Stability of Interconnections of ISS Systems

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Abstract– Interconnections of several nonlinear systems with inputs are considered in this paper. Each system is assumed to be input-to-state stable (ISS). However an interconnection of such systems is not stable in general. We provide a stability condition for interconnections of such systems. Some interpretations of this condition are given. Moreover we show how an ISS-Lyapunov function for such a network can be constructed explicitly if this condition is satisfied. Further we give the idea of numerical verification of the small gain condition for networks.

Key Words: nonlinear systems, nonlinear gains, interconnections, networks, input-to-state stability, small gain condition

I. Introduction

Nonlinear systems appear frequently in applications and their stability properties are important for design and performance of such systems. We consider several nonlinear stable systems and refer to the question, whether an interconnection of these systems is also stable.

First consider one nonlinear system

$$\dot{x} = f(x,u) \tag{1}$$

where $x \in \mathbb{R}^N$ denotes the state and $u \in \mathbb{R}^M$ is input. Function f is such that there exists a unique solution of this equation for any initial condition and arbitrary measurable input $u \in L_\infty$. An appropriate stability notion for such systems was introduced by Eduardo Sontag in [11] and is called input-to-state stability (ISS), see the definition below. It is known that feed forward connections of such systems are ISS again. However a feedback interconnection of two ISS systems

$$x_i = f_i(x_1, x_2, u_i), \quad i = 1,2$$

is in general not stable. Stability conditions for interconnections of two ISS systems were derived in [9], see also [7] for interconnection of ISS and integral ISS systems, where the latter type of systems is more general and contains ISS systems as subset.

It is known that ISS property is equivalent to the existence of an ISS-Lyapunov function, see the definition below. A construction of an ISS-Lyapunov function for interconnection of two ISS systems on the base of ISS-Lyapunov functions of each system was given in [8].

Now consider a network of n ISS systems

$$x_i = f_i(x_1, \ldots, x_n, u_i),\ x \in \mathbb{R}^{n_i},\ u \in \mathbb{R}^{m_i}. \tag{2}$$

In this paper we collect some known stability results recently obtained for such interconnections.

II. Definitions

Let $\mathbb{R}_+$ denote the set of nonnegative real numbers and $\mathbb{R}^n_+$ be the positive orthant in $\mathbb{R}^n$. Recall that $\mathcal{K}$ denotes the class of strictly increasing functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$. A subset of such functions which are unbounded is denoted by $\mathcal{K}_\infty$. Function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the $\mathcal{KL}$ class if $\beta(\cdot, t) \in \mathcal{K}$ for any $t \geq 0$ and $\beta(s, \cdot)$ is decreasing with $\lim_{t\to\infty} \beta(s,t) = 0, \forall s \geq 0$.

Definition 2.1: System (1) is called ISS from u to x if for any initial state $x(0)$ and any measurable input u there exist $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that

$$|x(t)| < \max\{\beta(|x(0)|, t), \gamma(||u||_\infty)\},\ t \geq 0, \tag{3}$$

where $|\cdot|$ denotes Euclidean norm in a corresponding space and $||\cdot||_\infty$ is the standard norm in L_∞. Function γ is then called nonlinear gain.

Remark 2.2: Inequality (3) in the above definition can be equivalently replaced by

$$|x(t)| < \beta(|x(0)|, t) + \gamma(||u||_\infty),\ t \geq 0, \tag{4}$$

with some different β and γ.

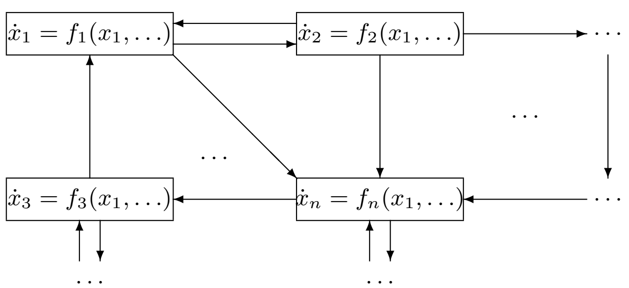
Definition 2.3: Function $V : \mathbb{R}^N \to \mathbb{R}_+$ is called ISS-Lyapunov function for (1) if there exist $\psi_1, \psi_2, \chi \in \mathcal{K}_\infty$ and a positive definite function $\alpha \in \mathrm{pdf}$ such that for any initial condition and any measurable input u the following holds

