Estimating the refraction index from the knowledge of interior transmission eigenvalues

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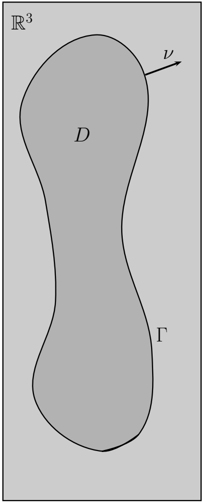
General setup

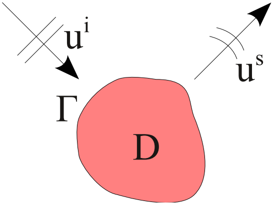
- D bounded open region in $\mathbb{R}^3$.
- Boundary Γ consists of a finite number of disjoint, closed, bounded surfaces belonging to class C^2.
- Complement $\mathbb{R}^3 \backslash \overline{D}$ is connected.
- κ given wave number.
- ν denotes normal pointing in the exterior.
- $n > 1$ is the index of refraction.



Scattering by an inhomogeneous media

- Solve:

$$
\begin{aligned}
\Delta u + \kappa^2 \ u &= 0 && \text{in } \mathbb{R}^3 \backslash \overline{D}\,, \\
\Delta u + \kappa^2 n u &= 0 && \text{in } D\,, \\
u^+ &= u^- && \text{on } \Gamma\,, \\
(\partial_\nu u)^+ &= (\partial_\nu u)^- && \text{on } \Gamma\,, \\
\lim_{r\to\infty} r\left(\partial_r u^s - \mathrm{i}\kappa u^s\right) &= 0\,, && r = |x|\,.
\end{aligned}
$$



- Total wave is $u = u^s + u^i$ with planar incident wave $u^i = \exp\{\mathrm{i}\kappa x \cdot d\}$.
- Question: Is there an incident wave that does not scatter?

Transmission eigenvalue problem — What?

- Question is related to the interior transmission problem (ITP).
- If u^i is given such that $u^s = 0$, then setting $w = u|_D$ and $v = u^i|_D$ yields the following problem:
- Find a solution $(v, w) \neq (0, 0)$ to the ITP given by

$$
\begin{aligned}
\Delta w + \kappa^2 n w &= 0 \quad \text{in } D\,,\\
\Delta v + \kappa^2 \;\; v &= 0 \quad \text{in } D\,,\\
v &= w \quad \text{on } \Gamma\,,\\
\partial_\nu v &= \partial_\nu w \text{ on } \Gamma\,.
\end{aligned}
$$

- Then $\kappa \in \mathbb{C}$ will be a transmission eigenvalue (TE).

Transmission eigenvalue problem — Who?

- ITP appears first in Kirsch in 1986 and Colton & Monk in 1988.
- Factorization Method and Linear Sampling Method do not work for TE, since the far-field operator is not injective with dense range.
- Both method are inverse solvers.

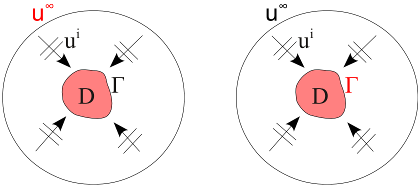


Figure 1: Left: Direct problem. Right: Inverse problem.

- Far-field: $u^s(x) = \frac{e^{i\kappa r}}{r} u^\infty(\hat{x}, d, \kappa) + \mathcal{O}\left(\frac{1}{r^2}\right)$, $r \to \infty$. Far-field operator $F_\kappa : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$: $(F_\kappa g)(\hat{x}) = \int_{\mathbb{S}^2} u^\infty(\hat{x}, d, \kappa) g(d)\, \mathrm{d}s(d)$.

Transmission eigenvalue problem Who?

- Discreteness of TE:
  Colton-Kirsch-Päivärinta in 1989, Rynne-Sleeman in 1991, Cakoni-Haddar and Colton-Päivärinta-Sylvester in 2007, Kirsch, Cakoni-Haddar in 2009 and Hickmann in 2012.
- Existence of TE:
  Päivärinta-Sylvester and Kirsch in 2009, Cakoni-Gintides-Haddar, Cakoni-Haddar and Cakoni-Kirsch in 2011, and Bellis-Cakoni-Guzina and Cossonnière in 2011.
- And many more.

Transmission eigenvalue problem — Why?

