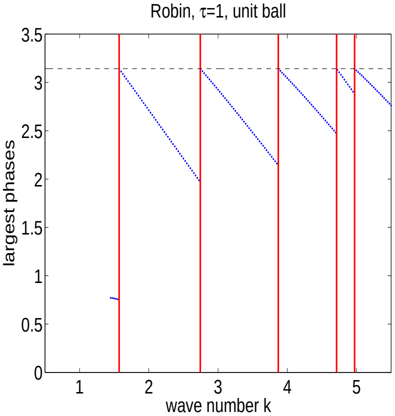
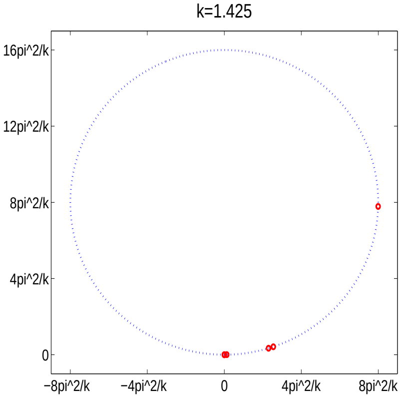


Straightforward approximation of interior eigenvalues

Eigenvalues.



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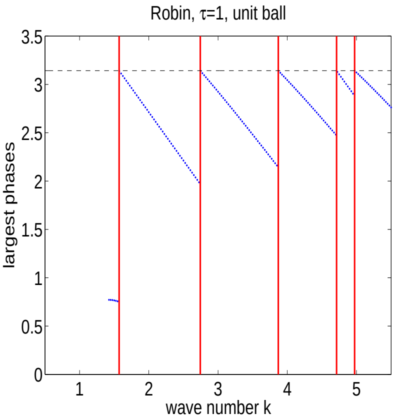

Straightforward approximation of interior eigenvalues

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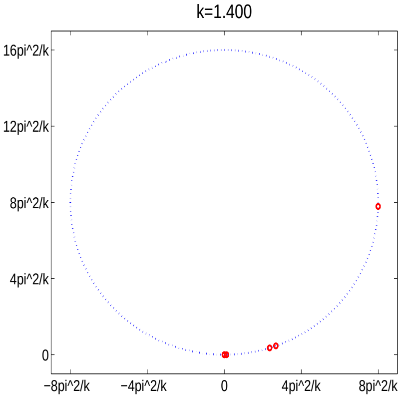

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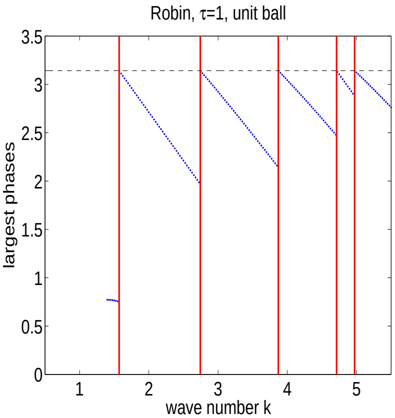

Straightforward approximation of interior eigenvalues

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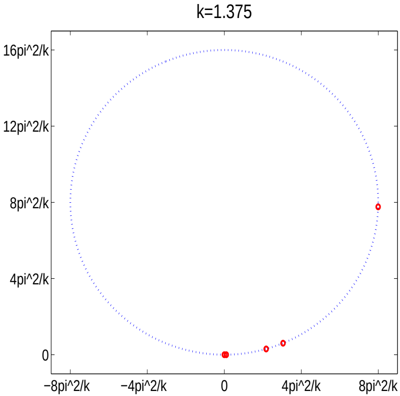

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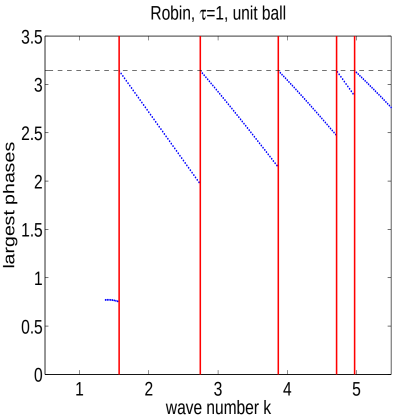

Straightforward approximation of interior eigenvalues

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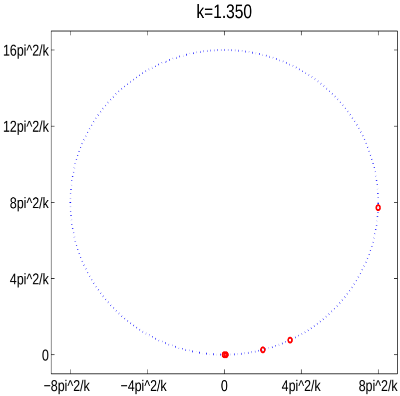

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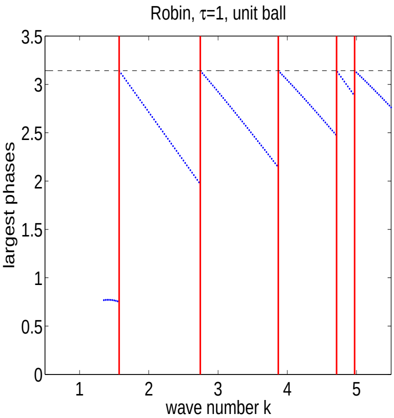
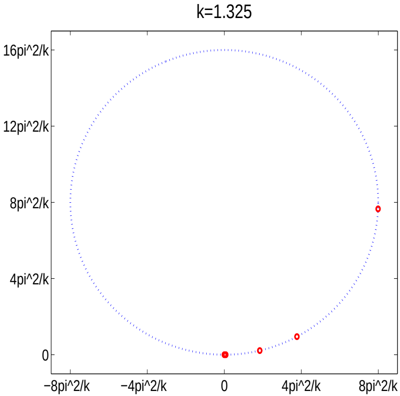


Straightforward approximation of interior eigenvalues

Eigenvalues.



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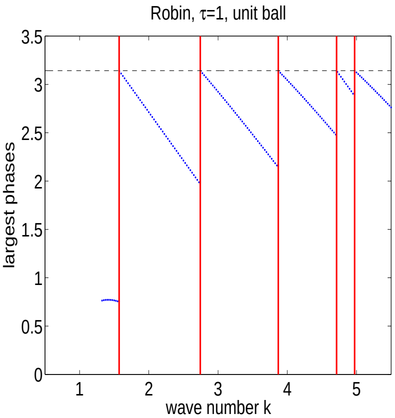
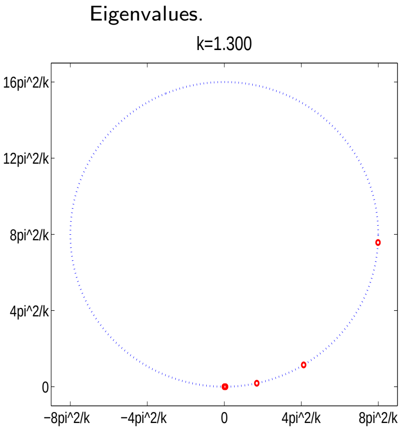
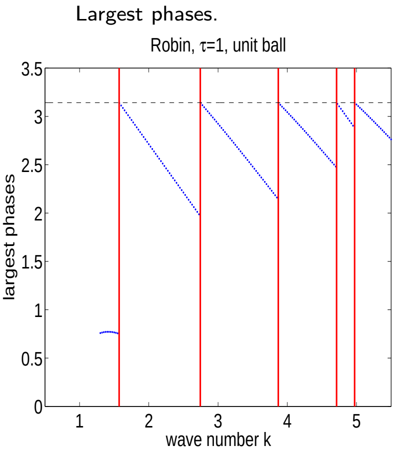


Straightforward approximation of interior eigenvalues





Straightforward approximation of interior eigenvalues

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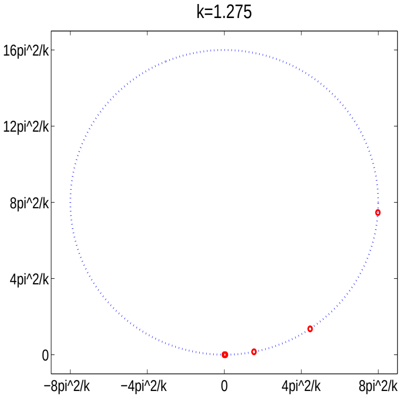

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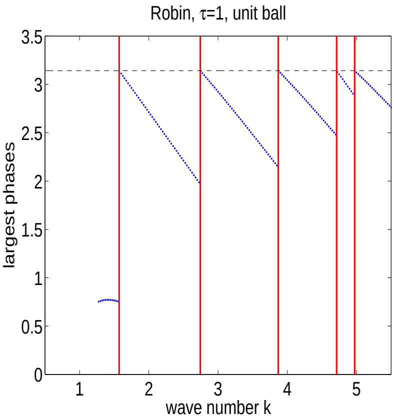

