Time Domain Linear Sampling Method

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Outline

This talk will be about time domain and multifrequency Linear Sampling Methods (LSM):

- Scattering from a sound soft object.[1]
- Time domain for penetrable media.[2]
- Multifrequency and time domain numerical results[3]

See also [HLM14][4]

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[1] Chen Q, Haddar H, Lechtleiter A, Monk P. A sampling method for inverse scattering in the time domain. Inverse Problems. 2010;26; 085001.

[2] Guo Y, Monk P, Colton D. Toward a time domain approach to the linear sampling method. Inverse Problems. 2013;29; 095016.

[3] Guo Y, Monk P, Colton D, (2015) In press

[4] Haddar H, Lechleiter A, Marmorat S. An improved time domain linear sampling method for Robin and Neumann obstacles. Applicable Analysis. 2014;93:369–390.

Time Domain Acoustic Wave Equation

Let D be a bounded domain (scatterer) with connected complement, boundary ∂D and unit outward normal $\boldsymbol{\nu}$.

Suppose u^i is a given incident wave (solution of the wave equation vanishing near ∂D for $t < 0$). The scattered pressure field $u^s = u^s(x,t)$ and total field u satisfy

$$\begin{aligned}
\frac{1}{c^2}\ddot{u} - \Delta u &= 0 \text{ in } \Omega = \mathbb{R}^3 \setminus \overline{D} \text{ for } t > 0, \\
u &= u^i + u^s \text{ in } \Omega = \mathbb{R}^3 \setminus \overline{D} \text{ for } t > 0, \\
u &= 0 \text{ on } \partial D \text{ for } t > 0 \\
u^s = \dot{u}^s &= 0 \text{ at } t = 0 \text{ in } \Omega.
\end{aligned}$$

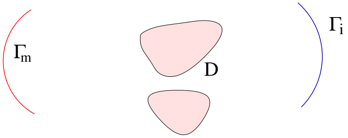
Here $\ddot{u} = \partial^2 u/\partial t^2$ and c is the constant wave speed.

The Dirichlet inverse problem

We probe the unknown scatterer D with fields due to point sources at points x_0 on a surface Γ_i. “Measurements" are the scattered field u^s a surface Γ_m. In particular we assume that $u^s = u^s(x, t; x_0)$ satisfies

$$\begin{aligned} \frac{1}{c^2}\ddot{u}^s - \Delta u &= 0 \quad \text{in } \Omega \times \mathbb{R} \\ u^s &= 0 \text{ for } t \leq 0 \\ u^s(\boldsymbol{x}, t) &= -\Phi(\boldsymbol{x} - \boldsymbol{x}_0, t) \quad \text{for } \boldsymbol{x} \in \partial D \text{ and } t \in \mathbb{R}. \end{aligned}$$

From this data, we want to determine the shape of D.



$$\Phi(\boldsymbol{x}, t) = \frac{\delta(t - \|\boldsymbol{x}\|/c)}{4\pi\|\boldsymbol{x}\|}$$

The near field operator

The method we shall use is the direct analogue of the Linear Sampling Method of Colton and Kirsch in the time domain[5].

The time domain near field operator N is given by

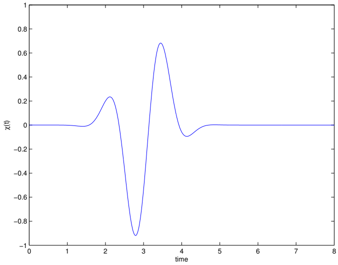
$$(\mathrm{N}g)(x,t) = \int_{\mathbb{R}}\int_{\Gamma_i} u^s(x,t-t_0;x_0)g(x_0,t_0)\,dx_0\,dt_0.$$

[5] D. Colton and A. Kirsch, *Inverse Problems*, **12** (1996), pp. 383-93.