- TE carry information of material properties
  - ⇒ Can be used to say something about the presence of abnormalities inside an homogeneous media.
  - ⇒ Can be used to test the integrity of a material (nondestructive testing).
- Many open questions.
  - ⇒ Can Faber-Krahn type inequalities be established for higher TE?
  - ⇒ **What does the first TE tell us about the inhomogeneous media $n(x)$?**

Estimating refraction index — How?

- Method 1: Use Faber-Krahn-type inequality (FKTI)

$$n > \frac{\lambda_1(D)}{\kappa_1^2(D)}\,,$$

  where $\lambda_1(D)$ is the first Dirichlet eigenvalue.
- Very coarse estimation.
- Example unit sphere: $\lambda_1(D) = 3.141593^2$ and $\kappa_1^2(D) = 3.141593^2$ for $n = 4$.
- We get $\hat{n}_{\mathrm{FKTI}} > 1$.

Estimating refraction index How?

- Method 2: Bisection method[1]
- Need computation of 10–20 “direct problems” of the form

$$\mu_D(n) = \kappa_1(D)\,.$$

- Method 3 (new): Combine boundary integral equation method and complex-valued contour integrals.[2]
- Computational cost: 1–3 “direct problems”.

[1] J. Sun, Estimation of transmission eigenvalues and the index of refraction from Cauchy data, Inverse Problems 27 (2011) 015009 (11pp).

[2] A. Kleefeld, A numerical method to compute interior transmission eigenvalues, Inverse Problems, 29 (2013), 104012 (20pp).

Solving the ITP

How?

Integral operators:

$$
\begin{aligned}
\mathrm{SL}_\kappa(\varphi)(P) &= \int_\Gamma \Phi_\kappa(P,q)\varphi(q)\,\mathrm{d}s(q)\,, & P\in D\,,\\
\mathrm{DL}_\kappa(\varphi)(P) &= \int_\Gamma \partial_{\nu(q)}\Phi_\kappa(P,q)\varphi(q)\,\mathrm{d}s(q)\,, & P\in D\,,\\
\\
\mathrm{S}_\kappa(\varphi)(p) &= \int_\Gamma \Phi_\kappa(p,q)\varphi(q)\,\mathrm{d}s(q)\,, & p\in\Gamma\,,\\
\mathrm{K}_\kappa(\varphi)(p) &= \int_\Gamma \partial_{\nu(q)}\Phi_\kappa(p,q)\varphi(q)\,\mathrm{d}s(q)\,, & p\in\Gamma\,,\\
\mathrm{K}'_\kappa(\varphi)(p) &= \int_\Gamma \partial_{\nu(p)}\Phi_\kappa(p,q)\varphi(q)\,\mathrm{d}s(q)\,, & p\in\Gamma\,,\\
\mathrm{T}_\kappa(\varphi)(p) &= \partial_{\nu(p)}\int_\Gamma \partial_{\nu(q)}\Phi_\kappa(p,q)\varphi(q)\,\mathrm{d}s(q)\,, & p\in\Gamma\,,
\end{aligned}
$$

and $\Phi_\kappa(p,q)=\mathrm{e}^{\mathrm{i}\kappa r}/4\pi r$ with $r=|p-q|$ and $p\neq q$.

Solving the ITP

How?

- Use boundary integral equations (see Cossonnière & Haddar).
- We have using Green's representation theorem

$$\begin{aligned} v &= \mathrm{SL}_{\kappa} a - \mathrm{DL}_{\kappa} b\,, && \text{in } D\,, \\ w &= \mathrm{SL}_{\kappa\sqrt{n}} a - \mathrm{DL}_{\kappa\sqrt{n}} b\,, && \text{in } D\,, \end{aligned}$$

  with $a = \partial_\nu v|_\Gamma = \partial_\nu w|_\Gamma$ and $b = v|_\Gamma = w|_\Gamma$.
- On the boundary holds

$$0 = w - v = \mathrm{S}_{\kappa\sqrt{n}} a - \mathrm{S}_{\kappa} a - \mathrm{K}_{\kappa\sqrt{n}} b + \mathrm{K}_{\kappa} b\,.$$

- Additionally,

$$0 = \partial_\nu w - \partial_\nu v = \mathrm{K}'_{\kappa\sqrt{n}} a - \mathrm{K}'_{\kappa} a - \mathrm{T}_{\kappa\sqrt{n}} b + \mathrm{T}_{\kappa} b\,.$$

Solving the ITP

How?

