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Stochastic Galerkin finite element method for electrical impedance tomography

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Connection to waves

A more general name for this talk could be, e.g.,

Utilizing solution techniques designed for parameter-dependent PDEs in inverse boundary value problems.

EIT is just one example.

Outline

- Problem setting (CEM, SCEM).
- Karhunen-Loève expansion.
- Polynomial chaos expansion.
- Bayesian inverse problem.
- Numerical examples.
- Real data.

Complete electrode model (CEM)

We assume that

- $D \subset \mathbb{R}^2$ is a bounded domain with a smooth enough boundary,
- ∂D is partially covered with $M \in \mathbb{N} \setminus \{1\}$ well-separated connected electrodes $\{E_m\}_{m=1}^M$,
- $\sigma \in L_+^\infty(D)$ models the conductivity of D,
- $z_1, \ldots, z_M \geq c > 0$ are the contact resistances between the electrodes and the domain, and
- $I = [I_m]_{m=1}^M$ and $U = [U_m]_{m=1}^M$ of $\mathbb{R}_\diamond^M$ represent the net current and voltage patterns on the electrodes.

Complete electrode model (CEM)

Deterministic forward problem: Find

$$(u, U) \in \mathcal{H} := H^1(D) \oplus \mathbb{R}^M_\diamond$$

such that the following equations hold:

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 & \text{in } D, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial D \setminus \overline{\cup_m E_m}, \\ u + z_m \sigma \dfrac{\partial u}{\partial \nu} = U_m & \text{on } E_m, \quad m = 1, \dots, M, \\ \displaystyle\int_{E_m} \sigma \frac{\partial u}{\partial \nu} \, \mathrm{d}S = I_m, & m = 1, \dots, M, \end{cases}$$

for a given electrode current pattern $I \in \mathbb{R}^M_\diamond$.

Stochastic complete electrode model (SCEM)

Modified assumptions:

- (Ω, Σ, P) is a probability space,
- $\sigma : \Omega \times D \to \mathbb{R}$ is a random conductivity field in $L^\infty(\Omega \times D)$,
- σ is uniformly strictly positive.

Stochastic complete electrode model (SCEM)

Stochastic forward problem: Find

$$(u, U) \in L^2_P(\Omega; \mathcal{H}) \simeq L^2_P(\Omega) \otimes \mathcal{H}, \quad \mathcal{H} := H^1(D) \oplus \mathbb{R}^M_\diamond,$$

such that the following equations hold P-almost surely:

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 & \text{in } D, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial D \setminus \overline{\cup_m E_m}, \\ u + z_m \sigma \dfrac{\partial u}{\partial \nu} = U_m & \text{on } E_m, \quad m = 1, \dots, M, \\ \displaystyle\int_{E_m} \sigma \frac{\partial u}{\partial \nu} \, \mathrm{d}S = I_m, & m = 1, \dots, M, \end{cases}$$

for a given (deterministic) electrode current pattern $I \in \mathbb{R}^M_\diamond$.

Variational formulation for SCEM

Find $(u, U) \in L^2_P(\Omega; \mathcal{H})$ such that

$$\mathbb{E}\big[B\big((u,U),(v,V)\big)\big] \;=\; I \cdot \mathbb{E}[V] \qquad \text{for all } (v, V) \in L^2_P(\Omega; \mathcal{H}),$$

where

$$B\big((u,U),(v,V)\big) \;=\; \int_D \sigma \nabla u \cdot \nabla v \,\mathrm{d}x + \sum_{m=1}^{M} \frac{1}{z_m} \int_{E_m} (U_m - u)(V_m - v)\,\mathrm{d}S.$$

The unique solvability of this stochastic CEM forward problem follows from the same line of reasoning as its deterministic counterpart under the above assumptions on σ and $z_1, \ldots, z_M$ (cf. [Somersalo92]).

Log-normal random field (not the best choice?)

Conductivity σ is assumed to be a *log-normal random field*

- $\sigma(\omega, \mathrm{x}) = \exp(g(\omega, \mathrm{x}))$,
- $g(\cdot, \mathrm{x})$ is Gaussian for all $\mathrm{x} \in D$.

