Inverse scattering by impenetrable cavities

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Joint work with
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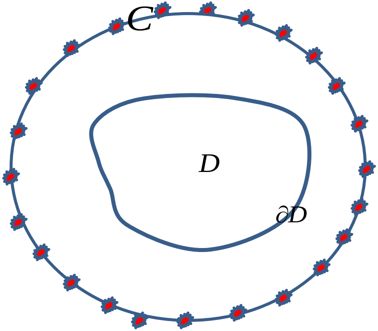
Haihua Qin China University of Mining and Technology.

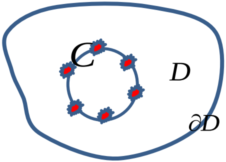
The workshop on Inverse Problems in Wave Propagation, Bremen

Exterior Problems v.s. Interior Problems

C : Measurement surface/curve.



(1) Exterior problems.



(2) Interior problems

Outline

- Problem setting
- The factorization method for cavities
- Uniqueness by a single measurement
- Summary and outlook

The simplest case: Dirichlet boundary condition

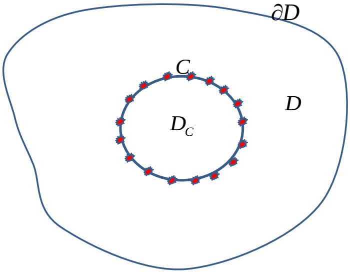


Figure: Cavity D and the curve C where measurements are taken.

Mathematical model:

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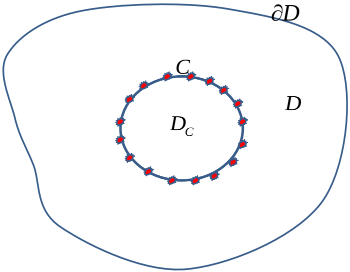
\begin{aligned}
&\Delta u^s + k^2 u^s = 0 \quad \text{in } D,\\
&u = u^i(\cdot, z) + u^s \quad \text{in } D\backslash\{z\},\\
&u = 0 \quad \text{on } \partial D.
\end{aligned}
$$

- D: bounded simply connected domain with Lipschitz boundary ∂D;
- C: measurement surface with its interior denoted by D_C;
- k: wave number.

Direct Problem and Inverse Problem



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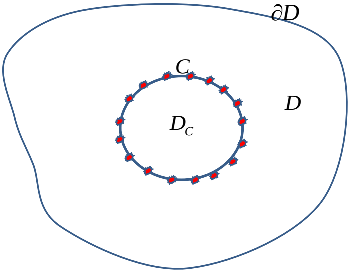

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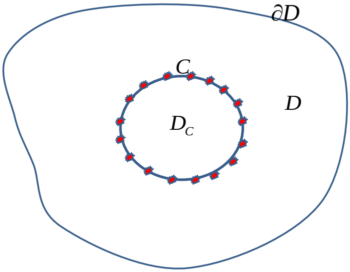

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- Direct problem: Give D, look for the scattered fields $u^s \in H^1(D)$;
- Inverse problem: Reconstruct ∂D from $u^s(x)$, $x \in C$.

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   One may solve this by adding an artificial obstacle with impedance boundary condition into the scattering system, see e.g.,
   H. Qin and X. Liu, The interior inverse scattering problem for cavities with an artificial obstacle, Appl. Numer. Math. 88, (2015), 18-30.

