Reconstruction of an inhomogeneity in planar waveguide using finite elements

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Inverse Problems in Wave Propagation
Apr. 07-10, 2015, University of Bremen

1 Introduction

2 The Inverse Problems

3 Numerical Algorithm

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Problem Setting

Let $\Omega = \mathbb{R} \times [0, H]$, $H > 0$, $\Sigma_- = \mathbb{R} \times \{0\}$ and $\Sigma_- = \mathbb{R} \times \{H\}$. Let D be a star-shaped penetrable obstacle with Lipschitz continuous boundary ∂D

$$\partial D := \{(z_1,\, z_2) + r(t)(\cos t,\, \sin t) |\, 0 \leqslant t < 2\pi\}.$$

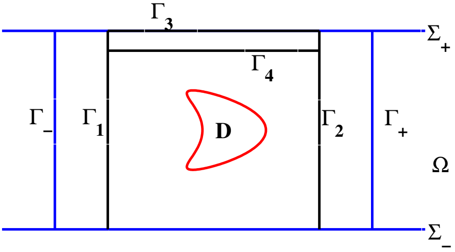

Figure: Explicative figure.

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Green's Function

Let k_n and θ_n be given by

$$k_n = \frac{n\pi}{H}, \quad \theta_n(x_2) = \begin{cases} \sqrt{\frac{1}{H}}, & n = 0, \\ \sqrt{\frac{2}{H}} \cos\left(\frac{n\pi}{H} x_2\right), & n \geqslant 1. \end{cases}$$

The Green's function is defined by (see [Bourgeois and Lunéville 08])

$$G(x, y) = \sum_{n\in\mathbb{N}} \frac{e^{i\beta_n|x_1-y_1|}}{2i\beta_n}\theta_n(x_2)\theta_n(y_2),$$

where $\beta_n = \sqrt{k^2 - k_n^2}$, $\mathrm{Re}\,\beta_n \geqslant 0$, $\mathrm{Im}\,\beta_n \geqslant 0$.

Dirichlet-to-Neumann map

Let Γ_- (Γ_+) be given by

$$\Gamma_\pm = \{(\pm s, x_2),\ x_2 \in [0, H]\}, \quad \text{where } s > \max_{t\in[0,2\pi)} \{|z_1 + r(t)\cos t|\}.$$

$\Omega_0 = [-s, s] \times [0, H]$ is a rectangle containing D, n is the piecewise constant refractive index such that

$$n = \begin{cases} n_1, & \text{in } \Omega \setminus D, \\ n_2, & \text{in } D, \end{cases}$$

where $n_1 = 1$ and $\mathrm{Re}\ n_2 > 0$, $\mathrm{Im}\ n_2 \geqslant 0$. $T_\pm$ is the DtN map

$$T_\pm v = \sum_{n\in\mathbb{N}} i\beta_n (v, \theta_n)_{\Gamma_\pm} \theta_n.$$

The Scattering Problem

Let u be the total field and u^s ($u^s = u - u^i$) be the scattered field.

$$\begin{aligned} \triangle u^s + k^2 n u^s &= k^2(1-n)u^i\,, && \text{in } \Omega_0, && (1.1)\\ \frac{\partial u^s}{\partial x_2} &= 0\,, && \text{on } \Sigma_\pm, && (1.2)\\ \frac{\partial u^s}{\partial x_1} &= -T_- u^s\,, && \text{on } \Gamma_-, && (1.3)\\ \frac{\partial u^s}{\partial x_1} &= T_+ u^s\,, && \text{on } \Gamma_+. && (1.4) \end{aligned}$$

The variational form can be written as

$$-\int_{\Omega_0} \nabla u^s \cdot \overline{\nabla\phi} + k^2 \int_{\Omega_0} n u^s \overline{\phi} + \int_{\Gamma_-} T_- u^s \overline{\phi}\, ds + \int_{\Gamma_+} T_+ u^s \overline{\phi}\, ds = k^2 \int_D g\overline{\phi}$$

where $g = (1-n)u^i$.

Scattering Operator

Let $\mathbf{C} = (c_0, c_1, \dots, c_{2N}) \in \mathbb{R}^{2N+1}$ and

$$r(t) = \sum_{n=0}^{2N} c_n \phi_n,$$

where ϕ_n the trigonometric basis.
For any fixed point $(z_1, z_2) \in \Omega_0$, define the operator

$$f: \quad \mathbf{C} \longrightarrow r(t)(\cos t, \sin t).$$

Then $r \in X_N := \{f(\mathbf{C}),\ t \in [0, 2\pi)\}$. For any given incident field u^i, define the operator S

$$\begin{aligned} S: \quad X_N &\longrightarrow H^1(\Omega_0), \\ g &\mapsto u^s|_{\Omega_0}. \end{aligned}$$

Fréchet differentiability

Lemma

The scattering operator S is Fréchet differentiable, and its Fréchet derivative, denoted by FS, is defined as

$$FS(g)h = v|_{\Omega_0}, \quad \text{where } g, h \in X_N.$$

v satisfies the equations (1.1)-(1.4) and the following boundary conditions on ∂D, i.e.,

$$v|_- - v|_+ = 0, \quad \frac{\partial v}{\partial \nu}\Big|_- - \frac{\partial v}{\partial \nu}\Big|_+ = k^2(n_2 - 1)u(h \cdot \nu), \tag{1.5}$$

where u is the total field of the direct problem.

Fréchet derivatives of N_j

For $j = 1, 2, 3$, denoting by P_j the trace operators that from $H^1(\Omega_0)$ to $H^{1/2}(\Gamma_j)$. We can define the near-field operators

$$
\begin{array}{rcl}
N_j: \quad \mathbb{R}^{2N+1} & \longrightarrow & H^{1/2}(\Gamma_j) \\
\mathbf{C} = (c_0, c_1, \dots, c_{2N}) & \mapsto & P_j S\big[f(c_0, c_1, \dots, c_{2N})\big].
\end{array}
$$

N_j is an operator defined on the finite dimensional space, and the Fréchet derivatives of N_j satisfy

$$
\begin{aligned}
\frac{\partial}{\partial c_l} N_j(\mathbf{C}) &= P_j FS\big[f(\mathbf{C})\big] \frac{\partial}{\partial c_l} f(\mathbf{C}) \\
&= P_j FS\big[f(\mathbf{C})\big]\big[\phi_l(\cos t, \sin t)\big].
\end{aligned}
$$

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The Inverse Problem

Denote the measured data by $U_j := u^s|_{\Gamma_j}$. We define

$$Q_j(\mathbf{C}) = \|N_j(\mathbf{C}) - U_j\|^2_{L^2(\Gamma_j)}.$$

and

$$\begin{aligned}
F_1(\mathbf{C};\gamma) &= Q_1(\mathbf{C}) + Q_1(\mathbf{C}) + \gamma\|\mathbf{C}\|^2_{l^2},\\
F_2(\mathbf{C};\gamma) &= Q_1(\mathbf{C}) + \gamma\|\mathbf{C}\|^2_{l^2},\\
F_3(\mathbf{C};\gamma) &= Q_2(\mathbf{C}) + \gamma\|\mathbf{C}\|^2_{l^2},\\
F_4(\mathbf{C};\gamma) &= Q_3(\mathbf{C}) + \gamma\|\mathbf{C}\|^2_{l^2},
\end{aligned}$$

where γ is the regularization parameter.
Optimization Problem (OP): Find $\mathbf{C}^0 \in \mathbb{R}^{2N+1}$, such that

$$\mathbf{C}^0 = \min_{\mathbf{C}\in\mathbb{R}^{2N+1}} F_j(\mathbf{C}),$$

Lemma

The gradient of Q_j is defined by

$$\frac{\partial}{\partial c_l} Q_j(\mathbf{C}) = 2\mathrm{Re}\Big(M_j[N_j(\mathbf{C}) - U_j] \cdot (\cos t, \sin t), \phi_l \Big),$$

where $M_j = (P_j DS[f(\mathbf{C})])^*$ is the adjoint operator of $P_j FS[f(\mathbf{C})]$.