The random field σ can be characterized by defining

- the mean field $\mathbb{E}_g$, and
- the covariance function V_g

of the underlying Gaussian random field g.

Remark

The requirement for σ to be uniformly strictly positive and bounded does not hold for a log-normal random field. See [Charrier12, Gittelson10] for suitable relaxations.

The Karhunen–Loève expansion

According to the Karhunen–Loève theorem, the Gaussian random field g allows the expansion

$$g(\omega, \mathsf{x}) = \mathbb{E}_g[\mathsf{x}] + \sum_{l=1}^{\infty} \sqrt{\lambda_l} Y_l(\omega)\phi_l(\mathsf{x}),$$

where

- $\{Y_l\}_{l\geq 1}$ follow standard normal distribution and are mutually independent,
- $\{\lambda_l\}_{l\geq 1}$ and $\{\phi_l\}_{l\geq 1}$ are the eigenvalues and eigenfunctions of the covariance operator defined by V_g, respectively.

The exponential Karhunen–Loève expansion

For numerical computations σ is approximated as

$$\sigma(\omega, \mathsf{x}) \approx \sigma_L(\omega, \mathsf{x}) = \exp\left(\mathbb{E}_g[\mathsf{x}] + \sum_{l=1}^{L} \sqrt{\lambda_l} Y_l(\omega)\phi_l(\mathsf{x})\right).$$

The behaviour of the random coefficients $\{Y_l\}_{l=1}^L$ is parametrized using a vector $\mathbf{y} = (y_1, \ldots, y_L) \in \mathbb{R}^L$, with the (prior) probability distribution

$$\mathrm{d}P_{\mathbf{Y}} = \prod_{i=1}^{L} \rho(y_i)\mathrm{d}\mathbf{y} =: \pi(\mathbf{y})\mathrm{d}\mathbf{y},$$

where $\rho : \mathbb{R} \to \mathbb{R}_+$ denotes the standard normal density.

Wiener polynomial chaos expansion

- The mth univariate Hermite polynomial:

$$h_m(x) := (-1)^m \exp(x^2/2) \frac{d^m}{dx^m} \exp(-x^2/2).$$

- Each $\mu \in (\mathbb{N}_0^\infty)_c$ determines a multivariate Hermite polynomial (Chaos polynomial) via

$$H_\mu(\mathbf{Y}) := \prod_{m=1}^{\infty} h_{\mu_m}(Y_m) = \prod_{m \in \operatorname{supp} \mu} h_{\mu_m}(Y_m),$$

where $\mathbf{Y} : \Omega \to \mathbb{R}^\infty$ is an (infinite) vector consisting of independent standard normals.

- The set $\mathcal{P}(\mathbf{Y}) := \{H_\mu(\mathbf{Y}) \mid \mu \in (\mathbb{N}_0^\infty)_c\}$ is an orthogonal basis of $L_P^2(\Omega)$ (under appropriate assumptions).

Discrete approximation of (u, U)

Since $(u, U) \in L^2_P(\Omega; \mathcal{H}) \simeq L^2_P(\Omega) \otimes (H^1(D) \oplus \mathbb{R}^M_\diamond)$, it is naturally approximated as

$$u(\mathbf{Y}, \mathrm{x}) \approx \tilde{u}(\mathbf{Y}, \mathrm{x}) = \sum_{j=1}^{N_D} \sum_{\mu \in \Lambda^L(k)} \alpha_{j,\mu} H_\mu(\mathbf{Y}) \varphi_j(\mathrm{x}),$$

$$U(\mathbf{Y}) \approx \tilde{U}(\mathbf{Y}) = \sum_{i=1}^{M-1} \sum_{\mu \in \Lambda^L(k)} \beta_{i,\mu} H_\mu(\mathbf{Y}) \mathrm{v}^i,$$

where

- $\{\alpha_{j,\mu}\}$ and $\{\beta_{i,\mu}\}$ are to-be-determined coefficients,
- $\{\varphi_j\}_{j=1}^{N_D} \subset H^1(D)$ is a FEM basis,
- $\{\mathrm{v}^i\}_{i=1}^{M-1}$ is a basis of $\mathbb{R}^M_\diamond$,
- $\Lambda^L(k) \subset (\mathbb{N}_0^L)_c$ is a suitable set of multi-indices.