$$\psi_1(|x|) < V(x) < \psi_2(|x|),\ x \in \mathbb{R}^N \tag{5}$$

$$V(x) > \chi(|u|) \ \Rightarrow\ \nabla V(x) f(x,u) < -\alpha(V(x)) \tag{6}$$

Function χ is called Lyapunov gain in this case.

Interconnection (2) can be seen as a network or a directed graph, with n nodes representing the systems and edges



between the nodes. There is an edge from the node i to the node j if the j-th systems has x_i as an input coming from the i-th system. A network is called strongly connected if the corresponding graph is strongly connected.

We assume that each system in (2) is ISS, i.e., for any solution of the i-th equation with arbitrary initial condition and any measurable input u the following holds for all $t \geq 0$

$$|x_i(t)| < \max\{\beta_i(|x_i(0)|, t), \max_{j\neq i}\{\gamma_{ij}(||x_j||_\infty)\}, \gamma_i(||u_i||_\infty)\}. \tag{7}$$

Note that we equivalently may assume

$$|x_i(t)| < \beta_i(|x_i(0)|, t) + \sum_{j\neq i}\gamma_{ij}(||x_j||_\infty) + \gamma_i(||u_i||_\infty) \tag{8}$$

instead of (7) with some different gains. Motivated by the expressions on the right hand side of (7) and (8) we define the following nonlinear mappings from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$

$$\Gamma_{\max}(s) = (\max_j\{\gamma_{1j}(s_j)\}, \dots, \max_j\{\gamma_{nj}(s_j)\})^T \tag{9}$$

$$\Gamma_{\sum}(s) = \Big(\sum_j \gamma_{1j}(s_j), \dots, \sum_j \gamma_{nj}(s_j)\Big)^T \tag{10}$$

where x^T denotes the transposition of vector $x \in \mathbb{R}^n$.

It is appropriate to collect the gains γ_{ij} in a matrix

$$\Gamma = \begin{bmatrix} 0 & \gamma_{12} & \dots & \dots & \gamma_{1n} \\ \gamma_{21} & 0 & \gamma_{23} & \dots & \gamma_{2n} \\ \vdots & & & & \vdots \\ \gamma_{n-1,1} & \dots & \gamma_{n-1,n-2} & 0 & \gamma_{n-1,n} \\ \gamma_{n1} & \dots & \dots & \gamma_{n,n-1} & 0 \end{bmatrix} \tag{11}$$

defining $\gamma_{ii} \equiv 0$ for completeness. Then (10) looks similar to the multiplication of the matrix Γ and vector s. Γ may be seen as an adjacency matrix of the corresponding graph representing the interconnection.

For $a, b \in \mathbb{R}^n$ we say that $a < b$ if $a_i < b_i$ for all $i = 1, \dots, n$. Similarly we define $a > b$ and $a \geq b$. The negation of the latter is denoted by $a \not\geq b$ and means that there exists $i \in \{1, \dots, n\}$ such that $a_i < b_i$. For to operators A and B mapping $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$ we say that $A < B$ if $A(s) < B(s)$ for all $s \in \mathbb{R}^n_+$. Relations $>$ $, \geq$ for operators are defined similarly. $A \not\geq B$ means that for some $i \in \{1, \dots, n\}$ the i-th component of $A(s)$ is less then the i-th component of $B(s)$. $A \circ B$ denotes the composition of operators A and B. By id we denote the identity operator on $\mathbb{R}^n_+$ or other appropriate space. Note that both operators $\Gamma_{\max}$ and $\Gamma_{\sum}$ are monotone in the sense that for any $a \geq b \in \mathbb{R}^n_+$ it holds $\Gamma_{\max}(a) \geq \Gamma_{\max}(b)$ and the same for $\Gamma_{\sum}$.