Lemma

Suppose w is in the space $H^{1/2}(\Gamma_\pm)$, then $(T_\pm)^*$ satisfies

$$\overline{(T_\pm)^* w} = (T_\pm)\overline{w}.$$

Lemma

Let $\Omega_1 = [-s_0, s_0] \times [0, H]$*. The operator* M_1 *is defined by*

$$M_1(\phi) = -\overline{k^2(n_2 - 1)uw}|_{\partial D}\nu,$$

where w *satisfies the boundary value problem in* Ω_1

$$\triangle w + k^2 nw = 0, \qquad in\ \Omega_1, \tag{2.6}$$

$$\frac{\partial w}{\partial x_2} = 0, \qquad on\ \Sigma_\pm, \tag{2.7}$$

$$T_+ w - \frac{\partial w}{\partial x_1} = 0, \qquad on\ \Gamma_2, \tag{2.8}$$

$$T_- w + \frac{\partial w}{\partial x_1} = \overline{\phi}, \qquad on\ \Gamma_1. \tag{2.9}$$

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Finite Element Method

The finite element method of the direct problem can be written in the matrix form

$$(A+B)U = F.$$

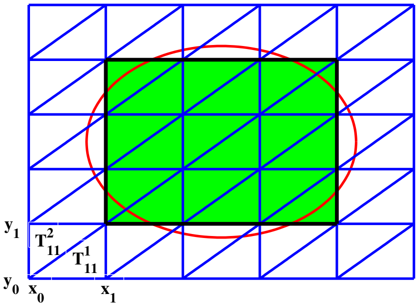

Figure: Finite Element Mesh for Reconstruction.

Initial Guess

LSM: consider the integral equation

$$\int_{\Gamma_2} u^s(z, z_0) g_2(z_0, z_*) dz_0 = G(z, z_*), \quad z \text{ on } \Gamma_2, \tag{3.10}$$

For any $\epsilon > 0$, find the approximate solution $h_3^\epsilon(\cdot, z_*)$ satisfy

$$\left\| \int_{\Gamma_2} u^s(z, z_0) h_3^\epsilon(z_0, z_*) dz_0 - G(z, z_*) \right\|_{L^2(\Gamma_2)} \leqslant \epsilon$$

We define the indicator function

$$E_3^\epsilon(z_*) := \|h_3^\epsilon(\cdot, z_*)\|^2_{L^2(\Gamma_2)},$$

[Xu et.al. 2000, Bourgeois and Lunéville 2008]

Quasi-Newton Method

Let $k_j, j = 1, 2, \ldots, K$ be different wave numbers, $z_l, l = 1, 2, \ldots, Z$ be different locations, corresponding to the four cases. The incident fields are given by

$$u^i_{l+(j-1)K} := G_{k_j}(\cdot, z_l).$$

1. Initialization: for the wave number k_0, find a center (z_1^0, z_2^0) and radius r_0 using the linear sampling method. Set the initial guess $C_0 = (r_0, 0, \ldots, 0)$. Choose a parameter $0 < \eta < 1$.
2. Iteration: for $j_0 = 1 : KZ$. Set $l = 1$. Start iteration with $u^i_{j_0}$.

Quasi-Newton Method (continued)

(2.a) If $l = 1$, obtain the numerical approximation of $\nabla F_j(C_0; \gamma)$. If $l \geqslant 2$, $\nabla F_j(C_{l-1}; \gamma)$ is obtained in the $l-1$-th step.

(2.b) Set a search direction $\xi_l = -H_{l-1} \nabla F_j(C_{l-1}; \gamma)$, where $\nabla F_j(C_{l-1}; \gamma)$ is obtained in the l-th step, $H_{l-1} = I$ is obtained in the $l-1$-th step, $H_{l-1} = I$ if $l = 1$.

(2.c) For $\alpha_l^s = 2^s, s \in \{-5, -4, \ldots, 5\}$, define the function $h_l^s = \alpha_l^s \xi_l$. Find the largest number $s_0 \in \{-5, -4, \ldots, 5\}$ and define $h_l = h_l^{s_0}$ such that the Wolfe's condition is satisfied:

$$F_j(C_{l-1} + h_l; \gamma) \leqslant F_j(C_{l-1}; \gamma) + \eta h_l \cdot \nabla F_j(C_{l-1}; \gamma).$$

(2.d) The new coefficients in the l-th step is given by $C_l = C_{l-1} + h_l$.

Quasi-Newton Method (continued)

(2.e) Compute $\nabla F_j(C_l;\gamma)$. Set $\zeta = \nabla F_j(C_l;\gamma) - \nabla F_j(f_{C-1};\gamma)$ and use the BFGS method to update the approximate inverse Hessian matrix H_l.

(2.f) Check that if the following conditions are satisfied:

- the maximum number of iterations (20 in our algorithm) is reached;
- $|F_j(C_{l-1};\gamma) - F_j(C_l;\gamma)| < 10^{-8}$;
- $\|h_l\|_{L^2[0,L]} < 10^{-7}$.

1. Stop if any of the conditions is satisfied;
2. Go to (2.a) if none of the four conditions is satisfied.

Set $C_0 = C_l$.

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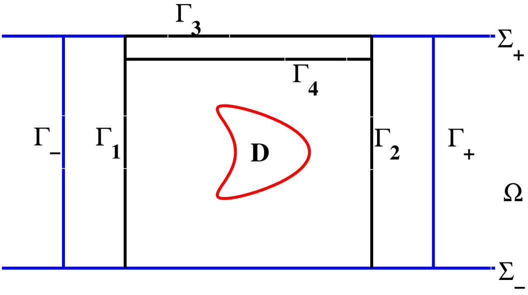
Measurements

1. The point sources are located on both Γ_1 and Γ_2, and data are measured on Γ_1 and Γ_2.
2. The point sources are located on Γ_1, and data are also measured on Γ_1.
3. The point sources are located on Γ_2, and data are also measured on Γ_2.
4. The point sources are located on Γ_4, and data are measured on Γ_3.



Settings

$N_x = N_y = 200$, $s = 0.4$, $H = 1$, $\rho = 0.9$, $M_x = M_y = 40$, $N = 8$, $\epsilon = 10^{-10}$.

The regularization parameter is chosen as $\gamma = 10^{-4}$. When using the linear sampling method to obtain the initial guess, we fix $k_0 = 1$, and place 40 point sources uniformly located on each edge. During the iteration scheme, we use seven wave numbers $k = 1, 7, 13, 19, 25, 31, 37$, and use 2 point sources located on Γ_1 or Γ_2, and 4 point sources on Γ_4:

$$\begin{aligned}&(0.25, -0.4), (0.75, -0.4) \quad \text{on } \Gamma_1,\\ &(0.25, 0.4), (0.75, 0.4) \quad \text{on } \Gamma_2,\\ &(-0.3, 0.9), (-0.1, 0.9), (0.1, 0.9), (0.3, 0.9) \quad \text{on } \Gamma_4.\end{aligned}$$

The data are measured on Γ_1, Γ_2 and Γ_4.

