Tikhonov regularization in L^p applied to inverse medium scattering

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Abstract. This paper presents Tikhonov- and iterated soft-shrinkage regularization methods for non-linear inverse medium scattering problems. Motivated by recent sparsity-promoting reconstruction schemes for inverse problems, we assume that the contrast of the medium is supported within a small subdomain of a known search domain and minimize Tikhonov functionals with sparsity-promoting penalty terms based on L^p-norms. Analytically, this is based on scattering theory for the Helmholtz equation with refractive index in L^p, $1 < p < \infty$, and on crucial continuity and compactness properties of the contrast-to-measurement operator. Algorithmically, we use an iterated soft-shrinkage scheme combined with the differentiability of the forward operator in L^p to approximate the minimizer of the Tikhonov functional. The feasibility of this approach together with the quality of the obtained reconstructions is demonstrated via numerical examples.

1. Introduction

We consider time-harmonic inverse scattering of either electromagnetic waves in transverse magnetic polarization from a penetrable non-magnetic material, or of acoustic waves from an inhomogeneous medium with constant density. The model describing such waves with time-dependence $\exp(-\mathrm{i}\omega t)$ is the Helmholtz equation [1, 8]

$$\Delta u + k^2 n^2 u = 0 \qquad \text{in } \mathbb{R}^d,\ d = 2 \text{ or } 3. \tag{1}$$

The wave number k is positive and the refractive index function n equals one outside a bounded and open set $D \subset \mathbb{R}^d$. Inside the scattering object D the refractive index is different from one. We define the contrast function $q : \mathbb{R}^d \to \mathbb{C}$, supported in $\overline{D}$, by

$$q := n^2 - 1 \qquad \text{in } \mathbb{R}^d.$$

The aim of this paper is to establish a regularization scheme in Banach spaces for the inverse scattering problem to reconstruct q from multi-static measurements of scattered waves solving (1). This algorithm is motivated by recent sparsity(-promoting) reconstruction techniques for linear and non-linear operator equations in Banach spaces [2, 3, 4]. We illustrate the reconstruction quality of our reconstruction scheme for inverse scattering problems by numerical examples for "sparse" contrasts, that is, for contrasts with small support within the search domain.

When an incident time-harmonic wave u^i, that is, a solution to the Helmholtz equation $\Delta u^i + k^2 u^i = 0$, illuminates the inclusion D, then the total wave u satisfies (1) in $\mathbb{R}^d$, subject to Sommerfeld's radiation condition for the scattered wave $u^s = u - u^i$,

$$\lim_{|x|\to\infty} |x|^{\frac{d-1}{2}} \left(\frac{\partial}{\partial |x|} - ik \right) u^s(x) = 0 \qquad \text{uniformly in all directions } \hat{x} = x/|x|. \tag{2}$$

Denote by Φ the radiating fundamental solution of the Helmholtz equation,

$$\Phi(x, y) = \begin{cases} \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|), & x, y \in \mathbb{R}^2,\ x \neq y, \\ \frac{\exp(\mathrm{i}k|x-y|)}{4\pi|x-y|}, & x, y \in \mathbb{R}^3,\ x \neq y, \end{cases}$$

and (formally) define the radiating volume potential by

$$V(f)(x) := \int_D \Phi(x, y) f(y) \,\mathrm{d}y, \qquad x \in \mathbb{R}^d.$$

The scattered field $u^s = u - u^i$ from (2) can then be found as solution to the Lippmann-Schwinger integral equation

$$u^s - k^2 V(q u^s) = k^2 V(q u^i) \qquad \text{in } D. \tag{3}$$

There are several possible choices for the spaces in which one considers this integral equation. For $q \in L^\infty(D)$, the natural (and easiest) choice is to solve for $u \in L^2(D)$. However, we are interested in a reconstruction scheme that exploits the *a-priori* information that the contrast has small support within the search domain. Hence, for any Tikhonov-type regularization approach, a penalty term based on L^p-norms for small p seems most appropriate, since, roughly speaking, small values of the reconstruction are strongly penalized. Since we would like to work with reflexive function spaces, we restrict ourselves to $p \in (1, \infty)$, for simplicity. Obviously, such an L^p-based Tikhonov regularization approach requires solution theory for the Helmholtz equation (1) or the Lippmann-Schwinger equation (3) that is able to deal with contrasts in L^p-spaces. Such theory is also established in [5] for the Schrödinger equation $\Delta u + (k^2 + q)u = 0$ for q in distributional Sobolev spaces $W^{-\varepsilon,p}_{\text{comp}}(\mathbb{R}^d)$ with $\varepsilon > 0$ small enough and $p > d/2$. The solution to the integral equation is then found in $W^{\varepsilon,2p'}(\mathbb{R}^d)$ where $p' = p/(p-1)$. In principle, we could use the results from [5] for our paper. However, the proofs in [5] use, e.g., sophisticated multiplier estimates in the Sobolev spaces $W^{\varepsilon,p}(\mathbb{R}^d)$. We prefer in this paper to provide a solution theory for (1) or, equivalently, for the integral equation (3), in Sobolev spaces $W^{2,t}$ with contrast $q \in L^p$ that uses comparatively elementary tools: the well-known Sobolev spaces $W^{m,p}$ for $m \in \mathbb{N}$, Sobolev embeddings, and bounds for the volume potential.

As in [5], our bound in the Lebesgue index p for the contrast $q \in L^p$ is $p > d/2$. This is due to a unique continuation argument that is needed to establish uniqueness of a distributional solution to (1). The most general unique continuation result seems to be contained in [6]. However, the proof of this result is deep and involved. We also present an independent elementary proof of the unique continuation result generalizing the Fourier series technique from [7] to a refractive index in L^p for $p > d$. Despite the bound in p of this proof is not optimal, we believe again that an elementary proof, that may still be optimized in p, has its own interest.

The analytic results for scattering with L^p-contrasts serve to prove several continuity results for the contrast-to-measurement operator $\mathcal{N}$ that maps $q \in L^p$ to the multi-static near-field measurements. Amongst others, we prove continuity, compactness and weak sequential closedness of this mapping. These properties are sufficient to show convergence of a non-linear Tikhonov regularization in L^p, $d/2 < p < \infty$, applied to the inverse problem. Since $\mathcal{N}$ is Fréchet differentiable, this allows to use a shrinked, non-linear Landweber iteration to minimize the Tikhonov functional numerically. We illustrate the resulting scheme by numerical examples.

Inverse scattering problems are among the most popular and well-studied non-linear ill-posed problems with a rich and mathematically deep history. We refer to [8, 9] for an overview of theoretical and numerical methods for (direct and) inverse scattering. Inverse scattering problems are on the one hand challenging due to their ill-posedness and non-linearity, but on the other hand also crucial for many important problems in science and industry. Examples include SONAR and RADAR, light scattering from nano-structured surfaces (e.g., solar cells), or inverse scattering problems related to spectroscopy measurements with tunable lasers occurring in production processes. Known techniques to tackle such problems include high- or low-frequency approximations as for instance the Born or geometric optics approximation. These approximations linearize the inverse problem and cannot be applied in the important resonance region where the inverse problem is truly non-linear. (In our numerical examples later on, the wave number will be chosen that large that the problem is set in the resonance region, i.e., $v \mapsto k^2 V(qv)$ is not a contraction.) Let us note here that the paper [10] applies a sparsity-promoting ℓ^1-penalty approach to tackle a linearized inverse scattering problem for small scatterers when dealing with intensity measurements. Further, [11] studies a two-stage approach for the reconstruction of a sparse contrast, where a direct method is coupled with a semi-smooth Newton method for minimizing a combined L^1- and H^1-Tikhonov functional.

If one cannot avoid to cope with the non-linearity of the inverse problem, Newton-like schemes [12] are powerful and accurate methods to solve inverse medium scattering problems. This class of methods is probably the closest to our technique, since we also exploit the Fréchet differentiability of the contrast-to-measurement operator. Of course, the resulting disadvantage, as for all Newton-like methods, is that computing such Fréchet derivatives is time-consuming since it requires to solve differential equations. Last but not least, decomposition methods are popular, in particular in the engineering community. Examples of such techniques are for instance the contrast source inversion method [13] or the approximative inverse [14] applied to inverse medium scattering.

We finish this introduction with a couple of further remarks on the setting of this paper. We entirely consider point measurements of the scattered fields taken a finite distance away from the scattering objects. However, we are not aware of any theoretical obstacle to extend the present work to a far-field setting. Note that in our numerical examples we take the measurements several wavelengths away from the scattering object, corresponding effectively to far-field measurements.

A couple of points will remain open in this paper. First, we will not give a proper numerical analysis of the sparsity-promoting reconstruction method that we propose, and we will not consider complex-valued contrasts in our numerical examples. Second, we will only consider Hilbert spaces as image spaces for the contrast-to-measurement operator, but not Banach spaces.

The structure of this paper is as follows: In Section 2 we present solution theory for the Lippmann-Schwinger integral equation with contrast in L^p for $p > d/2$. Being able to solve this integral equation, we can define a contrast-to-measurement map in Section 3 and analyze crucial analytic properties of this nonlinear operator. Section 4 extends these results to multi-static measurements and provides convergence results for a non-linear Tikhonov regularization scheme. In Section 5 we use the analytic properties of the contrast-to-measurement operator to construct sparsity-promoting reconstruction schemes and present numerical examples. The appendices contain mostly well-known auxiliary results on Sobolev embeddings, collectively compact operator theory, Hilbert-Schmidt operators, nonlinear Tikhonov regularization, and differentiability of the forward operator in L^p that are necessary to prove the main results of this work.

2. Solving the Scattering Problem via Integral Equations

To tackle the scattering problem via integral equations, let us define the volume potential

$$(Vf)(x) = \int_{B_R} \Phi(x, y) f(y) \, \mathrm{d}y, \qquad x \in \mathbb{R}^d, \ d = 2, 3, \tag{4}$$

for smooth functions $f \in C_0^\infty(B_R)$ with compact support in $B_R = \{x \in \mathbb{R}^d,\ |x| < R\}$.