Stability condition of the small gain type for the interconnections of systems satisfying (7) (or (8)) are given in the next section.

III. Stability Results

A. Small gain condition for networks

Let us denote $x = (x_1, \dots, x_n)^T$, $f(x, u) = (f_1(x, u_1), \dots f_n(x, u_n))^T$, $N = \sum_{i=1}^n n_i$. Then the interconnection (2) takes the form (1). Consider interconnection (1) of systems (2) satisfying (7).

Theorem 3.1: Let each system of (2) be ISS satisfying (7). Let $\Gamma_{\max}$ be defined as in (9). If

$$\Gamma_{\max} \not\geq \text{id on } \mathbb{R}^n_+ \setminus 0 \tag{12}$$

then the interconnection (1) is ISS form u to x.
This result was firstly obtained in [2] for the case of ISS definition with $\sum$ as in (8), see [3], [10] for an alternative and more detailed proof for both cases, similar results for the asymptotic gain property and global stability was also obtained there for such interconnections.

Note that if we would use the definition of ISS with $\sum$ on the places of max, i.e., (8) instead of (7) than this theorem would hold with the following small gain condition

$$\Gamma_{\sum} \circ D \not\geq \text{id}, \text{ on } \mathbb{R}^n_+ \setminus 0 \tag{13}$$

for some $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by $D(s) = (s_1 + \alpha(s_1), \dots, s_n + \alpha(s_n))^T$ with $\alpha_i \in \mathcal{K}_\infty$. In case of linear gains γ_{ij} the operator $\Gamma_{\sum}$ is linear on $\mathbb{R}^n_+$ and (13) is equivalent to $\rho(\Gamma_{\sum}) < 1$, where ρ denotes the spectral radius.

Remark 3.2: For $n = 2$ it holds $\Gamma_{\max} = \Gamma_{\sum}$, the condition (12) is equivalent to $\gamma_{12} \circ \gamma_{21} < \text{id}$ which can be found in [8], and (13) is equivalent to existence of $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\gamma_{12} \circ (\text{id} + \alpha_1) \circ \gamma_{21} \circ (\text{id} + \alpha_2) < \text{id}$. The last small gain condition was derived in [9]. Hence we see that the small gain condition for networks generalizes the known small gain results.

The condition (12) (or (13)) is called small gain condition for networks. It says that $\Gamma_{\max}$ cannot "expand" in all components simultaneously, since it states that there is at least one $i \in \{1, \dots, n\}$ such that the i-th component of $\Gamma_{\max}(s)$ is less then s_i. Moreover it can be shown that there exists an open unbounded domain Ω in $\mathbb{R}^n_+$ such that for any $s \in \Omega$ it holds $\Gamma_{\max}(s) < s$. We quote several results from [3] to show several geometrical consequences from the small gain condition for the operator $\Gamma_{\sum}$. Let $\Gamma_{\sum}$ be such that the corresponding small gain condition (13) is satisfied. Hence we know that for any $s \in \mathbb{R}^n_+$ there is always some i such that $s_i < \Big(\Gamma_{\sum}(s)\Big)_i$. This motivates the introduction of the following domains in the positive orthant:

$$\Omega_i := \left\{ s \in \mathbb{R}^n_+ : \; s_i > \sum_{j=1}^n \gamma_{ij}(s_j) \right\}, \quad i = 1, \dots, n.$$

Further, let Δ_r denote a simplex defined as the intersection of $\mathbb{R}^n_+$ and the hyperplane $s_1 + \dots + s_n = r > 0$.

Proposition 3.3: Consider $\Gamma_{\sum}$ as above. The weaker condition

$$\Gamma_{\sum} \not\geq \text{id on } \mathbb{R}^n_+ \setminus 0$$

is equivalent to $\bigcup_{i=1}^n \Omega_i = \mathbb{R}^n_+ \setminus \{0\}$. Furthermore this condition implies that for all $r > 0$

$$\Delta_r \cap \bigcap_{i=1}^n \Omega_i \neq \emptyset. \tag{14}$$

Let us denote $\Omega := \cap_{i=1}^n \Omega_i$. Note that for any point s in Ω it holds $s > \Gamma_{\sum}(s)$. The last proposition shows that this domain is nonempty and unbounded. It can be also shown, that this domain is pathwise connected, see [4]. Next proposition shows some invariance property of Ω:

Proposition 3.4: Consider operator $\Gamma_{\sum}$ in a matrix form as above. Assume that Γ has no zero row. Assume

that it satisfies the same condition as in previous proposition. If $s \in \Omega$ then $\Gamma_{\sum}(s) \in \Omega$.
These geometrical properties were used in [4] to construct an ISS-Lyapunov function for an interconnection. Similar results are available for $\Gamma_{\max}$ and for even more general monotone operators, see [10].

Another interesting result related to the small-gain condition is the following. Let Γ be $\Gamma_{\max}$ or $\Gamma_{\sum}$. Consider a discrete dynamical system defined by

$$s_{k+1} = \Gamma(s_k), \quad k = 1, 2, \dots \tag{15}$$

with initial state $s_0 \in \mathbb{R}^n_+$.

Theorem 3.5: Let network (2) of ISS systems be strongly connected. Let Γ denote $\Gamma_{\max}$ or $\Gamma_{\sum}$ as above. The small gain condition

$$\gamma \not\geq \text{id on } \mathbb{R}^n_+ \setminus 0 \tag{16}$$

is equivalent to the global asymptotic stability of (15).
This theorem reduces the investigation of stability of the interconnection (2) to the stability investigation of a simpler system (15), which has dimension n that is in many cases much smaller then the dimension $N = \sum_{i=1}^n n_i$ of the network (2).

Similar small gain condition as in Theorem 3.1 holds also for discrete time systems. Consider the interconnected discrete time system

$$\begin{aligned} \Sigma_1 &: x_1(k+1) = f_1(x_1(k), \dots, x_n(k), u(k)) \\ &\vdots \\ \Sigma_n &: x_n(k+1) = f_n(x_1(k), \dots, x_n(k), u(k)) \end{aligned} \tag{17}$$

for $k \in \mathbb{N}$, where $x_i(k) \in \mathbb{R}^{N_i}$, $u(k) \in \mathbb{R}^M$, and $f_i : \mathbb{R}^{\sum_{j=1}^n N_j + M} \to \mathbb{R}^{N_i}$ is continuous.

System Σ_i is ISS, if there exists $\beta \in \mathcal{KL}$ and $\gamma_{ij}, \gamma \in \mathcal{K} \cup \{0\}$ with $\gamma_{ii} = 0$, such that every solution $x_i : \mathbb{N} \to \mathbb{R}^{N_i}$ of (17) satisfies

$$|x_i(k)| \leq \beta_i(|x_i(0)|, k) + \sum_{j=1}^n \gamma_{ij}(||x_{j\,[0,k]}||_\infty) + \gamma(||u||_\infty) \tag{18}$$

for all inputs $x_j : \mathbb{N} \to \mathbb{R}^{N_j}$, $j = 1, \dots, n, j \neq i$, and $u : \mathbb{N} \to \mathbb{R}^M$, where we denote by $[0,k]$ the set $\{0, \dots, k\}$ and by $\|x\|_\infty = \sup_{l \in \mathbb{N}}\{x_l\}$ for functions $x : \mathbb{N} \to \mathbb{R}^N$. The following result was obtained in [3], Proposition 15.

Theorem 3.6: Consider the systems Σ_i in (17) and suppose that each subsystem is ISS, i.e., condition (18) holds for all $i = 1, \dots, n$. Let $\Gamma_{\sum}$ be defined similarly as in (10). If there exists a mapping D as in (13), such that

$$(\Gamma_{\sum} \circ D)(s) \not\geq s, \qquad \forall s \in \mathbb{R}^n_+ \setminus 0,$$

then the interconnection (17) is ISS from u to x.

Remark 3.7: In case of definition of ISS with $\max$ on the place of $\sum$ in (18) the small gain condition should be changed to $\Gamma_{\max} \not\geq$ id on $\mathbb{R}^n_+ \setminus 0$.