Summary of the research on the interior problems

1. P. Jakubik and R. Potthast, Testing the integrity of some cavity-the Cauchy problem and the range test, Appl. Numer. Math. 58 (2008) 899-914.
2. H. Qin and D. Colton, The inverse scattering problem for cavities, Appl. Numer. Math. 62 (2012) 699-708.
3. H. Qin and D. Colton, The inverse scattering problem for cavities with impedance boundary condition, Adv. Comput. Math. 36 (2012) 157-174.
4. H. Qin and F. Cakoni, Nonlinear integral equations for shape reconstruction in the inverse interior scattering problem, Inverse Probl. 27 (2011) 035005.
5. H. Qin and J. Liu, Reconstruction for cavities with impedance boundary condition, J. Integral Equ. Appl. 25 (2013) 431-454.
6. F. Zeng, F. Cakoni and J. Sun, An inverse electromagnetic scattering problem for a cavity, Inverse Probl. 27 (2011) 125002.
7. F. Zeng, P. Suarez and J. Sun, A Decomposition method for an interior inverse scattering problem, Inverse Probl. Imag. 7 (2013) 291-303.
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   ☹ *Surprisingly, the quality of the numerical reconstructions are notably poorer than that obtained by using the same method for the analogue exterior problem.*
2. ☹ The **classical Factorization method** fails to work for the exterior problem with near field measurements. As shown in Guanghui's talk yesterday, some modification has to be considered.

The Factorization Method

Whether the Factorization Method works for the interior problems?

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Yes

Near field operator

Define the near field operator $N : L^2(C) \to L^2(C)$ by

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Real and imaginary parts of N:

$$\Re(N) = \frac{N + N^*}{2} \quad \text{and} \quad \Im(N) = \frac{N - N^*}{2i}.$$

Positive self-adjoint operator:

$$N_\sharp := |\Re(N)| + |\Im(N)|$$

The factorization method: mathematical basis

Theorem: (Liu, IP015006, 2015)
For any $z \in \mathbb{R}^2 \backslash \overline{D_C}$ define $\phi_z \in L^2(C)$ by

$$\phi_z(x) := H_0^{(1)}(k|x - z|), \quad x \in C.$$

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Remark: For the points $z \in \overline{D_C}$, we are not clear about the relation between the function ϕ_z and operator $N_\sharp^{1/2}$. However, since $\overline{D_C} \subset D$, we are not interested in those points located in D_C.

The factorization method: Indicator function

$$W(z) := \left[\sum_{n=1}^{\infty} \frac{|\langle \phi_z, \psi_n \rangle_{L^2(C)}|^2}{|\sigma_n|}\right]^{-1},$$

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Features:

- Makes no use of boundary conditions of D;
- Simple to computational implementation;

An explicit example in 2D: $C = \partial B_r$, $D = B_R$ with $r < R$

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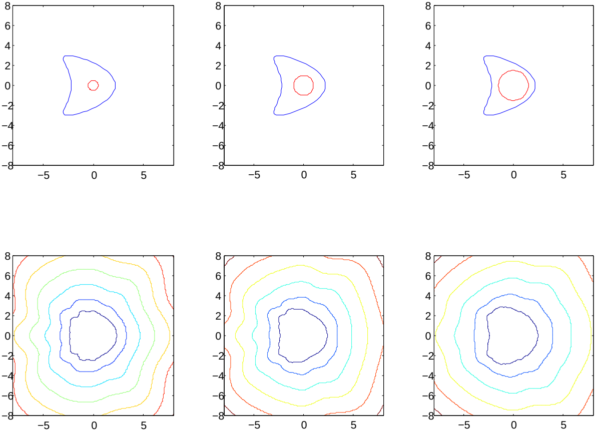
The series converges if, and only if, $|z| > R$, i.e., z is in the exterior of D.

The factorization method: Scheme

1) **Collect** the scattered fields on C by sending point sources located on the same curve C.
2) **Select** a mesh $\mathcal{G}$ of sampling points in a region Ω which contains D.
3) **Calculate** the indicator function $W(z)$ for each sampling point z of $\mathcal{G}$. Here we deliberately set $W(z) = 0$ for the sampling points $z \in \overline{D_C}$.
4) **Plot** $W(z)$, e.g., by using the MATLAB routine $contour(z_x, z_y, W(z))$.

Numerical results: $u = 0$ on ∂D



Different circle $C = \{x \in \mathbb{R}^2 : |x| = r.\}$. Left: $r = 0.5$; Middle: $r = 1$; Right: $r = 1.5$.

