University of Bremen
Center for Industrial Mathematics
AWI

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Inverse modeling of basal conditions of the East Antarctic Ice Sheet

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Background & motivation

- Effect of melting temperate glaciers in the Antarctica is crucial for sea level rise
- Queen Maud Land (QML) and Wilkes Land (WL) considered (see Fig. 1)
- Basal friction plays a major role for flow speed
- Friction coefficient on a glacier's base not observable → Inverse methods:

- Regularization: prevent over-fitting & enhance convexity in inverse methods
- L-Curve analysis: determine optimal level of regularization

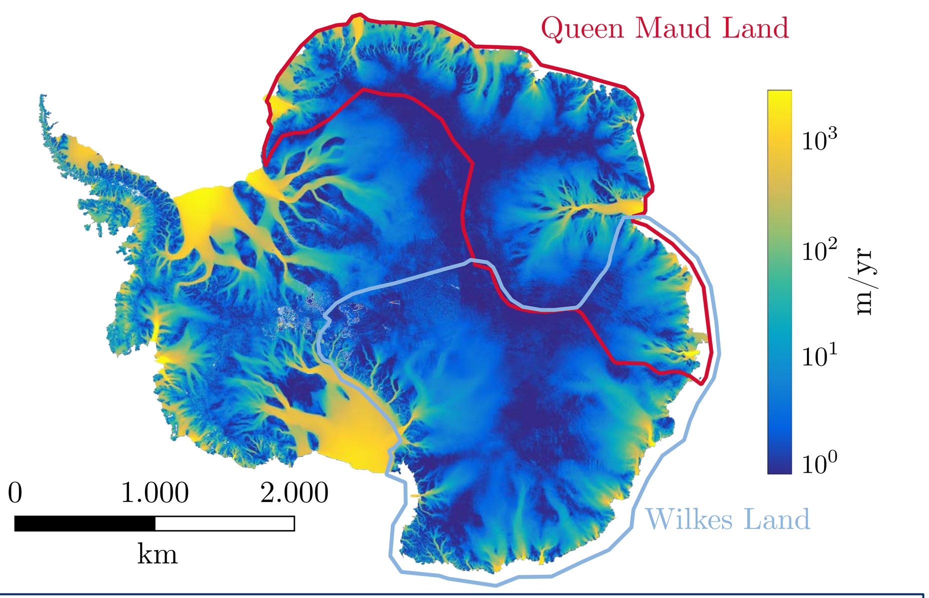


Figure 1: Observed ice flow velocities in the Antarctica [1].

Challenge

Basal conditions are hard to observe but crucial for the ice flow.

Aim

Infer basal drag coefficient and it's spatial distribution using an inverse approach.

Methodology & approach

- NASA's Ice-sheet and Sea-level System Model (ISSM, [2]) used to solve the Higher-Order approximation [3]

$$\frac{\partial}{\partial x_1}\left(4\eta\frac{\partial u_1}{\partial x_1}+2\eta\frac{\partial u_2}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(\eta\frac{\partial u_1}{\partial x_2}+\eta\frac{\partial u_2}{\partial x_1}\right)+\frac{\partial}{\partial x_3}\left(\eta\frac{\partial u_1}{\partial x_3}\right)=\rho_i g\frac{\partial h_s}{\partial x_1},$$

$$\frac{\partial}{\partial x_1}\left(\eta\frac{\partial u_1}{\partial x_2}+\eta\frac{\partial u_2}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(4\eta\frac{\partial u_2}{\partial x_2}+2\eta\frac{\partial u_1}{\partial x_1}\right)+\frac{\partial}{\partial x_3}\left(\eta\frac{\partial u_2}{\partial x_3}\right)=\rho_i g\frac{\partial h_s}{\partial x_2},$$

$$\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}=0.$$

Base boundary condition: linear sliding law

$$(\boldsymbol{\sigma}\cdot\boldsymbol{n})\cdot\boldsymbol{t}_i=\tau_{b,i}=-\alpha^2 u_{b,i}=-k^2 N u_{b,i},\quad i=1,2 \text{ on } \Gamma_{b,gr}$$