Remark 3.8: In the discrete time context Teel [12] proves that if we have maximum ISS estimates of the type (7) for each system and if for each cycle (or equivalently, each minimal cycle) in the matrix Γ we have

$$\gamma_{k_1 k_2} \circ \gamma_{k_2 k_3} \circ \dots \circ \gamma_{k_{p-1} k_p} < \text{id},$$

for all $(k_1, \dots, k_p) \in \{1, \dots, n\}^p$ where $k_1 = k_p$, then the network under consideration is input-to-state stable. This result extends in a straightforward manner to continuous time systems. It is an easy exercise to show that the cycle condition and the statement

$$\Gamma_{\max}(s) \not\geq s, \quad \forall s \in \mathbb{R}^n_+, s \neq 0,$$

are equivalent. Note that this equivalence does not hold for $\Gamma_{\sum}$.

B. Construction of an ISS-Lyapunov function for networks

Since the ISS property is equivalent to the existence of an ISS-Lyapunov function, the next natural question arises: Can we use the ISS-Lyapunov function of systems in (2) to construct an ISS-Lyapunov function of their interconnection? Before we answer this question we recall an interesting consequence of the above small gain condition, see [4], [10].

Theorem 3.9: Let $\Gamma_{\max}$ and $\Gamma_{\sum}$ be as above and satisfy (12) and (13) respectively. Then for each of both operators there exist some $\sigma_1, \dots, \sigma_n \in \mathcal{K}_\infty$ such that for any $t > 0$ vector $\sigma(t) = (\sigma_1(t), \dots, \sigma_n(t))$ satisfies

$$\Gamma_{\max}(\sigma(t)) < \sigma(t) \tag{19}$$

and respectively

$$\Gamma_{\sum}(\sigma(t)) < \sigma(t). \tag{20}$$

$\sigma_1, \dots \sigma_n$ will be used to re-scale corresponding ISS-Lyapunov function of each system of the interconnection and construct an ISS-Lyapunov function for the overall system. Note that $\sigma = (\sigma_1, \dots, \sigma_n)$ parameterizes a continuous curve in the positive orthant $\mathbb{R}^n_+$ such that each its point $\sigma(t) = (\sigma_1(t), \dots, \sigma_n(t)) \in \Omega$, for all $t > 0$. See the definition of Ω above. The proof of the last theorem is based on the geometrical properties of Γ considered in the previous subsection.

Returning back to the stated question, let V_i be an ISS-Lyapunov function of the i-th system in (2), i.e., we assume that

$$\psi_{i1}(|x_i|) < V_i(x_i) < \psi_{i2}(|x_i|) \tag{21}$$

$$\begin{aligned} V_i(x_i) \geq \max\{\chi_i(\|u_i\|), \max_{j \neq i} \gamma_{ij}(V_j(x_j))\} \\ \Longrightarrow \nabla V_i(x_i) \cdot f_i(x, u_i) \leq -\alpha_i(V_i(x_i)), \end{aligned} \tag{22}$$

for some $\psi_{1i}, \psi_{2i}, \chi_i \in \mathcal{K}_\infty$ and $\alpha_i(V_i(x_i)) \in$ pdf.

Theorem 3.10: Let network (2) be strongly connected. Let V_i be an ISS-Lyapunov function for the i-th system of (2) as above. Let $\Gamma_{\max}$ be defined as in (9) with Lyapunov gains from (22) and satisfy the small gain condition (12). Let $\sigma = (\sigma_1, \dots, \sigma_n)$ be as in Theorem 3.9. Then the interconnection (2) is ISS and

$$V(x) = \max_i \sigma_i^{-1}(V_i(x_i)) \tag{23}$$

is an ISS-Lyapunov function for the overall system.
Note that σ_i is always invertible as a $\mathcal{K}_\infty$ function and its inverse belongs again to the $\mathcal{K}_\infty$ class. For the proof of the theorem we refer to [4], [10].

Remark 3.11: The construction in (23) is non-smooth since $\max$ is a non-smooth operator. However function

V is a Lipschitz continuous function as a maximum of smooth functions. Hence V is differentiable almost every where. Methods of non-smooth analysis [1] were used to show the ISS property of the network.

