Karlsruhe Institute of Technology

Characterizing non-scattering inhomogeneities for electromagnetic waves

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Disclaimer

non-scattering media $\longrightarrow$ cloaking, Harry Potter's cloak, ...

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Questions:

- What can we say about non-scattering media?
- How diverse are such media?
- How can we construct them mathematically?
- ... in Born approximation

Not all questions will be answered.

Outline

- the Setting
- Transformation Optics
- Deriving a Characterizing Equation
- Representation Formulae and their Consequences
- Open Questions

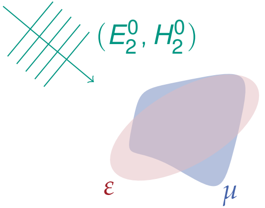
The Setting

$$\begin{aligned} \nabla \times E - \mathrm{i}\, k\, \mu\, H &= 0 \\ \nabla \times H + \mathrm{i}\, k\, \varepsilon\, E &= 0 \end{aligned} \qquad \text{in} \quad \mathbb{R}^3 \tag{1}$$

anisotropic medium:

- $\varepsilon, \mu : \mathbb{R}^3 \to \mathbb{C}^{3\times 3}$ symmetric
- both are smooth, compactly supported perturbations:

$$\operatorname{supp}(I_3 - \varepsilon)\,, \quad \operatorname{supp}(I_3 - \mu) \subseteq B$$



Fields separate into entire and scattered part:

$$E = E_2^0 + E_2^s \qquad H = H_2^0 + H_2^s$$

Question: Can we characterize ε, μ such that far field pattern of (E_2^s, H_2^s) is zero for all (E_2^0, H_2^0)?

Far fields, Scattering Kernel

The scattered fields are supposed to satisfy the Silver-Müller Radiation condition:

$$E_2^s(x) - H_2^s(x) \times \hat{x} = \mathrm{O}\left(\frac{1}{|x|^2}\right), \qquad H_2^s(x) + E_2^s(x) \times \hat{x} = \mathrm{O}\left(\frac{1}{|x|^2}\right)$$

Asymptotic expansion gives far field patterns

$$\begin{aligned} E_2^s(x) &= \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|}\left(I_2^\infty(\hat{x}) + \mathrm{O}\left(\frac{1}{|x|}\right)\right) \\ H_2^s(x) &= \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|}\left(\hat{x} \times I_2^\infty(\hat{x}) + \mathrm{O}\left(\frac{1}{|x|}\right)\right) \end{aligned} \qquad |x| \to \infty$$

Assume incident plane wave with direction $\hat{\theta} \in \mathbb{S}^2$ and amplitude $I_2^0 \in \mathbb{C}^3$

$$E_2^0(x) = I_2^0 \, \mathrm{e}^{\mathrm{i}k\,\hat{\theta}\cdot x}, \qquad H_2^0(x) = \hat{\theta} \times I_2^0 \, \mathrm{e}^{\mathrm{i}k\,\hat{\theta}\cdot x}.$$

Scattering kernel: $\quad I_2^\infty(\hat{x}) = \mathbf{S}(\hat{x}, \hat{\theta})\, I_2^0$

Transformation optics

Consider solution (E, H) of

$$\nabla \times E - \mathrm{i}\, k\, H = 0\,, \qquad \nabla \times H + \mathrm{i}\, k\, E = 0\,. \tag{2}$$

Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ denote a diffeomorphism with $\mathrm{supp}(\mathrm{id} - \Phi)$ compact

transformed fields $\tilde{E}$, $\tilde{H}$:

$$E(x) = \big(\Phi'(x)\big)^\top \tilde{E}(\Phi(x))\,, \qquad H(x) = \big(\Phi'(x)\big)^\top \tilde{H}(\Phi(x))$$

Then

$$\Phi'(x)\, \nabla \times E(x) = \det(\Phi'(x))\, \big(\nabla \times \tilde{E}\big)\, (\Phi(x))$$

Thus

$$\det(\Phi'(x))\, \big(\nabla \times \tilde{E}\big)\, (\Phi(x)) - ik\, \Phi'(x)\, \big(\Phi'(x)\big)^\top \tilde{H}(\Phi(x)) = 0$$