- Drag coefficient k and friction coefficient $\alpha = N^{1/2}k$ are crucial → quantity of interest
- Effective pressure N: accounts for the presence of water lubricating the ice-base interface
- Derive the inverse (minimization) problem [4]:
  - Interpret $\boldsymbol{u}$ as a function of k, $\boldsymbol{u}=\boldsymbol{u}(k)$
  - Modify base BC k to fit $\boldsymbol{u}(k)$ to $\boldsymbol{u}^{obs}$
  - Penalize gradient of k (regularization)

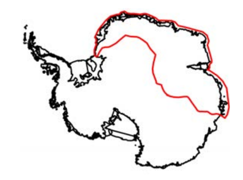
Inverse problem

$$\min_k \underbrace{\frac{1}{2}\int_{\Gamma_s}(u_1(k)-u_1^{obs})^2+(u_2(k)-u_2^{obs})^2\,\mathrm{d}x}_{:=J^{obs}(\boldsymbol{u}(k))} + \lambda\underbrace{\frac{1}{2}\int_{\Gamma_{b,gr}}\|\nabla k\|^2\,\mathrm{d}x}_{:=J^{reg}(k)}$$

- Tikhonov regularization term $\lambda>0$:
  - More realistic spatial k-distributions
  - Enhance convexity
  - Determined through L-Curve analysis

Results & discussion

Queen Maud Land (QML)



- Mesh with element sizes of 0.5 km to 64 km used (approx. 350 k elements)
- Characteristic L-shape in the $(\ln J^{obs}, \ln J^{reg})$-space
- Regularization cost increases as λ decreases and data cost increases as λ does
- Trade-off curve close to the individual runs (no outliers)

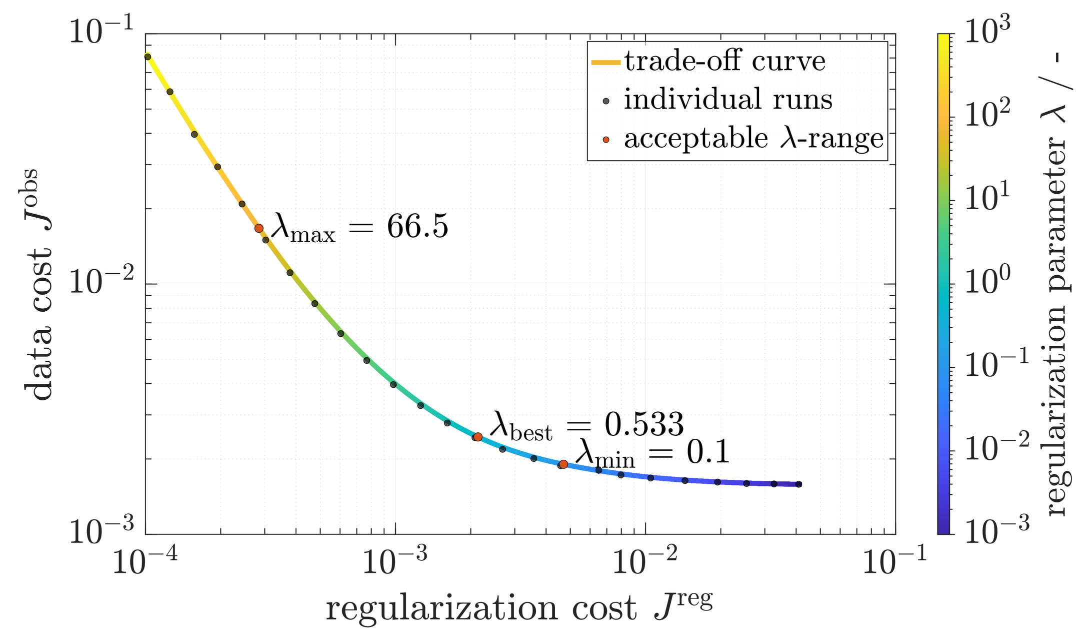


Figure 2: L-Curve results for Queen Maud Land. The red dots identify the best (max. curvature of trade-off), min/max (half of max. curvature of trade-off) value for λ and thus mark the corner region of the curve.