Remark 3.12: The construction of an ISS-Lyapunov function works similarly for the case of ISS systems with Lyapunov gains from the definition with $\sum$ on the place of $\max$ in (22). One have to use the corresponding σ in (23).

For application of the presented results we refer to [6], where a logistic network was considered and investigated on stability.

C. Verification of the small gain condition

The last question that we are going to discuss in this paper is the following. The small gain condition $\Gamma \not\geq \mathrm{id}$ looks very compact, however it is not obvious how to check this condition in $\mathbb{R}^n_+$ especially for large n. A numerical procedure was developed in [5] for this purpose. The notion local ISS (LISS) was used there. Subsystem i in (2) is LISS, provided there exist $\rho_i > 0$, $\gamma_{ij}, \gamma_i \in \mathcal{K}_\infty$, and a $\beta_i \in \mathcal{KL}$, such that for all $\|\xi_i\| \leq \rho_i$, $\|u_i\|_\infty \leq \rho_i$

$$\|x_i(t, \xi_i, x_j : j \neq i, u_i)\| \leq \beta_i(\|\xi_i\|, t) + \sum_{j \neq i} \gamma_{ij}(\|x_j\|_\infty) + \gamma_i(\|u_i\|_\infty) \quad \forall\, t \geq 0. \quad (24)$$

Remark 3.13: Instead of (24) we could also write

$$\|x_i(t, \xi_i, x_j : j \neq i, u_i)\| \leq \max\{\beta_i(\|\xi_i\|, t), \max_{j \neq i} \gamma_{ij}(\|x_j\|_\infty), \gamma_i(\|u_i\|_\infty)\} \quad \forall\, t \geq 0, \quad (25)$$

which is qualitatively equivalent. Of course the gains in (24) and (25) are in general different.

The following local small gain condition was used for such interconnections: Γ satisfies LSGC on the set $0 \leq s \leq w^*$ if

$$\Gamma(w^*) < w^* \text{ and } \Gamma(s) \not\geq s, \ \forall\, 0 \leq s \leq w^*, \ s \neq 0. \quad \text{(LSGC)}$$

The following results were proven in [5]:

Theorem 3.14: Let all subsystems (2), $i = 1, \dots, n$, satisfy (24). Suppose Γ satisfies (LSGC). Then there exists a $\rho > 0$, a $\beta \in \mathcal{KL}$, and a $\gamma \in \mathcal{K}_\infty$, such that interconnection (2) satisfies

$$\|x(t, \xi, u)\| \leq \beta(\|\xi\|, t) + \gamma(\|u\|_\infty) \quad \forall\, t \geq 0, \quad \text{(LISS)}$$

for all $\|\xi\| \leq \rho$, $\|u\|_\infty \leq \rho$, i.e., is LISS.

Lemma 3.15: Let Γ be a gain matrix as above. For any $w^* \in \mathbb{R}^n_+$ consider the trajectory $\{w(k)\}$ of the discrete monotone system $w(k+1) = \Gamma(w(k))$, $k = 0, 1, 2 \dots$ with $w(0) = w^*$. If $w(k) \to 0$ for $k \to \infty$ then Γ satisfies the small gain condition (LSGC) on $[0, w^*]$.

This is the key lemma for the numerical verification of the LSGC. An example that shows the application of the numerical procedure was given in this paper. ρ, β and γ were constructed numerically. However the obtained result were rather conservative in the sense that the obtained ρ was essentially smaller then $\min_i\{\rho_i\}$. This suggests the investigation of the dependence of ρ on ρ_i and w^* which we are going to undertake in the nearest future.

IV. Conclusions

We have considered interconnections of arbitrary amount of stable nonlinear systems. In general an interconnection is not stable. Stability conditions were described for such networks. Several interpretations and connections to known results were discussed. We have shown, how an ISS-Lyapunov function can be explicitly constructed for a given interconnection if the small gain condition is satisfied. These results can be used for investigation of stability of large nonlinear systems with inputs. Furthermore an approach for the numerical verification of the small gain condition was given. However this method needs some improvements and further developments.

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