- Leads to

$$\mathrm{M}(\kappa)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ on } \Gamma$$

with

$$\mathrm{M}(\kappa) = \begin{pmatrix} \mathrm{S}_{\kappa\sqrt{n}} - \mathrm{S}_\kappa & -\mathrm{K}_{\kappa\sqrt{n}} + \mathrm{K}_\kappa \\ \mathrm{K}'_{\kappa\sqrt{n}} - \mathrm{K}'_\kappa & -\mathrm{T}_{\kappa\sqrt{n}} + \mathrm{T}_\kappa \end{pmatrix}.$$

- $\mathrm{M}(\kappa) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is Fredholm of index zero and analytic on $\mathbb{C}\backslash\mathbb{R}^-$ and $\mathrm{M}(\mathrm{i}\kappa)$ is coercive for real κ.

Solving the ITP

How?

- Discretize $\mathrm{M}(\kappa)X = 0$.
- Compute eigenvalues of $\mathbf{M}(\kappa) \in \mathbb{C}^{m\times m}$ and look for κ for which the smallest eigenvalue is close to zero.
- Problem: Eigenvalues cluster around zero.
- Workaround: Solve generalized eigenvalue problem

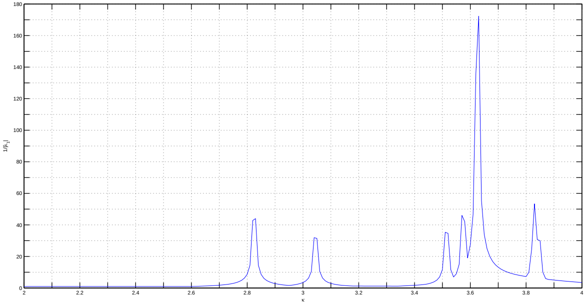
$$\mathbf{M}(\kappa)X = \lambda\mathbf{M}(\mathrm{i}\kappa)X\,.$$

- Has to be calculated for a lot of wave numbers, say $N = 200$ (at least).
- Generalized eigenvalue has to be solved for large m.

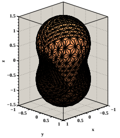
$\Rightarrow$ Method is expensive if one wants many highly accurate TE values.

Solving the ITP

How?



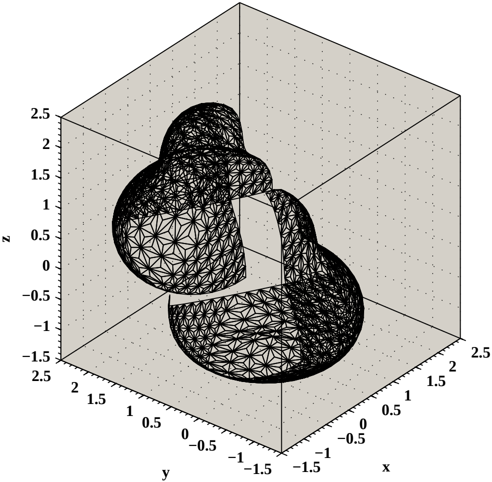
(a) Cossonnière's method for the peanut-shaped obstacle $\mathscr{P}$.



(b) Peanut $\mathscr{P}$.

Discretizing the integral equation

- Triangulation of Γ is $\mathscr{T}_n = \{\Delta_1, \ldots, \Delta_n\}$.
- Right picture shows triangulation of an acorn.



Discretizing the integral equation

- Integral equation

$$\int_\Gamma K_\kappa(P,Q)u(Q)\,\mathrm{d}s(Q) = f(P)\,, \qquad P \in \Gamma$$

can be written as

$$\sum_{k=1}^{n} \int_{\Delta_k} K_\kappa(P,Q)u(Q)\,\mathrm{d}s(Q) = f(P)\,.$$

- Using the map $m_k : \sigma \to \Delta_k$, we have

$$\sum_{k=1}^{n} \int_{\sigma} K_\kappa(P,m_k(s,t))u(m_k(s,t))\,|(\partial_s m_k \times \partial_t m_k)(s,t)|\,\mathrm{d}\sigma = f(P)\,.$$

Discretizing the integral equation

- With the approximation of $u(m_k(s,t))$ via constant interpolation

$$u(m_k(s,t)) \approx u(m_k(1/3,1/3))\cdot 1$$

and the requirement that we have equality at the nodes $P = m_i(1/3,1/3) \equiv v_i$, $i = 1,\ldots,n$, leads to solving the following linear system of equations:

$$\sum_{k=1}^{n} \int_{\sigma} K_\kappa(v_i, m_k(s,t))\,|(\partial_s m_k \times \partial_t m_k)(s,t)|\,\mathrm{d}\sigma u(v_k) = f(v_i)\,.$$

- Abstractly, leads to

$$M(\kappa)\vec{u} = \vec{f}\,.$$

- Entries of the matrix are 2D integrals that have to be approximated.