Straightforward approximation of interior eigenvalues

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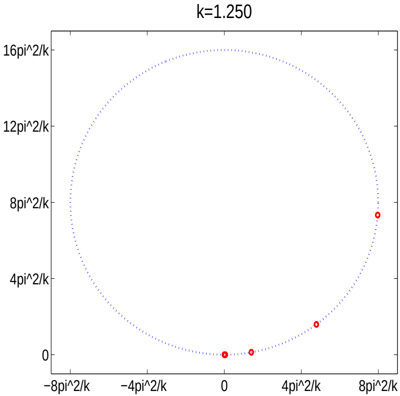

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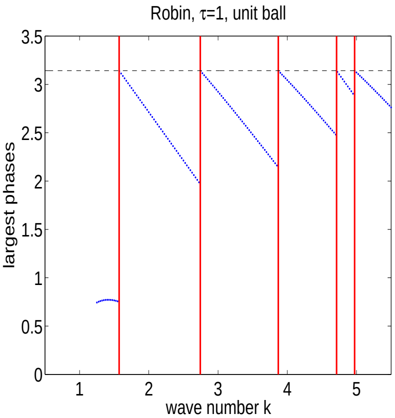
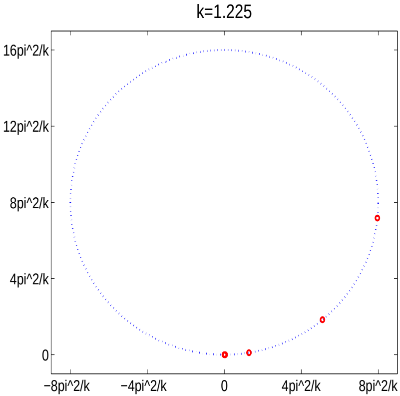


Straightforward approximation of interior eigenvalues

Eigenvalues.



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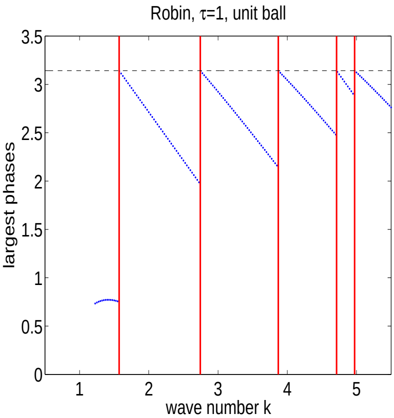

Straightforward approximation of interior eigenvalues

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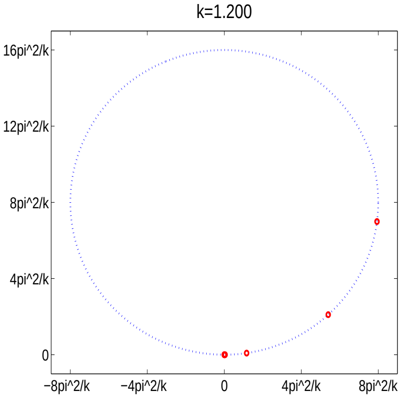

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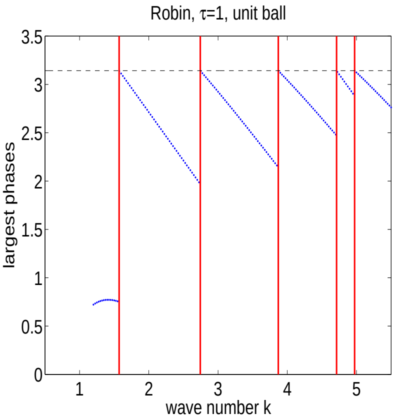

Straightforward approximation of interior eigenvalues

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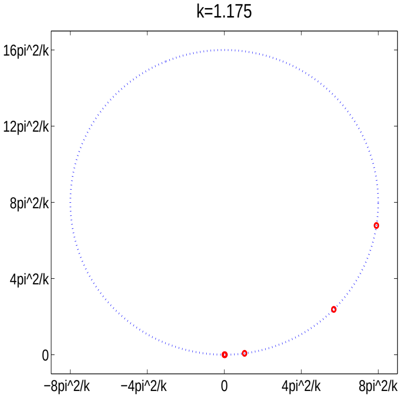

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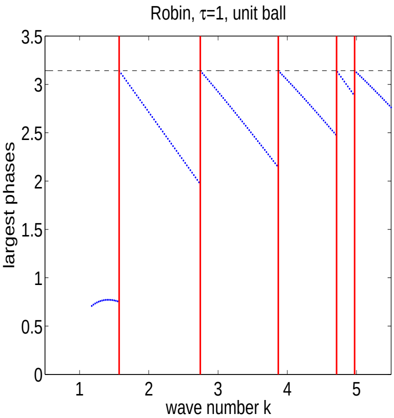
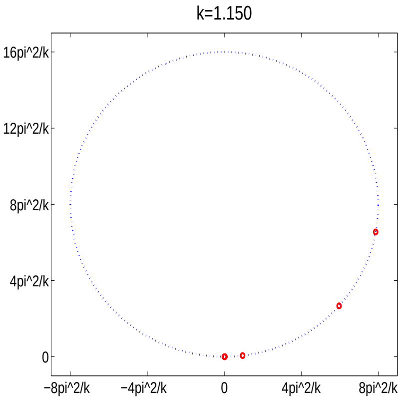
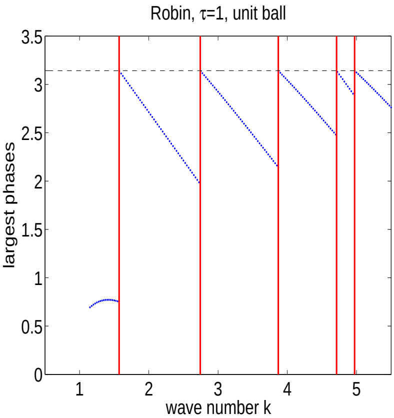


Straightforward approximation of interior eigenvalues

Eigenvalues.



Largest phases.



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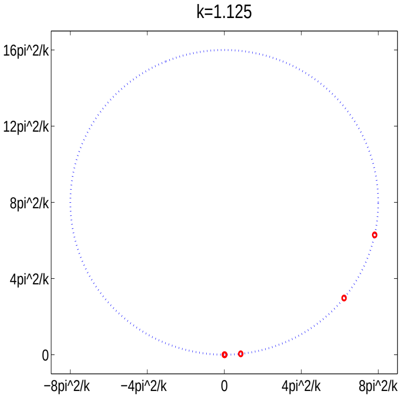

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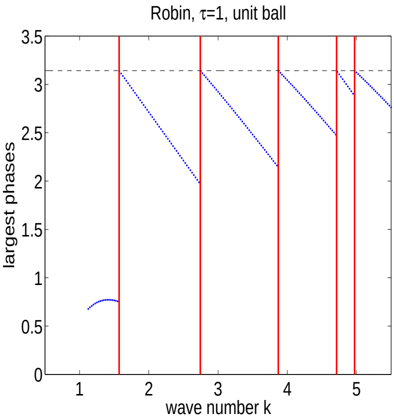

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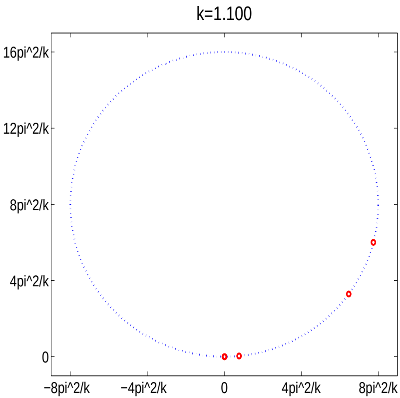

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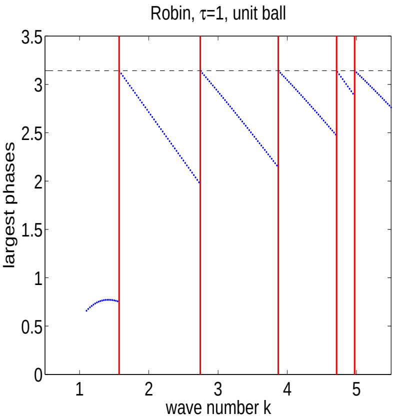

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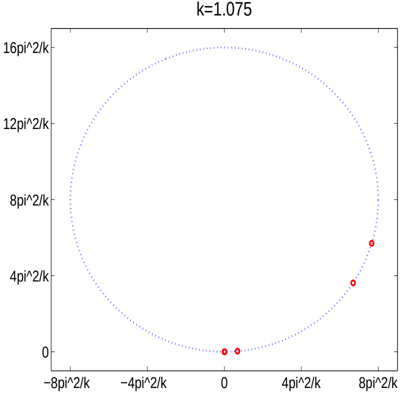

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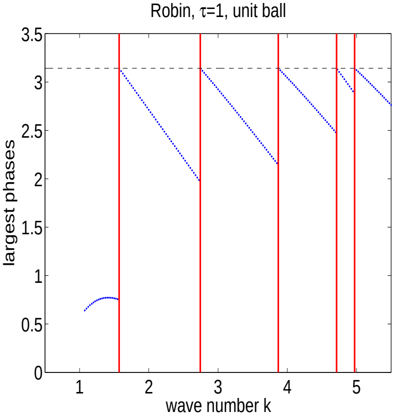