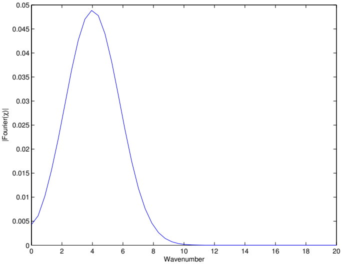
Auxiliary acoustic source

Given a profile $\chi \in C_0^\infty(\mathbb{R})$, a point $z \in \mathbb{R}^3$ and a $\tau \in \mathbb{R}$, we will need the auxiliary field

$$\Phi_{\boldsymbol{z},\tau}(x,t) = \int_{\mathbb{R}} \Phi(x-z, t-\tau-t_0)\chi(t_0)dt_0 = \frac{\chi(t-\tau-\|x-z\|)}{4\pi\|x-z\|}$$



Source $\chi(t)$



Spectrum

Basic idea (hope) of linear sampling

1. Choose points z and parameter τ and solve a regularized version of
$$\mathrm{N}g_{z,\tau} = \Phi_{z,\tau}$$
2. If z is inside D then a suitable norm of $g_{z,\tau}$ should be small and should blow up as z approaches the boundary of D. This norm of $g_{z,\tau}$ gives an indicator function for the domain D.
3. There are some technical points that need to be considered (e.g. $\mathrm{N}g_{z,\tau} = \phi_{z,\tau}$ may not have a solution).

Implementation of the method

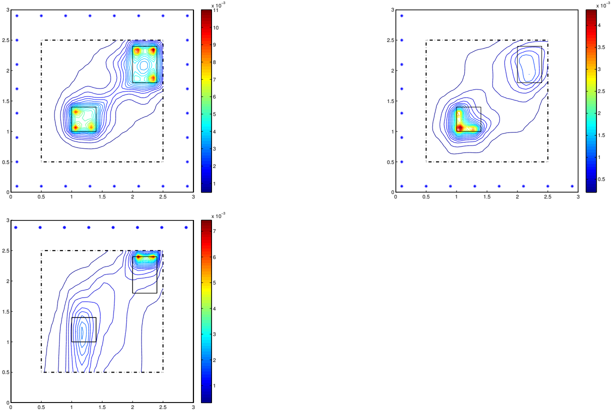
1. We assume $\Gamma_m = \Gamma_i$ and we have $u(x_i, n\Delta t; x_j, 0)$ at discrete points $x_i, x_j \in \Gamma_m$, $1 \leq i, j \leq N$ and $n \leq N_T$ such that $u \approx 0$ at $n = N_t$.
2. The discrete near field operator is

$$\mathrm{N}_d(i,n) = \sum_{0 \leq m \leq n} \sum_j u(x_i, (n-m)\Delta t; x_j, 0)\phi(j,n)$$

3. We solve $\mathrm{N}_d g_{z,\tau} = \Phi_{z,\tau}$ by truncated SVD
4. We plot $1/\|g_{z,\tau}\|_{L^2(\mathbb{R};L^2(\Gamma_i))}$
5. Numerical results are in 2D.

Later we use the method of [HLM] to implement SVD calculations using `eigs`.

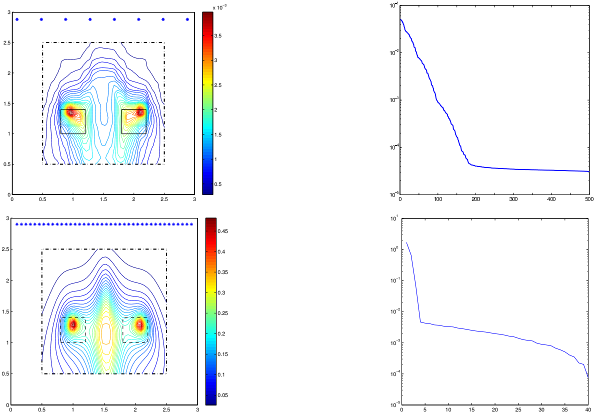
Numerical example in $\mathbb{R}^2$



Time domain inverse problem: different arrangements of sources and measurements.

Numerical example in $\mathbb{R}^2$ (continued)

Reconstruction Singular values



Top: time domain inversion. Bottom: frequency domain inverse at the center frequency.