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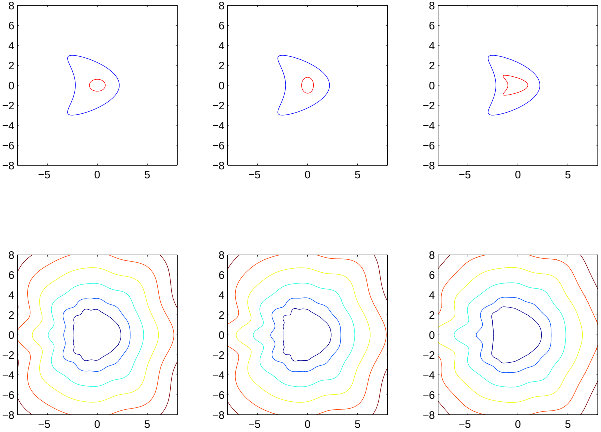

Different curves C. Left: ellipse with axes $0.8, 0.6$; Middle: ellipse with axes $0.6, 0.8$; Right: kite parameterized by $x(t) = (\cos t + 0.65\cos 2t - 0.65,\ \sin t)$.

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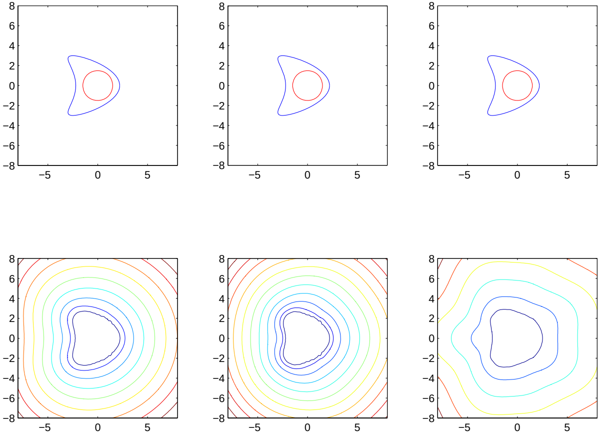
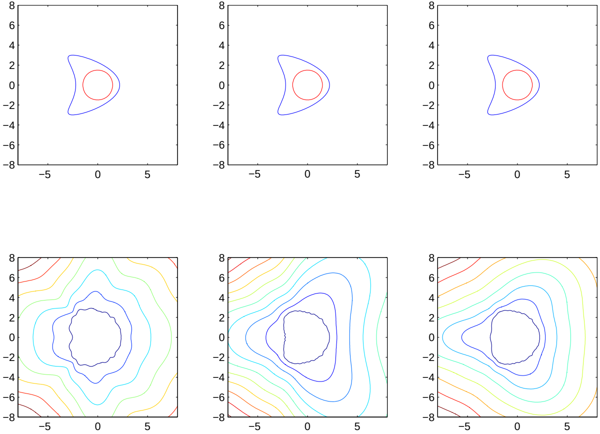


Different wave numbers k. Left: $k = 0.5$; Middle: $k = 1$; Right: $k = 1.5$.

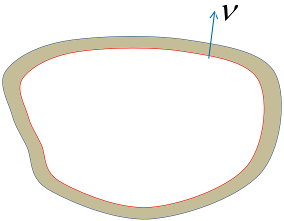
Numerical results: $\partial u/\partial \nu + \lambda u = 0$ on ∂D



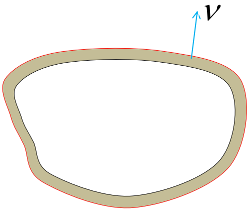
Different impedance boundary conditions. Left: $\lambda = 0$; Middle: $\lambda = i$; Right: $\lambda = 1 + i$.

Remark on the FM for the impedance boundary conditions

Let $\lambda \in L^\infty(\partial D)$ be the impedance function with $\Re(\lambda) \geq 0$ and $\Im(\lambda) \geq 0$.