Lemma 1. *The volume potential V extends to a bounded operator from $L^t(B_R)$ into $W^{2,t}(B_R)$ for all $t \in (1, \infty)$ and all $R > 0$.*

Proof. For $t = 2$, this is a well-known result (see, e.g., [8]). For $t \neq 2$, the above bound follows from the Calderon-Zygmund decomposition, e.g., the one stated in [15, Theorem 9.9]. In detail, for $f \in C_0^\infty(B_R)$ it is well-known that $u(x) = (Vf)(x)$, $x \in \mathbb{R}^d$, solves $\Delta u + k^2 u = -f$ in $\mathbb{R}^d$ (where f is extended by zero to $\mathbb{R}^d$), that is, $\Delta u = -(k^2 u + f)$. Introducing an arbitrary smooth cut-off function χ with compact support in B_{2R} that equals one in B_R, we set $w := \chi u$, a smooth function with compact support. Hence, [15, Theorem 9.9] states that

$$\sum_{i,j=1}^{d} \left\| \frac{\partial^2 w}{\partial x_i \partial x_j} \right\|_{L^t(\mathbb{R}^d)} \leq C \| \chi(k^2 u + f) - 2\nabla\chi \cdot \nabla u - u\Delta\chi \|_{L^t(\mathbb{R}^d)}$$
$$\leq C(\chi) \left[\|u\|_{L^t(B_{2R})} + \|f\|_{L^t(B_{B_R})} + \|\nabla u\|_{L^t(B_{2R})^d} \right]. \quad (5)$$

It is moreover obvious that $\|u\|_{L^t(B_{2R})} \leq C\|f\|_{L^t(B_R)}$ since the integral operator V has a weakly singular kernel; the bound $\|\nabla u\|_{L^t(B_{2R})^d} \leq C\|f\|_{L^t(B_R)}$ follows by the same argument. Together with (5), this shows that

$$\sum_{i,j=1}^{d} \left\| \frac{\partial^2 w}{\partial x_i \partial x_j} \right\|_{L^t(\mathbb{R}^d)} \leq C(\chi) \|f\|_{L^t(B_R)}.$$

However, since χ equals to one in B_R, we showed in particular that $\|\partial^2 u/(\partial x_i \partial x_j)\|_{L^t(B_R)} \leq C(\chi)\|f\|_{L^t(B_R)}$ for $i, j = 1, \dots, d$. We conclude that $\|u\|_{W^{2,t}(B_R)} \leq C\|f\|_{L^t(B_R)}$ for all $f \in C_0^\infty(B_R)$. The claim now follows from the density of these functions in $L^t(B_R)$. □

As we already discussed, the Lippmann-Schwinger integral equation describes the scattered field in terms of the incident field restricted to the scatterer D. In the rest of the paper we assume that

$$\overline{D} = \operatorname{supp}(q) \subset B_R.$$

It is then obvious that we can consider the Lippmann-Schwinger equation (3), originally acting on functions defined in D, as an equation acting on functions defined in B_R. We denote the corresponding volume potential by $V_{B_R \to B_R}$ (to distinguish it from other potentials needed later on). Hence, the Lippmann-Schwinger integral equation becomes

$$u^s - k^2 V_{B_R \to B_R}(q u^s) = k^2 V_{B_R \to B_R}(q u^i) \qquad \text{in } B_R.$$

As preparation for the proof of the next proposition, let us note that the boundedness of the volume potential V from $L^{tp/(t+p)}(B_R)$ into $W^{2,tp/(t+p)}(B_R)$ and the generalized Hölder inequality (A.1) implies the bound

$$\|V_{B_R \to B_R}(qu)\|_{W^{2,tp/(t+p)}(B_R)} \leq C\|qu\|_{L^{tp/(t+p)}(B_R)} \leq C\|q\|_{L^p(B_R)}\|u\|_{L^t(B_R)}, \quad (6)$$

whenever $tp/(t+p) > 1$ and $t, p > 1$. Loosely speaking, we next show that $u \mapsto V(qu)$ is compact on $L^t(B_R)$ if t is big enough with respect to the Lebesgue index p of $q \in L^p(B_R)$.

Proposition 2. *Let $p > d/2$ and $q \in L^p(B_R)$.*
(a) If $\frac{p}{p-1} < t < \frac{dp}{2p-d}$, then the operator $u \mapsto V_{B_R \to B_R}(qu)$ is compact on $L^t(B_R)$.
(b) If $t = \frac{dp}{2p-d}$, then $u \mapsto V_{B_R \to B_R}(qu)$ is compact from $L^t(B_R)$ to $L^r(B_R)$ for $r \in [1, \infty)$.
(c) If $t > \frac{dp}{2p-d}$, then $u \mapsto V_{B_R \to B_R}(qu)$ is compact from $L^t(B_R)$ to $L^r(B_R)$ for $r \in [1, \infty]$.

Proof. (a) Note that $t > p/(p-1)$ implies that $tp/(t+p) > 1$. We want to exploit (6) and, to this end, note that the compact Sobolev embedding (see Lemma 12(a))

$$W^{2,tp/(t+p)}(B_R) \hookrightarrow L^r(B_R) \quad \text{for} \quad 1 \leq r < \frac{d\,tp/(t+p)}{d-2\,tp/(t+p)}$$

implies that $u \mapsto V_{B_R \to B_R}(qu)$ is compact on $L^t(B_R)$ if

$$t < \frac{d\,tp/(t+p)}{d-2\,tp/(t+p)}, \quad \text{that is, if} \quad 1 < \frac{dp/(t+p)}{d-2\,tp/(t+p)}.$$

The last inequality is equivalent to $d - 2tp/(t+p) < dp/(t+p)$, that is, to our general assumption $d/2 < p$.

(b) If $t = dp/(2p-d)$, then $d = 2tp/(t+p)$. The latter inequality implies (by Lemma 12(b)) that $W^{2,tp/(t+p)}(B_R) = W^{2,d/2}(B_R)$ is compactly embedded in $L^r(B_R)$ for all $r \in [1, \infty)$. Together with (6), this implies the claimed boundedness of $u \mapsto V(qu)$.

(c) If $t > dp/(2p-d)$, then $d < 2tp/(t+p)$. In this case, Sobolev's embedding lemma (see again Lemma 12(c)) states that $W^{2,tp/(t+p)}(B_R)$ is compactly embedded in $L^r(B_R)$ for all $r \in [1, \infty]$. The bound (6) is then again sufficient to conclude. □

The next result is a unique continuation property for L^p-solutions to the Helmholtz equation. To state this result, we use the spaces

$$W^{m,r}_{\mathrm{loc}}(\mathbb{R}^d) := \{v : \mathbb{R}^d \to \mathbb{C},\, v \in W^{m,r}(B_R) \text{ for all } R > 0\}, \qquad m \in \mathbb{N}_0,\ r \geq 1.$$

Lemma 3. *Let $q \in L^p(B_R)$ for $p > d/2$ such that $\mathrm{Im}\,(q) \geq 0$ in B_R. Assume that $u \in W^{2,r}_{\mathrm{loc}}(\mathbb{R}^d)$ with*

$$\frac{2d}{d+2} < r < \infty$$

solves $\Delta u + k^2 n^2 u = 0$ in the distributional sense in $\mathbb{R}^d$, subject to the Sommerfeld radiation condition (2). Then u vanishes in $\mathbb{R}^d$.

Proof. Since the Helmholtz equation $\Delta u + k^2 n^2 u = 0$ has constant coefficients in the complement of B_R, the solution u is a real-analytic function outside B_R (see Theorem 9.19 in [15]) and the radiation condition (2) is well-defined. If $u \in W^{2,r}_{\mathrm{loc}}(\mathbb{R}^d)$ for $2d/(2+d) < r < \infty$, Sobolev's embedding theorem (see Lemma 12) implies that u belongs to $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d)$, since $\|u\|_{W^{1,2}(B_R)} \leq C(R)\|u\|_{W^{2,r}(B_R)}$ for all $R > 0$.

Integrating $\Delta u + k^2 n^2 u = 0$ in B_R against $\overline{u}$ and partial integration shows that

$$\int_{B_R} \left(|\nabla u|^2 - k^2 n^2 |u|^2 \right) \mathrm{d}x = \int_{B_R} \overline{u} \frac{\partial u}{\partial \nu} \, \mathrm{d}S.$$

Since $\mathrm{Im}\,(q) = \mathrm{Im}\,(n^2) > 0$ by assumption, $\int_{B_R} u \; \partial \overline{u}/\partial \nu \, \mathrm{d}S \geq 0$ and Rellich's lemma (see [8]) implies that u vanishes outside a ball of radius R such that $D \subset B_R$. Since solutions to the homogeneous Helmholtz equation are analytic, u even vanishes in the complement of $\overline{B_R}$. The partial differential equation $\Delta u + k^2 n^2 u = 0$ that is satisfied almost everywhere in $\mathbb{R}^d$ implies that

$$|\Delta u(x)| \leq k^2 |n^2(x)|\, |u(x)| = k^2(1+|q|)\, |u| \qquad \text{almost everywhere in } \mathbb{R}^d. \tag{7}$$

The above assumptions fit to Theorem 6.3 in [6] (see also Remark 6.7 in that reference), yielding that u vanishes entirely in $\mathbb{R}^d$. □

Remark 4. *The results from [6] require proofs that are far from elementary. For our problem, easier Fourier series techniques from [7] can be employed. In an L^p-setting, these techniques yield, however, suboptimal results in the Lebesgue coefficient p. We nevertheless sketch these results here, since we employ them when proving strong convergence of the Tikhonov regularized solutions for the inverse problem in Section 4.*

Assume hence that $u \in W^{2,r}_{\mathrm{loc}}(\mathbb{R}^d)$ with $r > d/2$ is a weak solution of $\Delta u + k^2 n^2 u = \Delta u + k^2(1+q)u = 0$ for $q \in L^p(B_R)$ with $p > d$. We already noted in the proof of Lemma 3 that such solutions u automatically belong to $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d)$ and that they vanish outside of B_R. For $R' > R$ and $t > 0$ set $\zeta_t = (t, \mathrm{i}t)^\top$ for $d = 2$ and $\zeta_t = (t, \mathrm{i}t, 0)^\top$ for $d = 3$. Then

$$w_t(x) = \exp(-\mathrm{i}\zeta_t \cdot x) u(x), \quad x \in Q := [-R', R']^d,$$

can be extended to a $2R'$-periodic function in $\mathbb{R}^d$. Hence, w_t belongs to the periodic Sobolev space $H^1_{\mathrm{per}}(Q)$, defined by

$$H^s_{\mathrm{per}}(Q) = \Big\{ v(x) = \sum_{j\in\mathbb{Z}^3} \hat{v}_n e^{\mathrm{i}n\pi/Rx},\ \sum_{j\in\mathbb{Z}^3} (1+|n|^2)^s |\hat{v}_n|^2 < \infty \Big\}, \quad s \in \mathbb{R}. \tag{8}$$