Analysis of the TD-LSM

We establish properties of the solution of the wave equation using the Fourier-Laplace transform approach of Bamberger and Ha-Duong[6]. Let

$$\hat{u}(x,s) = \int_0^\infty u(x,t)\mathrm{e}^{-st}\,dt$$

where $s = \sigma - \mathrm{i}\omega$ for $\sigma > \sigma_0 > 0$, and $\omega \in \mathbb{R}$. Then $\hat{u}$ satisfies

$$\begin{aligned} \frac{s^2}{c^2}\hat{u} - \Delta\hat{u} &= 0 \quad \text{in } \mathbb{R}^3 \setminus \partial D \\ \hat{u} &= \hat{g} \text{ on } \partial D, \text{ and } \hat{u} \text{ bounded at infinity.} \end{aligned}$$

The fundamental solution for this problem $\hat{\Phi}$ is bounded at infinity and satisfies

$$\frac{s^2}{c^2}\hat{\Phi} - \Delta\hat{\Phi} = \delta_{\boldsymbol{y}} \quad \text{in } \mathbb{R}^3$$

[6] Bamberger, A., Ha-Duong, T., 1986, *Math. Methods Appl. Sci.* **8** 405-35

Indirect representation

Let $\hat{\xi} = [\![\partial \hat{u}/\partial\nu]\!]$ then

$$\hat{u}(x) = \int_{\partial D} \hat{\Phi}(x,y)\hat{\xi}(y)\, dA(y) := \widehat{\mathrm{SL}}\hat{\xi}.$$

where $\widehat{\mathrm{SL}}$ is the single layer potential operator. Letting $x \to \partial D$ gives, for $x \in \partial D$,

$$\hat{g}(x) = \int_{\partial D} \hat{\Phi}(x,y)\hat{\xi}(y)\, dA(y) := \hat{\mathrm{S}}\hat{\xi}$$

Solving the single layer equation $\hat{\mathrm{S}}\hat{\xi} = \hat{g}$ gives formally $\hat{u} = \widehat{\mathrm{SL}}\hat{\mathrm{S}}^{-1}\hat{g}$.

Time domain single layer and potential operators

Mapping to the time domain by the inverse Fourier-Laplace transform (using $s = \sigma - \mathrm{i}\omega$, $\omega \in \mathbb{R}$) gives

$$\mathrm{SL}\xi = \int_0^\infty \int_{\partial D} \Phi(x-y, t-\tau)\xi(y,\tau)\, dA(y)\, d\tau.$$

where $\Phi(x-y, t-\tau) = \delta(t-\tau-\|x-y\|/c)/(4\pi\|x-y\|)$ and for $x \in \partial D$

$$\mathrm{S}\xi = \int_0^\infty \int_{\partial D} \Phi(x-y, t-\tau)\xi(y,\tau)\, dA(y)\, d\tau.$$

Now $u = \mathrm{SLS}^{-1} g$.

Mapping properties

Using Bamberger and Ha-Duong's technique, we can obtain properties of the time domain operators.
Define the $H^p_\sigma(\mathbb{R}_{>0}, X)$ norm on functions f such that $f = 0$ for $t < 0$ by

$$\|f\|^2_{H^p_\sigma(\mathbb{R}_{>0},X)} = \sum_{q=0}^{p} \|\exp(-\sigma t) f^{(q)}(t)\|^2_{L^2(\mathbb{R},X)}.$$

Lemma

The following maps are bounded:

1. $\mathrm{SL} : H^p_\sigma(\mathbb{R}_{>0}, H^{-1/2}(\partial D)) \to H^{p-1}_\sigma(\mathbb{R}_{>0}, H^1(\Omega))$.
2. $\mathrm{S} : H^p_\sigma(\mathbb{R}_{>0}, H^{-1/2}(\partial D)) \to H^{p-1}_\sigma(\mathbb{R}_{>0}, H^{1/2}(\partial D))$.
3. $\mathrm{S}^{-1} : H^p_\sigma(\mathbb{R}_{>0}, H^{1/2}(\partial D)) \to H^{p-2}_\sigma(\mathbb{R}_{>0}, H^{-1/2}(\partial D))$.

Sketch of proof

Let $\hat{u} = \widehat{\mathrm{SL}}\hat{\xi}$. Key observation: Green's theorem implies

$$\int_{\Omega \cup D} |\nabla \hat{u}|^2 + s^2 |\hat{u}|^2 \, dV = \int_{\partial D} [\![\partial \hat{u}/\partial \nu]\!] \, \overline{\hat{u}} \, dA$$

But $[\![\partial \hat{u}/\partial \nu]\!] = \hat{\xi}$. Then

$$\int_{\Omega \cup D} \overline{s} |\nabla \hat{u}|^2 + s|s|^2 |\hat{u}|^2 \, dV = \int_{\partial D} \hat{\xi} \, (\overline{s\widehat{\mathrm{S}}\hat{\xi}}) \, dA$$

So, since $s = \sigma + \mathrm{i}\omega$, $\sigma > \sigma_0 > 0$ and $\omega \in \mathbb{R}$,

$$\min(\sigma, \sigma^3) \|\hat{u}\|^2_{H^1(\Omega \cup D)} \leq \left| \int_{\partial D} \overline{s} \hat{\xi} \, \overline{\widehat{\mathrm{S}}\hat{\xi}} \, dA \right|$$

The trace theorem and Parseval's Theorem completes the proof of the first claim.