Exterior Problems: Coated outside



Interior Problems: Coated intside

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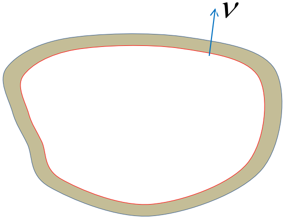

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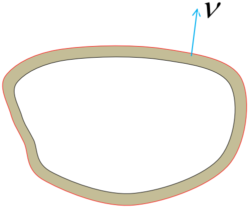

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$$(1)\ \frac{\partial u}{\partial \nu} + \lambda u = 0 \text{ on } \partial D$$

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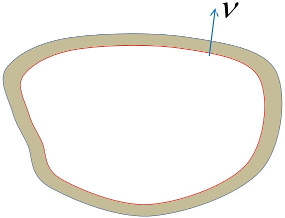

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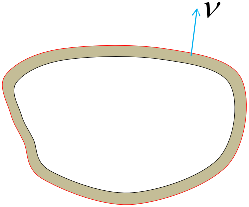

Interior Problems: Coated intside

$$(2)\ \frac{\partial u}{\partial \nu} - \lambda u = 0 \text{ on } \partial D$$

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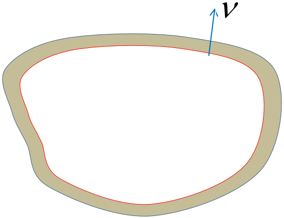

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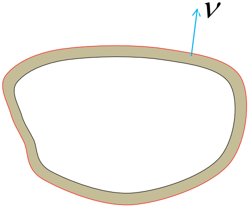

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$$(1)\ \frac{\partial u}{\partial \nu} + \lambda u = 0 \text{ on } \partial D \qquad (2)\ \frac{\partial u}{\partial \nu} - \lambda u = 0 \text{ on } \partial D$$

The FM for the impedance boundary condition (2) is still not justified.

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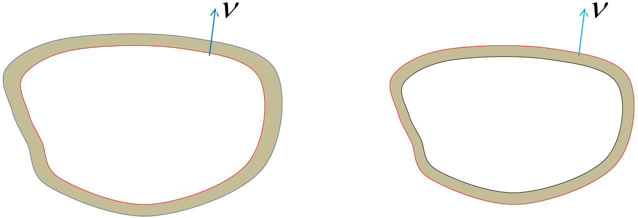

Exterior Problems: Coated outside **Interior Problems: Coated intside**

$(1)\ \frac{\partial u}{\partial \nu} + \lambda u = 0$ on ∂D $(2)\ \frac{\partial u}{\partial \nu} - \lambda u = 0$ on ∂D

The FM for the impedance boundary condition (2) is still not justified. One may solve this mathematically by using nonphysical incident point source $H_0^{(2)}(k|\cdot - z|)$.

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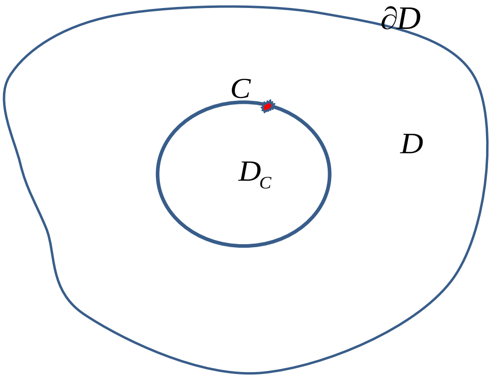
- The mathematical basis of the FM provides a constructive proof of the uniqueness on the inverse problems.

- The implementation of the FM needs full data, i.e., the measurements $u^s(\cdot, z)$ are taken for all $z \in C$.

Inverse scattering by a single measurement



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   Uniqueness is given if D is ball with given center.