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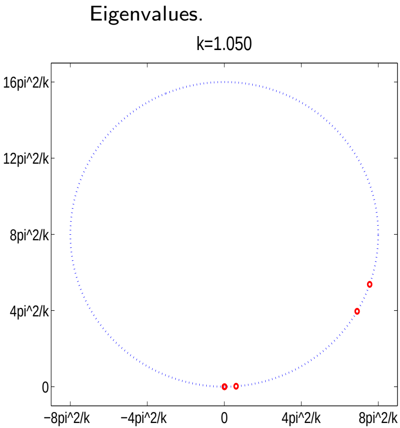

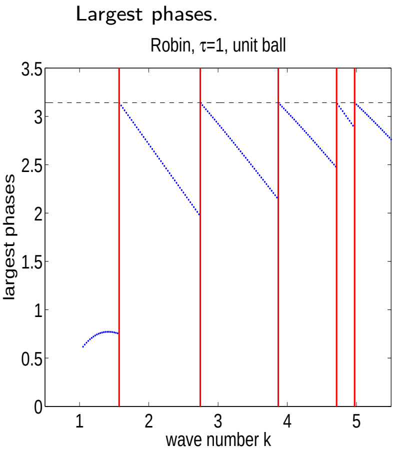

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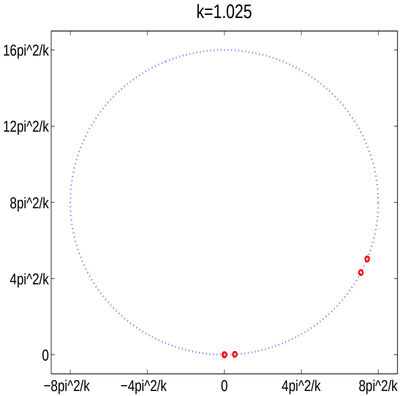

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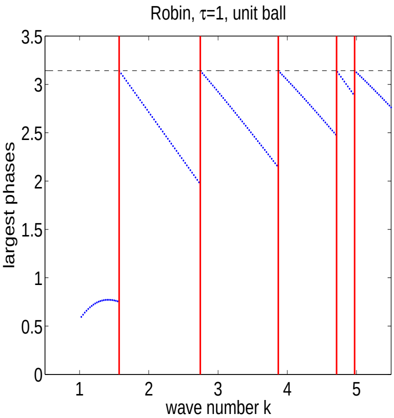

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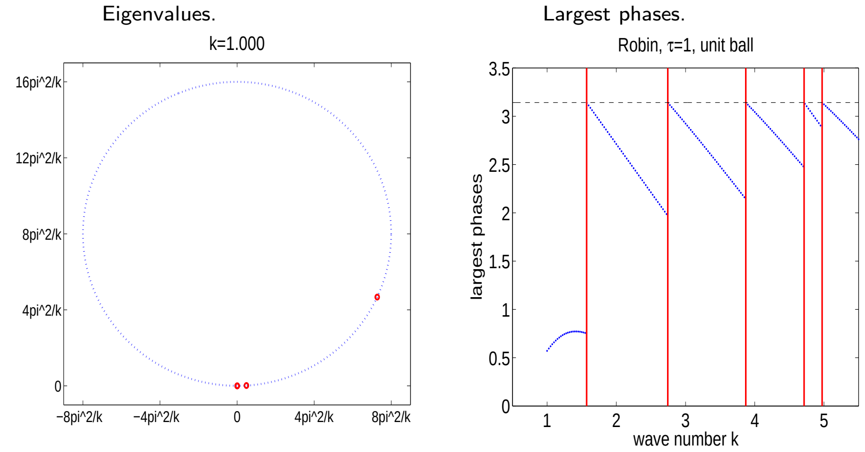

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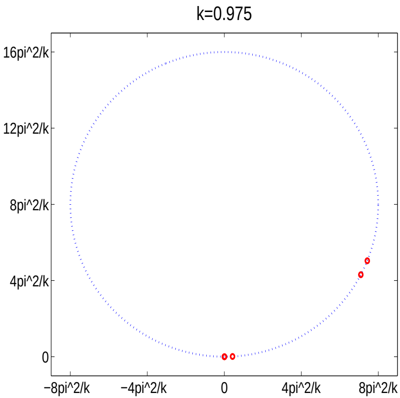

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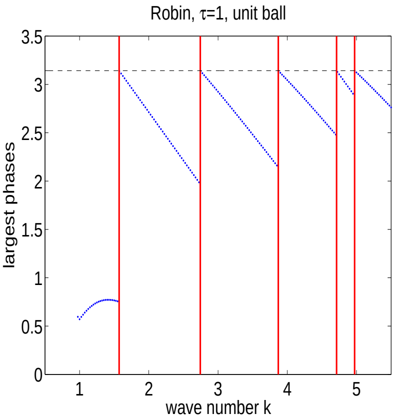

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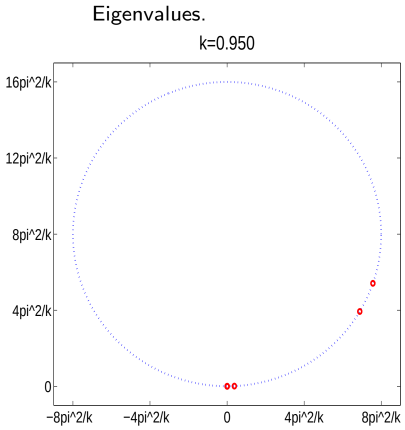

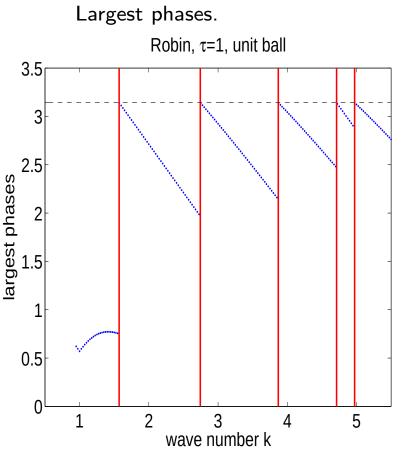

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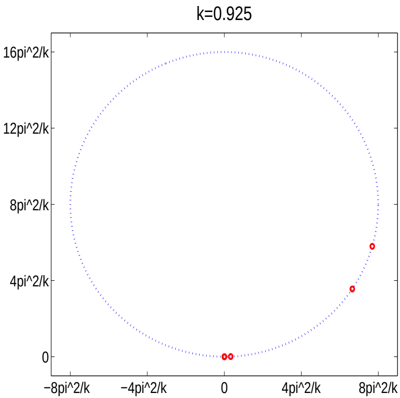

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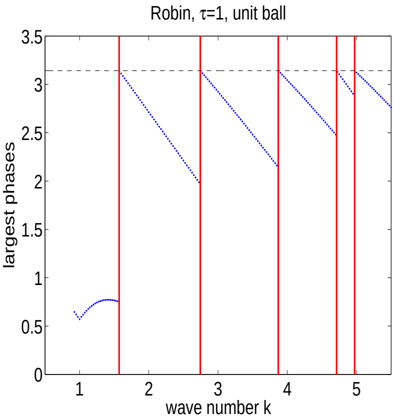

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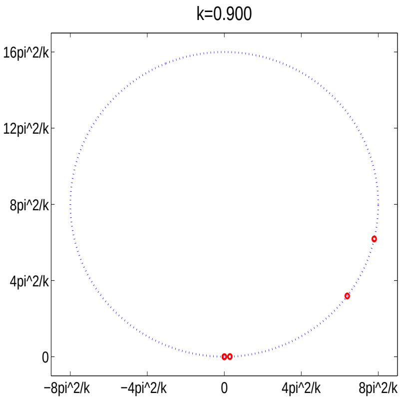

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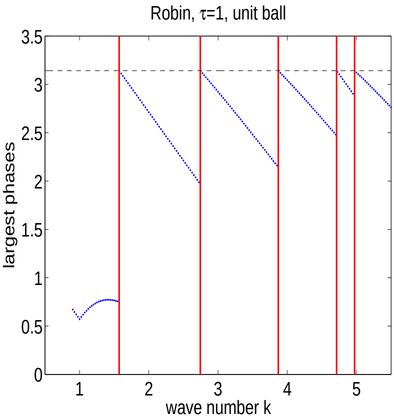

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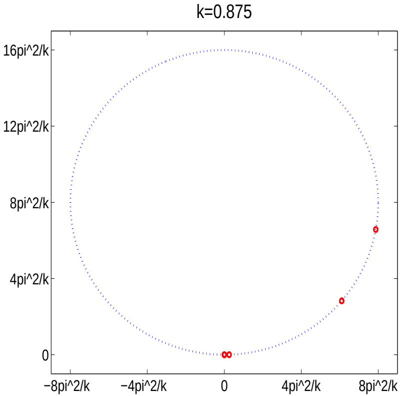