The near field operator (again)

The scattered field u_ϕ due to an incident field $\mathrm{SL}_{\Gamma_i}\phi$ is denoted $u_\phi = \mathrm{N}\phi$. Recall that the near field operator N is given by

$$(\mathrm{N}g)(x,t) \quad = \quad \int_{\mathbb{R}} \int_{\Gamma_i} u^s(x, t - t_0; x_0) g(x_0, t_0)\, dx_0\, dt_0.$$

Note that $\mathrm{N} = \mathrm{SLS}^{-1}\mathrm{SL}_{\Gamma_i}$

Lemma

$\mathrm{N} : H^p_\sigma(\mathbb{R}, \tilde{H}^{-1/2}(\Gamma_i)) \to H^{p-4}_\sigma(\mathbb{R}, H^{1/2}(\Gamma_m))$ *is injective with dense range (so are the potential operators defined previously with appropriate spaces).*

Basic theorem of linear sampling

Theorem

Suppose we solve $\mathrm{N}g_{\mathbf{z},\tau} = \phi_{\mathbf{z},\tau}$ using Tikhonov regularization.

- *For $z \in D$, $\tau \in \mathbb{R}$, there is a solution $g_{\mathbf{z},\tau,\epsilon} \in H^4_\sigma(\mathbb{R}, \tilde{H}^{-1/2}(\Gamma_i))$ such that*

$$\begin{aligned}\lim_{\epsilon\to 0} \|\mathrm{N}g_{\mathbf{z},\tau,\epsilon} - \phi_{\mathbf{z},\tau}\|_{L^2_\sigma(\mathbb{R},H^{1/2}(\Gamma_m))} &= 0,\\ \lim_{\mathbf{z}\to\partial D} \|g_{\mathbf{z},\tau,\epsilon}\|_{H^4_\sigma(\mathbb{R},\tilde{H}^{-1/2}(\Gamma_i))} &= \infty.\end{aligned}$$

- *Let $z \in \Omega$, $\tau \in \mathbb{R}$. For any $g_{\mathbf{z},\tau,\epsilon}$ such that*

$$\lim_{\epsilon\to 0} \|\mathrm{N}g_{\mathbf{z},\tau,\epsilon} - \phi_{\mathbf{z},\tau}\|_{L^2_\sigma(\mathbb{R},H^{1/2}(\Gamma_m))} = 0,$$

  we have

$$\lim_{\epsilon\to 0} \|g_{\mathbf{z},\tau,\epsilon}\|_{H^4_\sigma(\mathbb{R},\tilde{H}^{-1/2}(\Gamma_i))} = \infty.$$

Acoustic Wave Equation: Penetrable Medium

Let D be a bounded domain (scatterer) with connected complement, boundary ∂D and unit outward normal $\boldsymbol{\nu}$. Let $c = c(\boldsymbol{x}) > 0$ and assume $c(\boldsymbol{x}) = c_0$ if $\|\boldsymbol{x}\| > R$ for some R.

Suppose u^i is a given incident wave (solution of the background wave equation vanishing near ∂D for $t < 0$). The scattered pressure field $u = u(x,t)$ satisfies

$$\begin{aligned} \frac{1}{c(\boldsymbol{x})^2}\ddot{u} - \Delta u &= 0 \text{ in } \mathbb{R}^3 \text{ for } t > 0, \\ u &= u^s + u^i \text{ in } \mathbb{R}^3 \text{ for } t > 0 \\ u^s = \dot{u}^s &= 0 \text{ at } t = 0 \text{ in } \mathbb{R}^3 \end{aligned}$$

The Near Field LSM

Formally, this is the same as for the impenetrable case.

Penetrable medium: what can we prove?