Uniqueness by a single point source

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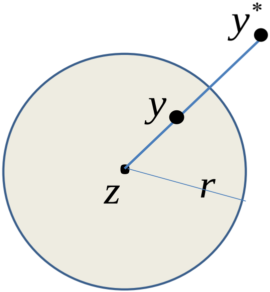
Symmetric point with respect to a sphere



$$|y - z| \times |y^* - z| = r^2$$

Reflection principle with respect to a sphere

Lemma 1. (Hu and Liu, IP065010, 2014) Let $u(\cdot; y)$ be a solution to the boundary value problem

$$\begin{array}{rcll} \Delta u(x) + k^2 u(x) & = & -\delta(x-y) & \text{in} \quad \mathbb{R}^N \backslash \overline{B_r}, \\ u & = & 0 & \text{on} \quad \partial B_r, \end{array}$$

where y is a fixed point in $\mathbb{R}^N \backslash \overline{B_r}$. Then $u(\cdot; y)$ can be analytically extended into the interior of B_r except for the point $y^* := (r/|y|)^2\, y$.

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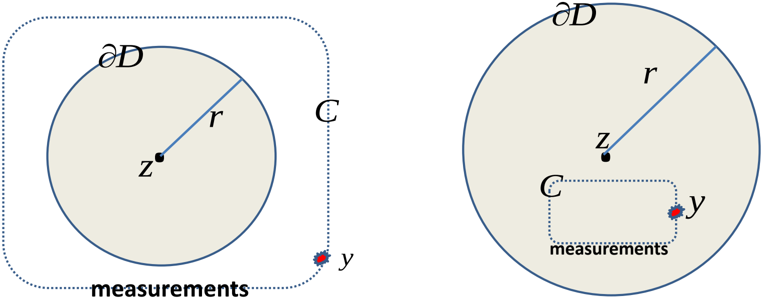
where y is a fixed point in $\mathbb{R}^N \backslash \overline{B_r}$. Then $u(\cdot; y)$ can be analytically extended into the interior of B_r except for the point $y^* := (r/|y|)^2 \, y$. Furthermore, the extension of $u(\cdot; y)$ in B_r solves the following interior boundary value problem

$$\begin{aligned} \Delta u(x) + k^2 u(x) &= (r/|y^*|)^{N+2} \delta(x - y^*) && \text{in} \quad B_r, \\ u &= 0 && \text{on} \quad \partial B_r. \end{aligned}$$

Uniqueness by a single point source

Theorem 1. (Hu and Liu, IP065010, 2014) Both the center z and the radius r of a ball are uniquely determined by a single measurement $u^s(\cdot, y)$ taken on the curve C.



Uniqueness by a single point source: exterior problem

Notations:

- $D_l := B_{r_l}(z_l)$, $l = 1, 2$ are two sound-soft balls;
- $y \in C$ is the source point; y_l^* is the symmetric point w.r.t. the sphere ∂D_l, $l = 1, 2$;
- $u^i(\cdot; y)$ denotes the point source at fixed $y \in C$.
- $u_l^s(\cdot; y)$ denotes the scattered fields due to D_l.

Aim: $u_1^{sc}(\cdot; y) = u_2^{sc}(\cdot; y)$ on C implies $D_1 = D_2$.

Proof: $y_1^* = y_2^* = y^*$

By the reflection principle, the total field $u_l(\cdot; y) = u^{in}(\cdot; y) + u_l^{sc}(\cdot; y)$ can be analytically extended into D_l except for the point

$$y_l^* := z_l + \frac{r_l^2}{|y - z_l|^2}(y - z_l), \quad l = 1, 2.$$

We claim that $y_1^* = y_2^* = y^*$ and therefore

$$|z_l - y|\,|z_l - y^*| = r_l^2, \quad l = 1, 2.$$

In addition, one can readily conclude that z_1, z_2, y and y^* are collinear points and y^* is located between $z_j\,(j = 1, 2)$ and y.

Proof: Connection with the scattering of plane waves

$u_l^\infty(\cdot; y)$: far-field pattern of the scattered field $u_l^{sc}(\cdot; y)$;
$u_l^{sc}(\cdot; -\hat{x})$: scattered field corresponds to the inc. plane wave $u_l^{in}(\cdot; -\hat{x})$.