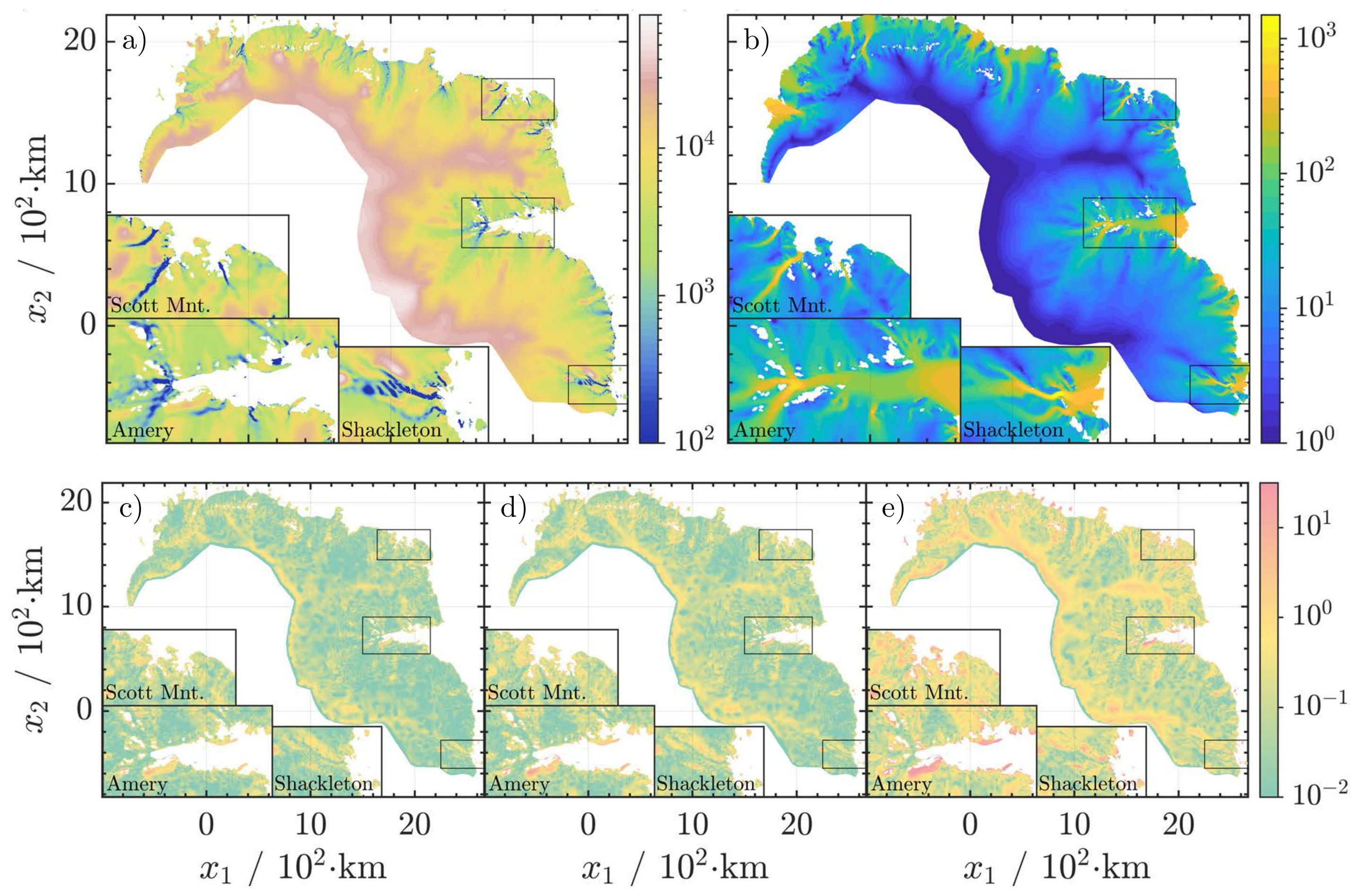


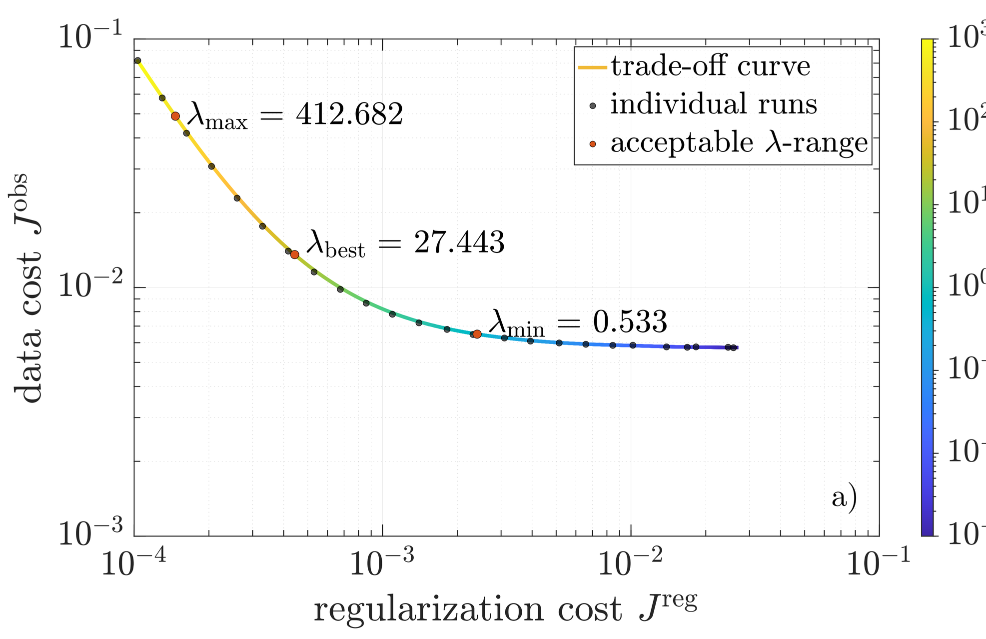
Figure 3: Best estimates for a) the friction coefficient in Pa·(m/yr)-1 and b) the velocity in m/yr for QML. Plots c)-e) show the relative velocity error for min/best/max values of λ respectively.

- Linear sliding law reflected: fast-flowing regions show small values for α and inversely (see Fig. 3a)).
- Complex distributions of α in fast–flowing regions (Amery and Shackleton ice shelfs)
- Visual coherence of computed velocity (Fig. 3b) and the observed velocity (Fig. 1)
- Relative velocity misfit (Fig. 3c)-e)) unsurprisingly shows smaller errors for smaller λ values and vice-versa
- Overall distribution remains the same across min-/best-/max-λ

Wilkes Land (WL)

- Low-res mesh in WL (approx. 120 k elements)
- Higher max-λ and larger range as well as overall higher data cost in Fig. 4a) compared to QML
- Results for QML and WL coincide
- Linear sliding law: fast-flowing regions show small values for α and inversely (see Fig. 4b))

- Convergence issues
  - High-res. Mesh of approx. 350 k elements
  - Lake Vostok was set to grounded area
  - Mountain ranges in Oates Land and Ross Shelf results in badly covered glaciers
  - Cropping of land tongues and islands lead to some improvement