Analogously to the periodization of w_t, we restrict the refractive index n^2 to Q and extend it to a $2R'$-periodic function. Since u satisfies the Helmholtz equation in $L^2(Q)$, the product rule yields that

$$\Delta w_t + 2\mathrm{i}\zeta_t \cdot \nabla w_t + k^2 n^2 w_t = -k^2 n^2 w_t = -k^2(1+q) w_t \quad \text{in } L^2(Q).$$

In Theorem 1 and the subsequent remark in [7], it is shown that the solution operator G_t to this periodic partial differential equation satisfies $\|G_t\|_{L^2(Q)\to L^2(Q)} \leq C/t$ and $\|G_t\|_{L^2(Q)\to H^1_{\mathrm{per}}(Q)} \leq C$. Standard interpolation theory for periodic Sobolev spaces hence yields that $\|G_t\|_{L^2(Q)\to H^s_{\mathrm{per}}(Q)} \leq C t^{s-1}$ for $0 \leq s \leq 1$. Hence,

$$\begin{aligned} \|w_t\|_{H^s_{\mathrm{per}}(Q)} &= k^2 \|G_t(n^2 w_t)\|_{H^s_{\mathrm{per}}(Q)} \leq C t^{s-1} \|n^2 w_t\|_{L^2(Q)} \\ &\leq C t^{s-1} \|n^2\|_{L^p(D)} \|w_t\|_{L^{2p/(p-2)}(Q)} \\ &\leq C t^{s-1} \|n^2\|_{L^p(D)} \|w_t\|_{H^s_{\mathrm{per}}(Q)} \quad \text{if } d < sp. \end{aligned} \tag{9}$$

If $d < sp$, the last inequality follows from Lemma 13, stating that $H^s(Q)$ is compactly embedded in $L^{2p/(p-2)}(Q)$, since $d < sp$ implies that $2p/(p-2) < 2d/(d-2s)$. Indeed,

$$\tfrac{2p}{p-2} < \tfrac{2d}{d-2s} \Leftrightarrow 2p(d-2s) < 2d(p-2) \Leftrightarrow -4ps < -4d \Leftrightarrow d < sp.$$

By our assumption $d < p$, there is some $s \in (0,1)$ such that $d < sp$. Finally choosing t large enough, we conclude from (9) that w_t, and hence also u, must vanish.

In the remainder of this paper, we always choose $q \in L^p(D)$ for $p > d/2$, and then determine a Lebesgue index $t > 1$ depending on p such that the Lippmann-Schwinger integral equation is uniquely solvable in $L^t(B_R)$.

Assumption 5 (Choice of p and t)**.** *We fix $p > d/2\,(\geq 1)$ to work with contrasts $q \in L^p(B_R)$ and choose*

$$t > \max\left\{\tfrac{p}{p-1},\, \tfrac{2d}{d+2}\right\}, \tag{10}$$

which guarantees that $t > 1$. Theorem 2 implies that the Lippmann-Schwinger equation is then well-defined in $L^t(B_R)$. To be able to apply the unique continuation result stated in Lemma 3, we additionally need that

$$\tfrac{tp}{t+p} > \tfrac{2d}{d+2}. \tag{11}$$

(In dimension $d = 2$, condition (10) is equivalent to $t > p/(p-1)$ which furthermore implies (11).)

Combining the last two results with the well-know Riesz theory (see [17]) yields solvability of the Lippmann-Schwinger equation

$$v - k^2 V_{B_R \to B_R}(qv) = f \qquad \text{in } L^t(B_R) \tag{12}$$

for contrasts in $L^p(D)$: Under Assumption 5, compactness of $v \mapsto V_{B_R \to B_R}(qv)$ on $L^t(B_R)$ follows from Proposition 2, and uniqueness of solution follows from Lemma 3.

Theorem 6. *Let $q \in L^p(B_R)$ for $p > d/2$ such that* $\mathrm{Im}\,(q) \geq 0$ *in B_R and choose $t > 1$ according to Assumption 5. Then (12) has a unique solution $v \in L^t(B_R)$ and $\|v\|_{L^t(B_R)} \leq C\|f\|_{L^t(B_R)}$. If $f = k^2 V_{B_R \to B_R}(qu^i)$ for some incident field $u^i \in L^t(B_R)$, then $u^s = k^2 V_{B_R \to B_R}(q(v + u^i))$ defines a radiating solution $u^s \in W^{2,tp/(t+p)}_{\mathrm{loc}}(\mathbb{R}^d)$ to the Helmholtz equation $\Delta u^s + k^2 n^2 u^s = -k^2 q u^i$ in $L^t_{\mathrm{loc}}(\mathbb{R}^d)$.*

Once the scattered field u^s is known in D, u^s can be evaluated everywhere in $\mathbb{R}^d$ using the integral equation,

$$u^s(x) = k^2 \int_D \Phi(x, y) q(y)(u^s(y) + u^i(y))\,\mathrm{d}y, \quad x \in \mathbb{R}^d.$$

3. Properties of the Contrast-to-Measurement Map

The solution theory established in Theorem 6 allows to associate to a contrast $q \in L^p(B_R)$ and an incident field $u^i \in L^t(B_R)$ a unique solution of the scattering problem (1, 2). The inverse problem we consider in this paper is the determination of q from measurements of u^s taken a finite distance away from the scattering object. We assume that the measurements are given as point measurements of u^s on a non-empty, closed Lipschitz surface Γ_{m}. (See [16] for the definition of a Lipschitz surface.) For simplicity, we suppose that Γ_{m} and $\overline{B_R}$ do not intersect. Under this assumption, the evaluation $V_{B_R \to \Gamma_{\mathrm{m}}}$ of the volume potential defined in B_R on the surface Γ_{m} is an integral operator with smooth kernel and hence compact between any reasonable Sobolev function spaces.

The operator mapping (q, u^i) to $u^s|_{\Gamma_m}$ is called the (mono-static) contrast-to-measurement operator in the sequel. We start the analysis of the inverse problem by proving important boundedness and continuity properties of this operator, and then extend these results to multi-static data. To this end, we assume in this section that the Lebesgue indices $p > d/2$ and $t > 1$ are always chosen according to Assumption 5, such that Theorem 6 is applicable.

Let us fix an incident field $u^i \in L^t(B_R)$, a smooth solution of the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in B_R. Consider contrasts q that belong to the closed and convex set

$$L^p_{\mathrm{Im}\geq 0}(B_R) := \{q \in L^p(B_R) : \mathrm{Im}\,(q) \geq 0 \text{ in } B_R\} \subset L^p(B_R).$$

For $q \in L^p_{\mathrm{Im}\geq 0}(B_R)$ the inverse

$$T_q := (I - k^2 V_{B_R\to B_R}(q\,\cdot))^{-1} \tag{13}$$

is a bounded operator on $L^t(B_R)$ due to Assumption 5 and Theorem 6. In consequence, the non-linear contrast-to-measurement operator

$$S : L^p_{\mathrm{Im}\geq 0}(B_R) \subset L^p(B_R) \times L^p(B_R) \to L^2(\Gamma_m), \quad (q, u^i) \mapsto u^s(q, u^i)\big|_{\Gamma_m}$$

is well-defined. Explicitly,

$$\begin{aligned} S(q, u^i) &= k^2 V_{B_R\to\Gamma_m}(q(u^s(q) + u^i)) \\ &= k^2 V_{B_R\to\Gamma_m}(qu^i + k^2 q(I - k^2 V_{B_R\to B_R}(q\,\cdot))^{-1} V_{B_R\to B_R}(qu^i)) \qquad (14) \\ &= k^2 V_{B_R\to\Gamma_m}[q(I - k^2 V_{B_R\to B_R}(q\,\cdot))^{-1} u^i] = k^2 V_{B_R\to\Gamma_m}[qT_q u^i]. \end{aligned}$$

Lemma 7. *Assume that $p > d/2$ and $t > 1$ satisfy Assumption 5.*
(a) If $\{q_n\}_{n\in\mathbb{N}} \subset L^p_{\mathrm{Im}\geq 0}(B_R)$ is weakly convergent in $L^p(B_R)$ to q, then q belongs to $L^p_{\mathrm{Im}\geq 0}(B_R)$ and $T_{q_n} v \to T_q v$ as $n \to \infty$ in $L^t(B_R)$ pointwise for all $v \in L^t(B_R)$.
(b) Under the assumptions of part (a), let $R : X \to L^t(B_R)$ be a compact operator from a Banach space X into $L^t(B_R)$. Then $\|(T_{q_n} - T_q)R\|_{X\to L^t(B_R)} \to 0$ as $n \to \infty$.
(c) The mapping $q \mapsto T_q$ from $L^p_{\mathrm{Im}\geq 0}(B_R) \subset L^p(B_R)$ into the space of linear bounded operators on $L^t(B_R)$ is uniformly bounded on each bounded subset of $L^p_{\mathrm{Im}\geq 0}(B_R)$.

Proof. Our proof relies on collectively compact operator theory, see Appendix B. To this end, we abbreviate $K := k^2 V_{B_R\to B_R}(q\,\cdot)$ and $K_n := V_{B_R\to B_R}(q_n\,\cdot)$.

(a) Since $q_n \rightharpoonup q \in L^p$, the norms $\|q_n\|_{L^p(B_R)}$ are uniformly bounded in n. Moreover, for each real-valued and positive $\psi \in C_0^\infty(B_R)$ $0 \leq \int_{B_R} \mathrm{Im}\,(q_n)\psi \,\mathrm{d}x \to \int_{B_R} \mathrm{Im}\,(q)\psi \,\mathrm{d}x$, which implies that $q \in L^p_{\mathrm{Im}\geq 0}(B_R)$. Recall from (6) that

$$\|K_n u\|_{L^t(B_R)} \leq C\|K_n u\|_{W^{2,tp/(t+p)}(B_R)} \leq C\|q_n\|_{L^p(B_R)}\|u\|_{L^t(B_R)} \leq CC^*\|u\|_{L^t(B_R)}.$$

Due to Proposition 2, the embedding $W^{2,tp/(t+p)}(B_R) \hookrightarrow L^t(B_R)$ is compact. Hence, for each $B > 0$, the set $\{K_n u : u \in L^t(B_R), \|u\|_{L^t(B_R)} < B, n \in \mathbb{N}\}$ is pre-compact in $L^t(B_R)$. This means that $\{K_n\}$ is collectively compact on $L^t(B_R)$.

