Transmission Eigenvalues for Spherically Stratified Media

David Colton

Department of Mathematical Sciences, University of Delaware

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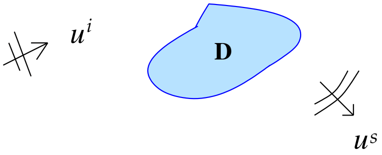
Outline

Scattering by a Spherically Stratified Medium



$$\Delta_3 u + k^2 n(r) u = 0 \qquad \text{in } \mathbb{R}^3$$
$$u = u^i + u^s$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

where $u^i = e^{ikx\cdot d}$, $|d| = 1$, $k > 0$ is the wave number, $r = |x|$ and $n(r)$ is such that $n(r) > 0$ and $n(r) = 1$ for $r \geq a$.

Scattering by a Spherically Stratified Medium

It can be shown that

$$u^s(x) = \frac{e^{ikr}}{r} u_\infty(\hat{x}, d, k) + O\left(\frac{1}{r^2}\right)$$

where $\hat{x} = x/|x|$ and $\hat{x}, d \in S^2 := \{x : \, |x| = 1\}$.

The function u_∞ is called the far field pattern of the scattered field u^s.

It is observed that there exist values of $k > 0$ such that

$$\int_{S^2} u_\infty(\hat{x}, d, k) ds(\hat{x}) = 0.$$

Such values of k are called transmission eigenvalues.

How can we analytically characterize transmission eigenvalues?

Scattering by a Spherically Stratified Medium

Let $y(r)$ be the unique solution of the ODE

$$y'' + k^2 n(r) y = 0$$
$$y(0) = 0, \quad y'(0) = 1$$

Then separation of variables and elementary calculations show that k is a transmission eigenvalue if and only if $y(r)$ and the solution of

$$y_0'' + k^2 y_0 = 0$$
$$y_0(0) = 0, \quad y_0'(0) = 1$$

are such that $y(a) = y_0(a)$, $y'(a) = y_0'(a)$ i.e.

$$d(k) := Det \begin{vmatrix} y(a) & -\dfrac{\sin ka}{k} \\ y'(a) & -\cos ka \end{vmatrix} = 0.$$

Scattering by a Spherically Stratified Medium

$d(k)$ is an entire function of k that is real for real k and is bounded on the real axis. Hence if $d(k)$ is not a constant then there exist a countably infinite set of transmission eigenvalues.

Theorem (Aktosun-Gintides-Papanicolaou)

If $d(k) \equiv 0$ *then* $n(r) \equiv 1$.

We now assume that $n(r)$ is not identically one. If $n(a) = 1$ and $n'(a) = 0$ an asymptotic analysis shows that

$$d(k) = \frac{1}{k[n(0)]^{1/4}} \left\{ \sin k \left(a - \int_0^a \sqrt{n(\rho)} d\rho \right) + O\left(\frac{1}{k}\right) \right\}, \qquad k \to \infty$$

and hence if $0 < n(r) < 1$ or $n(r) > 1$ there exist an infinite number of positive transmission eigenvalues. It can be shown that this is also true if $n(a) \neq 1$ and $n'(a) \neq 0$.

Complex Transmission Eigenvalues

Theorem (Laguerre)

Let $f(z)$ be an entire function of order less than two that is real for real z and has only real zeros. Then the zeros of $f'(z)$ are also all real and are separated from each other by zeros of $f(z)$.

Examples: Laguerre's theorem is not true in general for entire functions of order two. For example, if

$$f(z) = ze^{z^2}$$

then $f'(z) = (2z^2 + 1)e^{z^2}$ and the zeros of $f'(z)$ are complex. On the other hand if

$$f(z) = (z^2 - 4)e^{z^2/3}$$

then $f'(z) = \frac{2}{3}z(z^2 - 1)e^{z^2/3}$ and the zeros of $f'(z)$ are real but not separated by those of $f(z)$.

Complex Transmission Eigenvalues

Theorem (Colton-Leung)

Let $n(r) = n_0^2$ where n_0 is a positive constant not equal to one. Then if n_0 is an integer or the reciprocal of an integer all the transmission eigenvalues are real. If n_0 is not an integer or the reciprocal of an integer then there are infinitely many real and infinitely many complex transmission eigenvalues.