Since $u_1^{sc}(\cdot; y) = u_2^{sc}(\cdot; y)$ on C, we know $u_1^\infty(\hat{x}; y) = u_2^\infty(\hat{x}; y)$ and thus by the well-known mixed reciprocity relation we conclude that $u_1^{sc}(y; -\hat{x}) = u_2^{sc}(y; -\hat{x})$ for all $\hat{x} \in S^{N-1}$. The explicit representation of u_l^{sc} in 3D is given by

$$u_l^{sc}(y; -\hat{x}) = -\sum_{n=0}^{\infty} i^n \,(2n+1) \frac{j_n(kr_l)}{h_n^{(1)}(kr_l)}\, h_n^{(1)}(k|z_l - y|) P_n(\cos\varphi_l), \quad l = 1, 2.$$

Here, φ_l is the angle between $(y - z_l)/|y - z_l|$ and $-\hat{x}$. $\varphi_1 = \varphi_2$!!!

$$\frac{j_n(kr_l)}{h_n^{(1)}(kr_l)}\, h_n^{(1)}(k|z_l - y|) \sim \frac{k^n}{(2n+1)!!} \frac{r_l^{2n+1}}{|z_l - y|^{n+1}} \quad \text{as} \quad n \to \infty.$$

Proof: $r_1 = r_2, \quad z_1 = z_2$

Hence, it follows from $u_1^{sc}(y; -\hat{x}) = u_2^{sc}(y; -\hat{x})$ and $\varphi_1 = \varphi_2$ for all $\hat{x} \in S^{N-1}$ that

$$\frac{r_1^{2n+1}}{|z_1 - y|^{n+1}} = \frac{r_2^{2n+1}}{|z_2 - y|^{n+1}} \quad \text{for sufficiently large} \quad n \in \mathbb{N},$$

from which we conclude that

$$\frac{r_1^2}{|z_1 - y|} = \frac{r_2^2}{|z_2 - y|}. \tag{3.1}$$

Recalling that $|z_l - y|\,|z_l - y^*| = r_l^2, \quad l = 1, 2$, we obtain $|y^* - z_1| = |y^* - z_2|$, implying that $z_1 = z_2$. Finally, the relation $r_1 = r_2$ follows immediately.

Uniqueness by a single point source: interior problem

Notations:

- $D_l := B_{r_l}(z_l)$, $l = 1, 2$ are two sound-soft balls;
- $y \in C$ is the source point; y_l^* is the symmetric point w.r.t. the sphere ∂D_l, $l = 1, 2$;
- $u^i(\cdot; y)$ denotes the point source at fixed $y \in C$.
- $u_l^s(\cdot; y)$ denotes the scattered fields due to D_l.

Recall that k^2 is neither a Dirichlet eigenvalue of $-\Delta$ in D nor in D_C.

Aim: $u_1^{sc}(\cdot; y) = u_2^{sc}(\cdot; y)$ on C implies $D_1 = D_2$.

Proof.

Radiating solution $w_l(\cdot; y_l^*)$:

$$\begin{aligned} \Delta w_l + k^2 w_l &= -\delta(\cdot - y_l^*) \qquad \text{in}\, \mathbb{R}^N \backslash \overline{D_l}, \\ w_l &= 0 \qquad \text{on}\, \partial D_l. \end{aligned}$$

By Lemma 1, we have the analytically extension:

$$\begin{aligned} \Delta w_l + k^2 w_l &= -\frac{1}{\alpha_l}\delta(\cdot - y) \qquad \text{in} \quad D_l, \\ w_l &= 0 \qquad \text{on} \quad \partial D_l. \end{aligned}$$

Here, $\alpha_l := -(|y - z_l|/r_l)^{N+2}$ is a constant.