Example 1: Case 1

Let $n_2 = 0.5$. The obstacle is an ellipse defined by

$$\frac{40}{9}x^2 + 100(y - 0.5)^2 = 1.$$

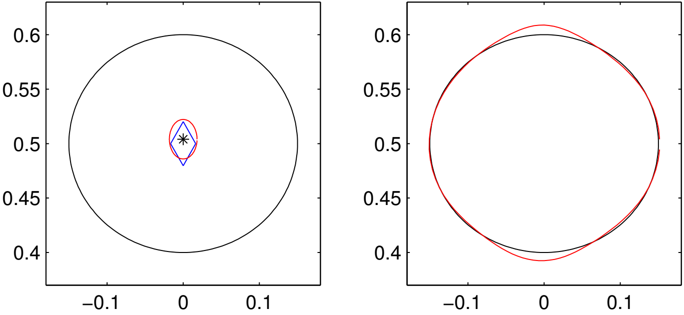


Figure: Case (1) with 5% noise.

Example 1: Case 2

Let $n_2 = 0.5$. The obstacle is an ellipse defined by

$$\frac{40}{9}x^2 + 100(y - 0.5)^2 = 1.$$

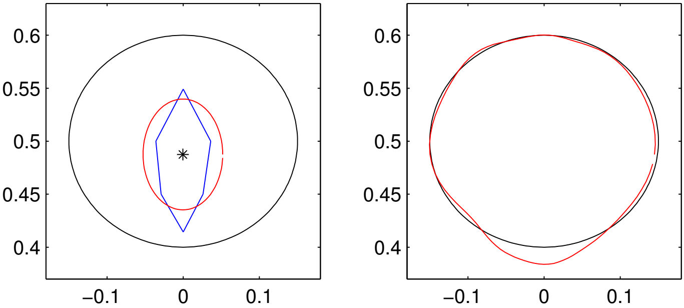


Figure: Case (2) with 5% noise.

Example 1: Case 3

Let $n_2 = 0.5$. The obstacle is an ellipse defined by

$$\frac{40}{9}x^2 + 100(y - 0.5)^2 = 1.$$

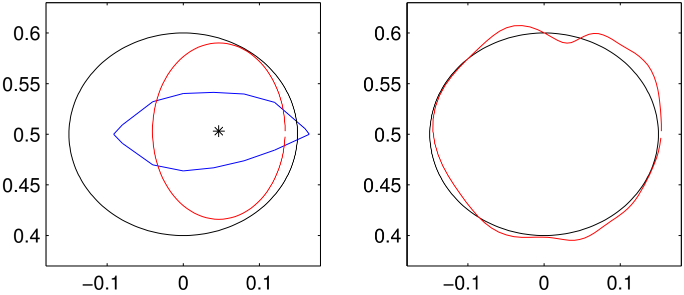


Figure: Case (3) with 5% noise.

Example 1: Case 4

Let $n_2 = 0.5$. The obstacle is an ellipse defined by

$$\frac{40}{9}x^2 + 100(y - 0.5)^2 = 1.$$

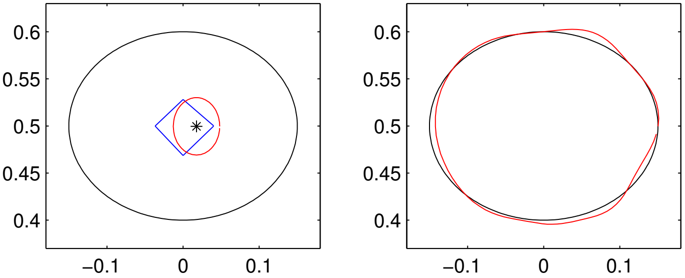


Figure: Case (4) with 5% noise.

Example 2: Case 1

Let $n_2 = 0.5$. The obstacle is a rectangle defined by vertices $(-0.05, 0.35)$, $(0.15, 0.35)$, $(0.15, 0.55)$ and $(-0.05, 0.55)$.

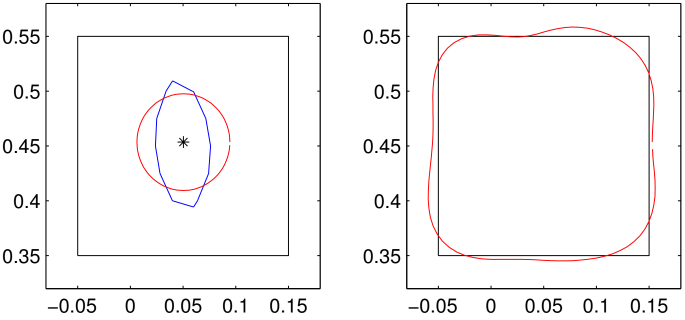


Figure: Case (1) with 5% noise.

Example 2: Case 2

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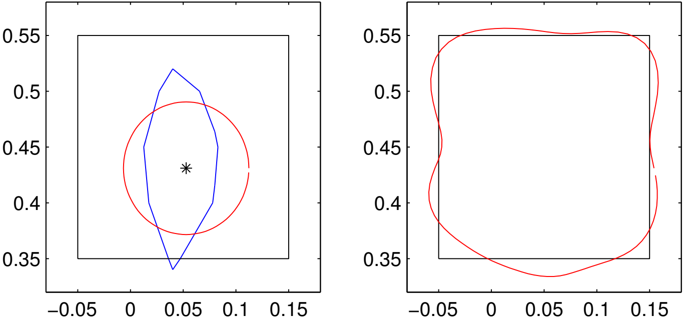

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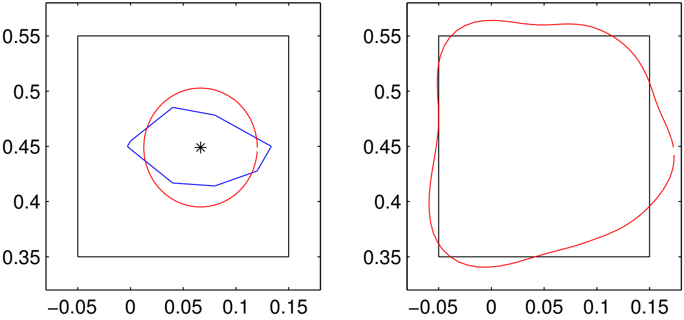


Figure: Case (3) with 5% noise.

Example 2: Case 4

Let $n_2 = 0.5$. The obstacle is a rectangle defined by vertices $(-0.05, 0.35)$, $(0.15, 0.35)$, $(0.15, 0.55)$ and $(-0.05, 0.55)$.

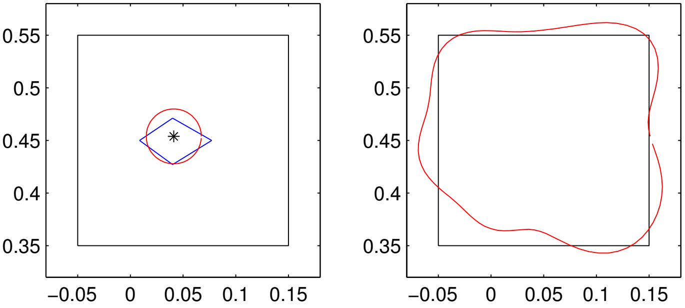
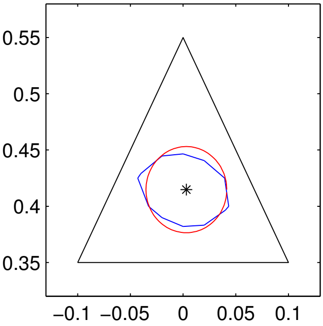


Figure: Case (4) with 5% noise.