Proof (for $n_0 > 1$ an integer): If $n(r) = n_0^2$ then

$$y(r) = \frac{1}{kn_0}\sin(kn_0 r).$$

If $n_0 > 1$ is an integer, the nonzero roots of $d(k) = 0$ are the critical points of the entire function

$$\frac{\sin(n_0 ka)}{\sin(ka)}.$$

Hence, by Laguerre's theorem, all zeros of $d(k)$ are real.

Complex Transmission Eigenvalues

Example (Aktosun-Gintides-Papanicolaou)

Let $n(r) = n_0^2$. When $n_0 = 1/2$ we have that

$$d(k) = -\frac{2}{k}\sin^3\left(\frac{ka}{2}\right)$$

and hence $d(k)$ has an infinite set of real zeros and no complex zeros. When $n_0 = 2/3$ we have that

$$d(k) = -\frac{1}{k}\sin^3\left(\frac{ka}{2}\right)\left[3 + 2\cos\left(\frac{2ka}{3}\right)\right]$$

and hence $d(k)$ has an infinite set of real and complex zeros.

Question: If complex transmission eigenvalues exist, where do they lie in the complex plane?

Complex Transmission Eigenvalues

Let $n \in C^2[0, a]$ and let

$$\delta := \int_0^a \sqrt{n(\rho)} d\rho, \qquad \delta \neq a$$

$$A := \frac{1 + \sqrt{n(a)}}{1 - \sqrt{n(a)}}, \quad n(a) \neq 1.$$

Theorem (Colton-Leung)

Let $\delta = \ell / m$, where ℓ and m are integers, $\ell > m$ and either $|A| > 1 + \delta$ or $|A| > (\delta + 1)/(\delta - 1)$. Then there exist infinitely many real and infinitely many complex transmission eigenvalues.

Theorem (Colton-Leung)

Assume that $n(a) \neq 1$. Then, if complex transmission eigenvalues exist, they all lie in a strip parallel to the real axis.

Complex Transmission Eigenvalues

Now assume that $n(a) = 1$ and $n'(a) = 0$.

1. $d(k)$ is an even entire function of k of order (at most) one.
2. If $0 < n(r) < 1$ then $d(k)$ has a zero of order two at the origin.

Thus, by the Hadamard factorization theorem, we have that

$$d(k) = c\, k^2 \prod_{j=1}^{\infty} \left(1 - k^2/k_j^2\right)$$

where $\{k_j\}$ are the zeros of $d(k)$ (including multiplicities) and c is a constant. From

$$d(k) = \frac{1}{k[n(0)]^{1/4}} \left\{ \sin\, k \left(a - \int_0^a \sqrt{n(\rho)}\, d\rho \right) + O\left(\frac{1}{k}\right) \right\}$$

as $k \to \infty$ along the positive real axis we have that if $n(0)$ is known then so is c. Hence, under the above assumptions, the transmission eigenvalues (real and complex!) determine $d(k)$.

The Inverse Spectral Problem

As we have just seen, under appropriate assumptions the transmission eigenvalues determine $d(k)$. In order to determine $n(r)$ from $d(k)$ we need an integral representation of the solution to

$$y'' + k^2 n(r) y = 0$$
$$y(0) = 0, \quad y'(0) = 1.$$

Using the Liouville transformation

$$\xi := \int_0^r \sqrt{n(\rho)}\, d\rho$$
$$z(\xi) := [n(r)]^{1/4} y(r)$$

We arrive at

$$z'' + [k^2 - p(\xi)] z = 0$$
$$z(0) = 0, \quad z'(0) = [n(0)]^{-1/4}$$

where

$$p(\xi) := \frac{n''(r)}{4[n(r)]^2} - \frac{5}{16} \frac{[n'(r)]^2}{[n(r)]^3}.$$

The Inverse Spectral Problem

The solution of

$$z'' + [k^2 - p(\xi)]z = 0$$
$$z(0) = 0, \quad z'(0) = [n(0)]^{-1/4}$$

can be represented in the form

$$z(\xi) = \frac{1}{[n(0)]^{1/4}} \left[\frac{\sin k\xi}{k} + \int_0^\xi K(\xi, t) \frac{\sin kt}{k} \, dt \right]$$

for $0 \leq \xi \leq \delta$ where

$$\delta = \int_0^a \sqrt{n(\rho)} \, d\rho$$

and $K(\xi, t)$ is the unique solution of the Goursat problem

$$K_{\xi\xi} - K_{tt} - p(\xi)K = 0, \qquad 0 < t < \xi < \delta$$
$$K(\xi, 0) = 0, \quad 0 \leq \xi \leq \delta$$
$$K(\xi, \xi) = \frac{1}{2} \int_0^\xi p(s) \, ds, \;\; 0 \leq \xi \leq \delta$$

The Inverse Spectral Problem

Theorem (Rundell-Sacks)

Let $K(\xi, t)$ satisfy the above Goursat problem. Then $p \in C^1[0, \delta]$ is uniquely determined by the Cauchy data $K(\delta, t)$, $K_\xi(\delta, t)$.