Transformation optics

Consider solution (E, H) of

$$\nabla \times E - \mathrm{i}\, k\, H = 0\,, \qquad \nabla \times H + \mathrm{i}\, k\, E = 0\,. \tag{2}$$

If

$$\tilde{\varepsilon}(y) = \tilde{\mu}(y) = \left. \frac{\Phi'(x)\, \Phi'(x)^\top}{\det \Phi'(x)} \right|_{x=\Phi^{-1}(y)}\,, \qquad y \in \mathbb{R}^3\,.$$

then

$$\nabla \times \tilde{E} - \mathrm{i}\, k\, \tilde{\mu}\, \tilde{H} = 0\,, \qquad \nabla \times \tilde{H} + \mathrm{i}\, k\, \tilde{\varepsilon}\, \tilde{E} = 0\,. \tag{3}$$

Note: $(E, H) = (\tilde{E}, \tilde{H})$ outside $\mathrm{supp}(\mathrm{id} - \Phi)$

$\longrightarrow$ $(\tilde{E}, \tilde{H})$ have zero far field pattern

The Born approximation

Consider family of media:

- ε, μ smoothly depend on some parameter $\tau \in \mathbb{R}$
- $\varepsilon(\tau), \mu(\tau) \longrightarrow I_3 \ (\tau \to 0)$

$$\varepsilon = I_3 + \tau\,\varepsilon_1 + \mathrm{o}(\tau) \qquad \mu = I_3 + \tau\,\mu_1 + \mathrm{o}(\tau) \qquad (\tau \to 0)$$

Born approximation of fields: Linearization of fields with respect to τ

$$E = E_2^0 + E_2^s \approx E_2^0 + \tau\,E_2^B = E_2^0 + \tau\,\frac{\mathrm{d}E_2^s}{\mathrm{d}\tau}\bigg|_{\tau=0}$$

Similarly for H field.

$$\nabla \times E_2^b - \mathrm{i}\,k\,H_2^b = \mathrm{i}k\mu_1\,H_2^0\,, \qquad \nabla \times H_2^b + \mathrm{i}\,k\,E_2^b = -\mathrm{i}k\varepsilon_1\,E_2^0\,.$$

The Born Approximation (2)

Linearize **S** in the same way:

$$\mathbf{S}(\hat{x},\hat{\theta}) = \tau \left.\frac{\mathrm{d}\mathbf{S}(\hat{x},\hat{\theta})}{\mathrm{d}\tau}\right|_{\tau=0} + \mathrm{o}(\tau) = \tau\,\mathbf{B}(\hat{x},\hat{\theta}) + \mathrm{o}(\tau) \qquad (\tau \to 0)$$

No scattering in Born approximation: $\mathbf{B} = 0$

The Example from Transformation Optics

Suppose that there is $w \in C^2(\mathbb{R}^3;\mathbb{R}^3)$, $\operatorname{supp} w \subseteq B$

$$\Phi^{-1}(y) = y - \tau\, w(y)$$

Then

$$\tilde{\varepsilon}(y) = \left.\frac{\Phi'(x)\,\Phi'(x)^\top}{\det \Phi'(x)}\right|_{x=\Phi^{-1}(y)} = \frac{[I_3 - \tau\, w'(y)]^{-1}\,[I_3 - \tau\, w'(y)]^{-\top}}{\det\left([I_3 - \tau\, w'(y)]^{-1}\right)}$$

Geometric series:

$$[I_3 - \tau\, w'(y)]^{-1} = I_3 + \tau\, w'(y) + \mathrm{O}\left(\tau^2\right)$$

Simple calculation

$$\frac{1}{\det\left([I_3 - \tau\, w'(y)]^{-1}\right)} = 1 - \tau\, \operatorname{tr} w'(y) + \mathrm{O}\left(\tau^2\right)$$

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Linearization:

$$\tilde{\varepsilon}(y) = \tilde{\mu}(y) = I_3 + \tau\left(w'(y) + w'(y)^\top - \operatorname{tr} w'(y)\, I_3\right) + \mathrm{O}\left(\tau^2\right)$$