It is straightforward to see that $q_n \rightharpoonup q$ implies that $K_n v$ converges to Kv in $L^t(B_R)$, pointwise for all $v \in L^t(B_R)$. Indeed, since $tp/(t + p) > 1$ it follows that $q_n v \rightharpoonup qv$ in $L^{tp/(t+p)}(B_R)$. Since $v \mapsto k^2 V_{B_R\to B_R} v$ is compact on $L^t(B_R)$, the sequence $k^2 V_{B_R\to B_R}(q_n v) = K_n(v)$ converges hence strongly in $L^t(B_R)$.

Let now $v_n - K_n v_n = f$ and $v - Kv = f$ in $L^t(B_R)$ for some f in $L^t(B_R)$. Due to the collective compactness and the pointwise convergence of the operators K_n, we can apply Theorem 14 to obtain the error estimate

$$\|v_n - v\|_{L^t(B_R)} \le C\|(K_n - K)v\|_{L^t(B_R)} = C\|V_{B_R\to B_R}((q_n - q)v)\|_{L^t(B_R)}.$$

As above, $q_n \rightharpoonup q$ yields that $V_{B_R\to B_R}((q_n - q)v) \to 0$ in $L^t(B_R)$. Hence, $v_n \to v$ in $L^t(B_R)$, or equivalently, $T_{q_n} f \to T_q f$ in $L^t(B_R)$. We have hence shown pointwise convergence of T_{q_n} to T_q.

While part (b) follows directly from Theorem 14, we briefly prove (c) by contradiction: If the assertion does not hold, then there is a bounded sequence $\{q_n\}_{n\in\mathbb{N}} \subset L^p_{\mathrm{Im}\ge 0}(B_R)$ such that the operator norms $\|T_{q_n}\|_{L^t(B_R)\to L^t(B_R)}$ are unbounded. Due to the boundedness of $\{q_n\}$ we can extract a weakly convergent subsequence, $q_n \rightharpoonup q \in L^p_{\mathrm{Im}\ge 0}(B_R)$. Since $\{K_n\}$ is collectively compact, Theorem 14 states that

$$\|T_{q_n}\|_{L^t(B_R)\to L^t(B_R)} = \|[I - K_n]^{-1}\|_{L^t(B_R)\to L^t(B_R)} \le C \quad \text{uniformly in } n \in \mathbb{N}.$$

□

In the next corollary we exploit that the contrast-to-measurement operator $q \mapsto S(q,f) = k^2 V_{B_R\to\Gamma_{\mathrm{m}}}[qT_q(f)]$ is a composition of T_q with the compact operator $v \mapsto k^2 V_{B_R\to\Gamma_{\mathrm{m}}}[qv]$.

Corollary 8. *Assume that $p > d/2$ and $t > 1$ satisfy Assumption 5. For fixed $f \in L^t(B_R)$ the contrast-to-measurement operator $q \mapsto S(q,f)$ is continuous, compact, and weakly sequentially closed from $L^p_{\mathrm{Im}\ge 0}(B_R) \subset L^p(B_R)$ into $L^2(\Gamma_{\mathrm{m}})$.*

4. Multi-Static Data and Tikhonov Regularization in L^p

The aim of this paper is to reconstruct a contrast function in a Banach space $L^p(B_R)$ from near-field measurements. Unique determination of the contrast in terms of the measured data can only hold for multi-static scattering data: We use incident point sources on a closed Lipschitz surface $\Gamma_{\mathrm{i}} \subset \mathbb{R}^d$ enclosing $\overline{B_R}$ and measure the resulting scattered fields on the measurement surface Γ_{m} (introduced in the last section). Due to the superposition principle, we can equivalently use single layer potentials as incident fields,

$$\mathrm{SL}_{\Gamma_{\mathrm{i}}}\varphi = \int_{\Gamma_{\mathrm{i}}} \Phi(\cdot,y)\varphi(y)\,\mathrm{d}y \quad \text{in } \mathbb{R}^d \setminus \Gamma_{\mathrm{i}}.$$

We will always assume that $\Gamma_{\mathrm{i}} \cap \overline{B_R} = \emptyset$ since, in this case, $\mathrm{SL}_{\Gamma_{\mathrm{i}}}$ is an integral operator with smooth kernel that is hence compact from $L^2(\Gamma_{\mathrm{i}})$ into $L^t(B_R)$ for all $t \in [1,\infty]$. The scattered field corresponding to the incident field $\mathrm{SL}_{\Gamma_{\mathrm{i}}}\varphi$ is then recorded on the measurement surface $\Gamma_{\mathrm{m}} \subset \mathbb{R}^d$ that we already used in the last section. By construction of the forward operator $S(q,\cdot)$, this field equals equals $S(q, \mathrm{SL}_{\Gamma_{\mathrm{i}}}\varphi)$. Hence, we define a multi-static contrast-to-measurement operator as follows,

$$\mathcal{N} :\, L^p_{\mathrm{Im}\ge 0}(B_R) \subset L^p(B_R) \to \mathrm{HS}\left(L^2(\Gamma_{\mathrm{i}}), L^2(\Gamma_{\mathrm{m}})\right), \quad q \mapsto N_q \tag{15}$$

where $N_q : L^2(\Gamma_\mathrm{i}) \to L^2(\Gamma_\mathrm{m})$ is defined by $\varphi \mapsto S(q, \mathrm{SL}_{\Gamma_\mathrm{i}}\varphi)$. Here, $\mathrm{HS}\,(L^2(\Gamma_\mathrm{i}), L^2(\Gamma_\mathrm{m}))$ denotes the space of Hilbert-Schmidt operators from $L^2(\Gamma_\mathrm{i})$ into $L^2(\Gamma_\mathrm{m})$ (see Appendix C). Let us first explain why N_q is a Hilbert-Schmidt operator. The superposition principle for linear differential equations implies that this operator can be represented as a linear integral operator with kernel $u_q^s(x, y) = S(q, \Phi(\cdot, y))(x)$ for $x \in \Gamma_\mathrm{m}$ and $y \in \Gamma_\mathrm{i}$. More precisely,

$$N_q : \varphi \mapsto \int_{\Gamma_\mathrm{i}} u_q^s(\cdot, y)\varphi(y)\,\mathrm{d}s(y). \tag{16}$$

It is easy to see that this kernel is square-integral in both variables, due to the C^∞-smoothness of the incident and scattered fields outside the scatterer. Hence, Lemma 15 implies that $N_q \in \mathrm{HS}\,(L^2(\Gamma_\mathrm{i}), L^2(\Gamma_\mathrm{m}))$.

Theorem 9. *Assume that $p > d/2$. Then the mapping $\mathcal{N}$ is continuous, compact, and weakly sequentially closed from $L^p_{\mathrm{Im}\geq 0}(B_R) \subset L^p(B_R)$ into* $\mathrm{HS}\,(L^2(\Gamma_\mathrm{i}), L^2(\Gamma_\mathrm{m}))$.

Proof. For the entire proof, we assume that the Lebesgue index t for the solution space $L^t(B_R)$ of the Lippmann-Schwinger integral equation is chosen as in Assumption 5. The basic ingredient of the proof is that the class of Hilbert-Schmidt operators is stable under multiplication with bounded linear operators, see (C.1). Since $V_{B_R\to\Gamma_\mathrm{m}}$ is an integral operator with smooth kernel, this operator is bounded from $L^r(B_R)$ for arbitrary $r \in (0, \infty)$ into all Sobolev spaces $H^m(\Gamma_\mathrm{m}) = W^{m,2}(\Gamma_\mathrm{m})$ (see [16] for a definition of these spaces). If $m > 0$ is chosen large enough – $m = 2$ is sufficient – then it is well-known that the embedding operator from $H^m(\Gamma_\mathrm{m})$ into $L^2(\Gamma_\mathrm{m})$ is Hilbert-Schmidt (see, e.g., [18]). Hence, $\mathcal{N}(q)[\varphi] = S(q, \mathrm{SL}_{\Gamma_\mathrm{i}}\varphi) = k^2 V_{B_R\to\Gamma_\mathrm{m}}[qT_q(\mathrm{SL}_{\Gamma_\mathrm{i}}\varphi)]$ is the composition of the Hilbert-Schmidt embedding from $H^m(\Gamma_\mathrm{m})$ into $L^2(\Gamma_\mathrm{m})$ with the compact linear operator

$$L^{tp/(t+p)}(B_R) \ni v \mapsto k^2 V_{B_R\to\Gamma_\mathrm{m}} v \in H^m(\Gamma_\mathrm{m})$$

and with the bounded linear operator $\varphi \mapsto qT_q(\mathrm{SL}_{\Gamma_\mathrm{i}}\varphi)$. Hence, (C.1) implies

$$\begin{aligned}\|\mathcal{N}(q)\|_{\mathrm{HS}(L^2(\Gamma_\mathrm{i}),L^2(\Gamma_\mathrm{m}))} \leq k^2 \|I\|_{\mathrm{HS}(L^2(\Gamma_\mathrm{m}),H^m(\Gamma_\mathrm{m}))}\, \|V_{B_R\to\Gamma_\mathrm{m}}\|_{L^{tp/(t+p)}(B_R)\to H^m(\Gamma_\mathrm{m})} \\ \|q\|_{L^p(B_R)}\, \|\varphi \mapsto T_q(\mathrm{SL}_{\Gamma_\mathrm{i}}\varphi)\|_{L^2(\Gamma_\mathrm{i})\to L^t(B_R)}.\end{aligned} \tag{17}$$

To prove compactness of $q \mapsto \mathcal{N}(q)$ it is now sufficient to note that the smoothness of the kernel of $\mathrm{SL}_{\Gamma_\mathrm{i}}$ implies that this operator is compact from $L^2(\Gamma_\mathrm{i})$ into $L^t(B_R)$. Moreover, Lemma 7(b,c) implies that $q \mapsto [\varphi \mapsto T_q(\mathrm{SL}_{\Gamma_\mathrm{i}}\varphi)]$ is continuous and bounded from $L^p(B_R)$ into the space of bounded and linear operators from $L^2(\Gamma_\mathrm{i})$ into $L^t(B_R)$.