Selection of polynomial basis (multi-indices)

Here we employ *total degree* (TD) polynomial space:

$$\Lambda^L(k) = \left\{ \mu \in \mathbb{N}_0^L \;\middle|\; \sum_{l=1}^{L} \mu_l \leq k \right\}.$$

The dimension of the corresponding polynomial basis is

$$\#\Lambda^L(k) = \binom{L+k}{k} =: N_\Omega,$$

which demonstrates that the size of polynomial chaos explodes as L and/or k grows.

Thus, the parameters L and k must be restricted to relatively small values in practical computations.

sGFEM for the CEM (preprocessing)

The sGFEM-discretized SCEM problem is altogether as follows:

Find $\{\alpha_{j,\mu}\} \subset \mathbb{R}$ and $\{\beta_{i,\mu}\} \subset \mathbb{R}$ such that

$$\mathbb{E}\big[B\big((\tilde{u}, \tilde{U}), (\tilde{v}, \tilde{V})\big)\big] = I \cdot \mathbb{E}[\tilde{V}], \tag{1}$$

where $\tilde{v} = H_{\mu'}(\mathbf{y})\varphi_{j'}(\mathrm{x})$ and $\tilde{V} = H_{\mu'}(\mathbf{y})\mathrm{v}^{i'}$, holds for all $\mu' \in \Lambda^L(k)$, $j' = 1, \ldots, N_D$, and $i' = 1, \ldots, M-1$.

This corresponds to a linear system

$$\mathbf{Ad} = \mathbf{J}$$

with a *sparse* $\mathbf{A} \in \mathbb{R}^{N \times N}$ and $N = N_\Omega(N_D + M - 1)$.

EIT with CEM

- Apply linearly independent current patterns $I^1, \ldots, I^{M-1} \in \mathbb{R}^M_\diamond$ through the electrodes.
- Measure the corresponding noisy potentials $V^1, \ldots, V^{M-1} \in \mathbb{R}^M$ on the electrodes.
- Try to reconstruct (usefull information about) the conductivity.

We denote

- $\mathcal{I} = [(I^1)^\mathsf{T}, \ldots, (I^{M-1})^\mathsf{T}]^\mathsf{T}$
- $\mathcal{V} = [(V^1)^\mathsf{T}, \ldots, (V^{M-1})^\mathsf{T}]^\mathsf{T}$,

and suppose the noise process contaminating $\mathcal{V}$ is

- an additive and mean-free Gaussian with a (known) diagonal covariance matrix Γ.

Bayesian solution of EIT with CEM

The likelihood function can be *approximated* as

$$\pi(\mathcal{V}\,|\,\mathbf{y}) \propto \exp\Big(-\frac{1}{2}\big(\mathcal{V}-\tilde{\mathcal{U}}(\mathbf{y})\big)^{\mathsf{T}}\Gamma^{-1}\big(\mathcal{V}-\tilde{\mathcal{U}}(\mathbf{y})\big)\Big),$$

where

- $\tilde{\mathcal{U}}(\mathbf{y}) = [\tilde{U}^1(\mathbf{y})^{\mathsf{T}}, \ldots, \tilde{U}^{M-1}(\mathbf{y})^{\mathsf{T}}]^{\mathsf{T}}$,
- $\tilde{U}^m(\mathbf{y})$ is the sGFEM approximation corresponding to the current pattern $I = I^m$.

Using Bayes' formula, we obtain the (approximate) posterior density for **y**

$$\pi(\mathbf{y}\,|\,\mathcal{V}) \propto \pi(\mathcal{V}\,|\,\mathbf{y})\pi(\mathbf{y}),$$

where

- $\pi(\mathbf{y}) \sim \mathcal{N}(0, \mathbf{I})$ is the prior density for **y**.