A characterizing equation

Consider plane wave

$$E^0(x) = I^0\, \mathrm{e}^{\mathrm{i}k\,\hat\theta\cdot x}\,, \qquad H^0(x) = \hat\theta \times I^0\, \mathrm{e}^{\mathrm{i}k\,\hat\theta\cdot x}\,.$$

amplitude $I^0 \in \mathbb{C}^3$, direction of incidence $\hat\theta \in \mathbb{S}^2$, $I^0 \cdot \hat\theta = 0$

Setting $\delta\varepsilon = \varepsilon - I_3$, $\delta\mu = \mu - I_3$, from (1) we obtain

$$\int_B \left(H^0 \cdot \nabla \times E - \mathrm{i}\, k\, H^0 \cdot \mu\, H \right) \mathrm{d}x = 0$$
$$\int_B \left(E^0 \cdot \nabla \times H + \mathrm{i}\, k\, E^0 \cdot \varepsilon\, E \right) \mathrm{d}x = 0$$

Integrate by parts and use (2)

$$\frac{1}{\mathrm{i}\,k} \int_{\partial B} (E \times H^0 + H \times E^0) \cdot \nu \,\mathrm{d}s(x) = \int_B \left[E^0 \cdot \delta\varepsilon\, E - H^0 \cdot \delta\mu\, H \right] \mathrm{d}x\,.$$

A characterizing equation (2)

With

$$\int_{\partial B} (E_2^0 \times H^0 + H_2^0 \times E^0) \cdot \nu \, \mathrm{d}s(x) = 0\,,$$

we obtain

$$\frac{1}{\mathrm{i}\,k} \int_{\partial B} (E_2^s \times H^0 + H_2^s \times E^0) \cdot \nu \, \mathrm{d}s(x) \\ = \int_B \left[E^0 \cdot \delta\varepsilon \, (E_2^0 + E_2^s) - H^0 \cdot \delta\mu \, (H_2^0 + H_2^s) \right] \mathrm{d}x$$

A characterizing equation (3)

Use far field asymptotics:

$$E_2^s(x) = \frac{e^{ik|x|}}{|x|}\left(I_2^\infty(\hat{x}) + \mathrm{O}\left(\frac{1}{|x|^2}\right)\right)$$
$$H_2^s(x) = \frac{e^{ik|x|}}{|x|}\left(\hat{x} \times I_2^\infty(\hat{x}) + \mathrm{O}\left(\frac{1}{|x|^2}\right)\right) \qquad |x| \to \infty$$

Replacing B by $B_R(0)$:

$$\lim_{R\to\infty} \int_{\partial B_R(0)} (E_2^s \times H^0 + H_2^s \times E^0) \cdot \nu \, \mathrm{d}s(x) = -\frac{4\pi\,\mathrm{i}}{k} I_2^\infty(-\hat{\theta}) \cdot I^0$$

Thus

$$-\frac{4\pi}{k^2} I_2^\infty(-\hat{\theta}) \cdot I^0 = \int_B \left[E^0 \cdot \delta\varepsilon\,(E_2^0 + E_2^s) - H^0 \cdot \delta\mu\,(H_2^0 + H_2^s)\right] \mathrm{d}x$$

A characterizing equation (4)

$$-\frac{4\pi}{k^2}\, I_2^s(-\hat{\theta}) \cdot I^0 = \int_B \left[E^0 \cdot \delta\varepsilon\, (E_2^0 + E_2^s) - H^0 \cdot \delta\mu\, (H_2^0 + H_2^s) \right] \mathrm{d}x$$