Finally, the weak sequential closedness of $q \mapsto \mathcal{N}(q)$ follows analogously: If $L^p_{\mathrm{Im}\geq 0}(B_R) \ni q_n \rightharpoonup q$ in $L^p(B_R)$, then $q \in L^p_{\mathrm{Im}\geq 0}(B_R)$ by Lemma 7(a) and $\|[T_{q_n} - T_q](\mathrm{SL}_{\Gamma_\mathrm{i}}\cdot)\|_{L^2(\Gamma_\mathrm{i})\to L^t(B_R)} \to 0$ due to Lemma 7(b). Since multiplication by q is a bounded and linear operation from $L^t(B_R)$ into $L^{tp/(t+p)}(B_R)$ and since $V_{B_R\to\Gamma_\mathrm{m}}$ does not depend on q, the decomposition exploited in (17) shows that $\mathcal{N}(q_n) \rightharpoonup \mathcal{N}(q)$. □

Now, assume that $q^\dagger \in L^p_{\mathrm{Im}\geq 0}(B_R)$ is the searched-for exact contrast corresponding to the exact near-field operator $N_{q^\dagger} := \mathcal{N}(q^\dagger)$, see (16). Assume further that for $\varepsilon > 0$ we possess noisy measured data $N^\varepsilon_{\mathrm{meas}} \in \mathrm{HS}\,(L^2(\Gamma_{\mathrm{i}}), L^2(\Gamma_{\mathrm{m}}))$ such that

$$\|\mathcal{N}(q^\dagger) - N^\varepsilon_{\mathrm{meas}}\|_{\mathrm{HS}(L^2(\Gamma_{\mathrm{i}}),L^2(\Gamma_{\mathrm{m}}))} \leq \varepsilon. \tag{18}$$

Typically, $N^\varepsilon_{\mathrm{meas}}$ is an integral operator with kernel given by noisy measurements of the the exact scattered fields $u^s_q(x,y)$ for $x \in \Gamma_{\mathrm{m}}$ and $y \in \Gamma_{\mathrm{i}}$. The equation

$$\mathcal{N}(q) = N_{q^\dagger}, \qquad q \in L^p_{\mathrm{Im}\geq 0}(B_R) \subset L^p(B_R), \tag{19}$$

for q is locally ill-posed about $q^\dagger$, see [4, Def. 3.15]: Indeed, for any *real-valued* sequence $\{e_n\}_{n\in\mathbb{N}}$ such that $\|e_n\|_{L^p(B_R)} = 1$ and $e_n \rightharpoonup 0$ as $n \to \infty$, and any radius $r > 0$, it holds that $q^\dagger + re_n \in L^p_{\mathrm{Im}\geq 0}(B_R)$ converges weakly to $q^\dagger$. However, the compactness of $q \mapsto \mathcal{N}(q)$ shown in Theorem 9 implies that $\mathcal{N}(q^\dagger + re_n) \to \mathcal{N}(q^\dagger)$ in $\mathrm{HS}\,(L^2(\Gamma_{\mathrm{i}}), L^2(\Gamma_{\mathrm{m}}))$, that is, (19) is locally-ill posed about (any) $q^\dagger$ in $L^p_{\mathrm{Im}\geq 0}(B_R)$. Hence, the inversion of (19) has to be regularized. For regularization we introduce the Tikhonov functional

$$\mathcal{J}^\varepsilon_\alpha(q) := \|\mathcal{N}(q) - N^\varepsilon_{\mathrm{meas}}\|^2_{\mathrm{HS}(L^2(\Gamma_{\mathrm{i}}),L^2(\Gamma_{\mathrm{m}}))} + \frac{\alpha}{p}\|q\|^p_{L^p(B_R)} \quad \text{on } L^p_{\mathrm{Im}\geq 0}(B_R) \subset L^p(B_R) \tag{20}$$

for parameters $\alpha > 0$, $\varepsilon \geq 0$ and $p > d/2$ and set $\mathcal{J}^\varepsilon_\alpha(q) = \infty$ if $q \in L^p(B_R) \setminus L^p_{\mathrm{Im}\geq 0}(B_R)$. (We sketch in Appendix E that a larger domain of definition of $\mathcal{J}^\varepsilon_\alpha$ is possible; all subsequent convergent results do also hold for the larger domain indicated in (E.1).)

Theorem 10. *Assume that $p > d/2$, that $q^\dagger \in L^p_{\mathrm{Im}\geq 0}(B_R)$, and that the family $\{N^\varepsilon_{\mathrm{meas}}\}_{\varepsilon>0} \subset \mathrm{HS}\,(L^2(\Gamma_{\mathrm{i}}), L^2(\Gamma_{\mathrm{m}}))$ satisfies (18). Choose $\alpha = \alpha(\varepsilon)$ such that*

$$0 < \alpha(\varepsilon) \to 0 \text{ and } 0 < \varepsilon^2/\alpha(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

Then there exists a minimizer $q^\varepsilon_{\alpha(\varepsilon)}$ of (20) in $L^p_{\mathrm{Im}\geq 0}(B_R)$ with $\alpha = \alpha(\varepsilon)$, for any $\varepsilon > 0$. If $0 < \varepsilon_n \to 0$ as $n \to \infty$, then $\{q^{\varepsilon_n}_{\alpha(\varepsilon_n)}\}_{n\in\mathbb{N}}$ contains an $L^p(B_R)$-convergent subsequence that converges to a norm-minimizing solution $q^* \in L^p_{\mathrm{Im}\geq 0}(B_R)$ of (19), i.e.,

$$\|q^*\|_{L^p(B_R)} \leq \|q\|_{L^p(B_R)} \quad \text{for all solutions } q \in L^p_{\mathrm{Im}\geq 0}(B_R) \text{ to (19).}$$

Proof. We apply Theorem 16 with $r = 2$, $F = \mathcal{N}$, $D(F) = L^p_{\mathrm{Im}\geq 0}(B_R)$, and $V = \mathrm{HS}\,(L^2(\Gamma_{\mathrm{i}}), L^2(\Gamma_{\mathrm{m}}))$. Due to Theorem 9, the assumptions of Theorem 16 are easy to check: For $\alpha, \varepsilon, M > 0$, the level set $L = \{q \in L^p(B_R) : \mathcal{J}^\varepsilon_\alpha(q) \leq M\}$ is obviously bounded in the L^p-norm by $(Mp/\alpha)^{1/p} > 0$ and contained in $L^p_{\mathrm{Im}\geq 0}(B_R)$ by definition of $\mathcal{J}^\varepsilon_\alpha$. Hence, L is weakly sequentially compact in $L^p(B_R)$. Suppose that $\{q_n\}_{n\in\mathbb{N}} \subset L^p_{\mathrm{Im}\geq 0}(B_R)$ is an arbitrary sequence satisfying $\mathcal{J}^\varepsilon_\alpha(q_n) \leq M$ and $q_n \rightharpoonup q \in L^p(B_R)$ as $n \to \infty$. Necessarily, $q \in L^p_{\mathrm{Im}\geq 0}(B_R)$. Moreover, $\mathcal{N}(q_n) \to \mathcal{N}(q)$ in $\mathrm{HS}\,(L^2(\Gamma_{\mathrm{i}}), L^2(\Gamma_{\mathrm{m}}))$ due to the compactness of $\mathcal{N}$ shown in Theorem 9. This means in particular that the restriction of $\mathcal{N}$ to L is weakly sequentially continuous. Since the L^p- and the Hilbert-Schmidt norm are both weakly lower semi-continuous, $\mathcal{J}^\varepsilon_\alpha(q) \leq \lim_{n\to\infty} \mathcal{J}^\varepsilon_\alpha(q_n) \leq M$. This implies that L is weakly sequentially closed in $L^p(B_R)$. ◻

Theorem 11. *Under the assumptions of Theorem 10, suppose additionally that the dimension d equals three and that $p > d = 3$. Then the solution $q^\dagger$ to (19) is unique and $q^\varepsilon_{\alpha(\varepsilon)} \to q^\dagger$ in $L^p(B_R)$ as $\varepsilon \to 0$.*

Proof. Due to Theorems 10 and 16, we merely need to show that $q^\dagger$ is the only solution to $\mathcal{N}(q) = N_{q^\dagger}$. This is based on unique determination results in dimension three, see [19, 20, 21]. These results are usually stated for $q \in L^\infty(B_R)$ and for far-field data, see, e.g., Section 6.4 in [9]. (An exception is, e.g., [20], considering the Helmholtz equation in a bounded domain.) Converting far-field data into near-field data and vice versa is based on well-known unique continuation results for the Helmholtz equation, see, e.g., [8], that are applicable due to our assumptions on the closed surfaces Γ_i and Γ_m (both enclose $\overline{B_R}$).

The treatment of contrasts in $L^p(B_R)$, $p > d = 3$, is less straightforward and relies essentially on Remark 4. We choose the Lebesgue index $t > 1$ for the solution to the scattering problem such that Assumption 5 is satisfied, and *additionally* assume that $tp/(t+p) > 3/2$. Then Theorem 12 implies that any function in $W^{2,tp/(t+p)}(B_R)$ belongs to $H^1(B_R) \cap C^0_\mathrm{B}(\overline{(}B_R))$. These two properties are essential to transfer the uniqueness proof of, e.g., [9, Sect. 6.4] to our setting.

Denote as in (16) by $u^s_q(\cdot, y) \in W^{2,tp/(t+p)}(B_R)$ the unique weak solution to the scattering problem (1, 2) for incident field $\Phi(\cdot, y)$, $y \in \Gamma_\mathrm{i}$. One first shows that the set of total fields

$$\{u_q(\cdot, y) : B_R \to \mathbb{C},\ u_q(\cdot, y) := \Phi(\cdot, y) + u^s_q(\cdot, y) \text{ for } y \in \Gamma_\mathrm{i}\} \subset W^{2,tp/(t+p)}(B_R),$$

is dense in $\{v \in W^{2,tp/(t+p)}(B_R),\ \Delta v + k^2 n^2 v = 0 \text{ in } B_R\}$ with respect to the $L^2(B_R)$-norm. Note that the Helmholtz equation is again understood in the distributional sense. All integrals in the proof of [9, Lem. 6.22] are well-defined due to the above-discussed embeddings of $W^{2,tp/(t+p)}(B_R)$ and the proof can be straightforwardly transferred.

