The Born Approximation for the Wave Equation

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Outline of the talk

Karlsruhe Institute of Technology

(A) Introduction

Born Approximation for Helmholtz equation: $u = u^i + u^s$ total field

$$\Delta u + k^2(1+q)u = 0\,;\ \text{that is, } \Delta u + k^2 u = -k^2 q\, u$$

$$\Delta u^s + k^2 u^s = -k^2 q u\,;\ \text{that is, } u^s(x) = k^2 \int_D q(y)u(y)\,\Phi_k(x,y)\,dy$$

Lippmann-Schwinger equation:

$$u(x) = u^i(x) \ + \ k^2 \int_D q(y)u(y)\,\Phi_k(x,y)\,dy\,,\ x \in D\,.$$

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Born series: $u_0 = u^i, \quad \Delta u_{j+1} + k^2 u_{j+1} = -k^2 q\, u_j,$ that is

$$u_{j+1}(x) = u^i(x) \;+\; k^2 \int_D q(y) u_j(y)\, \Phi_k(x,y)\, dy\,,\;\; x \in D\,.$$

L^2-convergence if $\quad k^4 \int_D \int_D q(y)^2\, \Phi_k(x,y)^2\, dy\, dx < 1$

Introduction, cont.

(Linearized) wave equation:

$$\frac{1}{\varrho\,\nu^2}\,\partial_t^2 u(x,t) \;-\; \nabla_x \cdot \left(\frac{1}{\varrho}\nabla_x u(x,t)\right) \;=\; f(x,t)\,, \quad (x,t) \in \Omega \times (0,T]\,,$$

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$\nu = \nu(x)$, $\varrho = \varrho(x)$ are wave speed and mass density, respectively.

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Set $a=1/(\varrho\,\nu^2)$ and $b=1/\varrho$, thus:

$$a(x)\,\partial_t^2 u(x,t)\;-\;\nabla_x\cdot\big(b(x)\nabla_x u(x,t)\big)\;=\;f(x,t)\,,\qquad (x,t)\in\Omega\times(0,T]\,.$$

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Boundary conditions: $u(\cdot,t)|_{\partial\Omega} \;=\; 0\,,$

Initial conditions: $u(\cdot,0) = u_0; \quad \partial_t u(\cdot,0) = u_1$ on $\Omega\,.$

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Inverse problem studies mapping $\mathcal{F} : (a,b) \mapsto \Psi u$ (for example $\Psi u = u|_D$ for some $D \subset \Omega$).

Introduction, cont.

Usually a, b are perturbations of known $\hat{a}$, $\hat{b}$; that is, $a = \hat{a} + \varepsilon\tilde{a}$ and $b = \hat{b} + \varepsilon\tilde{b}$. Formally, $u = \hat{u} + \varepsilon u' + o(\varepsilon)$ where

$$\hat{a}\,\partial_t^2 u' - \nabla_x \cdot \left(\hat{b}\nabla_x u'\right) \;=\; -\tilde{a}\,\partial_t^2 \hat{u} \;+\; \nabla_x \cdot \left(\tilde{b}\nabla_x \hat{u}\right), \quad (x,t) \in \Omega \times (0,T]\,,$$

where $\hat{u}$ is solution corresponding to $\hat{a}$ and $\hat{b}$.

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Goal of talk:

- Justify this rigorously as Fréchet derivative in proper function spaces
- Clarify why Born-*series* is not well defined
- Show that nonlinear inverse problem is improperly posed (in the sense of Hofmann 1997)

(B) The Initial Boundary Value Problem

Weak formulation (Green's first formula w.r.t. time and space variable):

$$\int_0^T \int_\Omega \left[b(x) \nabla_x u(x,t) \cdot \nabla_x \psi(x,t) - a(x)\, \partial_t u(x,t)\, \partial_t \psi(x,t) \right] dx\, dt$$

$$= \int_0^T \int_\Omega f(x,t)\, \psi(x,t)\, dx\, dt \quad \text{for all } \psi \in C^\infty(\overline{\Omega} \times [0,T]) \text{ with } \psi(\cdot, T) = 0\,.$$

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Identify $u : \Omega \times (0,T) \to \mathbb{R}$ with $u : (0,T) \to H_0^1(\Omega)$ and $\dot{u} = \partial_t u$. Then:

$$\int_0^T \big(b \nabla u(t), \nabla \psi(t)\big)_{L^2} - \big(a\, \dot{u}\,(t), \dot{\psi}\,(t)\big)_{L^2}\, dt \;=\; \int_0^T \big(f(t), \psi(t)\big)_{L^2}\, dt$$

for all $\psi \in C^\infty([0,T], C^\infty(\overline{\Omega}))$ with $\psi(0) = \psi(T) = 0$.