Now recall the determinant

$$d(k) := Det \begin{vmatrix} y(a) & -\dfrac{\sin ka}{k} \\ y'(a) & -\cos ka \end{vmatrix} = 0.$$

From the Liouville transformation and the representation for $z(\xi)$ we have that

$$y(a) = \frac{1}{[n(0)]^{1/4}} \left[\frac{\sin\, k\delta}{k} + \int_0^\delta K(\delta, t) \frac{\sin\, kt}{k}\, dt \right]$$

$$y'(a) = \frac{1}{[n(0)]^{1/4}} \left[\cos\, k\delta + \frac{\sin\, k\delta}{2k} \int_0^\delta p(s) ds + \int_0^\delta K_\xi(\delta, t) \frac{\sin\, kt}{k}\, dt \right]$$

Note that the asymptotic formulas for $d(k)$ gives us δ. The above formula now gives us

$$\frac{\ell\pi}{a} d\left(\frac{\ell\pi}{a}\right) = \frac{(-1)^{\ell+1}}{[n(0)]^{1/4}} \left[\sin\frac{\ell\pi\delta}{a} + \int_0^\delta K(\delta,t)\sin\frac{\ell\pi t}{a}\,dt\right] \qquad (1)$$

and

$$\begin{aligned}\frac{\ell\pi}{a} d\left(\frac{\ell\pi}{\delta}\right) &= -y(a)\frac{\ell\pi}{\delta}\cos\frac{\ell\pi a}{\delta} \\ &+ \frac{\sin\frac{\ell\pi a}{\delta}}{[n(0)]^{1/4}}\left[(-1)^\ell + \frac{\delta}{\ell\pi}\int_0^\delta K_\xi(\delta,t)\sin\frac{\ell\pi t}{\delta}\,dt\right] \qquad (2)\end{aligned}$$

The Inverse Spectral Problem

- Since $\left\{\sin \frac{\ell \pi t}{a}\right\}$ is complete in $L^2[0, \delta]$ if $\delta \leq a$ we have from (1) and the assumption that $n(0)$ is known that $K(\delta, t)$ (and hence $y(a)$) is known.
- From (2) and the completeness of $\sin \frac{\ell \pi t}{\delta}$ in $L^2[0, \delta]$ we have that $K_\xi(\delta, t)$ is known.

The Rundell-Sacks Theorem now implies that $p(\xi)$ is uniquely determined for $0 \leq \xi \leq \delta$ from a knowledge of $d(k)$.
From this we can now easily determine $n(r)$.

The Inverse Spectral Problem

Theorem (Colton-Leung)

Assume that $n \in C^3[0, a]$, $n(a) = 1$, $n'(a) = 0$ and $n(0)$ is given. Then if $0 < n(r) < 1$ for $0 < r < a$ the transmission eigenvalues (including multiplicity) uniquely determine $n(r)$.

The only extension of the above theorem to the case of more general domains D is the following :

Theorem (Cakoni-Colton-Gintides)

Let D be a bounded simply connected domain with piecewise C^1 boundary and corresponding constant index of refraction n.
Then n is uniquely determined from a knowledge of the smallest positive transmission eigenvalue provided it is known a priori that either $n > 1$ or $0 < n < 1$.

Complex Transmission Eigenvalues Again

The previous result on the inverse spectral problem requires that $n(a) = 1$ and $n'(a) = 0$. However, our previous results on the existence of complex transmission eigenvalues require that $n(a) \neq 1$ (Recall that if $0 < n(r) < 1$ or $n(r) > 1$ real transmission eigenvalues always exist). We now examine the existence of complex transmission eigenvalues when $n(a) = 1$ and $n'(a) = 0$.

Complex Transmission Eigenvalues Again

Definition

Let $M(r)$ denote the maximum modulus of the entire function $f(z)$ on $|z| = r$. Then $f(z)$ is of order ρ if

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \rho.$$

Definition

The entire function $f(z)$ of order $\rho = 1$ is called a function of exponential type τ if

$$\limsup_{r\to\infty} \frac{\log M(r)}{r} = \tau.$$

Complex Transmission Eigenvalues Again

Let $n_+(r)$ denote the number of zeros of an entire function $f(z)$ in the right half plane with $|z| < r$.