Second, one shows that if $\mathcal{N}(q_1) = \mathcal{N}(q_2)$ for $q_{1,2} \in L^p(B_R)$, then

$$\int_{B_R} v_1 v_2 [q_1 - q_2] \,\mathrm{d}x = 0 \tag{21}$$

for all solutions $v_{1,2} \in W^{2,tp/(t+p)}(B_R)$ to $\Delta v_{1,2} + k^2(1 + q_{1,2})v_{1,2} = 0$ in B_R. Since the equality $\mathcal{N}(q_1) = \mathcal{N}(q_2)$ implies that $u_{q_1}(\cdot, y) = u_{q_2}(\cdot, y)$ on Γ_m for all $y \in \Gamma_\mathrm{i}$, the proof of [9, Lem. 6.23] can again be directly transferred to our setting.

Third, one constructs distributional solutions $u_z \in L^2(B_R)$ of the form $u_z(x) = \exp(z \cdot x)(1 + v_z(x))$ to the Helmholtz equation $\Delta u_z + k^2(1 + q)u_z = 0$ that depend on a parameter $z \in \mathbb{C}^3$ with $z \cdot z = 0$. The crucial property of these solutions is that $\|v_z\|_{L^2(B_R)} \le C/|z|$ for all $z \in \mathbb{C}^3$ with $z \cdot z = 0$ and $|z|$ large enough. If $p > d$, then the construction of these solutions for $q \in L^p(B_R)$ works precisely as in Remark 4, and is the analogue to [9, Th. 6.24].

To prove the unique determination result, one finally plugs in solutions $u_{z^{1,2}}$ to $\Delta u_{z^{1,2}} + k^2(1 + q_{1,2})u_{z^{1,2}} = 0$ for two different parameters $z^{1,2} \in \mathbb{C}^3$ corresponding to the two contrasts q_1 and q_2 into (21). By a clever choice of $z^{1,2}$ (see, e.g., [9, Th. 6.25]) one finds that the continuous Fourier transform of $q_1 - q_2$ (which exists, since both functions have compact support) vanishes. Hence, $q_1 = q_2$. This last step works precisely as the proof of [9, Th. 6.25]. □

5. A Shrinked Landweber Scheme and Numerical Examples

Theorems 10 and 11 provide convergence results for a non-linear Tikhonov regularization in L^p-spaces applied to inverse medium scattering problems. In this section we discuss a numerical method to actually compute minimizers of the Tikhonov functional from (20),

$$\mathcal{J}_\alpha^\varepsilon(q) := \tfrac{1}{2}\|\mathcal{N}(q) - N_{\mathrm{meas}}^\varepsilon\|^2_{\mathrm{HS}(L^2(\Gamma_\mathrm{i}),L^2(\Gamma_\mathrm{m}))} + \tfrac{\alpha}{p}\|q\|^p_{L^p(B_R)}.$$

For simplicity, we will only consider real-valued contrasts in this section and denote the space of real-valued contrasts in $L^p(B_R)$ by $L^p_\mathbb{R}(B_R)$. By considering the (formal) first-order optimality conditions, we obtain that

$$[\mathcal{N}'(q)]^*(\mathcal{N}(q) - N_{\mathrm{meas}}^\varepsilon) + \alpha J_p(q) = 0.$$

The adjoint $[\mathcal{N}'(q)]^*$ of the Fréchet derivative $\mathcal{N}'(q)$ is discussed below; the mapping $J_p(\cdot)$ is the so-called duality mapping [4, 22]. For a Lebesgue index $p > 1$ one can check that

$$[J_p(q)](x) = |q(x)|^{p-1}\,\mathrm{sign}(q(x)), \quad x \in B_R.$$

For $p = 1$ (a case that is formally not included in our above analysis) the duality mapping is the set-valued sign function

$$[J_1(q)](x) = \mathrm{Sign}(q(x)) = \begin{cases} 1 & \text{if } q(x) > 0, \\ [-1,1] & \text{if } q(x) = 0, \\ -1 & \text{if } q(x) < 0, \end{cases} \quad x \in B_R.$$

Rearranging the terms in the (formal) optimality condition we get

$$(I + \alpha\mu J_p)(q) = q - \mu[\mathcal{N}'(q)]^*(\mathcal{N}(q) - N_{\mathrm{meas}}^\varepsilon), \qquad \mu > 0.$$

It turns out that the mapping $\mathbb{S}_{\alpha\mu,p} := (I + \alpha\mu J_p)^{-1}$ is well-defined. For $p = 1$ it is the well-known soft-shrinkage operator,

$$[\mathbb{S}_{\alpha,1}(q)](x) = \begin{cases} q(x) - \alpha & \text{if } q(x) \geq \alpha, \\ 0 & \text{if } |q(x)| < \alpha, \\ q(x) + \alpha & \text{if } q(x) \leq -\alpha, \end{cases} \qquad \alpha > 0,\, x \in B_R, \tag{22}$$

see, e.g., [2]. Hence, we arrive at the so-called shrinked Landweber method

$$q_{n+1} = \mathbb{S}_{\alpha\mu_n,p}\Big[q_n - \mu_n[\mathcal{N}'(q_n)]^*(\mathcal{N}(q_n) - N_{\mathrm{meas}}^\varepsilon)\Big], \qquad \mu_n > 0. \tag{23}$$

The numerical results below are obtained by using the Barzilai-Borwein rule [23] for the choice of the step sizes μ_n and by stopping the iteration using the standard discrepancy principle. Since we only treat real-valued contrasts, we additionally set the (unavoidable) imaginary component of q_n to zero in each step. Note that (23) is well-defined, since in our case the adjoint $[\mathcal{N}'(q_n)]^*$ is a bounded linear operator from HS $(L^2(\Gamma_\mathrm{i}), L^2(\Gamma_\mathrm{m}))$ into $L^r(B_R)$ for all $r \in [1, \infty]$, see Appendix E, in particular the explicit representation (E.3).

The shrinked Landweber iteration (23) requires to evaluate the adjoint $[\mathcal{N}'(q_n)]^*$ of the Fréchet derivative, evaluated in the direction $\mathcal{N}(q_n) - N^\varepsilon_\mathrm{meas}$. When dealing with j incident point sources in a discretized setting, computing the discretization of $\mathcal{N}(q)$ and of $[\mathcal{N}'(q_n)]^*$ requires to approximately solve j direct and adjoint Lippmann-Schwinger integral equations, respectively. In total, every iteration step of (23) requires to approximately solve $2j$ (direct and adjoint) integral equation, which is the main computational cost of the scheme. Since our integral equation solver provides point values of the solution on a grid, we apply the shrinkage operator $\mathbb{S}_{\alpha\mu_n,1}$ pointwise on the grid points.

For linear operators a convergence analysis for the iteration (23) (without step size control) was given in the seminal paper [2]. Further, it was shown in [24] that this iteration converges linearly if the respective (linear) operator has the so-called Finite Basis Injectivity property. For other minimization schemes consider [25, 26]. To the authors best knowledge the general convergence properties of the shrinkage iteration for non-linear problems are still an open problem. However for several special cases at least convergence to a stationary point can be shown, see, e.g., [27, 3, 28].

Our numerical results are preliminary in the sense that we only present two-dimensional reconstructions by the iteration (23) with $p = 1$, such that $\mathbb{S}_{\alpha\mu,1}$ is the soft-shrinkage operator. (Of course, in two dimensions we should, strictly speaking, choose $p > 1$ to match the assumptions of our theory.) For the numerical solution of the scattering problem (1–2) in two dimensions we approximate the solution to the Lippmann-Schwinger equation (3) by the fast Fourier transform (FFT) based volume integral equation approach from [29, 12]. This technique exploits that, after a suitable periodization, the integral operator can be diagonalized by trigonometric polynomials. We solve the resulting linear system by a GMRES iteration (without restart), preconditioned by the two-grid approach presented in [12]. Since the direct and adjoint linear problems to be solved in each iteration step of the shrinked Landweber iteration (23) merely differ in their right-hand side, using adapted preconditioners might provide some speed-up for the iteration, which we did not try so-far.

The contrasts $q_{1,2}$ that we consider for the numerical examples are plotted in Figures 1(a) and 2(a), respectively. Both contrasts have small support within the search domain, and both are piecewise constant ($q_1 = 4$ inside its support and $q_2 = 3$ inside its support). The wave number for the experiments with $q_{1,2}$ is always chosen as $k = \pi/0.09 \approx 34.9$ which corresponds to a wave length $\lambda = 0.18$. We use 32 transmitter/receiver pairs that are equidistributed on the unit circle (about five wave lengths away from the scatterers). With these parameters, one shrinked Landweber iteration in the reconstruction process of $q_{1,2}$ on a regular 512×512-grid took between 45 and 60 seconds on an Intel Core i7 processor (3,4 GHz, four cores, 16 GB RAM). The parameter τ for stopping the shrinked Landweber iteration via the discrepancy principle is chosen as 1.6 in all examples.

The reconstructions in Figures 1 and 2 show that the shrinked Landweber iteration is stable at high (relative) noise levels of 0.1 or 0.05 while producing accurate results for low (relative) noise level of 0.0005. All indicated relative errors are measured in discrete L^2-norms. In the last case, the numerical values of the contrast are well-approximated, in contrast to the reconstructions for higher noise levels that find the contrast shape well but do not even approximately reach the correct numerical values. As it is usual for the Landweber iteration, the reconstructions for low noise level are time-consuming due to the high number of iterations necessary to satisfy the discrepancy principle.

In Figures 1(i) and 2(i) we show reconstructions that are computed without using the shrinkage operator in the Landweber iteration, that is, for $p = 2$, or equivalently, using a "standard" Hilbert-space approach. This reconstruction has been computed from data with a noise level of $\varepsilon = 0.0005$, as for the sparsity reconstructions in Figures 1(g) and 2(g). Both the visual appearance of the reconstruction and the relative L^2-errors are comparable to the reconstructions of the shrinked Landweber scheme for the noise levels $\varepsilon = 0.005$ or even $\varepsilon = 0.01$. (Again, all errors are measured in discrete L^2-norms.) This advantage of the shrinked schemes is, according to all our numerical experiments, typical for the considered class of inverse medium scattering problems when the support of the contrast has small support within the search domain.