By the assumption that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D_l, we have

$$\alpha_l w_l(\cdot; y_l^*) = u_l(\cdot; y) \quad \text{in} \quad D_l.$$

Uniqueness by a single point source: interior problem

Define

$$v_l(x) := \alpha_l w_l(x; y_l^*) - u^i(x; y), \quad x \in \mathbb{R}^N \backslash \{y_l^*\}, \quad l = 1, 2.$$

Then, v_l is a radiating solution of the Helmholtz equation in $\mathbb{R}^N \backslash \{y_l^*\}$. On the measurement curve C, we have

$$v_1 = \alpha_1 w_1(\cdot; y_1^*) - u^i(\cdot; y) = u_1(x; y) - u^i(x; y) = u_1^s(\cdot; y) = u_2^s(\cdot; y) = v_2.$$

Since k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D_C, we obtain

$$v_1 = v_2 \quad \text{in} \quad D_C$$

and therefore in $\mathbb{R}^N \backslash \{y_1^*, y_2^*\}$ by analytic continuation. We claim that

$$y_l^* = y_2^* := y^*.$$

Uniqueness by a single point source: interior problem

The relation $v_1 = v_2$ in $\mathbb{R}^N \backslash \{y^*\}$ implies that

$$\alpha_1 w_1 = \alpha_2 w_2 \quad \text{in} \quad \mathbb{R}^N \backslash \{y^*, y\}.$$

We claim that $\alpha_1 = \alpha_2$. Actually, if $\alpha_1 \neq \alpha_2$, it follows from and the relation $w_l(\cdot; y_l^*) = u^i(\cdot; y_l^*) + w_l^s(\cdot; y_l^*)$ that

$$(\alpha_1 - \alpha_2) u^i(\cdot; y^*) = \alpha_2 w_2^s(\cdot; y^*) - \alpha_1 w_1^s(\cdot; y^*) \quad \text{in}\, \mathbb{R}^N \backslash \{y^*, y\}.$$

Clearly, $w_1^s(\cdot; y^*) = w_2^s(\cdot; y^*)$ on ∂B_R, where B_R is a large ball containing D_1 and D_2.

Uniqueness by a single point source: Polyhedra

A lot of works on uniqueness by a single plane wave, e.g.,

1. J. Cheng and M. Yamamoto, *Uniqueness in an inverse scattering problem with non-trapping polygonal obstacles with at most two incoming waves*, Inverse Problems 19 (2003): 1361-1384 (Corrigendum: Inverse Problems 21 (2005): 1193).
2. G. Alessandrini and L. Rondi, *Determining a sound-soft polyhedral scatterer by a single far-field measurement*, Proc. Amer. Math. Soc. 133 (2005): 1685–1691 (Corrigendum: arXiv: math/0601406v1).
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Basic ideas:

1. Using reflection principle for the Helmholtz equation to construct a Dirichlet line extending to infinity.
2. The scattered wave tends to 0 uniformly at infinity, while the incident plane wave has modulus 1 everywhere.

Uniqueness by a single point source: Polyhedra

Can we establish the corresponding uniqueness by a single point source?

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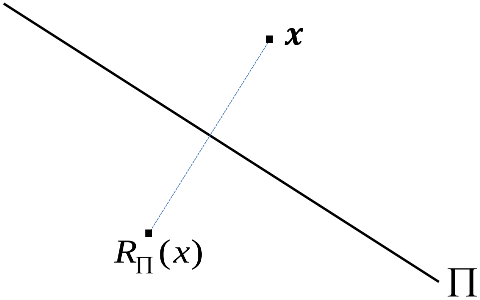
Yes

Symmetric point *w.r.t.* a hyperplane



Reflection principle *w.r.t.* Dirichlet planes

Lemma 2: (J. Cheng & M. Yamamoto, 2003) Suppose that $\Omega \subset \mathbb{R}^N$ is a symmetric connected domain with respect to an $(N-1)$-dimensional hyperplane Π and that $\Lambda = \Omega \cap \Pi \neq \emptyset$. Denote by Ω^+ and Ω^- the two connected subdomains of Ω separated by Λ. If $\Delta u + k^2 u = 0$ in Ω^+ and $u = 0$ on Λ, then u can be analytically extended into Ω^- by the formula

$$u(x) = -u(\mathcal{R}_\Pi(x)), \quad x \in \Omega^-,$$

where $\mathcal{R}_\Pi$ stands for the reflection with respect to Π.