Use plane incident wave

$$E_2^0(x) = I_2^0\, \mathrm{e}^{\mathrm{i}k\,\hat{\psi}\cdot x}\,, \qquad H_2^0(x) = \hat{\psi} \times I_2^0\, \mathrm{e}^{\mathrm{i}k\,\hat{\psi}\cdot x}\,, \qquad I_2^0 \cdot \hat{\psi} = 0$$

and apply Born approximation $\big((\delta\varepsilon = \tau\,\varepsilon_1 + \mathrm{O}(\tau^2)\,, \delta\mu = \tau\,\mu_1 + \mathrm{O}(\tau^2)\big)$:

$$\begin{aligned} -\frac{4\pi}{k^2}\, {I^0}^\top \mathbf{B}(\hat{\psi}, -\hat{\theta})\, I_2^0 \\ &= \int_B \left[E^0 \cdot \varepsilon_1\, E_2^0 - H^0 \cdot \mu_1\, H_2^0 \right] \mathrm{d}x \\ &= \int_B \left[{I^0}^\top \varepsilon_1\, I_2^0\, \mathrm{e}^{\mathrm{i}k(\hat{\theta}+\hat{\psi})\cdot x} - (\hat{\theta} \times I^0)^\top \mu_1\, (\hat{\psi} \times I_2^0)\, \mathrm{e}^{\mathrm{i}k(\hat{\theta}+\hat{\psi})\cdot x} \right] \mathrm{d}x \end{aligned}$$

A characterizing equation (5)

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Define the Fourier transforms

$$\widehat{\delta\varepsilon} = \mathcal{F}[\varepsilon_1]\big|_{\xi=k(\hat\theta+\hat\psi)}\,, \qquad \widehat{\delta\mu} = \mathcal{F}[\mu_1]\big|_{\xi=k(\hat\theta+\hat\psi)}$$

to obtain

$$-\frac{(2\pi)^{1/2}}{k^2}\, {I^0}^\top \mathbf{B}(\hat\psi, -\hat\theta)\, I_2^0 = {I^0}^\top \widehat{\delta\varepsilon}\, I_2^0 - (\hat\theta \times I^0)^\top \widehat{\delta\mu}\, (\hat\psi \times I_2^0)$$

No scattering in Born approximation $\iff$

$$\begin{cases} {I^0}^\top \widehat{\delta\varepsilon}\, I_2^0 - (\hat\theta \times I^0)^\top \widehat{\delta\mu}\, (\hat\psi \times I_2^0) = 0 \\ \quad \text{for all } \hat\theta, \hat\psi \in \mathbb{S}^2, \quad I^0, I_2^0 \in \mathbb{C}^3 \\ \quad \text{with } I^0 \cdot \hat\theta = 0, \quad I_2^0 \cdot \hat\psi = 0 \end{cases}$$

Rewrite the result

$$I^{0\top}\,\widehat{\delta\varepsilon}\,I_2^0 - (\hat{\theta}\times I^0)^\top\widehat{\delta\mu}\,(\hat{\psi}\times I_2^0) = 0$$

and also

$$(\hat{\theta}\times I^0)^\top\widehat{\delta\varepsilon}\,(\hat{\psi}\times I_2^0) - I^{0\top}\,\widehat{\delta\mu}\,I_2^0 = 0$$

Set $\hat{s} = \widehat{\delta\varepsilon} + \widehat{\delta\mu}$ $\qquad \widehat{d} = \widehat{\delta\varepsilon} - \widehat{\delta\mu}$

Then

$$0 = I^{0\top}\,\widehat{d}\,I_2^0 + (\hat{\theta}\times I^0)^\top\widehat{d}(\hat{\psi}\times I_2^0)$$

$$0 = I^{0\top}\,\hat{s}\,I_2^0 - (\hat{\theta}\times I^0)^\top\hat{s}(\hat{\psi}\times I_2^0)$$

A classification theorem

Theorem

For each $\hat{\theta}, \hat{\psi} \in \mathbb{S}^2$*, there exists an orthogonal matrix* M *and numbers* $s_1, s_2, s_3, d \in \mathbb{C}$ *such that*

$$M^\top \hat{s} M = \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & -s_1 & 0 \\ s_3 & 0 & -s_1 \end{pmatrix}$$

and

$$M^\top \hat{d} M = d \left[I + \begin{pmatrix} r_+^2 & 0 & 0 \\ 0 & -r_+^2 & 0 \\ 0 & 0 & -r_+^2 \end{pmatrix} \right],$$

where $r_+ = |(\hat{\theta} + \hat{\psi})|/2$

Proof of the classification theorem

Define $M = (b^1\, b^2\, b^3)$ with

$$b^1 = \frac{\hat{\theta} + \hat{\psi}}{|\hat{\theta} + \hat{\psi}|}, \qquad b^2 = \frac{\hat{\theta} - \hat{\psi}}{|\hat{\theta} - \hat{\psi}|}, \qquad b^3 = b^1 \times b^2$$