Postprocessing the posterior

The posterior density can be written as

$$\pi(\mathbf{y} \,|\, \mathcal{V}) \propto \exp\Big(-\frac{1}{2} P_{\mathcal{V}}(\mathbf{y}) \Big),$$

where $P_{\mathcal{V}}(\mathbf{y})$ is a polynomial in $\mathbf{y}$.
Hence,

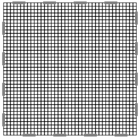
- finding the MAP estimate is a semidefinite polynomial minimization problem (fast to solve?),
- CM estimate can be computed as a high-dimensional integral with *an explicitly known integrand*.

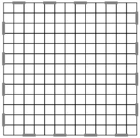
In particular, the computational complexities of these tasks are independent of the discretization of $H^1(D)$.

Numerical experiments

In the numerical experiments, we use

- $D = (-1, 1)^2$,
- $M = 16$ electrodes around D,
- $z_1, \dots, z_{16} = 1$,
- $\sigma(\omega, \mathrm{x}) = \exp(g(\omega, \mathrm{x}))$ with g defined through,
  - $\mathbb{E}_g[\mathrm{x}] \equiv 0$,
  - $V_g(\mathrm{x}, \mathrm{x}') = 0.16 \exp\Big(-|\mathrm{x} - \mathrm{x}'|^2/(2 \times 0.3^2)\Big)$,
- simplistic square meshes for data generation and for the inverse computations with quadratic spatial elements:



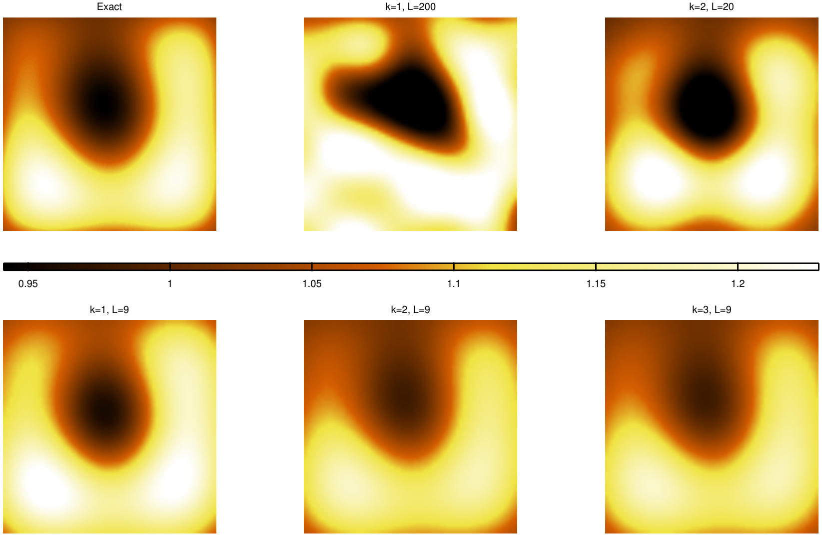


Numerical experiments

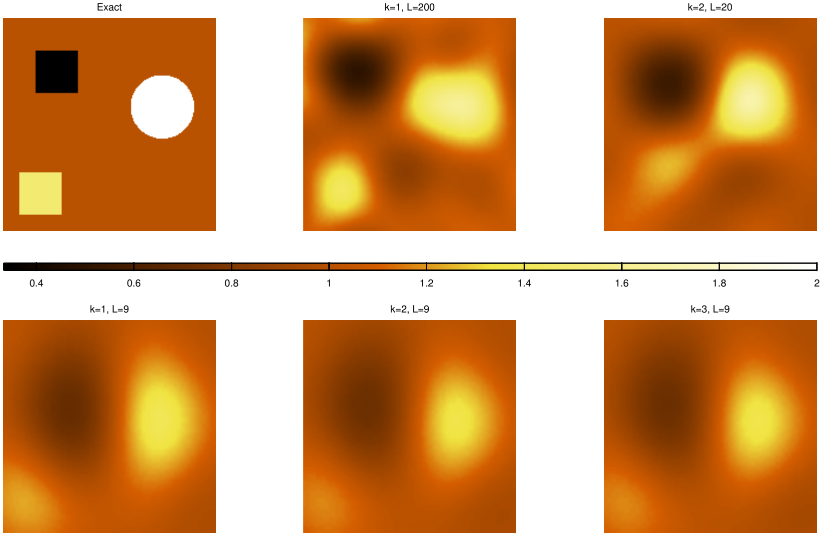
Figure arrangement:

Target conductivity	MAP estimate $k = 1, \quad \mathbf{L} = \mathbf{200}$ DoF = 149544	MAP estimate $k = 2, \quad \mathbf{L} = \mathbf{20}$ DoF = 171864
MAP estimate $k = 1, \quad \mathbf{L} = \mathbf{9}$ DoF = 7440	MAP estimate $k = 2, \quad \mathbf{L} = \mathbf{9}$ DoF = 40920	MAP estimate $k = 3, \quad \mathbf{L} = \mathbf{9}$ DoF = 163680

Example 1: Γ corresponds to 0.1% of noise



Example 2: Γ corresponds to 0.1% of noise

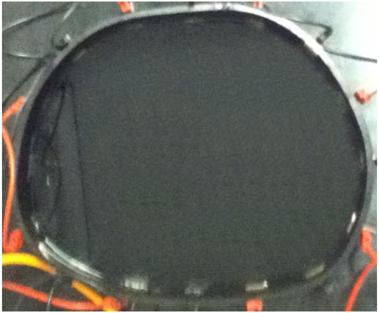


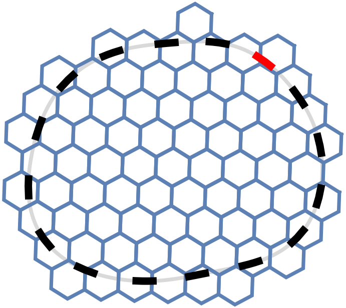
Real data

Modifications to increase the sparseness of the sGFEM system and to account for contact resistances:

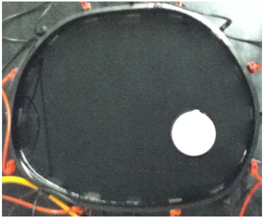
- The *a priori* mutually independent unknown $\mathbf{Y} : \Omega \to \mathbb{R}_{+}^{76+16}$ are
  - the *a priori* uniformly distributed pixel values of the conductivity, and
  - the *a priori* uniformly distributed contact conductances.
- Multivariate *Legendre polynomials* replace the *Hermite polynomials*.
- An additional smoothness prior is introduced in the postprocessing stage.

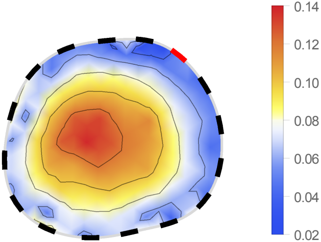
Real data

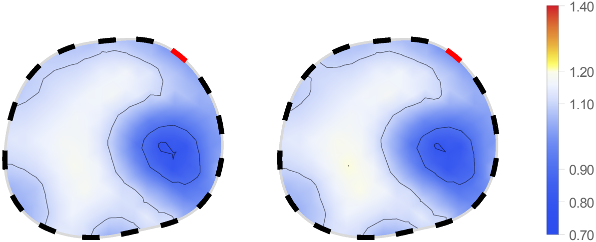




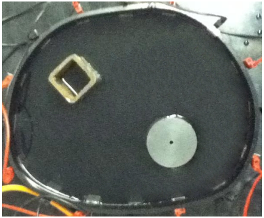
Real data

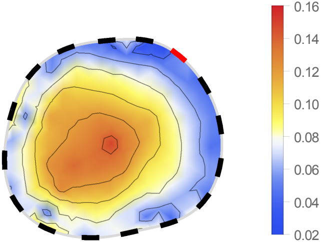


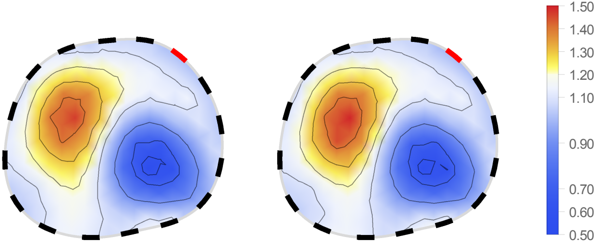




Real data




0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02


1.50
1.40
1.30
1.20
1.10
0.90
0.70
0.60
0.50

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