Solving the ITP

How?

- Consider the nonlinear eigenvalue problem of the form

$$\mathbf{M}(\kappa)v=0,\quad v\in\mathbb{C}^m,\quad v\neq 0,\quad \kappa\in\Omega\subset\mathbb{C}\,.$$

- Assume large scale problem $k \ll m$ (k is number of eigenvalues including multiplicities).
- Problem can be reduced to eigenvalue problem of dimension k (Keldysh's theorem).
- One has to use complex-valued contour integrals.
- See Beyn 2011.

Solving the ITP

How?

- $\mathscr{C} \subset \Omega$, compact and $\lambda_n \in \mathscr{C}$, $n = 1, \ldots, k$.
- Keldysh Theorem: There exist a neighborhood $\mathscr{U} \subset \Omega$ of $\mathscr{C}$ and a holomorphic function $\mathbf{R} : \mathscr{U} \to \mathbb{C}^{m\times m}$ such that

$$\mathbf{M}(z)^{-1} = \sum_{n=1}^{k} \frac{1}{z-\lambda_n} v_n w_n^{\mathrm{H}} + \mathbf{R}(z)$$

  for $z \in \mathscr{U} \setminus \{\lambda_1, \ldots, \lambda_k\}$.
- Contour $\Gamma \subset \Omega$ with $\Gamma \cap \sigma(\mathbf{M}) = \emptyset$, $f : \Omega \to \mathbb{C}$ holomorphic function, k is number of eigenvalues in the interior of Γ, then

$$\frac{1}{2\pi\mathrm{i}} \int_\Gamma f(z)\mathbf{M}(z)^{-1}\,\mathrm{d}z = \sum_{n=1}^{k} f(\lambda_n) v_n w_n^{\mathrm{H}}.$$

Solving the ITP — How?

- Choose a contour $\partial\Omega$ in $\mathbb{C}$ which might contain eigenvalues.
- For example an ellipse $\psi(t)=\mu+a\cos(t)+b\sin(t)\mathrm{i}$. Then $\psi'(t)=-a\sin(t)+b\cos(t)\mathrm{i}$
- Algorithm:

1. Choose an index $l\leq m$ and $\hat{V}\in\mathbb{C}^{m\times l}$ randomly.
2. Evaluate the contour integrals

$$A=\frac{1}{2\pi\mathrm{i}}\int_{\partial\Omega}\mathbf{M}(z)^{-1}\hat{V}\,\mathrm{d}z \quad \text{and} \quad B=\frac{1}{2\pi\mathrm{i}}\int_{\partial\Omega}z\mathbf{M}(z)^{-1}\hat{V}\,\mathrm{d}z$$

numerically with the trapezoidal rule. With $t_j=\frac{2\pi j}{N}$, $j=0,\dots,N$ ($\psi(t_0)=\psi(T_N)$) we have

$$A_N=\frac{1}{\mathrm{i}N}\sum_{j=0}^{N-1}\mathbf{M}(\psi(t_j))^{-1}\hat{V}\psi'(t_j)\,,\quad B_N=\frac{1}{\mathrm{i}N}\sum_{j=0}^{N-1}\mathbf{M}(\psi(t_j))^{-1}\hat{V}\psi(t_j)\psi'(t_j)\,.$$

- $N=50$ is more than enough.

Solving the ITP

How?