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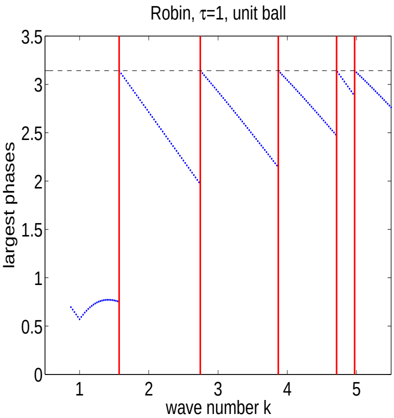

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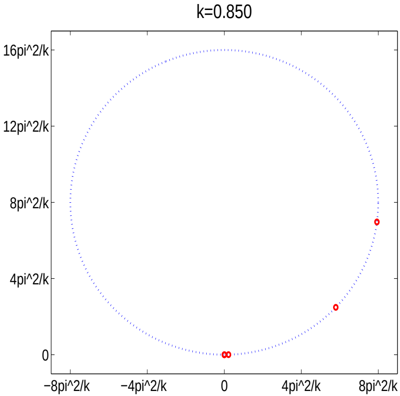

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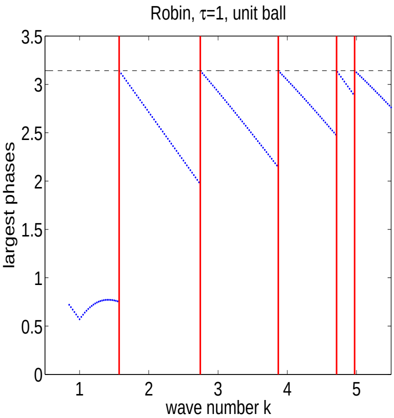

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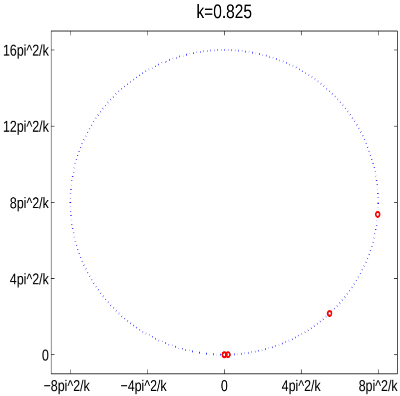

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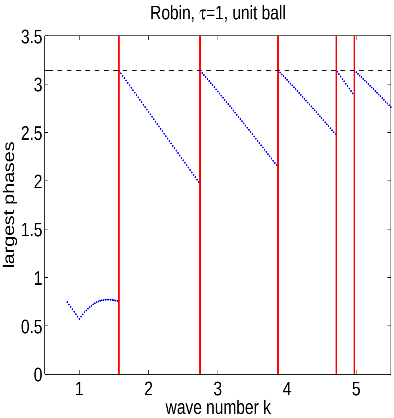

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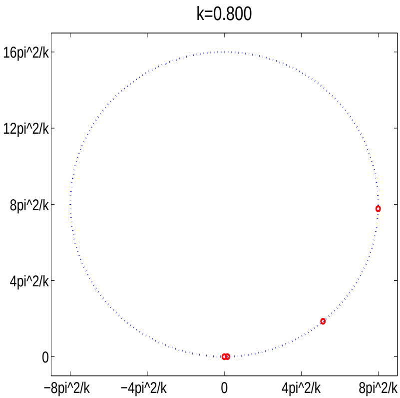

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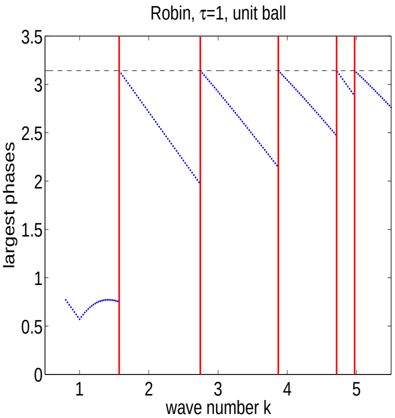

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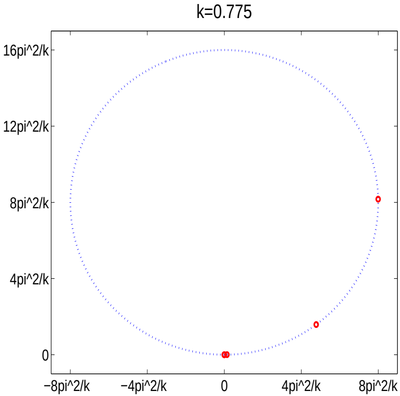

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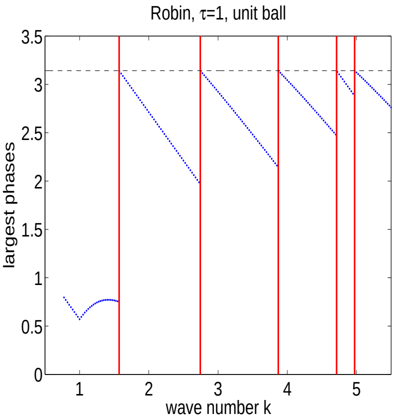

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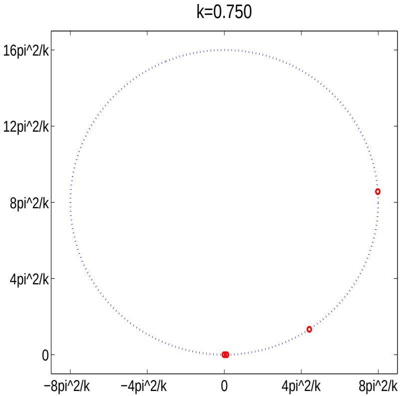

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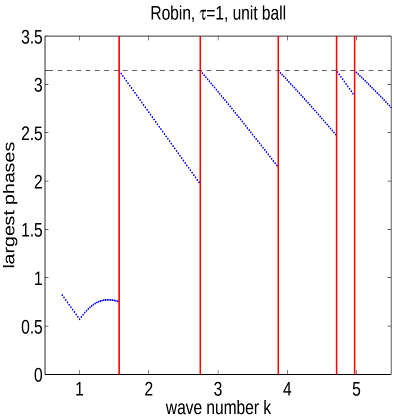

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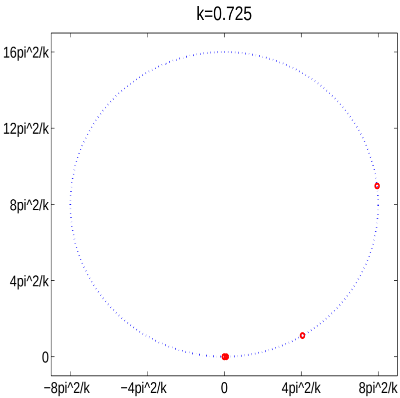

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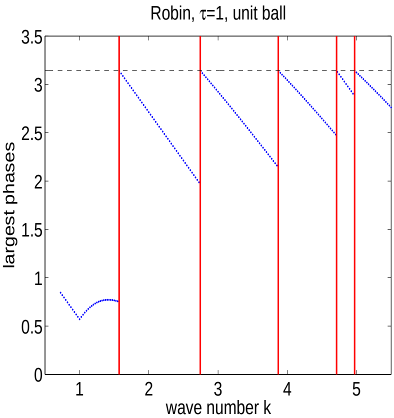

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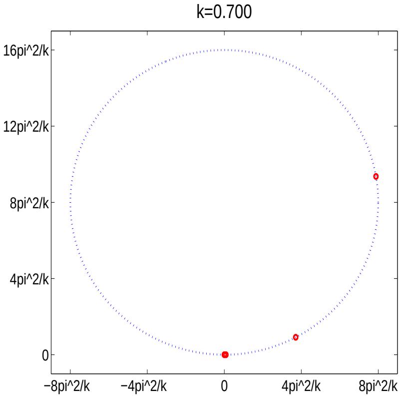

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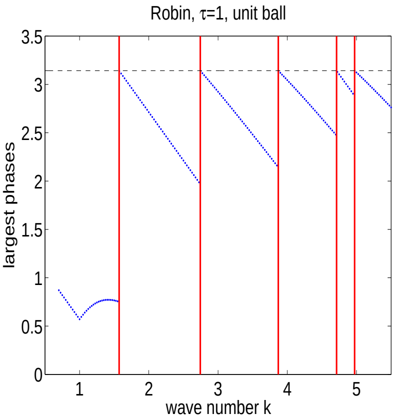

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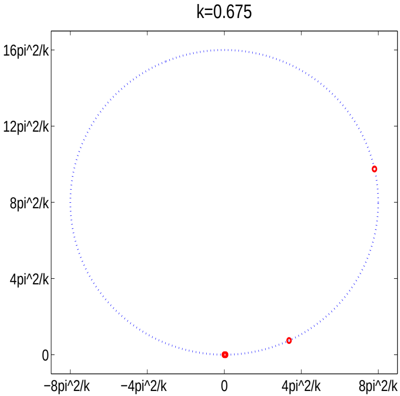