Appendix A. Inequalities and Embeddings

Here, $D \subset \mathbb{R}^d$, $d = 2, 3$ is an open set. The generalized Hölder inequality states that

$$\|uv\|_{L^{tp/(t+p)}(D)} \leq \|u\|_{L^p(D)} \|v\|_{L^t(D)} \quad \text{for } u \in L^p(D),\ v \in L^t(D),\ p, t \in (1, \infty). \qquad \text{(A.1)}$$

The following version of the Sobolev embedding lemma is from [30, Theorem 6.3].

Lemma 12. *Suppose that D is a Lipschitz domain, let $m \in \mathbb{N}$, and $1 \leq p < \infty$.*
(a) If $mp < d$, then $W^{m,p}(D)$ is compactly embedded in $L^q(D)$ for $1 \leq q < dp/(d - mp)$.
(b) If $mp = d$, then $W^{m,p}(D)$ is compactly embedded in $L^q(D)$ for $1 \leq q < \infty$.
(c) If $mp > d$, then $W^{m,p}(D)$ is compactly embedded in $C^0_{\mathrm{B}}(\overline{D})$, the space of bounded continuous functions on $\overline{D}$ equipped with the maximum norm on $\overline{D}$. Moreover, $W^{m,p}(D)$ is also compactly embedded in $L^q(D)$ for $1 \leq q \leq \infty$.

For non-integer values of the smoothness index of Sobolev spaces one has either to use interpolation techniques, see, e.g., [30], or to use Bessel potential techniques. For the periodic Sobolev spaces $H^s_{\mathrm{per}}(Q)$ (see (8) for a definition) these results hold analogously.

Lemma 13. *If $2s < d$, then $H^s_{\mathrm{per}}(Q)$ is continuously embedded in $L^{2d/(d-2s)}(Q)$, and compactly embedded in $L^r(Q)$ for $1 \leq r < 2d/(d - 2s)$.*

Appendix B. Collectively Compact Operators

We recall two results on pointwise convergent collectively compact operators from [17], see Corollary 10.8, Theorem 10.9 and Corollary 10.11. A sequence $\{K_n\}_{n\in\mathbb{N}}$ of operators

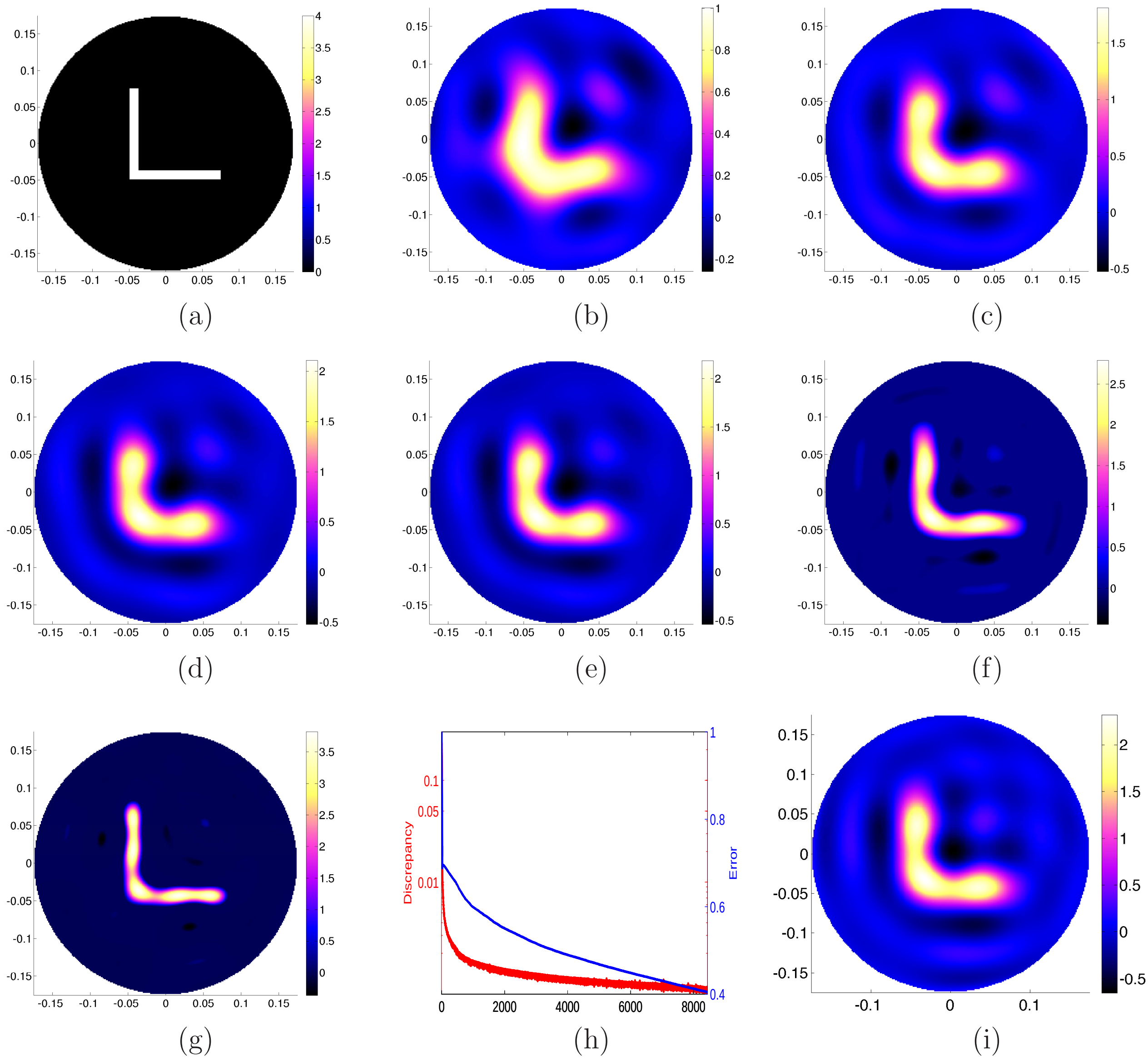

Figure 1. Reconstruction results for contrast q_1. (a) True contrast (b) $\varepsilon = 0.1$, 14 iter., rel. error= 0.806 (c) $\varepsilon = 0.05$, 16 iter., rel. error=0.737 (d) $\varepsilon = 0.01$, 27 iter., rel. error=0.693 (e) $\varepsilon = 0.005$, 46 iter., rel. error=0.695 (f) $\varepsilon = 0.001$, 1325 iter., rel. error=0.575 (g) $\varepsilon = 0.0005$, 8442 iter., rel. error=0.404 (h) Discrepancy and error plotted versus iteration index for $\varepsilon = 0.0005$ (i) L^2-Reconstruction without shrinkage, $\varepsilon = 0.0005$, 3467 iter., rel. error=0.682.

on a Banach space X is called collectively compact if the set $\{K_n\psi : \psi \in X, \|\psi\|_X < C, n \in \mathbb{N}\}$ is compact for arbitrary $C > 0$.

Theorem 14. *Assume that $\{K_n : X \to X\}_{n\in\mathbb{N}}$ is a sequence of collectively compact operators that converges pointwise to $K : X \to X$.*
(a) If $T : Y \to X$ is compact, then $\|(K_n - K)T\|_{X\to X} \to 0$ in the operator norm.
(b) If $I - K$ is injective, then the inverses of $I - K$ and $I - K_n$ exist as bounded operators on X for $n \geq N_0$, and the operator norms $\|(I - K_n)^{-1}\|_{X\to X}$ are uniformly bounded in