- Using the Laplace transform techniques we can still prove that N is injective with dense range.
- To prove the basic theorem of linear sampling, we need to define the solution operator for the interior transmission problem $(\hat{w}, \hat{v}) = \hat{K}(s)(\hat{f}, \hat{g})$ given by solving

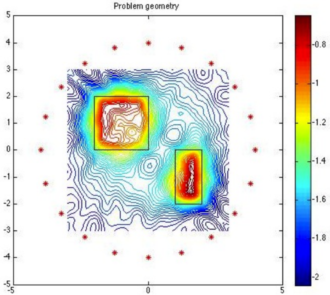
$$\begin{aligned}
-\Delta \hat{w} + \frac{s^2}{c^2(\boldsymbol{x})}\hat{w} &= 0 \text{ in } D \\
-\Delta \hat{z} + \frac{s^2}{c_0^2}\hat{z} &= 0 \text{ in } D \\
\hat{w} - \hat{z} &= \hat{f} \text{ on } \partial D \\
\frac{\partial}{\partial \nu}(\hat{w} - \hat{z}) &= \hat{g} \text{ on } \partial D
\end{aligned}$$

and prove that for $\sigma > \sigma_0 > 0$, $\hat{K}(s)$ is an analytic operator and $\|\hat{K}(s)\| \leq C|s|^{\mu}$. But there may be complex eigenvalues with arbitrarily large imaginary part.

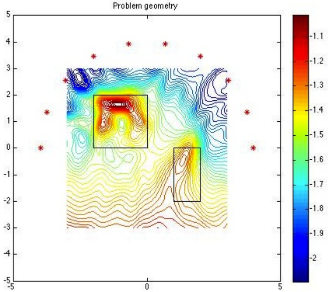
Numerical example - 2D Penetrable media

Results due to Dr. Y. Guo. We collect data for $0 \leq t \leq 16$ (252 timesteps using k-wave). Data is generated by k-Wave[7]

Full circle



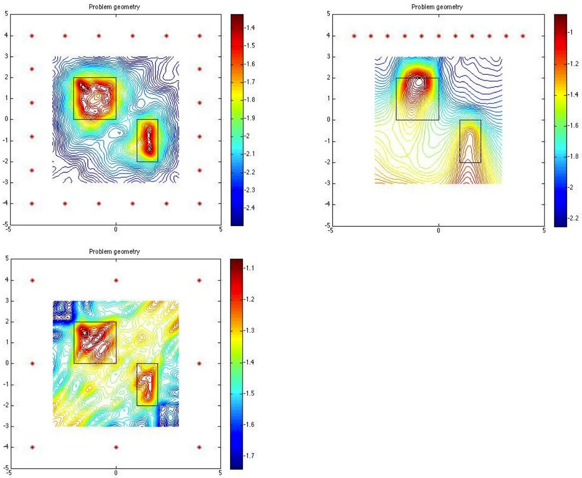
Half circle



[7] Treeby B, Cox B. k-Wave: MATLAB toolbox for the simulation and reconstruction of photoacoustic wave-fields. J Biomed Opt. 2010;15; 021314.

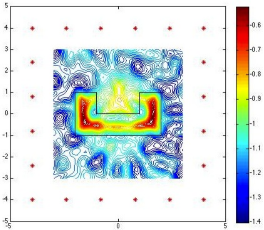
Numerical example - II 2D Penetrable media

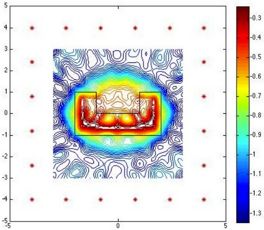
Results due to Dr. Y. Guo.

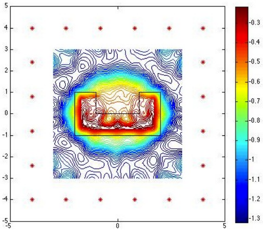


Numerical example - III 2D Penetrable media

Results due to Dr. Y. Guo (changing $\tau = 4, 8, 12$).