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for all $\psi \in C^\infty([0,T], C^\infty(\overline{\Omega}))$ with $\psi(0) = \psi(T) = 0$. Solution space:

$$X \;=\; C^1\big([0,T], L^2(\Omega)\big) \;\cap\; C\big([0,T], H_0^1(\Omega)\big)$$

with norm $$\|u\|_X \;:=\; \left(\max_{0\le t\le T} \|u(t)\|_{H^1}^2 \;+\; \max_{0\le t\le T} \|\,\dot{u}(t)\|_{L^2}^2\right)^{1/2}.$$

Then also test functions $\psi \in X$ with $\psi(0) = \psi(T) = 0$. In addition, initial conditions $u(0) = u_0$ and $\dot{u}(0) = u_1$.

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Assumptions: $a, b \in L^\infty(\Omega)$ with $\gamma \le a(x), b(x) \le \gamma^{-1}$ a.e. on Ω, $f \in L^2\big((0,T)\times\Omega\big) = L^2\big((0,T), L^2(\Omega)\big)$, $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$.

The Spectral Method

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simpler: expansion method; that is:

s.a. EVP: $\quad -\nabla \cdot (b \nabla v_n) = \lambda_n^2 \, a \, v_n$ in Ω, $\quad v_n = 0$ on $\partial\Omega$, $\quad$ that is,

$$v_n \in H_0^1(\Omega)\,, \quad \underbrace{\int_\Omega b \nabla v_n \cdot \nabla \psi \, dx}_{=\,(v_n,\psi)_{1,b}} = \lambda_n^2 \underbrace{\int_\Omega a \, v_n \, \psi \, dx}_{=\,(v_n,\psi)_{0,a}} \;\forall \psi \in H_0^1(\Omega)\,.$$

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Set $V = \big(H_0^1(\Omega), (\cdot,\cdot)_{1,b}\big)$. Normalize $v_n \in V$ such that $\|v_n\|_{1,b} = 1$.
Then: $\{v_n : n\in\mathbb{N}\}$ ONS in V and $\{\lambda_n v_n : n\in\mathbb{N}\}$ ONS in $L^2(\Omega, a\,dx)$.

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Also (because $\gamma b \le 1$ and $b \ge \gamma$) with Poincaré's constant c_Ω:

$$\gamma \|v\|_{1,b}^2 \;\le\; \|v\|_{H^1}^2 \;\le\; (1 + c_\Omega^2) \|\nabla v\|_{L^2}^2 \;\le\; \frac{1 + c_\Omega^2}{\gamma} \|v\|_{1,b}^2\,,$$

$$\gamma \|v\|_{0,a}^2 \;\le\; \|v\|_{L^2}^2 \;\le\; \frac{1}{\gamma} \|v\|_{0,a}^2\,.$$

The Spectral Method, cont.

Let $u_0 = \sum_n \alpha_n v_n$ in V and $u_1 = \sum_n \beta_n \lambda_n v_n$ in $L^2(\Omega, a\,dx)$ and $\frac{f(t)}{a} = \sum_n f_n(t)\,\lambda_n v_n$ in $L^2(\Omega, a\,dx)$.

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Define $$\eta_n(t) \;=\; \int_0^t \sin(\lambda_n(t-s))\, f_n(s)\, ds\,, \quad t \in [0,T]\,,\ n \in \mathbb{N}\,.$$

Then $$u(t) \;=\; \sum\nolimits_n \big[\alpha_n \cos(\lambda_n t) + \beta_n \sin(\lambda_n t) + \eta_n(t)\big]\, v_n$$

is the unique weak solution of $a\partial_t^2 u - \nabla\cdot(b\nabla u) = f$ in $\Omega\times(0,T)$ with $u(0) = u_0$ in Ω, $\partial_t u(0) = u_1$ in Ω, and $u = 0$ on $\partial\Omega\times(0,T)$

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$$\begin{aligned}
\|u(t)\|_{1,b}^2 + \|\dot{u}(t)\|_{0,a}^2 &\leq 2 \cdot 3 \sum_n \Big[\alpha_n^2 + \beta_n^2 + T\|f_n\|_{L^2(0,T)}^2\Big] \\
&\leq 6\big[\|u_0\|_{1,b}^2 + \|u_1\|_{0,a}^2 + (T/\gamma)\|f\|_{L^2(\Omega\times(0,T))}^2\big].
\end{aligned}$$