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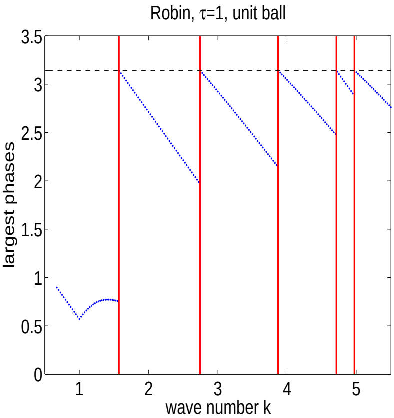

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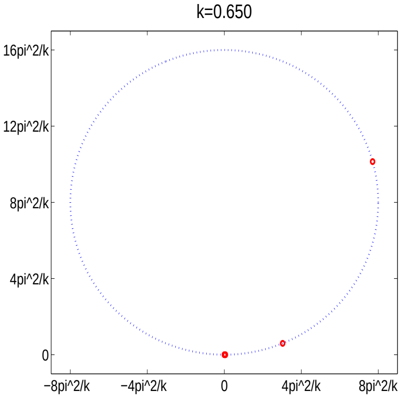

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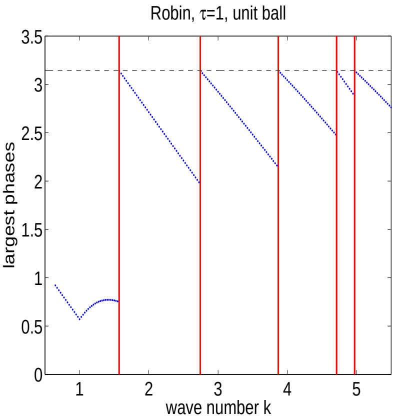

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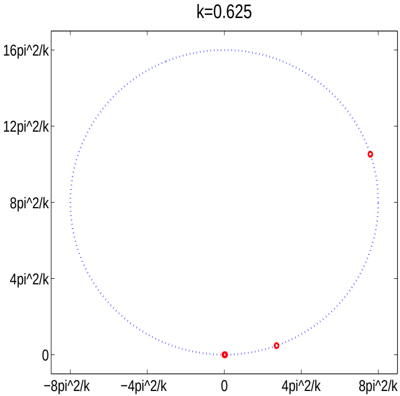

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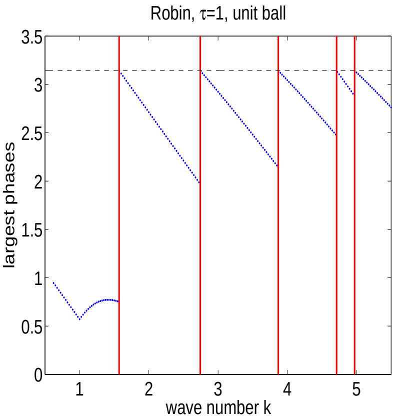

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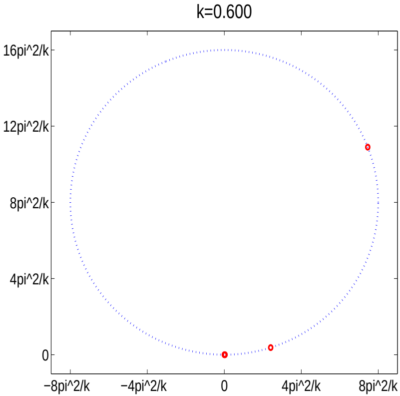

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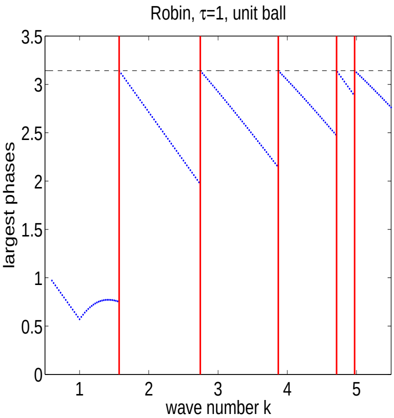

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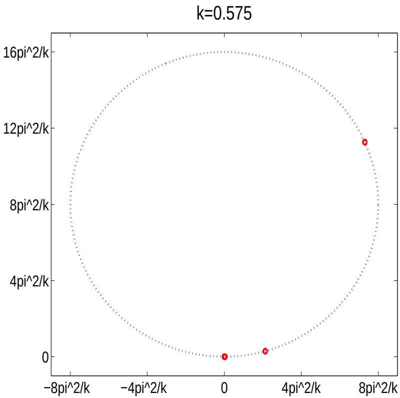

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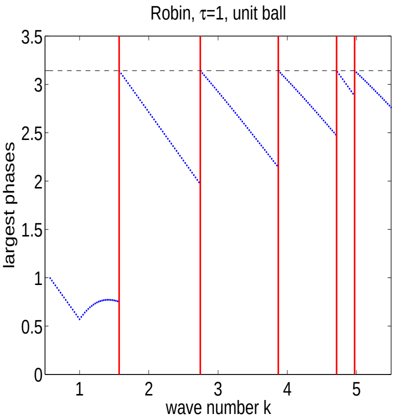

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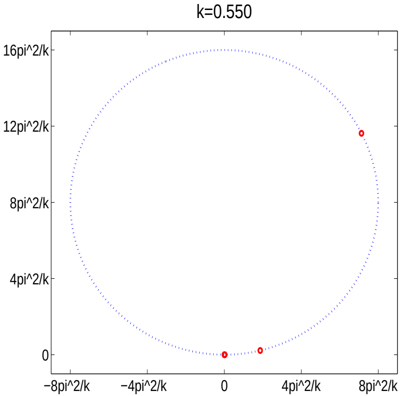

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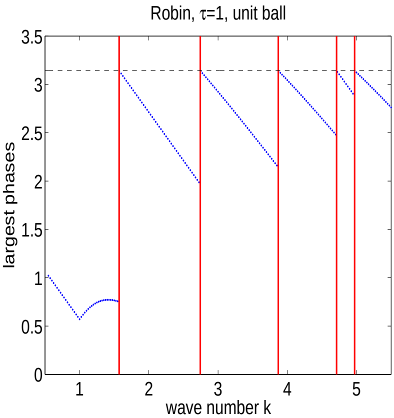

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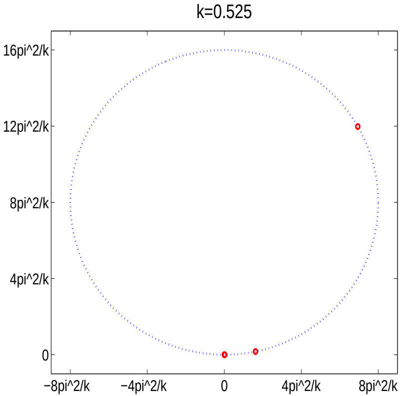

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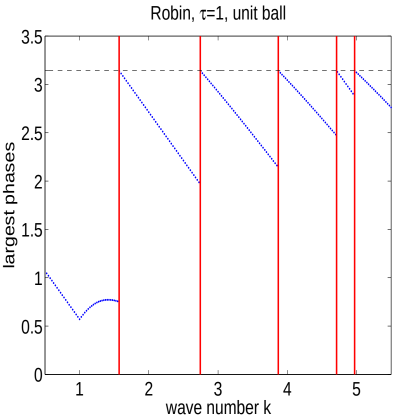

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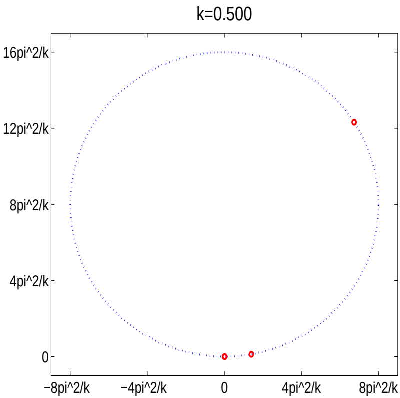

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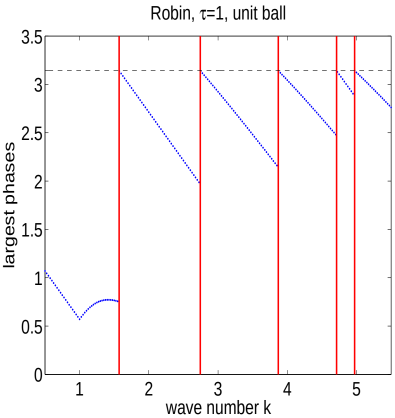
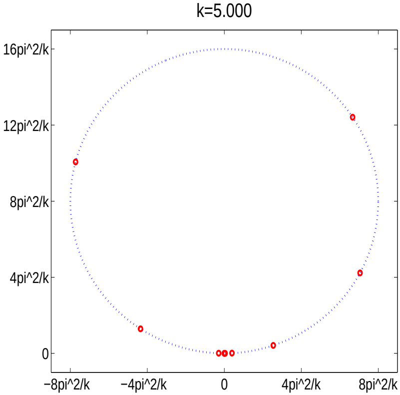


But may be inaccurate

Exact eigenvalues.