Example 3: Case 1

Let $n_2 = 2$. The obstacle is a triangle defined by vertices $(-0.1, 0.35)$, $(0.1, 0.35)$, and $(0, 0.55)$.



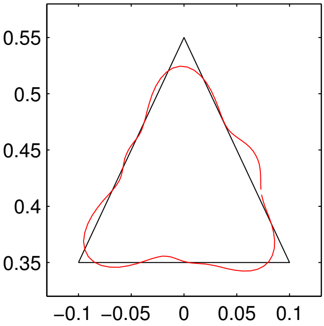


Figure: Case (1) with 5% noise.

Example 3: Case 2

Let $n_2 = 2$. The obstacle is a triangle defined by vertices $(-0.1, 0.35)$, $(0.1, 0.35)$, and $(0, 0.55)$.

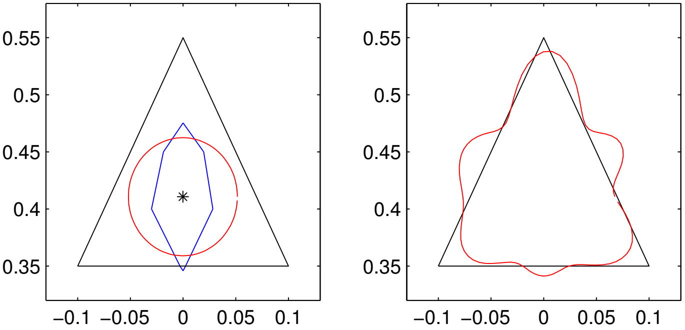


Figure: Case (2) with 5% noise.

Example 3: Case 3

Let $n_2 = 2$. The obstacle is a triangle defined by vertices $(-0.1, 0.35)$, $(0.1, 0.35)$, and $(0, 0.55)$.

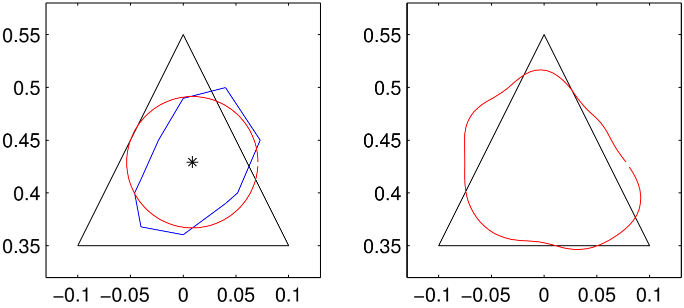


Figure: Case (3) with 5% noise.

Example 3: Case 4

Let $n_2 = 2$. The obstacle is a triangle defined by vertices $(-0.1, 0.35)$, $(0.1, 0.35)$, and $(0, 0.55)$.

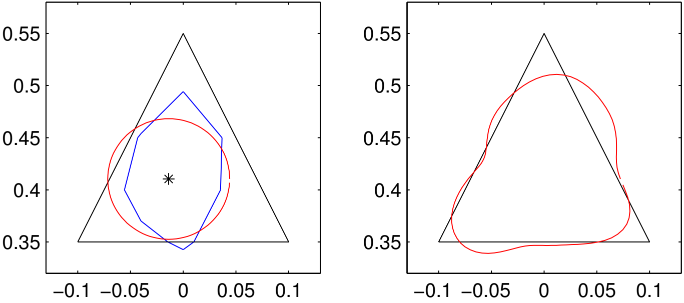


Figure: Case (4) with 5% noise.

Example 4: Case 1

Let $n_2 = 2$. The obstacle is a kite and its boundary is given by

$$(0.075\cos t + 0.04875\cos 2t - 0.05,\ 0.1\sin t + 0.5).$$

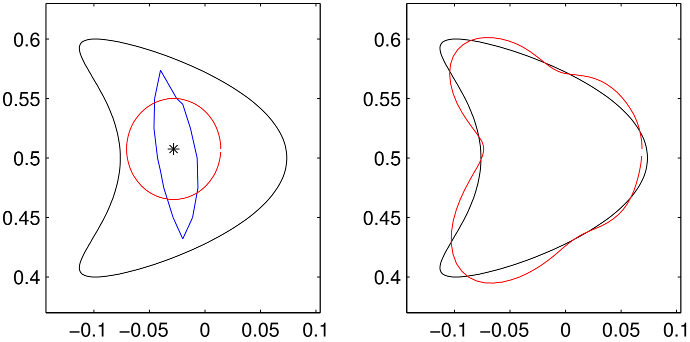


Figure: Case (1) with 5% noise.

Example 4: Case 2

Let $n_2 = 2$. The obstacle is a kite and its boundary is given by

$$(0.075\cos t + 0.04875\cos 2t - 0.05,\ 0.1\sin t + 0.5).$$

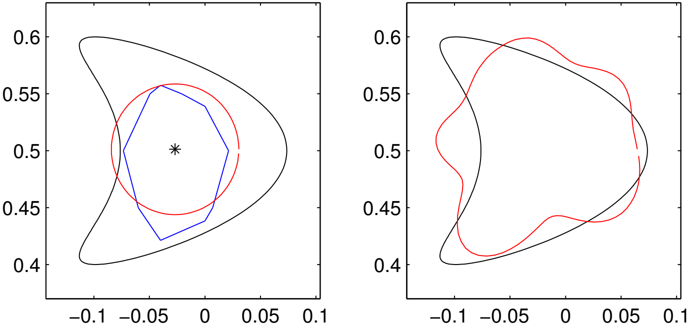


Figure: Case (2) with 5% noise.

Example 4: Case 3

Let $n_2 = 2$. The obstacle is a kite and its boundary is given by

$$(0.075\cos t + 0.04875\cos 2t - 0.05,\ 0.1\sin t + 0.5).$$

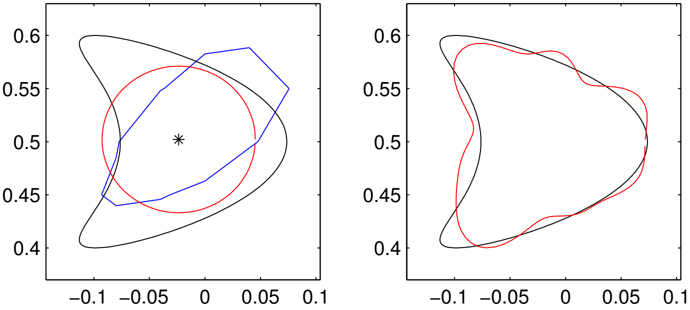


Figure: Case (3) with 5% noise.

Example 4: Case 4

Let $n_2 = 2$. The obstacle is a kite and its boundary is given by

$$(0.075\cos t + 0.04875\cos 2t - 0.05,\ 0.1\sin t + 0.5).$$

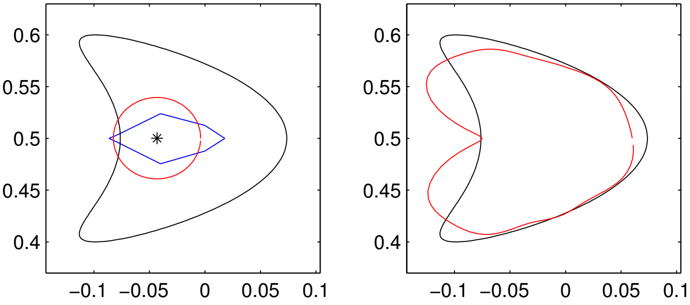


Figure: Case (4) with 5% noise.