Frequency Domain Penetrable Media

Let $c(x)$ denote the wave speed and $n(x) = c_0^2/c^2(x)$ where c_0 is the background sound speed. Let D denote the support of $1 - n(x)$. We also assume that $n(x) > 0$ for $x \in D$ and is piecewise continuous in $\overline{D}$. Given an incident field $\hat{u}^i$, $\hat{u}$ satisfies

$$\begin{aligned} \Delta\hat{u} + k^2 n(x)\hat{u} = 0 \quad &\text{in} \quad \mathbb{R}^3 \\ \hat{u} = \hat{u}^i + \hat{u}^s \quad &\text{in} \quad \mathbb{R}^3 \\ \lim_{r\to\infty} r\left(\frac{\partial\hat{u}^s}{\partial r} - ik\hat{u}^s\right) = 0. & \end{aligned}$$

Here $\hat{u}^i$ is the incident field due to a point source situated at $y \in \Gamma_i$, i.e. $\hat{u}^i(x) = \hat{\Phi}(x, y)$ where

$$\hat{\Phi}(x, y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x - y|},$$

Standard frequency domain LSM

Use Tikhonov regularization to solve the near field equation

$$(\hat{\mathrm{N}}\hat{g})(x) = \hat{\Phi}(x, z), \quad \boldsymbol{x} \in \Gamma_m$$

where $\hat{\mathrm{N}} : L^2(\Gamma_i) \to L^2(\Gamma_m)$ is defined by

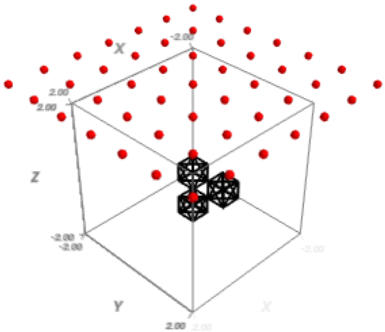
$$(\hat{\mathrm{N}}\hat{g})(x) := \int_\Gamma u^s(x, y) g(y, z)\, ds(y).$$

Here $\hat{u}^s(x, y)$ corresponds to $\hat{u}^i(x) = \hat{\Phi}(x, y)$ for $y \in \Gamma_i$. We use the indicator function

$$\mathcal{G}(z) := \frac{1}{\|g(\cdot, z)\|_{L^2(\Gamma_m)}}$$

Test problem

Target scatterer: three identical cubes with $n(x) = 1/4$ inside the cubes and $n(x) = 1$ outside.



The point sources and receivers are at the points in the grid above the scatterers. This grid is called A1.

Measurement/Source Arrays

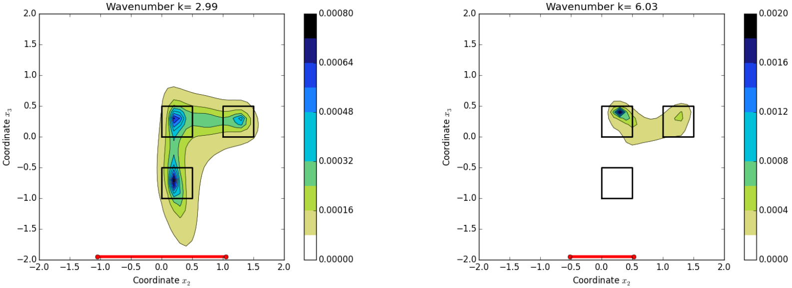
Using the BEM++[8] we can compute the scattered field from these cubes at a given frequency due to point sources on the measurement array.

Later we consider the effect of choosing various arrays of measurements and sources:

Array	Region	N points	Altitude
A1:	$[-2.5, 2.5]^2$	6	$z = 2.5$
A2:	$[-3, 3]^2$	7	$z = 2.5$
A3:	$[-4, 4]^2$	9	$z = 2.5$

[8] W. Śmigaj, S. Arridge, T. Betcke, J. Phillips and M. Schweiger, *Solving Boundary Integral Problems with BEM++*, submitted to *ACM Trans. Math. Software*, (2013)

Single frequency reconstructions



Cross sections of single frequency reconstructions in the plane $x_1 = 0.25$ using measurement array A1. Left: $k = 2.99$. Right: $k = 6.03$.

Multifrequency LSM

One possible remedy: use multiple frequencies in the LSM. This has been studied by Cakoni et al.[9].

If $g(x, k_j; z)$, $1 \leq j \leq N_k$, is the regularized solution of the near field equation for different wave numbers $k_1, k_2, \cdots, k_{N_k}$, then the indicator function $\mathcal{G}(z)$ is

$$\mathcal{G}(z) := \sum_{j=1}^{N_k} \frac{1}{\|g(\cdot, k_j; z)\|_{L^2(\Gamma)}}.$$

[9] Guzina B, Cakoni F, Bellis C. On the multi-frequency obstacle reconstruction via the linear sampling method. Inverse Problems. 2010;26; 125005.