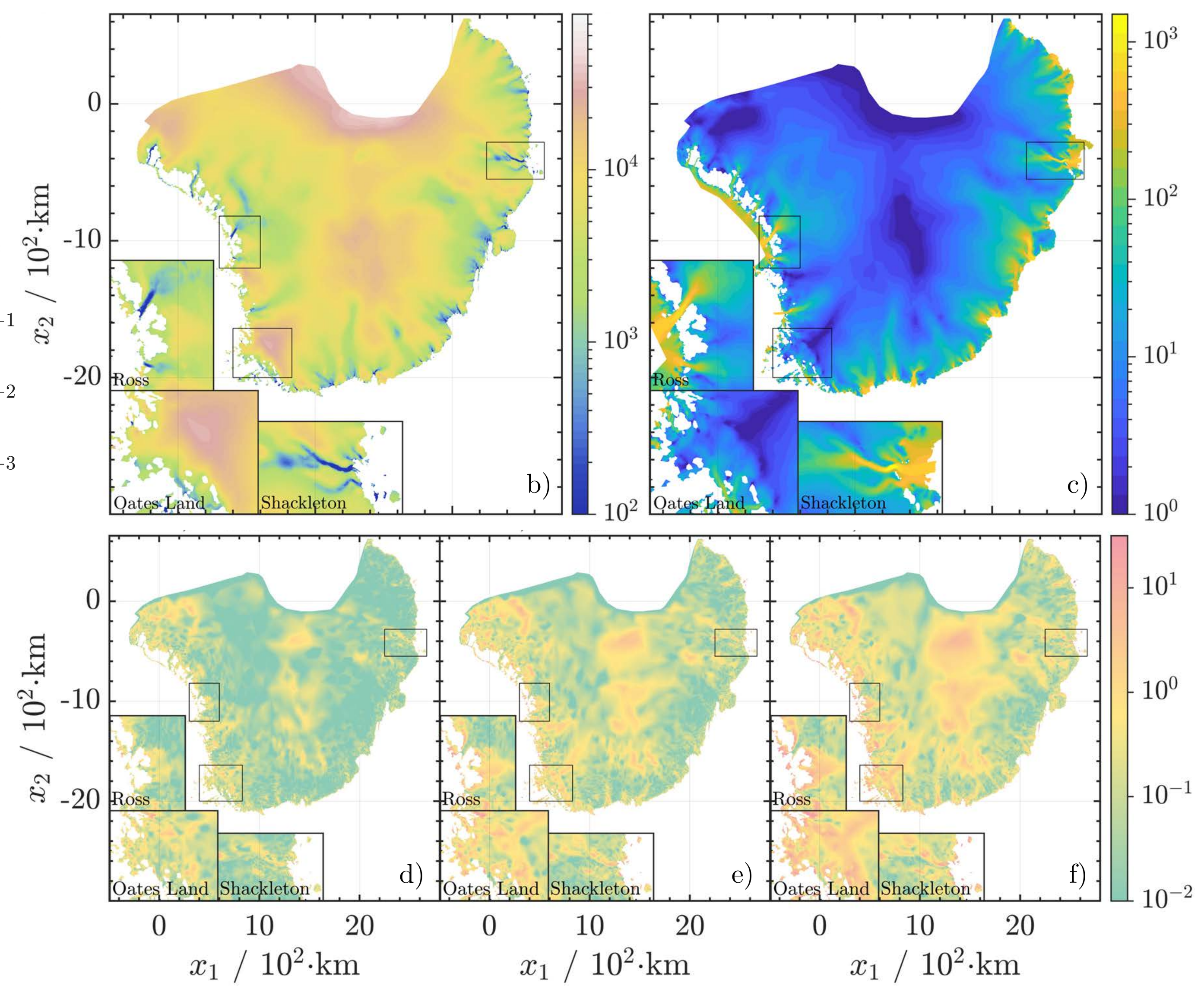


Figure 4: a) L-Curve for Wilkes Land (analogously to Fig. 2). Best estimates for b) the friction coefficient in Pa·(m/yr)-1 and c) the velocity in m/yr for Wilkes Land. Plots d)-f) show the relative velocity error for min/best/max values of λ respectively.

Conclusion

- Suitability of Higher-Order approximation and linear sliding law is domain-dependent
- L-Curve approach promising method for identifying the optimal level of regularization
- Optimal distributions for k and α could be found
- Overall, results for QML and WL coincide
- Convergence depends on the specific domain

Future work

- Get convergence for high-res. mesh in Wilkes Land
- Compare results in intersection of QML and WL
- Only the current state in East Antarctica considered (stationary):
  - Derive results for whole Antarctica
  - Expand model for time-depending problems

Project sea level rise!

References

[1] Data obtained from: Norsk-Polarinstitutt (2021). Quantarctica – a free gis for antarctica. https://www.npolar.no/quantarctica/
[2] Morlighem, M., Seroussi, H., and Larour, E. (2023). Ice sheet system model (4.22). California Institute of Technology.
[3] Larour, E., Seroussi, H., Morlighem, M., and Rignot, E. (2012). Continental scale, high order, high spatial resolution, ice sheet modeling using the ice sheet system model (issm). Journal of Geophysical Research: Earth Surface, 117(F1).
[4] Wolovick, M., Humbert, A., Kleiner, T., and Rückamp, M. (2023). Regularization and l-curves in ice sheet inverse models, a case study in the filchner-ronne catchment. EGU- sphere, 2023:1–46.

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