Thank you!

1. Introduction

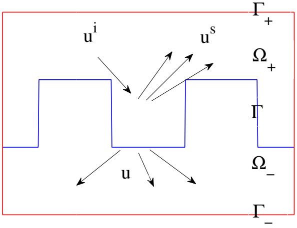


Figure: The physical configuration of the scattering problem.

Given the incident wave u^i, we have the scattered field in Ω_+ and the transmitted field u in Ω_-.

1. Introduction

Let X be the set of piecewise continuous 2π-periodic functions in $\mathbb{R}$. The grating profile Γ is defined by

$$\Gamma = \left\{(x, f(x)) : \, x \in \mathbb{R}\right\}$$

For a fixed $\alpha \in \mathbb{R}$, α-quasi-periodic function is defined by

$$g(x + 2\pi) = \exp(i2\pi\alpha) g(x)$$

If u^i is α-quasi-periodic, then u^s and u are all α-quasi-periodic.

1. Introduction

Notations & spaces:

$$
\begin{aligned}
\Omega_+ &= \{(x,y):\ y > f(x)\} \\
\Omega_- &= \{(x,y):\ y < f(x)\} \\
\Gamma_+ &= \{(x,h_+):\ h_+ > \max_{t\in\mathbb{R}} f(t)\} \\
\Gamma_- &= \{(x,h_-):\ h_- < \min_{t\in\mathbb{R}} f(t)\} \\
D &= [0,2\pi]\times[h_-,h_+] \\
H^m_{per}(D) &= \{u(\cdot,y) \text{ is } 2\pi\text{-periodic: } \ u|_D \in H^m(D)\} \\
H^m_\alpha(D) &= \{u(\cdot,y) \text{ is } \alpha\text{-quasi-periodic: } \ u|_D \in H^m(D)\} \\
L^2_{per}(D) &= H^0_{per}(D),\ L^2_\alpha(D) = H^0_\alpha(D)
\end{aligned}
$$

$H^m_{per}(\Gamma_\pm)$, $H^m_\alpha(\Gamma_\pm)$, $L^2_{per}(\Gamma_\pm)$, $L^2_\alpha(\Gamma_\pm)$ are defined similarly.

1. Introduction

$n(x, y)$ is the piecewise constant refractive index defined in $\mathbb{R}^2$.

$$n(x, y) = \begin{cases} n_1 & (x, y) \in \Omega_+ \\ n_2 & (x, y) \in \Omega_- \end{cases}$$

Let $n_1 = 1$ in Ω_+, and n_2 satisfies

$$\Re\, n_2 \geqslant 0,\ \Im\, n_2 \geqslant 0 \quad \text{in } \Omega_-$$

Problem: The measured data are defined by:

$$u^{meas} = u^s\big|_{\Gamma_+}$$

How to reconstruct the grating profile Γ (or f) from the measurement of the near-field scattered field u^{meas}?

Previous Works

- The factorization method in inverse scattering from periodic structures
  Arens and Kirsch '03
- Finite element method, for optimal design problem of binary gratings.
  J. Elschner & G. Schmidt '98
- Two-step optimization method based on Tikhonov regulation.
  G. Bruckner & J. Elschner '05
- Factorization method.
  A. Lechleiter '10
- Linear sampling method.
  J. Yang, B. Zhang & R. Zhang '13
- ...

2. Direct Scattering Problem

Incident angle: θ, define $\alpha = k \sin\theta$, $\beta = k\cos\theta$. Incident wave:

$$u^i(x,y) = \exp(i\alpha x - i\beta y)$$

Then the total field u ($u = u^i + u^s$ in Ω_+) is α-quasi-periodic:

$$u(x + 2\pi, y) = \exp(i2\pi\alpha)u(x,y)$$

and satisfies the Helmholtz equation:

$$\triangle u + k^2 n u = 0$$

2. Direct Scattering Problem

The scattered field u^s in Ω_+ and transmitted field u in Ω_- satisfies the Rayleigh expansion radiation conditions:

$$u^s(x,y) = \sum_{n\in\mathbb{Z}} u_n^+ \exp(i\alpha_n x + i\beta_n^{(1)} y), \; y > \max_{t\in\mathbb{R}}\{f(t)\}$$

$$u(x,y) = \sum_{n\in\mathbb{Z}} u_n^- \exp(i\alpha_n x - i\beta_n^{(2)} y), \; y < \min_{t\in\mathbb{R}}\{f(t)\}$$

where $\alpha_n = \alpha + n$ and $\beta_n^{(j)}$ is the square root of $(k^2 n_j - \alpha_n^2)$ such that $\Re\,\beta_n^{(j)} \geqslant 0$, $\Im\,\beta_n^{(j)} \geqslant 0$, for $j = 1, 2$.

2. Direct Scattering Problem

For $\phi^{\pm}(x, h_{\pm}) = \sum_{n\in\mathbb{Z}} \hat{\phi}_n^{\pm} e^{i\alpha_n x}$ defined on $\Gamma_{\pm}$, define Dirichlet to Neumann maps $T^{\pm}$

$$\begin{aligned} T^{+}(\phi^{+}) &= \sum_{n\in\mathbb{Z}} i\beta_n^{(1)} \hat{\phi}_n^{+} e^{i\alpha_n x}, \quad \text{on } \Gamma_{+}, \\ T^{-}(\phi^{-}) &= \sum_{n\in\mathbb{Z}} i\beta_n^{(2)} \hat{\phi}_n^{-} e^{i\alpha_n x}, \quad \text{on } \Gamma_{-}. \end{aligned}$$

Then u^s and u satisfy the following boundary conditions

$$\begin{aligned} \frac{\partial u^s}{\partial y} &= T^{+} u^s, \quad \text{on } \Gamma_{+}, \\ \frac{\partial u}{\partial y} &= -T^{-} u, \quad \text{on } \Gamma_{-}. \end{aligned}$$

Rayleigh expansions $\Longleftrightarrow$ boundary conditions above.

2. Direct Scattering Problem

Choose a proper smooth cutoff function $\mathcal{X}(x, y)$ and define:

$$v(x,y) = \begin{cases} u^s(x,y) + u^i(x,y)\mathcal{X}(x,y) & \text{in } \Omega_+, \\ u(x,y) \quad \text{in } \Omega_-. \end{cases}$$

such that for $\max_{t\in\mathbb{R}}\{f(t)\} < h_1 < h_2 < h_+$,

$$v(x,y) = \begin{cases} u(x,y), & y < h_1, \\ u^s(x,y), & y > h_2 \end{cases}$$

v satisfies the equation

$$\triangle v + k^2 v = g, \text{ where } g = 2\nabla u^i \cdot \nabla \mathcal{X} + u^i \triangle \mathcal{X}, \text{ in } \mathbb{R}^2$$

2. Direct Scattering Problem

Variational formulation

$$-\int_D \nabla v \cdot \overline{\nabla \phi} + k^2 \int_D n v \overline{\phi} + \int_{\Gamma_+} T^+ v \overline{\phi} + \int_{\Gamma_-} T^- v \overline{\phi} = \int_D g \overline{\phi}$$

for all $\phi \in H^1_\alpha(D)$.