TDLSM for penetrable media

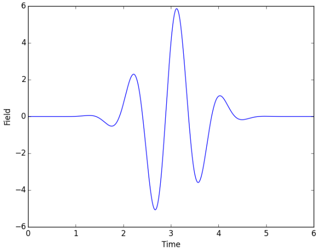
Another possible remedy is the Time Domain LSM that we have already described.

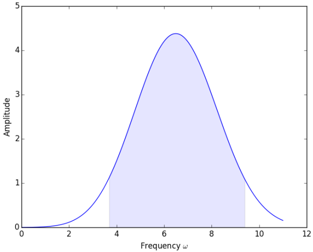
Some comments on the numerics

We choose

$$\chi(t) = (6\cos(6t) + (9.6 - 3.2t)\sin(6t))\exp(-1.6(t-3)^2)$$

While χ does not have compact support, the function becomes very small for $|t-3|$ large.

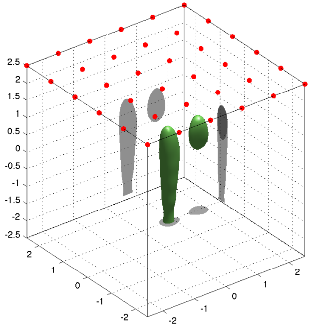


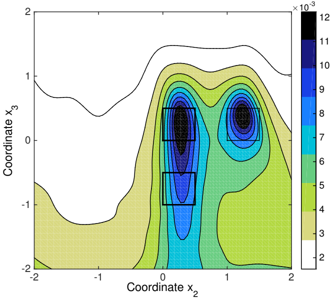


Left: Time course of the pulse function $\chi(t)$ as a function of t. Right: Magnitude of the Fourier transform $|\hat{\chi}(\omega)|$ as a function of ω.

Time domain, A1

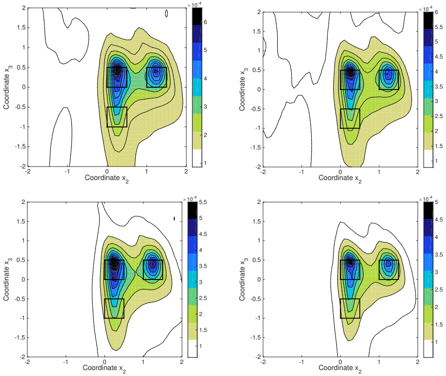
Using the k-Wave we can predict the scattered field for a given source and receiver, and invert the near field equation using the scheme from [HLM].





Multi-frequency from time domain, A1

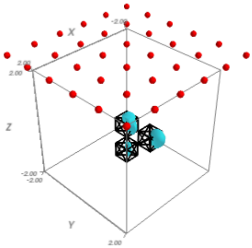
We can use FFT to convert windowed time domain data to the frequency domain.

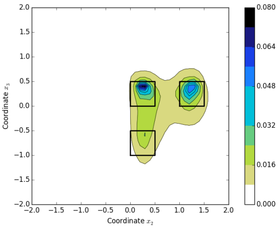


Top row: modes above 10% of maximum (27 frequencies)
Bottom row: modes above 20% of maximum (23 frequencies).
Left column: SVD tolerance of 10^{-3}. Right: 10^{-5}.

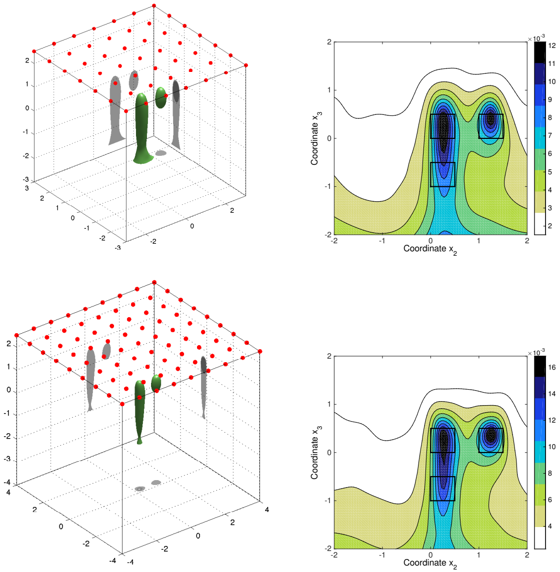
A1 multifrequency (accurate data)

Multi-frequency approach using 79 equally spaced frequencies in the interval $\omega \in [3.7, 9.4]$ using BEM++.