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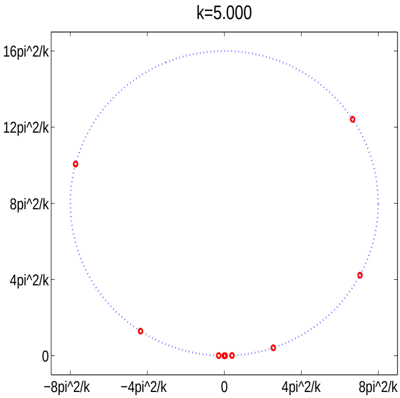
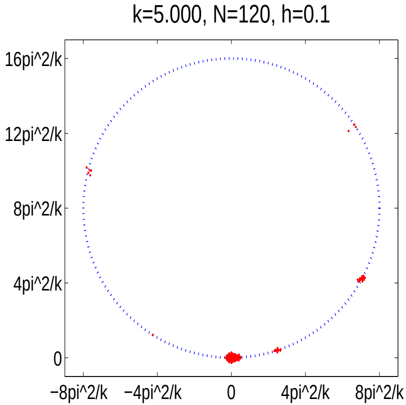


Numerical approximations.



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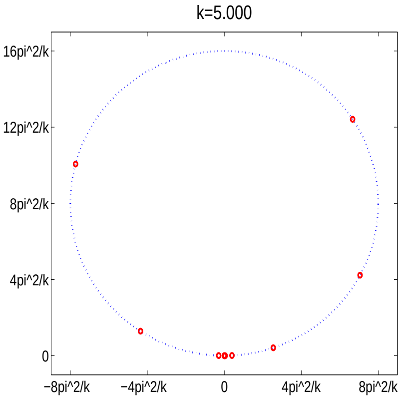

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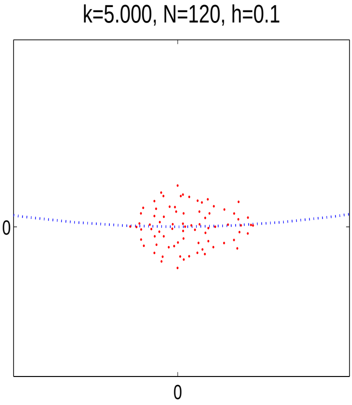

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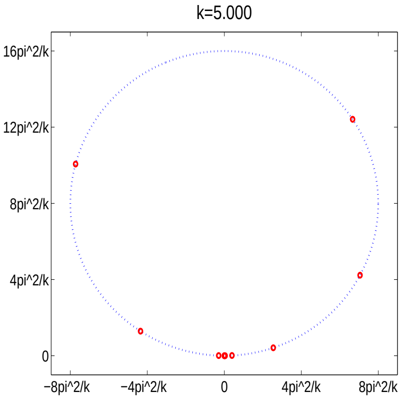

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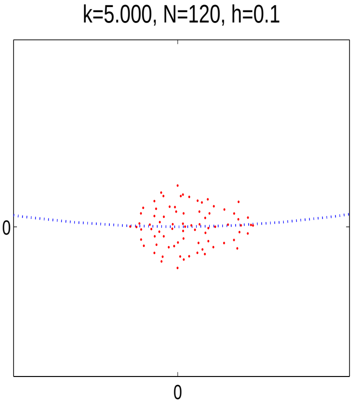

- Error estimate of approximations.
- Regularization.

Outline

1. Motivation
2. Error estimate
   - Approximation of far field operators
   - Approximation of eigenvalues and their phases of far field operators
3. Regularization
4. Numerical Examples
5. Conclusion

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1 Motivation

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Discrete far field data

Two sources of error

$$Fg(\hat{x}) := \int_{\mathbb{S}^2} u^\infty(\hat{x},\theta)g(\theta)\,\mathrm{d}S(\theta), \qquad \hat{x} \in \mathbb{S}^2,$$

Discrete far field data

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$$F_N \leftarrow \mathbb{F}_N := \left(u^\infty(\theta_N^{(j)}, \theta_N^{(\ell)})\right)_{j,\ell=1}^N, \quad N \in \mathbb{N}$$

- Interpolation on the sphere.

Discrete far field data

Two sources of error

$$
\begin{aligned}
Fg(\hat{x}) &:= \int_{\mathbb{S}^2} u^\infty(\hat{x},\theta) g(\theta)\,\mathrm{d}S(\theta), \qquad \hat{x} \in \mathbb{S}^2, \\
F_N \leftarrow \mathbb{F}_N &:= \left(u^\infty(\theta_N^{(j)}, \theta_N^{(\ell)})\right)_{j,\ell=1}^N, \quad N \in \mathbb{N} \\
F_N^\delta \leftarrow \mathbb{F}_N^\delta &:= \left(u_\delta^\infty(\theta_N^{(j)}, \theta_N^{(\ell)})\right)_{j,\ell=1}^N, \\
\|\mathbb{F}_N - \mathbb{F}_N^\delta\|_2 &= \left(\sum_{j,\ell=1}^N \left| u_\delta^\infty(\theta_N^{(j)}, \theta_N^{(\ell)}) - u^\infty(\theta_N^{(j)}, \theta_N^{(\ell)}) \right|^2 \right)^{1/2} \leq \delta.
\end{aligned}
$$

- Interpolation on the sphere.
- Noised data.

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$$F g(\hat{x}) := \int_{\mathbb{S}^2} u^\infty(\hat{x},\theta) g(\theta)\, \mathrm{d}S(\theta), \qquad \hat{x} \in \mathbb{S}^2,$$

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- Interpolation on the sphere.
- Noised data.

Inverse problem: $u_\delta^\infty(\theta_N^{(j)}, \theta_N^{(\ell)}) \to$ interior eigenvalues.

Regularity assumption on interpolation operators

- Pairwise different directions $\Theta_N := \{\theta_N^{(j)}\}_{j=1}^N \subset \mathbb{S}^2$ such that

$$h_N := \inf_{\theta \in \mathbb{S}^2} \left\{ \left| \theta - \theta_N^{(j)} \right|, 1 \leq j \leq N \right\} \to 0 \qquad \text{as } N \to \infty.$$

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- Associated linearly independent continuous functions $\Phi_N := \{\phi_N^{(j)}\}_{j=1}^N \subset C^0(\mathbb{S}^2)$ such that

$$A_{\Phi_N} = \left(\phi_N^{(\ell)} (\theta_N^{(j)}) \right)_{j,\ell=1}^N \in \mathbb{C}^{N \times N} \quad \text{is invertible.}$$

Regularity assumption on interpolation operators

- Pairwise different directions $\Theta_N := \{\theta_N^{(j)}\}_{j=1}^N \subset \mathbb{S}^2$ such that

$$h_N := \inf_{\theta \in \mathbb{S}^2} \left\{ \left|\theta - \theta_N^{(j)}\right|, \, 1 \le j \le N \right\} \to 0 \qquad \text{as } N \to \infty.$$

- Associated linearly independent continuous functions $\Phi_N := \{\phi_N^{(j)}\}_{j=1}^N \subset C^0(\mathbb{S}^2)$ such that

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- Interpolation operator $\mathcal{I}_N : C^0(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$

$$\mathcal{I}_N[g] = \sum_{j=1}^N \left[A_{\Phi_N}^{-1} \left(g(\theta_N^{(j)})\right)_{j=1}^N\right](j) \, \phi_N^{(j)}, \qquad \mathcal{I}_N[g](\theta_N^{(j)}) = g(\theta_N^{(j)}) \qquad \forall \theta_N^{(j)} \in \Theta_N.$$

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Assumption: for $\sigma = \sigma_{\mathcal{I}} \ge 0$ and $s > 1 - \sigma > 0$, there exists $C_{\mathcal{I}} = C_{\mathcal{I}}(s,\sigma) > 0$ such that

$$\begin{aligned} \|g - \mathcal{I}_N[g]\|_{L^2(\mathbb{S}^2)} &\le C_{\mathcal{I}} h_N^s \|g\|_{H^{s+\sigma}(\mathbb{S}^2)}, \qquad \forall g \in H^{s+\sigma}(\mathbb{S}^2), \forall N \in \mathbb{N}, \\ \|\mathcal{I}_N[g]\|_{L^2(\mathbb{S}^2)} &\le (1 + C_{\mathcal{I}})\|g\|_{H^{s+\sigma}(\mathbb{S}^2)}. \end{aligned} \tag{A1}$$

Discrete interpolation operators

$$\mathcal{I}_N : C^0(\mathbb{S}^2) \to L^2(\mathbb{S}^2), \qquad \mathcal{I}_N[g] = \sum_{j=1}^{N} \left[A_{\Phi_N}^{-1} \left(g(\theta_N^{(j)})\right)_{j=1}^{N}\right](j)\, \phi_N^{(j)}, \qquad \forall g \in C^0(\mathbb{S}^2),$$

$$Q_N : \mathbb{C}^N \to L^2(\mathbb{S}^2), \qquad Q_N g_N = \sum_{j=1}^{N} \left[A_{\Phi_N}^{-1} g_N\right](j)\, \phi_N^{(j)} \qquad \forall g_N \in \mathbb{C}^N.$$

Property: $\|Q_N\|^2_{\mathbb{C}^N \to L^2(\mathbb{S}^2)} \leq C_Q(N)$. $C_Q(N)$ is uniformly bounded in N for interpolations using indicator functions or piecewise polynomial and globally continuous functions.