Theorem

For any wave number k in $\mathbb{R}^+$ except for a discrete set, the variational problem is uniquely solvable in $H^1_\alpha(D)$, satisfies

$$\|v\|_{H^1_\alpha(D)} \leqslant C\|g\|_{L^2(D)}$$

Define Scattering operator S:

$$\begin{array}{rcl} S: \ X & \longrightarrow & H^{1/2}_\alpha(\Gamma_+) \\ f & \mapsto & v|_{\Gamma_+} (= u^s|_{\Gamma_+}) \end{array}$$

3. Inverse Scattering Problem

From the definition of u^{meas} and S,

$$u^{meas} := u^s|_{\Gamma_+} = S(f)$$

Inverse Problem (IP): Find $f^* \in X$ such that

$$S(f^*) = u^{meas},$$

or equivalently,

$$\|S(f^*) - u^{meas}\|_{L^2_{per}(\Gamma_+)} = 0$$

3. Inverse Scattering Problem

Define a functional with a regularization term $F(f;\gamma)$

$$F(f;\gamma) := \|S(f^*) - u^{meas}\|^2_{L^2_{per}(\Gamma_+)} + \gamma\|f\|^2_{L^2[0,2\pi]},$$

where γ is the regularization parameter.

Optimization Problem (OP): Find $f^* \in X$ such that

$$F(f^*;\gamma) = \min_{f \in X} F(f;\gamma)$$

3. Inverse Scattering Problem

M is a positive integer, define $x_j^M = 2\pi j/M$, $j = 0, 1, \ldots, M$, and

$$\phi_l^M = \begin{cases} 1, & \text{if } x \in [x_{l-1}^M, x_l^M), \\ 0, & \text{otherwise.} \end{cases}$$

where $l = 1, 2, \ldots, M$. Define

$$X^M = \mathrm{span}\left\{\phi_1^{\mathrm{M}}, \phi_2^{\mathrm{M}}, \ldots, \phi_{\mathrm{M}}^{\mathrm{M}}\right\}$$

Seek the numerical solution in the space X^M, i.e.,

$$f(x) = \sum_{j=1}^{M} \hat{f}_j \phi_j^M(x)$$

then $F(f;\gamma) = F(\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_M;\gamma)$.

3. Inverse Scattering Problem

Lemma

Let $h = \sum_{j=1}^{M} \hat{h}_j \phi_j^M$ be a function in X^M. The scattering operator $S(f)$ is Fréchet differentiable. Then

$$P(f)h := DS(f)h = w|_{\Gamma}$$

where $w \in H^1_\alpha(D)$ satisfies the following variational problem

$$-\int_D \nabla w \cdot \overline{\nabla \phi} + k^2 \int_D n w \overline{\phi} + \int_{\Gamma_+} T^+ w \overline{\phi} + T^- w \overline{\phi}$$
$$= k^2 (n_2 - n_1) \int_0^{2\pi} h(v \overline{\phi})|_{\Gamma}$$

for all $\phi \in H^1_\alpha(D)$.

3. Inverse Scattering Problem

Lemma

Denote $M(f) = (P(f))^*$. *If* $\phi \in L^2_\alpha(\Gamma_+)$, *then*

$$M(f)\phi = k^2(n_2 - n_1)(\overline{v}\psi)|_\Gamma$$

where $\psi \in H^1_\alpha(D)$ *satisfies*

$$\begin{aligned} \triangle\overline{\psi} + k^2 n\overline{\psi} &= 0, \\ T^+\overline{\psi} - \frac{\partial\overline{\psi}}{\partial y} &= \overline{\phi}, \\ T^-\overline{\psi} + \frac{\partial\overline{\psi}}{\partial y} &= 0. \end{aligned}$$

3. Inverse Scattering Problem

Theorem

The derivatives of $F(\ldots;\gamma)$ *are given by*

$$\frac{\partial F}{\partial \hat{f}_m} = 2k^2 \Re \left[(n_2 - n_1) \left((\overline{v}\psi)|_\Gamma, \phi_m^M \right)_{L^2[0,2\pi]} \right] + \frac{2\gamma L}{M} \hat{f}_m$$

where $\psi \in H^1_\alpha(D)$ *is the solution of the boundary value problem, in which* $\phi = S(f) - u^{meas}$.

3. Inverse Scattering Problem

Proof.

Recall the definition

$$F(\hat{f}_1, \hat{f}_2, \dots, \hat{f}_M; \gamma) = \left\| S(f) - u^{meas} \right\|^2_{L^2(\Gamma_+)} + \gamma \|f\|^2_{L^2[0,2\pi]}$$

The second term is trivial, then the first term

$$\begin{aligned}
\frac{\partial}{\partial \hat{f}_m} \left\| S(f) - u^{meas} \right\|^2_{L^2(\Gamma_+)} &= 2\Re\left(S(f) - u^{meas}, \frac{\partial}{\partial \hat{f}_m} S(f) \right) \\
&= 2\Re\left(S(f) - u^{meas}, P(f)\phi^M_m \right) \\
&= 2\Re\left(M(f)\left(S(f) - u^{meas}\right), \phi^M_m \right)
\end{aligned}$$

$\square$

3. Inverse Scattering Problem

If $\phi(x) = \sum_{n\in\mathbb{Z}} c_n \exp(i\alpha_n x)$, then the operator

$$Q\phi = \sum_{n\in\mathbb{Z}} \frac{c_n}{2i\beta_n} \exp\left[i\alpha_n x - i\beta_n^{(1)}(y - h_+)\right], \quad (x, y) \in D$$

defines an incident field propagating downward.

Lemma

The boundary value problem is equivalent to the following scattering problem

$$\begin{aligned} &\triangle\psi + k^2 n\psi = 0 \quad \text{in } D, \\ &\psi = \psi_1 + Q\phi \quad \text{in } D, \\ &T^{\pm}\psi_1 = \pm\frac{\partial\psi_1}{\partial y} \quad \text{on } \Gamma_{\pm}. \end{aligned}$$

where ψ_1 is the scattered field.

4. Numerical Method

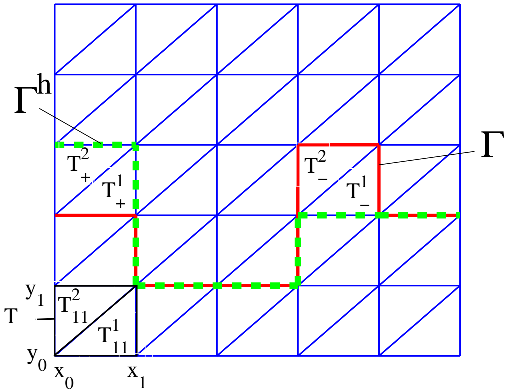
Use the finite element method to solve the direct problem by PML technique. (G. Bao, Z. Chen & H. Wu '03)
Define $D^{PML} = [0, 2\pi] \times [h_- - \delta, h_+ + \delta]$.

$$0 = x_0 < x_1 < \cdots < x_{N_x} = 2\pi$$
$$h_- - \delta = y_0 < y_1 < \cdots < y_{N_y} = h_+ + \delta.$$

Let N_x be divisible by M. We seek $f \in X^M$ and aligns with the mesh grid

$$f = \sum_{j=1}^{M} \hat{f}_j \phi_j^M, \; \hat{f}_j = y_{j_l}, \, j_l \in \{0, 1, \ldots, N_y\}$$

4. Numerical Method



4. Numerical Method

Define a PML function $s(y)$

$$s(y) = \begin{cases} 1 + \sigma \frac{(y - h_+ - \epsilon)^2}{(\delta - \epsilon)^2}, & y > h_+ + \epsilon \\ 1 + \sigma \frac{(y - h_- + \epsilon)^2}{(\delta - \epsilon)^2}, & y < h_- - \epsilon \\ 1, & \text{otherwise}, \end{cases}$$

where σ is a complex number, $0 < \epsilon < \delta$.