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Furthermore, $a\,\ddot{u} \in L^2((0,T), H^{-1}(\Omega))$ and

$$(b\nabla u(t), \nabla\psi)_{L^2} + \langle a\,\ddot{u}(t), \psi\rangle = (f(t), \psi)_{L^2} \quad \text{for all } \psi \in H_0^1(\Omega)\,,\ \text{a.e.}$$

Regularity

From this approach one gets easily regularity. Recall:

$$u(t) \;=\; \sum_n \left[\alpha_n \cos(\lambda_n t) + \beta_n \sin(\lambda_n t) + \eta_n(t)\right] v_n \quad \text{with}$$

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$$\dot{\eta}_n\,(t) \;=\; f_n(0)\sin(\lambda_n t) + \int_0^t \sin(\lambda_n(t-s))\;\dot{f}_n\,(s)\, ds$$

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Theorem

(a) If $\nabla \cdot (b \nabla u_0) \in L^2(\Omega)$, $u_1 \in H_0^1(\Omega)$, $f \in H^1((0,T), L^2(\Omega))$ then

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(b) If $\frac{1}{a}\left[\nabla \cdot (b\nabla u_0) - f(0)\right] \in H_0^1(\Omega)$, $\nabla \cdot (b\nabla u_1) \in L^2(\Omega)$, $f \in H^2((0,T), L^2(\Omega))$ then

$$u \in C^2\left([0,T], H_0^1(\Omega)\right) \cap C^3\left([0,T], L^2(\Omega)\right).$$

Example

$$\int_0^T \big(b\nabla u(t), \nabla\psi(t)\big)_{L^2} - \big(a\,\dot{u}\,(t), \dot{\psi}\,(t)\big)_{L^2}\,dt = \int_0^T \big(f(t), \psi(t)\big)_{L^2}\,dt$$

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Example: $\Omega = (0,\pi) \times (0,\pi)$, $a = b = 1$. Then

$$v_n(x_1, x_2) = \frac{2}{\pi|n|}\sin(n_1 x_1)\sin(n_2 x_2), \quad n = (n_1, n_2) \in \mathbb{N}^2,$$

is ONS in $H_0^1(\Omega)$ wrt $(\nabla u, \nabla v)_{L^2}$ and $\{|n| v_n : n \in \mathbb{N}^2\}$ is ONS in $L^2(\Omega)$.

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Remark: $f \in L^2((0,T), H^{-1}(\Omega))$ not sufficient for $u \in X$!

Example: $\Omega = (0,\pi) \times (0,\pi)$, $a = b = 1$. Then

$$v_n(x_1, x_2) \;=\; \frac{2}{\pi|n|}\,\sin(n_1 x_1)\sin(n_2 x_2)\,, \qquad n = (n_1, n_2) \in \mathbb{N}^2\,,$$

is ONS in $H_0^1(\Omega)$ wrt $(\nabla u, \nabla v)_{L^2}$ and $\{|n|v_n : n \in \mathbb{N}^2\}$ is ONS in $L^2(\Omega)$.

Define $\quad f(x,t) \;=\; \sum_{n\in\mathbb{N}^2} \rho_n |n|^2 \cos(|n|t)\, v_n(x) \quad$ with $\sum_n \rho_n^2 < \infty\,.$

Example

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Then $f \in L^2((0,T), H^{-1}(\Omega))$ and

$$u(t) = -\frac{1}{2}\sum_{n\in\mathbb{N}^2} \rho_n |n|\, t\sin(|n|t)\, v_n \in L^2(\Omega).$$

(C) Lipschitz-Continuity and Differentiability

$$\begin{aligned} a_j \, \ddot{u}_j(t) - \nabla \cdot \big(b_j \nabla u_j(t)\big) &= f(t)\,, \quad j = 1,2\,. \quad \text{Set } u := u_1 - u_2 : \\ a_2 \, \ddot{u}(t) - \nabla \cdot \big(b_2 \nabla u(t)\big) &= (a_2 - a_1)\, \ddot{u}_1(t) - \nabla \cdot \big((b_2 - b_1)\nabla u_1(t)\big) \\ &= (a_2 - a_1)\, \ddot{u}_1(t) - \nabla \cdot \big(b_2 \nabla w(t)\big) \end{aligned}$$

where $w(t) \in H_0^1(\Omega)$ solves $\nabla \cdot \big(b_2 \nabla w(t)\big) = \nabla \cdot \big((b_2 - b_1)\nabla u_1(t)\big)$.