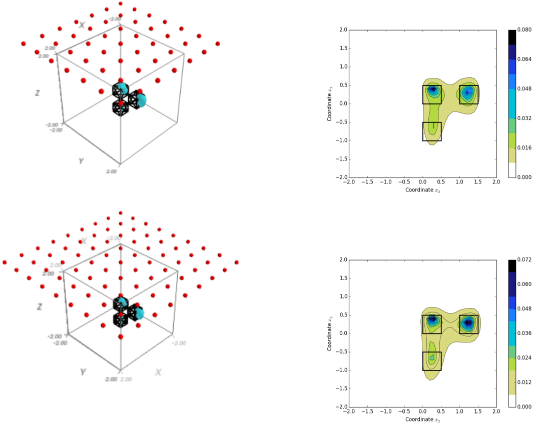




Large Sparse Arrays: A2/A3, time domain



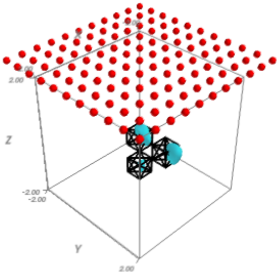
Multifrequency A2,A3

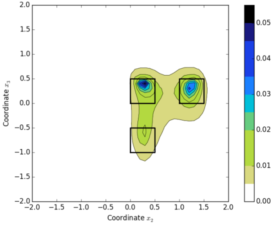


Multi frequency reconstruction using 79 equally spaced frequencies in $[3.7, 9.4]$.

Refined array A1f

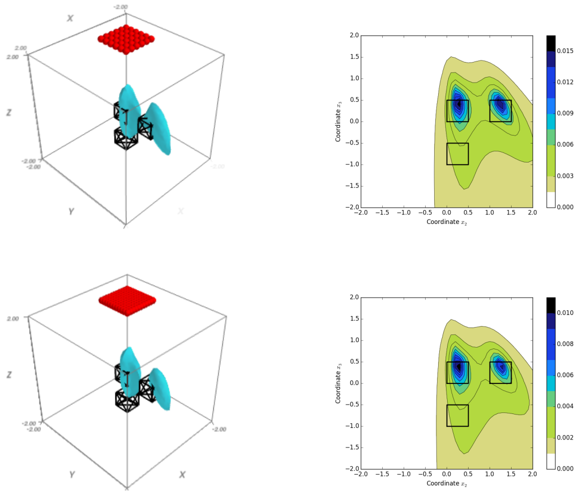
Array A1 but with $N = 11$ points in each direction.





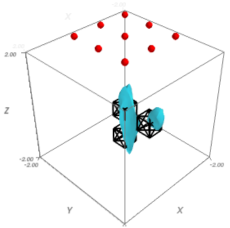
Multi-frequency frequency domain reconstructions using the finer measurement array A1f. We use 79 equally spaced frequencies in $[3.7, 9.4]$.

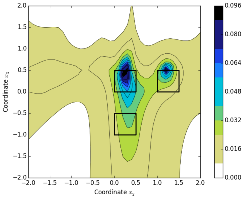
Small apertures I

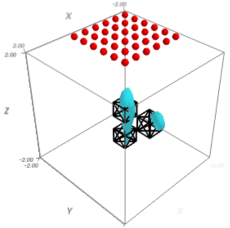


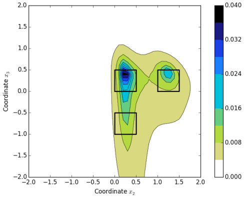
Refining the measurement grid doesn't help in this case.

Small apertures II









Refining a larger coarse array helps avoid artifacts.

Conclusion

Tentative conclusions:

1. Single frequency data may fail to reconstruct the scatterer.
2. Both time domain or frequency domain data may be used, and the methods give the same qualitative reconstruction when the data is of comparable quality.
3. Neither method can reconstruct the targets if the aperture is too small, and increasing the number of measurements in a small aperture does not help.
4. The measurement array has to be dense enough to sample the scattered field, but can be close to the Nyquist limit.

Open problem

- Do the complex transmission eigenvalues lie in a strip?
- More generally, how do we analyze the solution operator for the time domain interior transmission problem?