Define a differential operator:

$$\mathcal{L} := \frac{\partial}{\partial x}\left(s(y)\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{s(y)}\frac{\partial}{\partial y}\right) + k^2 n(x,y) s(y)$$

4. Numerical Method

Then we will solve the following boundary value problem

$$
\begin{aligned}
&\mathcal{L}\hat{u} = g = \mathcal{L}(u^i \mathcal{X}), \quad (x, y) \in D^{PML},\\
&\hat{u} = 0, \quad y = h_+ + \delta \text{ or } y = h_- - \delta,\\
&\hat{u}(2\pi, y) = \exp(2i\pi\alpha)\hat{u}(0, y), \quad h_- < y < h_+,
\end{aligned}
$$

Corresponding variational formulation

$$
\int_{D^{PML}} \left[s(y)\frac{\partial \hat{u}}{\partial x}\frac{\partial \overline{\psi}}{\partial x} + \frac{1}{s(y)}\frac{\partial \hat{u}}{\partial y}\frac{\partial \overline{\psi}}{\partial y} \right] - k^2 \int_{D^{PML}} s(y) n(x, y) \hat{u}\overline{\psi}
$$
$$
= -\int_{D^{PML}} g\overline{\psi}
$$

for all $\psi \in H^1_\alpha(D^{PML})$.

4. Numerical Method

$\{\psi_1, \psi_2, \ldots, \psi_N\}$ is the finite element basis in D^{PML}, the numerical solution $\tilde{u}$ can be represented by:

$$\tilde{u} = \sum_{j=1}^{N} \tilde{u}_j \psi_j$$

$A = (A_{jl})$: the stiffness matrix.

$$A_{jl} = \int_{D^{PML}} \left[s(y) \frac{\partial \psi_j}{\partial x} \frac{\partial \overline{\psi}_l}{\partial x} + \frac{1}{s(y)} \frac{\partial \psi_j}{\partial y} \frac{\partial \overline{\psi}_l}{\partial y} \right]$$

4. Numerical Method

$B = (B_{jl})$: the mass matrix.

$$B_{jl} = \int_{D^{PML}} s(y) n(x,y) \psi_j \overline{\psi}_l$$

From the definition of $\Omega_\pm$,

$$B_{jl} = n_1 \sum_{T \in \Omega_+} \int_T s(y) \psi_j \overline{\psi}_l + n_2 \sum_{T \in \Omega_-} \int_T s(y) \psi_j \overline{\psi}_l$$

4. Numerical Method

Suppose f is modified by $f + h$.

$$\Gamma^h = \{(x, f(x) + h(x)) : x \in \mathbb{R}\}$$
$$\Omega_+^h = \{(x, y) : y > f(x) + h(x)\}$$
$$\Omega_-^h = \{(x, y) : y < f(x) + h(x)\}$$

Refractive index: $n^h = n_1$ in Ω_+^h, $n^h = n_2$ in Ω_-^h.
Stiffness matrix: $A^h = A$.

4. Numerical Method

Mass matrix: $B^h = (B^h_{jl})$.

$$
\begin{aligned}
B^h_{jl} &= \int_{D^{PML}} s(y) n^h(x,y) \psi_j \overline{\psi}_l \\
&= n_1 \sum_{T \in \Omega^h_+} \int_T s(y) \psi_j \overline{\psi}_l + n_2 \sum_{T \in \Omega^h_-} \int_T s(y) \psi_j \overline{\psi}_l
\end{aligned}
$$

4. Numerical Method

Define $K^+ = \Omega_+ \setminus \Omega_+^h$ and $K^- = \Omega_+ \setminus \Omega_-^h$.
Calculate the difference between B_{jl} and B_{jl}^h

$$B_{jl}^h - B_{jl} = (n_1 - n_2)\left[\sum_{T\in K^-}\int_T s(y)\psi_j\overline{\psi}_l - \sum_{T\in K^+}\int_T s(y)\psi_j\overline{\psi}_l\right]$$

Cost of construct B: $O(N_x N_y)$.

Cost of modify B: $O(N_x)$

4. Numerical Method

Initial Guess

Define the interface $\Gamma^j := \{(x, y_j)\}$, $j = 0, 1, \ldots, N_y$.
The scattered field u_j^s corresponding to Γ^j is

$$u_j^s(x, h_+) = \frac{\beta_0^{(1)} - \beta_0^{(2)}}{\beta_0^{(1)} + \beta_0^{(2)}} \exp\left[i\alpha x + i\beta_0^{(1)}(h^+ - 2y_j)\right]$$

Choose the optimal j_0 such that

$$\left\| u^{meas} - u_{j_0}^s(x, h_+) \right\|_{L^2[0,2\pi]} = \min_{j=0,1,\ldots,N_y} \left\| u^{meas} - u_j^s(x, h_+) \right\|_{L^2[0,2\pi]}$$

The initial data is given by

$$f_0(x) = \sum_{m=1}^{M} y_{j_0} \phi_m^M; \; \Gamma_0 := \{(x, f_0(x))\}$$

4. Numerical Method

Quasi-Newton Method

- Use one fixed incident angle θ and multiple wave numbers k_j, $j = 1, \ldots, J$,

$$u_j^i = \exp(i\alpha^j x - i\beta^j y),\ \alpha^j = k_j \sin\theta,\ \beta^j k_j \cos\theta$$

- For each incident wave, use the quasi-Newton method to modify the function f, and the new function is used as the initial data for the quasi-Newton procedure with the next incident wave.

4. Numerical Method

1. Initialization: give an initial guess f_0.
2. Iteration:
   for j=1:J
   Denote by f_{j-1} the solution obtained in the $j-1$-th step.
   In the j-th step, use the BFGS method to obtain the modified function f_j.
   Remark: the modified functions f_j should in X^M and aligns with the mesh grid.

5. Numerical Examples

Parameters:

$k = 1, 2, \ldots, 8$

$M = 10, 20$

$N_x = N_y = 200$

$h_+ = 0.5$

$\gamma = 5 \times 10^{-3}$

Define the errors between the exact function f and the numerical solution $\tilde{f}^M$

$$err_M := \sum_{j=1}^{N_x} \left| f(x_j) - \tilde{f}^M(x_j) \right|^2$$

5. Numerical Examples

Example 1

The refractive index in Ω_- is $n_2 = 2$.

Use incident waves with one fixed incident angle $\theta = \frac{\pi}{3}$.

The profile Γ is defined by f:

$$f(x) = \frac{1}{8} \cos x.$$

5. Numerical Examples

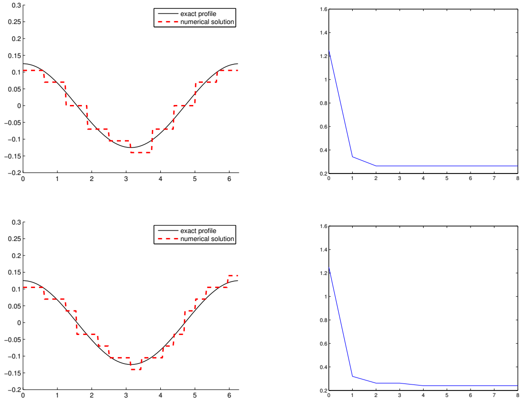


Figure: First row: $M = 10$; second row: $M = 20$.