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Theorem

If $\{\mathcal{I}_N\}_{N\in\mathbb{N}}$ satisfies (A1) for $\sigma = \sigma_{\mathcal{I}} \geq 0$ and $s > 1-\sigma > 0$ and if $\|\mathbb{F}_N - \mathbb{F}_N^\delta\|_2 \leq \delta$, then

$$\|F - F_N^\delta\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} \leq C h_N^s + C_Q(N)\delta, \qquad N \in \mathbb{N},$$

with C independent of $N \in \mathbb{N}$.

Idea of proof:

- $\|F - F_N\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} = \|F - \mathcal{I}_N F \mathcal{I}_N^*\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} \leq C h_N^s$ due to (A1).
- $\|F_N - F_N^\delta\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} = \|Q_N(\mathbb{F}_N - \mathbb{F}_N^\delta) Q_N^*\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} \leq C_Q(N)\delta$.

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Approximation of eigenvalues

Link condition between directions Θ_N and basis functions Φ_N:

- Vectors $g_{\Theta_N} := (g(\theta_N^{(j)}))_{j=1}^N \in \mathbb{C}^N$, $g_{\Phi_N} = ((g, \phi_N^{(j)})_{L^2(\mathbb{S}^2)})_{j=1}^N$.

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- Weights $w_N(j) := \|\phi_N^{(j)}\|_{L^2(\mathbb{S})}$ and weight matrix $\mathbb{W}_N = \mathrm{diag}(w_N(j))_{j=1}^N \in \mathbb{R}^{N\times N}$.

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- Assumption $(A2)$: $A^*_{\Phi_N}\left(\mathbb{W}_N^2\, g_{\Theta_N}\right) = g_{\Phi_N}$ for all $g \in \mathrm{span}\{\phi_N^{(1)}, \dots, \phi_N^{(j)}\}$.

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Lemma

If $(A2)$ is satisfied, then all eigenvalues of $\mathbb{W}_N\mathbb{F}_N\mathbb{W}_N$ and $\mathbb{W}_N\mathbb{F}_N^\delta\mathbb{W}_N$ are eigenvalues of F_N and F_N^δ respectively, and any additional eigenvalue of F_N and F_N^δ must vanish.

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Theorem

F has eigenvalues λ_j, $j \in \mathbb{N}$. For all eigenvalues λ_ℓ^N of F_N^δ it holds that

$$\min_{j\in\mathbb{N}} |\lambda_\ell^N - \lambda_j| \le \varepsilon := \|F_N^\delta - F\|, \qquad \ell = 1, \dots, \mathcal{J}(N).$$

If a finite subset $\sigma_1 \subset \sigma(F)$ satisfies that $\mathrm{dist}_H(\sigma_1, \sigma(F) \setminus \sigma_1) > 2\varepsilon$, then the ε-neighborhood of σ_1 contains at least one eigenvalue of F_N^δ.

Proof: Bauer-Fike theorem.

Approximation of eigenvalues phases

Theorem

Assume that $\mathcal{I}_N$, Φ_N and Θ_N satisfy $(A1)$, $(A2)$. Consider a sequence of perturbed far field matrices $\mathbb{F}_N^{\delta_N}$ with noise level δ_N and assume in addition that $C_Q(N)\delta_N \to 0$ as $N \to \infty$.

1. *If $\lambda_j \neq 0$ is a non-zero eigenvalue of F, then there exists $\{j'(N)\}_{N\in\mathbb{N}}$ such that*
$$|\lambda_{j'(N)}^N - \lambda_j| \leq \|F_N^\delta - F\| =: \varepsilon_N \leq Ch_N^s + C_Q(N)\delta_N \to 0 \qquad \text{as } N \to \infty.$$
The sum of the dimensions of the eigenspaces corresponding to all eigenvalues of $\mathbb{W}_N\mathbb{F}_N^\delta\mathbb{W}_N$ contained in $B(\lambda_j, \varepsilon_N)$ equals the multiplicity of λ_j.
2. *If $|\lambda_j| = r_j > \varepsilon_N = \|F_N^\delta - F\|$, then the phase of any eigenvalue $\lambda_{j'}^N \in B(\lambda_j, \varepsilon_N)$ of $\mathbb{W}_N\mathbb{F}_N^\delta\mathbb{W}_N$ satisfies*
$$|\vartheta_{j'}^N - \vartheta_j| < \arcsin(\varepsilon_N/r_j) \leq \frac{\pi\varepsilon_N}{2r_j} \leq \frac{\pi}{2r_j}(Ch_N^s + C_Q(N)\delta_N) \xrightarrow{N\to\infty} 0.$$

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Phases belong to $(0, \pi)$

- An eigenvalue λ_ℓ^N of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ satisfies $|\lambda_\ell^N| > 4\pi\left(\varepsilon_N/k\right)^{1/2} + \varepsilon_N$.

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- $\lambda_j = a + \mathrm{i}b$ lies on the circle $\{|z - 8\pi^2\mathrm{i}/k| = 8\pi^2/k\}$

$$a^2 + \left(b - \frac{8\pi^2}{k}\right)^2 = \left(\frac{8\pi^2}{k}\right)^2, \quad \text{that is,} \quad |\lambda_j|^2 = a^2 + b^2 = \frac{16\pi^2}{k} b.$$

So $b = \mathrm{Im}\,(\lambda_j) > \varepsilon_N$. Hence $\mathrm{Im}\,(\lambda_\ell^N) > 0$ and $\vartheta_\ell^N \in (0, \pi)$.

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- Assume that N is so large that $\varepsilon_N^{1/2} < 4\pi/k^{1/2}$, then $r_j > 4\pi\left(\varepsilon_N/k\right)^{1/2} > \varepsilon_N$ and

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Largest regularized discrete phase

$$\vartheta^{\divideontimes}(k, N) = \max\left\{\vartheta_j^N,\ \lambda_j^N \in \sigma(\mathbb{W}_N \mathbb{F}_N^{\delta_N}(k) \mathbb{W}_N),\ |\lambda_j^N| > 4\pi(\varepsilon_N/k)^{1/2} + \varepsilon_N\right\}.$$

Regularization

Assumptions:

- Link between directions Θ_N and basis functions Φ_N:

$$A^*_{\Phi_N}\left(\mathbb{W}^2_N\, g_{\Theta_N}\right) = g_{\Phi_N} \text{ for all } g \in \operatorname{span}\{\phi_N^{(1)}, \dots, \phi_N^{(j)}\}.$$

- Density of discrete wave numbers: $K = \{k_i\}_{i\in\mathbb{N}} \subset [k_{\min}, k_{\max}]$ such that $\overline{K} = [k_{\min}, k_{\max}]$.
- Uniform noise level: $\max_{k\in K} \|\mathbb{F}_N^{\delta_N}(k) - \mathbb{F}_N(k)\|_2 \le \delta_N,\ N \in \mathbb{N}$.
- $C_Q(N)\delta_N \to 0$ as $N \to \infty$.

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- $C_Q(N)\delta_N \to 0$ as $N \to \infty$.

Theorem

Under above assumptions, if $\{k_i\}_{i\in\mathbb{N}} \subset K$, $k_i \neq k \in (k_{\min}, k_{\max})$, $k_i \to k$ as $i \to \infty$, and denote the eigenvalue of $F(k_i)$ with largest phase by $\lambda^*(k_i)$ and its phase by $\vartheta^*(k_i)$, then

1. *If $\vartheta^{\divideontimes}(k_i, N_i) \to \pi$ for any sequence $\{N_i\}_{i\ge i_0} \subset \mathbb{N}$ with $N_i \to \infty$ as $i \to \infty$, then $\vartheta^*(k_i) \to \pi$ as $i \to \infty$ and k^2 is an interior eigenvalue of D.*
2. *If $\vartheta^*(k_i) \to \pi$ there is $i_0 \in \mathbb{N}$ and $\{N_i\}_{i\ge i_0} \subset \mathbb{N}$ with $N_i \to \infty$ as $i \to \infty$ such that*

$$4\pi\left(\frac{\varepsilon_{N_i}}{k_i}\right)^{1/2} + 2\varepsilon_{N_i} \le \min\left\{|\lambda^*(k_i)|, \operatorname{dist}\left[\lambda^*(k_i), \sigma\big(F(k_i)\big) \setminus \{\lambda^*(k_i)\}\right]\right\}, \qquad i \ge i_0,$$

and for any such sequence $\{N_i\}_{i>i_0}$ it holds that $\vartheta^{\divideontimes}(k_i, N_i) \to \pi$ as $i \to \infty$.