3. Compute SVD $A_N = V\Sigma W^{\mathrm{H}}$.
4. Perform a rank test for Σ, find $0 < k \le l$ such that

$$\sigma_1 \ge \ldots \ge \sigma_k > \mathrm{tol}_{\mathrm{rank}} > \sigma_{k+1} \approx 0 \approx \sigma_l \approx 0\,.$$

   If $k = l$, then increase l and go to step 1. Otherwise let $V_0 = V(1:m, 1:k)$, $W_0 = W(1:l, 1:k)$, and $\Sigma_0 = \mathrm{diag}(\sigma_1, \ldots, \sigma_k)$.
5. Compute $C = V_0^{\mathrm{H}} B_N W_0 \Sigma_0^{-1} \in \mathbb{C}^{k\times k}$.
6. Solve the eigenvalue problem for C.

Estimating refraction index

How?

- Assume that κ_1^2 is an interior transmission eigenvalue.
- Consider $\mathrm{M}(n)X = 0$ instead of $\mathrm{M}(\kappa)X = 0$.
- That is, replace κ with κ_1 (n is the unknown).
- Solve the nonlinear eigenvalue problem as before.
- For simplicity replace $\sqrt{n}$ with $\tilde{n}$.
- Problem: Result might not be unique.
- Remedy: Use more interior transmission eigenvalues.

Numerical results — Sphere

- We choose $n = 4$.
- Unit sphere $\mathbb{S}^2$ (centered at zero) has eigenvalues (EV)

$$\det\begin{pmatrix} j_m(\kappa) & -j_m(2\kappa) \\ j_m'(\kappa) & -2j_m'(2\kappa) \end{pmatrix} = 0$$

- Thus,

$$\kappa_{1,\mathbb{S}^2,4} \approx 3.14159[3]$$
$$\kappa_{2,\mathbb{S}^2,4} \approx 3.69245[5]$$
$$\kappa_{3,\mathbb{S}^2,4} \approx 4.26168[7]$$
$$\kappa_{4,\mathbb{S}^2,4} \approx 4.83186[9]$$

Numerical results — Sphere

- Using a circle as contour with center $(1.85, 0)$ and radius $1/2$ gives

$$1.999\,986[3]\,,\qquad 2.000\,000[1]\,,\qquad 2.306\,060[2]\,,\qquad 2.306\,081[3]\,.$$

- Result is not unique. Using second eigenvalue yields

$$1.801\,241[3]\,,\qquad 1.937\,814[1]\,,\qquad 1.999\,893[2]\,,\qquad 1.999\,910[3]\,,\qquad 2.278\,325[3]\,,\qquad 2.279\,122[3]\,,\qquad 2.279\,518[1]\,.$$

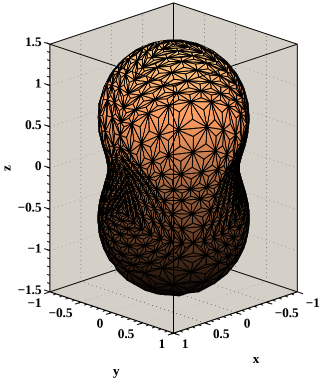
- "Unique" element is $\tilde{n} = 2.00 \rightarrow n = 4.00$.

Numerical results — Sphere

- Using the third eigenvalue gives

$$
\begin{array}{lll}
1.721816[3]\,, & 1.780663[1]\,, & 1.783978[5]\,, \\
1.999187[3]\,, & 1.999881[3]\,, & 2.000219[1]\,, \\
2.244925[2]\,, & 2.245649[1]\,, & 2.246163[3]\,, \\
2.247330[3]\,. & &
\end{array}
$$

Numerical results — Peanut



Peanut is given parametrically by

$$
\begin{aligned}
x &= \rho \sin(\phi)\cos(\theta), \\
y &= \rho \sin(\phi)\sin(\theta), \\
z &= \rho \cos(\phi), \\
\rho^2 &= 9\left\{\cos^2(\phi) + \sin^2(\phi)/4\right\}/4.
\end{aligned}
$$

EV	calculated EV
$\kappa_{1,\mathscr{P},4}$	2.825465 [1]
$\kappa_{2,\mathscr{P},4}$	3.044714 [1]
$\kappa_{3,\mathscr{P},4}$	3.515142 [2]
$\kappa_{4,\mathscr{P},4}$	3.574896 [2]

Dirichlet eigenvalue is 3.189591[1].