5. Numerical Examples

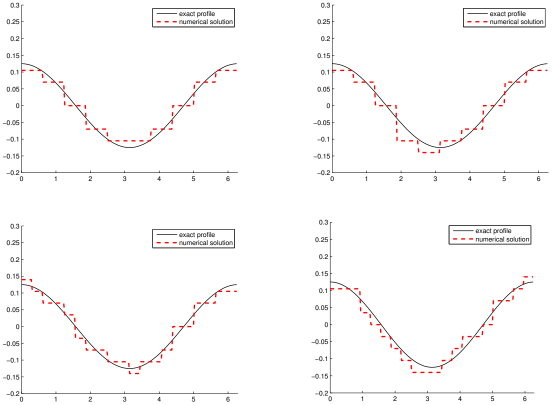


Figure: First row: $M = 10$; second row: $M = 20$. First column: 5% noise; second column: 10% noise.

5. Numerical Examples

Example 2

The refractive index in Ω_- is $n_2 = 0.8 + 0.1i$.

Use incident waves with one fixed incident angle $\theta = \frac{\pi}{4}$.

The profile Γ is defined by f:

$$f(x) = -0.1 + \frac{1}{6}\sin x - \frac{1}{8}\cos 2x.$$

5. Numerical Examples

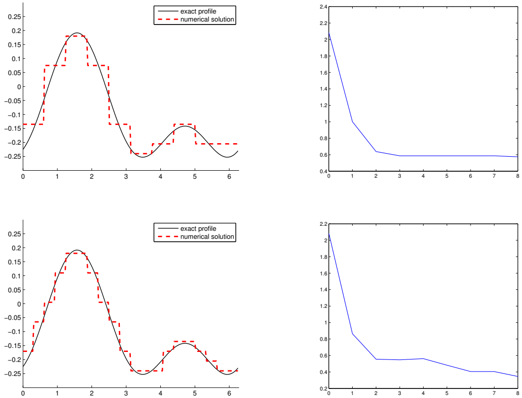


Figure: First row: $M = 10$; second row: $M = 20$.

5. Numerical Examples

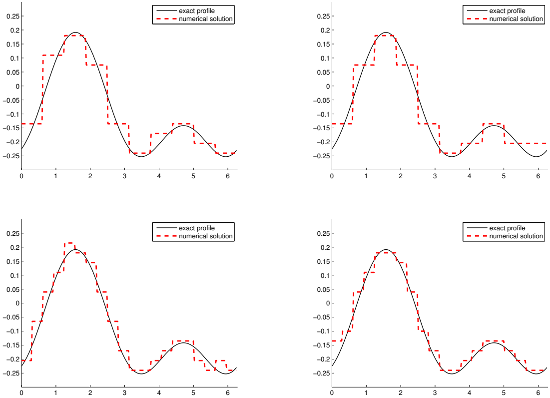


Figure: First row: $M = 10$; second row: $M = 20$. First column: 5% noise; second column: 10% noise.

5. Numerical Examples

Example 3

The refractive index in Ω_- is $n_2 = 2 + 0.1i$.

Use incident waves with one fixed incident angle $\theta = -\frac{\pi}{6}$.

The profile Γ is defined by f:

$$f(x) = \begin{cases} 0.25, & 0.5\pi < x < 1.5\pi, \\ 0, & \text{otherwise.} \end{cases}$$

Note that Γ dose not align with the mesh grid. $f \notin X^{10}$, $f \in X^{20}$.

5. Numerical Examples

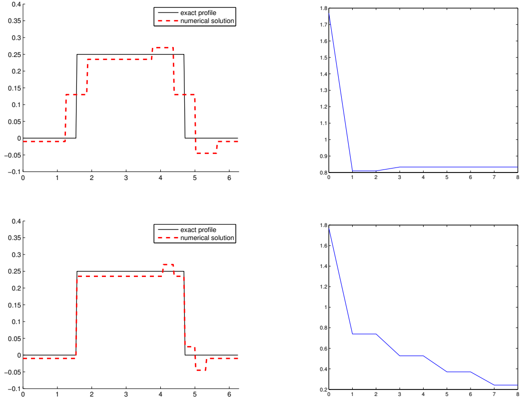


Figure: First row: $M = 10$; second row: $M = 20$.

5. Numerical Examples

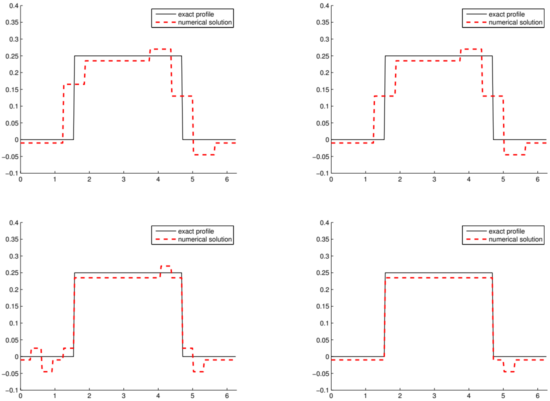


Figure: First row: $M = 10$; second row: $M = 20$. First column: 5% noise; second column: 10% noise.

5. Numerical Examples

Example 4

The refractive index in Ω_- is $n_2 = 2 + 0.1i$.

Use incident waves with one fixed incident angle $\theta = \frac{\pi}{3}$.

The profile Γ is defined by f:

$$f(x) = \begin{cases} 0.28, & 0.5\pi < x < \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Note that Γ aligns with the mesh grid. $f \notin X^{10}$, $f \in X^{20}$.

5. Numerical Examples

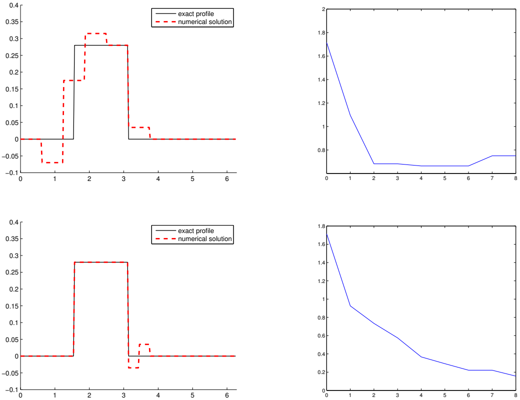


Figure: First row: $M = 10$; second row: $M = 20$.

5. Numerical Examples

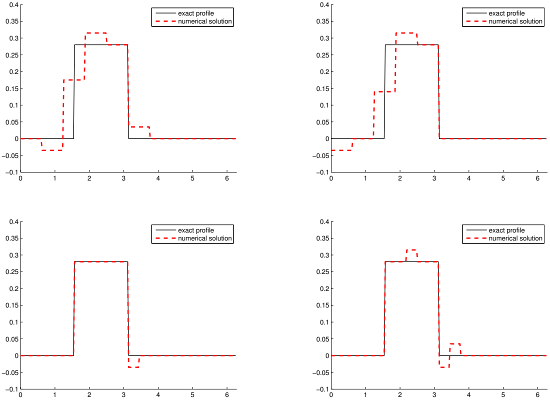


Figure: First row: $M = 10$; second row: $M = 20$. First column: 5% noise; second column: 10% noise.