Recall

$$0 = {l^0}^\top \hat{s}\, l_2^0 - (\hat{\theta} \times l^0)^\top \hat{s}(\hat{\psi} \times l_2^0)$$

Choosing $l^0 = b^3$, $l_2^0 = \hat{\psi} \times b^3$ gives

$$0 = \cdots = -2\, r_+\, {b^2}^\top \hat{s}\, b^3$$

$$M^\top \hat{s} M = \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & * & 0 \\ s_3 & 0 & * \end{pmatrix}$$

Proof of the classification theorem

Define $M = (b^1\, b^2\, b^3)$ with

$$b^1 = \frac{\hat\theta + \hat\psi}{|\hat\theta + \hat\psi|}, \qquad b^2 = \frac{\hat\theta - \hat\psi}{|\hat\theta - \hat\psi|}, \qquad b^3 = b^1 \times b^2$$

Recall

$$0 = {l^0}^\top \hat{s}\, l_2^0 - (\hat\theta \times l^0)^\top \hat{s}(\hat\psi \times l_2^0)$$

Choosing $l^0 = l_2^0 = b^3$ gives

$$0 = \cdots = {b^3}^\top \hat{s}\, b^3 + r_-^2\, {b^1}^\top \hat{s}\, b^1 - r_+^2\, {b^2}^\top \hat{s}\, b^2$$

where $r_- = |(\hat\theta - \hat\psi)|/2$

But $\hat{s}$ is invariant under rotations around the b^1-axis!

Proof of the classification theorem

Define $M = (b^1\, b^2\, b^3)$ with

$$b^1 = \frac{\hat\theta + \hat\psi}{|\hat\theta + \hat\psi|}, \qquad b^2 = \frac{\hat\theta - \hat\psi}{|\hat\theta - \hat\psi|}, \qquad b^3 = b^1 \times b^2$$

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and

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Proof of the classification theorem

Define $M = (b^1\, b^2\, b^3)$ with

$$b^1 = \frac{\hat{\theta} + \hat{\psi}}{|\hat{\theta} + \hat{\psi}|}, \qquad b^2 = \frac{\hat{\theta} - \hat{\psi}}{|\hat{\theta} - \hat{\psi}|}, \qquad b^3 = b^1 \times b^2$$

Recall

$$0 = l^{0\top} \hat{s}\, l_2^0 - (\hat{\theta} \times l^0)^\top \hat{s}(\hat{\psi} \times l_2^0)$$

Choosing $l^0 = l_2^0 = b^3$ gives

$$b^{1\top} \hat{s}\, b^1 = -b^{2\top} \hat{s}\, b^2 = -b^{3\top} \hat{s}\, b^3$$

$$M^\top \hat{s} M = \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & -s_1 & 0 \\ s_3 & 0 & -s_1 \end{pmatrix}$$

Simpler Representation

Corollary

There is a vector $v \in \mathbb{C}^3$ *and a number* $d \in \mathbb{C}$ *depending only on* $\xi = k\,(\hat\theta + \hat\psi)$ *such that*

$$\hat{s} = v\,\xi^\top + \xi\,v^\top - \mathrm{tr}(\xi\,v^\top)\,I$$

$$\widehat{d} = d\,\left[(1 - r_+^2)\,I + \frac{1}{2\,k^2}\,\xi\,\xi^\top\right]$$

Proof:

$$v = \frac{1}{k|\hat\theta + \hat\psi|}\,M\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

$$M\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = MM^\top \hat{s} M e_1 = \hat{s} b^1$$

A simple consequence

Corollary

$\delta\varepsilon = 0$ *if and only if* $\delta\mu = 0$

Proof: Suppose $\delta\mu = 0$ $\quad\Longrightarrow\quad \widehat{\delta\mu} = 0, \quad \hat{s} = \hat{d}$