Numerical results — Peanut

- Using first eigenvalue gives

$$1.999\,828[1]\,, 2.124\,571[1]\,, 2.201\,110[1]\,.$$

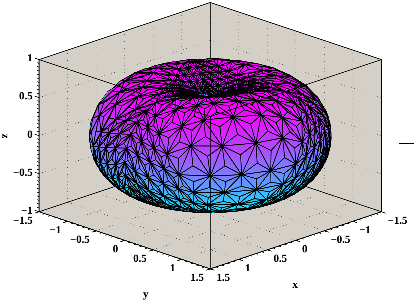
- Using second eigenvalue yields

$$1.916\,773[1]\,, 2.000\,124[1]\,, 2.176\,429[1]\,,$$

$$2.227\,661[1]\,, 2.254\,307[2]\,, 2.260\,086[2]\,.$$

- $\tilde{n} = 2.00 \rightarrow n = 4.00$.

Numerical results — Cushion



Cushion is given parametrically by

$$\begin{aligned} x &= \rho \sin(\phi)\cos(\theta)\,, \\ y &= \rho \sin(\phi)\sin(\theta)\,, \\ z &= \rho \cos(\phi)\,, \\ \rho &= 1 - \cos(2\phi)/2\,. \end{aligned}$$

EV	calculated EV
$\kappa_{1,\mathscr{C},4}$	2.941084 [2]
$\kappa_{2,\mathscr{C},4}$	2.962924 [2]
$\kappa_{3,\mathscr{C},4}$	3.192652 [2]
$\kappa_{4,\mathscr{C},4}$	3.234727 [1]

Dirichlet eigenvalue is 2.950220[1].

Numerical results — Cushion

- Using first eigenvalue gives

$$1.999\,334[2]\,,2.006\,331[2]\,,2.148\,752[1]\,,2.148\,908[1]\,,2.151\,459[1]\,.$$

- Using second eigenvalue yields

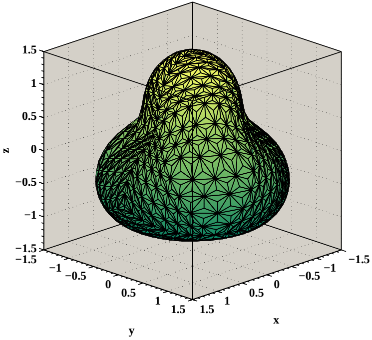
$$1.987\,971[2]\,,1.999\,188[2]\,,2.134\,612[1]\,,2.134\,766[1]\,,$$
$$2.138\,575[1]\,,2.335\,564[2]\,,2.348\,646[2]\,.$$

- Using third eigenvalue gives

$$1.881\,800[2]\,,1.937\,152[2]\,,1.999\,386[1]\,,1.999\,511[1]\,,$$
$$2.016\,798[1]\,,2.178\,440[2]\,,2.233\,806[2]\,,2.341\,157[2]\,.$$

- $\tilde{n} = 2.00 \rightarrow n = 4.00$.

Numerical results — Acorn



Acorn is given parametrically by

$$
\begin{aligned}
x &= \rho \sin(\phi) \cos(\theta)\,, \\
y &= \rho \sin(\phi) \sin(\theta)\,, \\
z &= \rho \cos(\phi)\,, \\
\rho^2 &= 9\left\{17/4 + 2\cos(3\phi)\right\}/25\,.
\end{aligned}
$$

EV	calculated EV
$\kappa_{1,\mathscr{A},4}$	2.706295 [1]
$\kappa_{2,\mathscr{A},4}$	2.718191 [2]
$\kappa_{3,\mathscr{A},4}$	2.940516 [1]
$\kappa_{4,\mathscr{A},4}$	2.994077 [2]

Dirichlet eigenvalue is 2.714524[1].

Numerical results — Acorn

- Using first eigenvalue gives

$$1.995548[1]\,,2.002731[2]\,,2.098831[1]\,,2.174251[2]\,.$$

- Using second eigenvalue yields

$$1.990991[1]\,,1.996762[2]\,,2.092468[1]\,,2.165971[2]\,.$$

- Using third eigenvalue gives

$$1.900786[2]\,,1.904686[1]\,,1.987280[1]\,,2.025899[2]\,,$$

$$2.225917[2]\,,2.228727[1]\,,2.238764[1]\,,2.243261[1]\,.$$

- $\tilde{n} = 2.00 \to n = 4.00$.

Summary and outlook

- Reviewed how to calculate numerically transmission eigenvalues.
- Presented an alternative method to calculate refraction index for various surfaces.
- Results are very accurate.

- Further investigation is needed for the “intersection” method.
- Can the use of more eigenvalues be better combined?