Dirichlet set of the total field

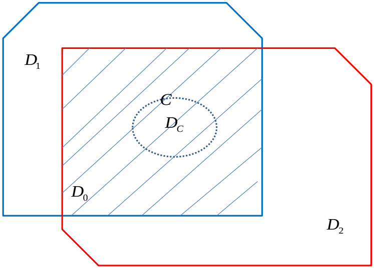
Let Π be a $(N-1)$-dimensional hyperplane in $\mathbb{R}^N$. A non-void open connected component $\Lambda \subset \Pi$ will be called a Dirichlet set of u if $u = 0$ on Λ.

Corollary 1. With the notations used in Lemma 2, we suppose that u is a solution to the Helmholtz equation in Ω vanishing on Λ.

1. If Λ_0 is a Dirichlet set of u in Ω^+, then $\mathcal{R}_\Pi(\Lambda_0) \subset \Omega^-$ is also a Dirichlet set of u.
2. If u is singular at $y \in \Omega$, then u is also singular at $\mathcal{R}_\Pi(y)$.

Notations:

- $D_l, l = 1, 2$ denotes two polygons;
- D_0 denotes the connected component of $D_1 \cap D_2$ containing D_C;
- $u^i(\cdot; y)$ denotes the point source at fixed $y \in C$.
- $u_l(\cdot; y)$ and $u_l^s(\cdot; y)$ denote the total field and scattered field due to D_l, respectively.



Proof. Assume that $u_1^s(\cdot;y) = u_2^s(\cdot;y)$ on C, we aim at proving $D_1 = D_2$. By the assumption that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D_C and the unique continuation of solutions to the Helmholtz equation, we see

$$u_1(\cdot;y) = u_2(\cdot;y) \quad \text{in} \quad D_0 \backslash \{y\}.$$

Uniqueness by a single point source: Polyhedra

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If $\partial D_1 \neq \partial D_2$, without loss of generality we may always assume there exists a Dirichlet set Λ of u_1 in D_1.

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If $\partial D_1 \neq \partial D_2$, without loss of generality we may always assume there exists a Dirichlet set Λ of u_1 in D_1.
Define

- $\mathcal{D} := \{\tilde{\Lambda} : \tilde{\Lambda}$ is a Dirichlet set of u_1 in $D_1\}$;
- $\gamma(t)$: continuous injective curve connecting $y_0 = \gamma(0) \in \Lambda$ and $y = \gamma(T)$;
- $\mathcal{M} = \{y_n : \exists\, \Lambda_n \in \mathcal{D}\,\&\, t_n \geq 0\, s.t.\, \Lambda_n \cap \gamma(t_n) = y_n\}$.

Uniqueness by a single point source: Polyhedra

$\mathcal{M}$ is bounded, closed in the sense that $y_n \to y'$ implies $y' \in \mathcal{M}$. Therefore we can find some $t^* > 0$ such that there exists $\Lambda^* \in \mathcal{D}$ intersecting with $\gamma(t)$ at $t = t^*$ and that

$$\gamma(t) \cap \mathcal{M} = \emptyset, \qquad \forall \quad T > t > t^*.$$

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$$\gamma(t) \cap \mathcal{M} = \emptyset, \qquad \forall \quad T > t > t^*.$$

The proof is finished by constructing a connected symmetric domain $\Omega \subset D_1$ with respect to the hyperplane containing Λ^* such that

$$(1)\, u_1 = 0 \quad \text{on}\, \partial\Omega; \qquad \text{and} \qquad (2)\, y \in \Omega.$$

Summary and outlook

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1. Uniqueness of the inverse problems for other boundary conditions and Maxwell's system;
2. More "direct" sampling methods;
3. It is time for coffee break!

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Thanks for your attention

- X. Liu, The factorization method for cavities, *Inverse Problems 30*, (2014), 015006.
- G. Hu and X. Liu, Unique determination of balls and polyhedral scatterers with a single point source wave, *Inverse Problems 30*, (2014), 065010.
- H. Qin and X. Liu, The interior inverse scattering problem for cavities with an artificial obstacle, *Appl. Numer. Math. 88*, (2015), 18-30.