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Assumption: $\nabla \cdot (b_1 \nabla u_0) \in L^2(\Omega)$, $u_1 \in H_0^1(\Omega)$, $f \in H^1((0, T), L^2(\Omega))$.

Then $u_1 \in C^1\big([0, T], H_0^1(\Omega)\big) \cap C^2\big([0, T], L^2(\Omega)\big)$.

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Then $u_1 \in C^1\big([0,T], H_0^1(\Omega)\big) \cap C^2\big([0,T], L^2(\Omega)\big)$. Thus:

$(a_2 - a_1)\, \ddot{u}_1 \in L^2((0,T) \times \Omega)$ and also $w \in C^1\big([0,T], H_0^1(\Omega)\big)$ and $\|w(t)\|_{1,b_2} \le \frac{1}{\gamma}\|b_2 - b_1\|_\infty \|u_1(t)\|_{1,b_2}$. Thus:

$$
\begin{aligned}
\|u_1 - u_2\|_X &\le \tilde{c}\left[\|(a_2 - a_1)\, \ddot{u}_1\|_{L^2((0,T)\times\Omega)} + \|w\|_{H^1((0,T),H^1(\Omega))}\right] \\
&\le c\left[\|a_2 - a_1\|_\infty + \|b_2 - b_1\|_\infty\right]
\end{aligned}
$$

where c depends only on T, Ω, γ, u_0, u_1, and f.

Differentiability

Theorem: Let $f \in H^2((0,T), L^2(\Omega))$. Set

$$U = \{v \in L^\infty(\Omega) : \gamma \le v \le \gamma^{-1} \text{ a.e. on } \Omega\}.$$

Furthermore, let $(\hat{a}, \hat{b}) \in \text{int}(U) \times \text{int}(U)$ and the initial data satisfy $\frac{1}{\hat{a}}\left[\nabla \cdot (\hat{b}\nabla u_0) - f(0)\right] \in H_0^1(\Omega)$ and $\nabla \cdot (\hat{b}\nabla u_1) \in L^2(\Omega)$.

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Theorem: Let $f \in H^2((0,T), L^2(\Omega))$. Set

$$U = \{v \in L^\infty(\Omega) : \gamma \le v \le \gamma^{-1} \text{ a.e. on } \Omega\}.$$

Furthermore, let $(\hat{a}, \hat{b}) \in \mathrm{int}(U) \times \mathrm{int}(U)$ and the initial data satisfy $\frac{1}{\hat{a}}\left[\nabla \cdot (\hat{b}\nabla u_0) - f(0)\right] \in H_0^1(\Omega)$ and $\nabla \cdot (\hat{b}\nabla u_1) \in L^2(\Omega)$. Then the mapping $\mathcal{F} : (a,b) \mapsto u$ is Fréchet-differentiable at $(\hat{a}, \hat{b})$ and $\mathcal{F}'(\hat{a}, \hat{b})(a,b) = u'$ for $a, b \in L^\infty(\Omega)$ where $u' \in X$ solves

$$\hat{a}\partial_t^2 u' - \nabla \cdot (\hat{b}\nabla u') = -a\partial_t^2 \hat{u} + \nabla \cdot (b\nabla \hat{u}) \quad \text{in } \Omega \times (0,T],$$

and $u'(0) = \partial_t u'(0) = 0$. Here, $\hat{u} = \mathcal{F}(\hat{a}, \hat{b})$.

(D) Remarks on the Born-Approximation

Let, for simplicity, $u_0 = u_1 = 0$. Recall: $a \hat{=} \hat{a} + a$ and $b \hat{=} \hat{b} + b$ and $u \in X$ satisfies

$$(\hat{a} + a)\,\partial_t^2 u - \nabla_x \cdot \big((\hat{b} + b)\nabla_x u\big) \;=\; f\,; \quad \text{that is,}$$

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Introduce linear solution operator $L : g \mapsto \hat{u}$ where $\hat{u} \in X$ solves

$$\hat{a}\,\partial_t^2 \hat{u} - \nabla_x \cdot \big(\hat{b}\nabla_x \hat{u}\big) \;=\; g\,.$$

Then u solves fixed point equation

$$u \;=\; L\big[f - a\partial_t^2 u + \nabla_x \cdot (b\nabla_x u)\big] \;=\; Lf \;-\; L\big[a\partial_t^2 u - \nabla_x \cdot (b\nabla_x u)\big]\,.$$

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Born series is the same as fixed point iteration; that is, $u_0 = Lf = \hat{u}$ and

$$u_{k+1} \;=\; \hat{u} \;-\; L\big[a\,\partial_t^2 u_k - \nabla_x \cdot (b\nabla_x u_k)\big]\,, \qquad k = 0, 1, 2, \ldots$$

Remarks on the Born-Approximation, cont.