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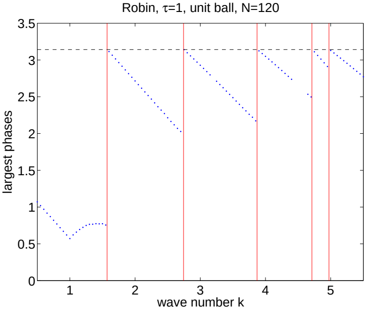
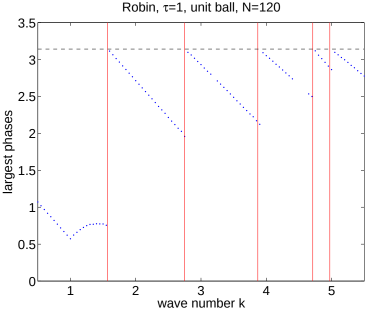
Scattering from a unit ball with Robin b.c.

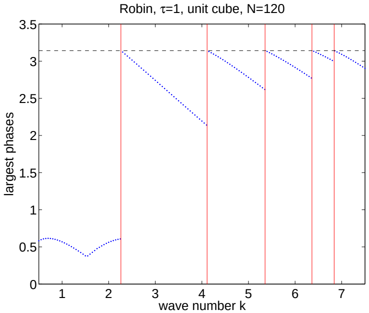


Largest regularised discrete phases, no artificial noise.



Largest regularised discrete phases, 5% artificial noise.

Scattering from a unit cube with Robin b.c.



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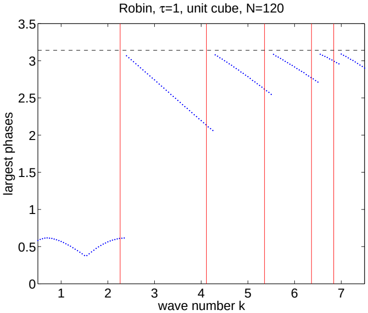


Largest regularised discrete phases, 5% artificial noise.

Convergence

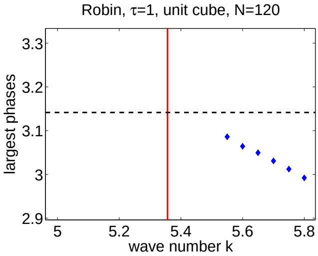
Roots of interior eigenvalues (ball)	1st	2nd	3rd	4th	5th
Exact value (4 digits)	1.571	2.743	3.870	4.712	4.973
$N = 48$, $\triangle k = 0.2$, no artificial noise	1.70	2.90	3.90	4.90	5.10
$N = 80$, $\triangle k = 0.1$, no artificial noise	1.60	2.80	3.90	4.80	5.00
$N = 120$, $\triangle k = 0.05$, no artificial noise	1.60	2.75	3.90	4.75	5.00
$N = 120$, $\triangle k = 0.05$, 1% artificial noise	1.60	2.80	3.90	4.75	5.00
$N = 120$, $\triangle k = 0.05$, 5% artificial noise	1.60	2.80	3.95	4.75	5.05

Table: Estimates of the square roots of the first five Robin eigenvalues of $-\Delta$ in the unit ball from far field data for different levels of artificial additive noise.

Roots of interior eigenvalues (cube)	1st	2nd	3rd	4th	5th
Exact value (4 digits)	2.263	4.112	5.357	6.362	6.839
$N = 48$, $\triangle k = 0.2$, no artificial noise	2.30	4.30	5.50	6.50	6.90
$N = 80$, $\triangle k = 0.1$, no artificial noise	2.30	4.20	5.40	6.40	6.90
$N = 120$, $\triangle k = 0.05$, no artificial noise	2.30	4.15	5.40	6.40	6.85
$N = 120$, $\triangle k = 0.05$, 1% artificial noise	2.30	4.15	5.40	6.40	6.90
$N = 120$, $\triangle k = 0.05$, 5% artificial noise	2.35	4.20	5.45	6.50	6.95

Table: Estimates of the square roots of the first five Robin eigenvalues of $-\Delta$ in the cube $(0,1)^3$ from far field data for different levels of artificial additive noise.

Better approximation using extrapolation



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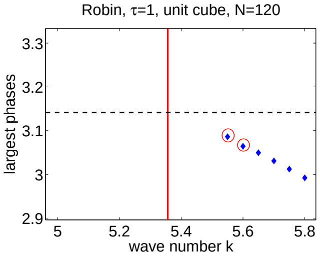

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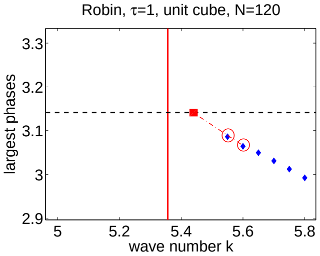

Extrapolation

$$k_{\text{appr}}^{\text{rob}} = k_* + \frac{\pi - \vartheta^{※}(k_*, N)}{\vartheta^{※}(k_* + \triangle k, N) - \vartheta^{※}(k_*, N)} \triangle k$$

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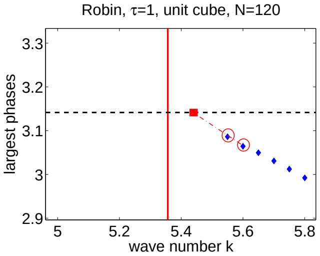

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Roots of interior eigenvalues	1st	2nd	3rd	4th	5th
Ball – Exact value (4 digits)	1.571	2.743	3.870	4.712	4.973
Before extrapolation	1.600	2.800	3.950	4.750	5.050
Extrapolation	1.575	2.763	3.884	4.719	4.988
Relative error	0.25%	0.73%	0.36%	0.14%	0.30%
Cube – Exact value (4 digits)	2.263	4.112	5.357	6.362	6.839
Before extrapolation	2.350	4.200	5.450	6.500	6.950
Extrapolation	2.267	4.133	5.307	6.406	6.827
Relative error	0.17%	0.51%	0.93%	0.69%	0.17%

Table: Estimates of the square roots of the first five Robin eigenvalues by the extrapolation procedure. The relative artificial noise level equals 5%. Fixed parameters are $N = 120$, $\triangle k = 0.05$

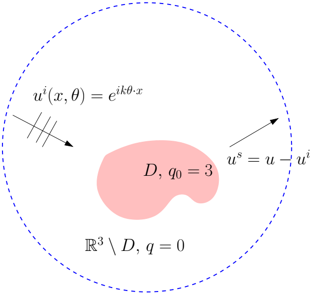
Approximating interior transmission eigenvalues of penetrable media

Scattering problem from an inhomogeneous medium

$$\Delta u + k^2(1+q)u = 0 \qquad \text{in } \mathbb{R}^3,$$

Interior transmission eigenvalue problem $(v, w) := (u, u^i)$

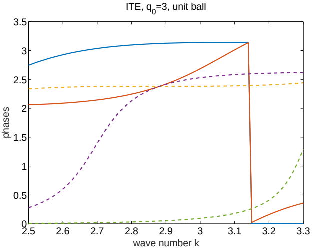
$$\begin{cases} \Delta v + k^2(1+q)v = 0 & \text{in} D, \\ \Delta w + k^2 w = 0 & \text{in } D, \\ v = w & \text{on } \partial D, \\ \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} & \text{on } \partial D. \end{cases}$$



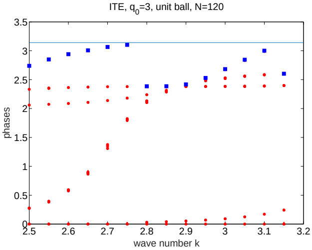
Inhomogeneous medium with contrast q_0 supported in D.

Roots of interior eigenvalues (ball)	1st		2nd	3rd	4th
Exact value (4 digits)	3.142		3.692	4.262	4.832
Estimate, no artificial noise	3.00		3.65	4.25	4.85
Extrapolation, no artificial noise	3.116		3.697	4.268	4.857
Relative error (extrapolation)	0.81%		0.14%	0.14%	0.52%
Estimate, 5% artificial noise	2.75	3.10	3.65	4.25	4.80
Extrapolation, 5% artificial noise	2.803	3.144	3.698	4.268	4.845
Relative error (extrapolation)	10.78%	0.08%	0.16%	0.14%	0.27%

Table: Estimates of the square roots of the first four interior transmission eigenvalues of the unit ball for various noise levels. Fixed parameters are $N = 120$, $\triangle k = 0.05$, $q_0 = 3$.



Exact phases of F.



Computed phases, 5% art. noise.

Outline

1 Motivation

2 Error estimate
- Approximation of far field operators
- Approximation of eigenvalues and their phases of far field operators

3 Regularization

4 Numerical Examples

5 Conclusion

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- Give estimate of error for the far field operator due to interpolation and discrete data noise.
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- Suggest a regularized inversion method to approximate interior eigenvalues of domains.
- Illustrate the feasibility and robustness with some examples.

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Thank you for your attention!