From classification theorem:

$$s_1 = d\,(1 + r_+^2) \qquad -s_1 = d\,(1 - r_+^2)\,.$$

Thus $d = 0$

$$\Longrightarrow \qquad \hat{d} = 0 \qquad \Longrightarrow \qquad \widehat{\delta\varepsilon} = \widehat{\delta\mu} = 0$$

Representations in real space (1)

Theorem

Suppose $\mathbf{B} = 0$. *Set*

$$w = \sqrt{\frac{2}{\pi}}\,(\varepsilon_1 + \mu_1) * \left(\nabla \frac{1}{|x|}\right) .$$

Then

$$\varepsilon_1 + \mu_1 = w' + w'^{\top} - \mathrm{tr}(w')\, I_3 .$$

Proof:

$$\mathcal{F}(w) = \sqrt{\frac{2}{\pi}}\,\hat{s}\,\mathcal{F}\left(\nabla \frac{1}{|x|}\right) = -\mathrm{i}\sqrt{\frac{2}{\pi}}\,\hat{s}\,\xi\,\mathcal{F}\left(\frac{1}{|x|}\right) = \mathrm{i}\,\frac{\hat{s}\xi}{|\xi|^2} = \mathrm{i}\,v$$

Representations in real space (1)

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Then

$$\varepsilon_1 + \mu_1 = w' + w'^{\top} - \operatorname{tr}(w')\, I_3 .$$

Remarks:

- $w = \nabla u$ with $\Delta u = 0$ outside $\operatorname{supp} \varepsilon_1 \cup \operatorname{supp} \mu_1$
- $\Delta w = 0$ and $\nabla \cdot w = 0$ outside $\operatorname{supp} \varepsilon_1 \cup \operatorname{supp} \mu_1$
- we may add $\tilde{w}$ to w with $\Delta \tilde{w} = 0$ and $\nabla \cdot \tilde{w} = 0$

Representations in real space (2)

Theorem

Suppose $\mathbf{B} = 0$. *Set*

$$f = \frac{1}{4k^2}\left(\varepsilon_1 - \mu_1\right) * H\left(\frac{1}{|x|} - h_0^{(1)}(2k\,|x|)\right) .$$

Then

$$\varepsilon_1 - \mu_1 = 2H(f) - (\Delta - 4k^2) f\, I_3.$$

Proof similar to previous theorem.

Remarks:

- f is uniquely determined
- f is dependent on frequency

The case $\widehat{d} = 0$

$$2\,\widehat{\delta\varepsilon}(\xi) = \hat{s}(\xi) = v(\xi)\,\xi^\top + \xi\,v(\xi)^\top - \mathrm{tr}(\xi\,v(\xi)^\top)\,I_3$$

Define w as in the representation theorem in real space (divide by 2)
Then

$$\delta\varepsilon = \delta\mu = w' + {w'}^\top - \mathrm{tr}(w')\,I_3$$

We have seen this before!

The case $\widehat{d} = 0$

$$2\,\widehat{\delta\varepsilon}(\xi) = \hat{s}(\xi) = v(\xi)\,\xi^\top + \xi\, v(\xi)^\top - \mathrm{tr}(\xi\, v(\xi)^\top)\, I_3$$

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We have seen this before!

Remarks:

- in general w does not have compact support
- optics transformation: there exist $\tilde{w}$ with $\tilde{w} = w$ outside $\mathrm{supp}\,\varepsilon_1 \cup \mathrm{supp}\,\mu_1$

Open Questions

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$$\varepsilon_1 + \mu_1 = w' + w'^\top - \mathrm{tr}(w')\, I_3$$
$$\varepsilon_1 - \mu_1 = 2H(f) - (\Delta - 4k^2) f\, I_3$$

- Can we interpret f?
  Note that if $w = \nabla u$ then

  $$\varepsilon_1 + \mu_1 = 2H(u) - \Delta u\, I_3$$

  But:
  – f does not have compact support
  – f is frequency dependent
- Can we relate further properties of w, f to ε_1, μ_1 and vice versa?
- Can we characterize w, f further?

Open Questions

$$\varepsilon_1 + \mu_1 = w' + w'^\top - \mathrm{tr}(w')\, I_3$$
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The End (for now)