Recall:

$$u_{k+1} \;=\; \hat{u} \;-\; L\big[a\partial_t^2 u_k - \nabla_x \cdot (b\nabla_x u_k)\big]\,, \quad k = 0, 1, 2, \ldots$$

First step $k = 0$:

$$u_1 \;=\; \hat{u} \;-\; L\big[a\partial_t^2 u_0 - \nabla_x \cdot (b\nabla_x u_0)\big] \;=\; \hat{u} \;+\; u'\,;$$

that is, first Born approximation coincides with Fréchet-linearization.

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Note that L is bounded from $L^2\big((0,T) \times \Omega\big)$ into
$X = C^1\big([0,T], L^2(\Omega)\big) \;\cap\; C\big([0,T], H_0^1(\Omega)\big)$ but

$$a\partial_t^2 u_k - \nabla_x \cdot \big(b\nabla_x u_k\big) \;\notin\; L^2\big((0,T) \times \Omega\big) \quad \text{for } u_k \in X\,!$$

Therefore, Born series is not even well-defined!

(E) Ill-Posedness of the Inverse Problem

Definition (Hofmann 1997) An equation $\mathcal{F}y = u$ is called locally ill-posed in $y^* \in D(\mathcal{F})$ with $\mathcal{F}y^* = u$ if in any neighborhood of y^* there is a sequence $y_j \in D(\mathcal{F})$ with $\mathcal{F}(y_j) \to \mathcal{F}(y^*)$ but $y_j \nrightarrow y^*$.

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In our case $\mathcal{F} : Y^* \to Z$ with $Y^* = L^\infty(\Omega) \times L^\infty(\Omega)$; that is, $Y = L^1(\Omega) \times L^1(\Omega)$, and $Z = L^2((0, t) \times \Omega)$.

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Proposition Let $\mathcal{F} : Y^* \to Z$ be compact and weak-$*$-to-weak continuous. Further, let $y^* \in D(\mathcal{F})$ satisfy $\mathcal{F}y^* = u$ and assume the existence of a sequence $e_j \in Y^*$ with $\|e_j\|_{Y^*} = 1$ and $e_j \to 0$ weakly-$*$ and $y^* + re_j \in D(\mathcal{F})$ for every $r \in [0,1]$ and $j \in \mathbb{N}$. Then the equation $\mathcal{F}y = u$ is locally ill-posed.

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Assumptions are satisfied in this case! Compactness of $\mathcal{F}$ by Theorem of Arzela-Ascoli:

$$\|u(t_2) - u(t_1)\|_{L^2} = \sup_{\|\psi\|_{L^2}=1} \left(u(t_2) - u(t_1), \psi\right)_{L^2} = \sup_{\|\psi\|_{L^2}=1} \int_{t_1}^{t_2} \left(\dot{u}(s), \psi\right)_{L^2} ds$$

$$\leq |t_2 - t_1| \, \|\dot{u}\|_{C([0,T], L^2(\Omega))}$$

Ill-Posedness of the Inverse Problem, cont.

Weak-$*$-to-weak continuity by variational formulation of differential equation.

Ill-Posedness of the Inverse Problem, cont.

Weak-$*$-to-weak continuity by variational formulation of differential equation.

Construction of $e_j \in L^\infty(\Omega)$ with $\|e_j\|_{L^\infty} = 1$ and $e_j \to 0$ weakly-$*$ and $y^* + re_j \in D(\mathcal{F})$:

Let $z \in \Omega$ and let $B_j = B(z, 1/j)$ be ball centered at z with radius $1/j$. Set $e_j = \chi_{B_j}$ characteristic function of B_j. Then $\|e_j\|_{L^\infty} = 1$ and $e_j \to 0$ weakly-$*$.

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Theorem Let the measurement operator $\Psi : L^2(\Omega) \to L^2(D)$ be linear and bounded. Then the problem to determine the parameters $a, b \in L^\infty(\Omega)$ from Ψu is ill-posed.

(F) Final Remarks

References:

- On the Linearization of Operators Related to the Full Waveform Inversion in Seismology. MMAS 2014
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Extensions

- Maxwell's equations analogously, data-to-solution operator $\mathcal{F} : (\varepsilon, \mu) \mapsto (E, H)$ not compact because $H(\mathrm{curl}, \Omega)$ not compactly imbedded in $L^2(\Omega)$.
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Thank you for your attention   and, in particular,

thank you, Armin, for this phantastic workshop!