Theorem (Cartwright-Levinson)

Let the entire function $f(z)$ of exponential type be such that

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1 + x^2}\, dx < \infty$$

and suppose that

$$\limsup_{y \to \pm\infty} \frac{\log^+ |f(iy)|}{|y|} = \tau.$$

Then

$$\lim_{r \to \infty} \frac{n_+(r)}{r} = \frac{\tau}{\pi}.$$

Complex Transmission Eigenvalues Again

Definition

The number τ/π is called the density of zeros in the right half plane.

We now again consider

$$y'' + k^2 n(r) y = 0, \qquad y(0) = 0, \quad y'(0) = 1$$

and use the Liouville transformation

$$\xi := \int_0^r \sqrt{n(\rho)}\, d\rho, \qquad z(\xi) := [n(r)]^{1/4} y(r).$$

As previously, we have the representation

$$z(\xi) = \frac{1}{[n(0)]^{1/4}} \left[\frac{\sin k\xi}{k} + \int_0^\xi K(\xi, t) \frac{\sin kt}{k}\, dt \right]$$

and again define

$$d(k) := Det \begin{vmatrix} y(a) & -\dfrac{\sin ka}{k} \\ y'(a) & -\cos ka \end{vmatrix}$$

Complex Transmission Eigenvalues Again

Integrating by parts in the expression for $z(\xi)$ now yields

$$d(k) = \frac{-1}{k n(0)^{1/4} n(a)^{1/4}} \left[\sin((\delta - a)k) - \frac{K(\delta,\delta)}{k} \cos((\delta - a)k) \right.$$
$$\left. + \frac{K_\tau(\delta,\delta) - K_\xi(\delta,\delta)}{2k^2} \sin((\delta - a)k) + \frac{n''(a)}{8k^2} \sin((\delta + a)k) + O\left(\frac{1}{k^3}\right) \right]$$

where again

$$\delta := \int_0^a \sqrt{n(\rho)}\, d\rho.$$

In particular, $d(k)$ is of type $(\delta + a)$ and the leading term $\sin((\delta - a)k)$ generates an infinite set of positive real zeros with density $|\delta - a|/\pi$. However, if $n''(a) \neq 0$, from the Cartwright-Levinson theorem the density of all zeros in the right half plane is $(\delta + a)/\pi$.

Complex Transmission Eigenvalues Again

Theorem (Colton-Leung-Meng)

Suppose that $n \in C^2[0, a]$ with $n(a) = 1$ and $n'(a) = 0$ and $\delta \neq 1$. Then, under the extra assumption that $n''(a) \neq 0$, there exist infinitely many real and infinitely many complex transmission eigenvalues.

An application of Rouche's theorem implies the following theorem:

Theorem (Colton-Leung-Meng)

Suppose that $n \in C^2[0, a]$ with $n(a) = 1$ and $n'(a) = 0$ and $\delta \neq 1$. Then if either $n'(a)$ or $n''(a)$ is non-zero, the zeros of $d(k)$ do not lie inside any fixed strip parallel to the real axis.

References

- J. McLaughlin and P.L. Polyakov, On the uniqueness of a spherically symmetric speed of sound from transmission eigenvalues, *Jour. Differential Equations*, **107** (1994), 351-382.
- T. Aktosun, D. Gintides and V.G. Papanicolaou, The uniqueness in the inverse problem for transmission eigenvalues for the spherically symmetric variable-speed wave equation, *Inverse Problems*, **27** (2011).
- T. Aktosun and V.G. Papanicolaou, Reconstruction of the wave speed from transmission eigenvalues for spherically symmetric variable-speed wave equation, *Inverse Problems*, **29** (2013).

References

- Y.J. LEUNG AND D. COLTON, Complex transmission eigenvalues for spherically stratified media, *Inverse Problems*, **28** (2012).
- D. COLTON AND Y.J. LEUNG, Complex eigenvalues and the inverse spectral problem for transmission eigenvalues, *Inverse Problems*, **29** (2013).
- D. COLTON, Y.J. LEUNG AND S. MENG, Distribution of complex transmission eigenvalues for spherically stratified media, *Inverse Problems*, **31** (2015).
- G. WEI AND H. XU, Inverse spectral analysis for the transmission eigenvalue problem, *Inverse Problems*, **29** (2013).