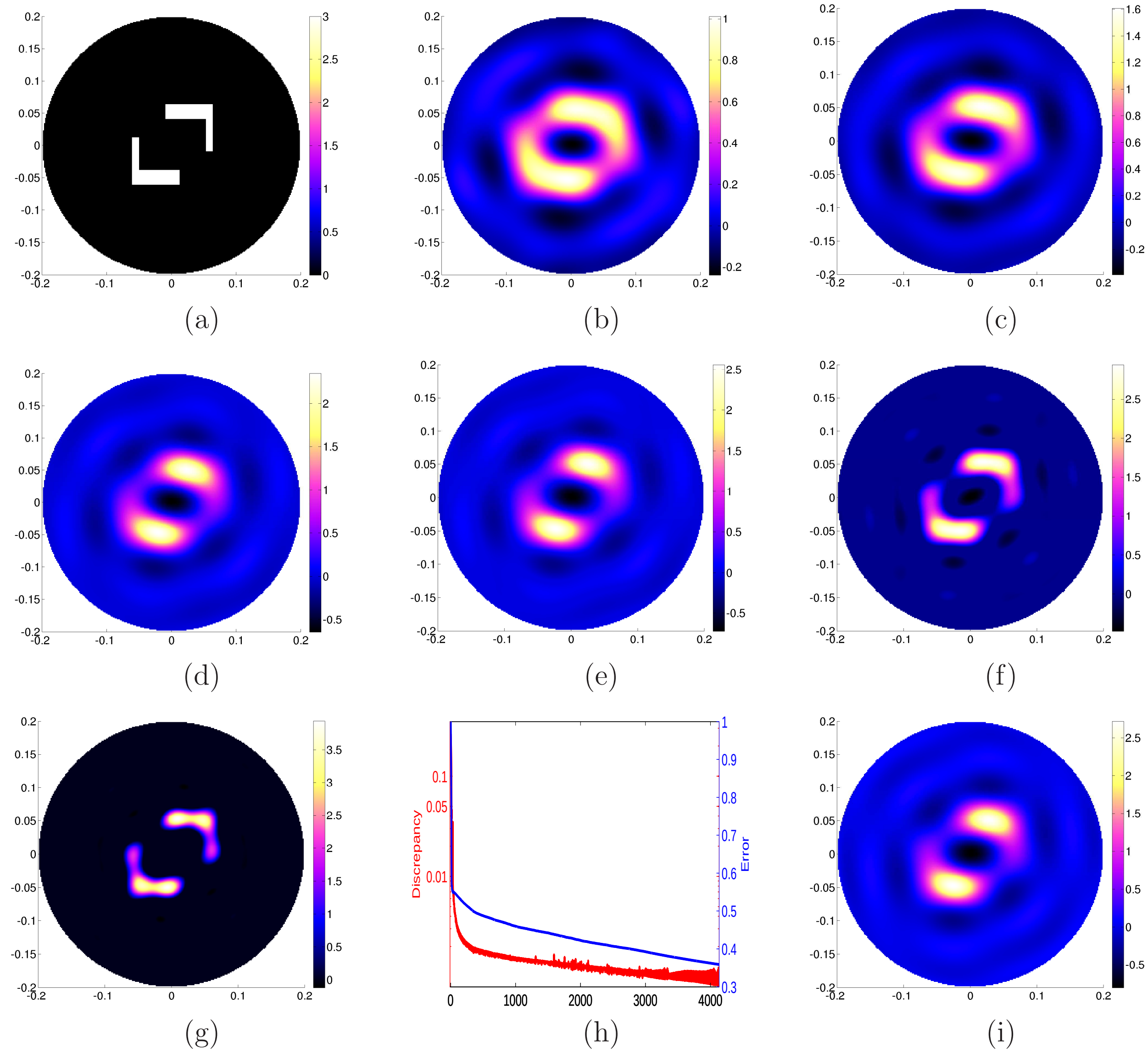

Figure 2. Reconstruction results for contrast q_2. (a) True contrast (b) $\varepsilon = 0.1$, 12 iter., rel. error=0.721 (c) $\varepsilon = 0.05$, 14 iter., rel. error=0.622 (d) $\varepsilon = 0.01$, 27 iter., rel. error=0.553 (e) $\varepsilon = 0.005$, 53 iter., rel. error=0.549 (f) $\varepsilon = 0.001$, 1280 iter., rel. error=0.451 (g) $\varepsilon = 0.0005$, 4134 iter., rel. error=0.358 (h) Discrepancy and error plotted versus iteration index for $\varepsilon = 0.0005$ (i) L^2-Reconstruction without shrinkage, $\varepsilon = 0.0005$, 3467 iter., rel. error=0.556.

$n \geq N_0$. If v and v_n solve $v - Kv = f$ and $v_n - Kv_n = f$ for $f \in X$, respectively, then

$$\|v_n - v\|_X \leq C\|(K - K_n)v\|_X, \qquad n \geq N_0,$$

holds for some constant $C > 0$ independent of n.

Appendix C. Hilbert-Schmidt Operators

Let H_1 and H_2 be separable Hilbert spaces, and assume that $\{\varphi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of H_1. A bounded, linear operator $T : H_1 \to H_2$ is called a Hilbert-Schmidt operator if

$$\|T\|_{\mathrm{HS}}^2 = \sum_{j\in\mathbb{N}} \|T\varphi_j\|_{H_2}^2 < \infty.$$

We denote this class of operators by $\mathrm{HS}\,(H_1, H_2)$. It turns out that $\|\cdot\|_{\mathrm{HS}}$ is independent of the choice of the orthonormal basis in H_1, see, e.g., [31, Section A.3] and that this operator norm is induced by an inner product, $(T_1, T_2)_{\mathrm{HS}} = \sum_{j\in\mathbb{N}}(T_1\varphi_j, T_2\varphi_j)_{H_2}$, making of $\mathrm{HS}\,(H_1, H_2)$ a Hilbert space. An essential class of Hilbert-Schmidt operators are integral operators with square-integrable kernels. Given open and non-empty submanifolds Γ_1 and Γ_2 of $\mathbb{R}^d$, consider the integral operator

$$N : \varphi \mapsto \int_{\Gamma_1} \kappa(\cdot, y)\varphi(y)\,\mathrm{d}y$$

from $L^2(\Gamma_1)$ into $L^2(\Gamma_2)$. The next result can be found in, e.g., [31, Prop. A.3.2].

Lemma 15. *The mapping $\kappa \mapsto N$ from $L^2(\Gamma_1 \times \Gamma_2)$ into $\mathrm{HS}\,(L^2(\Gamma_1), L^2(\Gamma_2))$ is an isometric isomorphism, i.e., $\|N\|_{\mathrm{HS}(L^2(\Gamma_1),L^2(\Gamma_2))} = \|\kappa\|_{L^2(\Gamma_1\times\Gamma_2)}$.*

Finally, Hilbert-Schmidt operators form a two-sided ideal in the algebra of all bounded linear operators: If, e.g., $T_1 : H_1 \to H_2$ and $T_2 : H_2 \to H_3$ is a bounded linear operator and a Hilbert-Schmidt operator between separable Hilbert spaces, respectively, then $T_2T_1 : H_1 \to H_3$ is a Hilbert-Schmidt operator and

$$\|T_2T_1\|_{\mathrm{HS}(H_1,H_3)} \le \|T_2\|_{\mathrm{HS}(H_2,H_3)}\,\|T_1\|_{H_1\to H_2}. \tag{C.1}$$

Appendix D. Tikhonov Regularization

We cite the main result of Tikhonov regularization in Banach spaces from Section 3.2 in [32] in a reduced form sufficient for our application. To this end, assume that $F : D(F) \subset L^p(B_R) \to V$ is a non-linear operator between $L^p(B_R)$, $1 < p < \infty$, and a reflexive Banach space V, defined on a non-empty domain $D(F) \subset L^p(B_R)$. It is well-known that the norm in $L^p(B_R)$ is convex, weakly sequentially lower semi-continuous, and satisfies the Radon-Riesz property: $u_m \rightharpoonup u$ weakly in $L^p(B_R)$ and $\|u_m\|_{L^p(B_R)} \to \|u\|_{L^p(B_R)}$ implies that $u_m \to u$ strongly in $L^p(B_R)$ as $m \to \infty$.

Define, for $v^\varepsilon \in V$, $\alpha > 0$, and $q \in (1, \infty)$,

$$J_{\alpha,v^\varepsilon}(u) := \|F(u) - v^\varepsilon\|_V^q + \alpha\|u\|_{L^p(B_R)}^p, \qquad u \in D(F),$$

and set $J_{\alpha,v^\varepsilon}(u) = \infty$ if $u \in L^p(B_R) \setminus D(F)$. If $F(u^\dagger) = v^\dagger$, then $u^\dagger$ is called a norm-minimizing solution of this equation if $\|u^\dagger\|_{L^p(B_R)} := \min\{\|u\|_{L^p(B_R)},\, u \in D(F),\, F(u) = v^\dagger\}$. Since the penalty term in the definition of J_{α,v^ε} uses the $L^p(B_R)$-norm, it is clear that the level sets $L_{\alpha,v}(M) := \{u \in L^p(B_R) : J_{\alpha,v}(u) \le M\}$ are weakly sequentially compact and weakly sequentially closed in $L^p(B_R)$ for every $\alpha > 0$ and $M > 0$.

Theorem 16 (Proposition 3.32 in [32]). *Assume that the restriction of F to $L_{\alpha,v}(M)$ is weakly sequentially continuous for all $\alpha > 0$, $M > 0$ and $v \in V$. Assume further that there exists a solution $u^\dagger \in D(F)$ to $F(u^\dagger) = v^\dagger \in V$, that $v^\varepsilon \in V$ satisfies $\|v^\varepsilon - v^\dagger\|_V \leq \varepsilon$ for $\varepsilon > 0$, and choose $\alpha : (0,\infty) \to (0,\infty)$ such that $\alpha(\varepsilon) \to 0$ and $\varepsilon^q/\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$. For any sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ such that $0 < \varepsilon_m \to 0$ as $m \to \infty$, set $v_m = v^{\varepsilon_m}$ and $\alpha_m = \alpha(\varepsilon_m)$.*

Then there exists a minimizer u_m of J_{α_m,v_m} for all $m \in \mathbb{N}$. For every sequence of minimizers $(u_m)_{m\in\mathbb{N}}$ there exists a subsequence $(u_{m'})$ and a norm-minimizing solution u of $F(u) = u^\dagger$ such that $\|u_{m'} - u\|_{L^p(B_R)} \to 0$ as $m' \to \infty$. If the norm-minimizing solution u is unique, then $u_m \to u$ as $m \to \infty$.

Appendix E. Fréchet Derivatives

The domain of definition $L^p_{\mathrm{Im}\geq 0}(B_R)$ of the contrast-to-measurement operator $\mathcal{N}$ is a closed subset of $L^p(B_R)$ with empty interior. Since we want to show that $\mathcal{N}$ is Fréchet differentiable (instead of studying directional derivatives as in, e.g. [4, 32]) we first indicate how to define $\mathcal{N}$ on an open subset of $L^p(B_R)$. We always assume in this section that the Lebesgue indices $p > d/2$ and $t > 1$ satisfy Assumption 5.

A standard Neumann series argument shows that for $q \in L^p_{\mathrm{Im}\geq 0}(B_R)$ there exists $\varepsilon = \varepsilon(q) > 0$ such that the solution operator T_{q+h} (see (13)) exists as a bounded operator on $L^t(B_R)$ for each $h \in L^p(B_R)$ with $\|h\|_{L^p(B_R)} < \varepsilon(q)$. This follows from Proposition 2, since $\|V_{B_R\to B_R}(qv) - V_{B_R\to B_R}((q+h)v)\|_{L^t(B_R)} \leq C\|h\|_{L^p(B_R)}\|v\|_{L^t(B_R)}$. Recall from (14) that $S(q,u^i) = k^2 V_{B_R\to\Gamma_\mathrm{m}}[qT_q u^i]$. Hence, we can extend the domain of definition of $S(\cdot,u^i)$ to

$$\mathcal{D}_p := \cup_{q\in L^p_{\mathrm{Im}\geq 0}(B_R)}\{q+h,\ \|h\|_{L^p(B_R)} < \varepsilon(q)\}. \tag{E.1}$$

Since the potential $V_{B_R\to\Gamma_\mathrm{m}}$ is a linear operator, and since $q \mapsto qT_q$ is Fréchet differentiable by the product rule [33, Prop. 4.10] in Banach spaces, the mapping $q \mapsto S(q,u^i)$ is Fréchet differentiable with derivative

$$S'(q,u^i)[h] = k^2 V_{B_R\to\Gamma_\mathrm{m}}\left(hT_q(u^i) + qk^2 T_q(V_{B_R\to B_R}(hu(q)))\right). \tag{E.2}$$

One can additionally prove that $\|S(q+h,u^i) - S(q,u^i) - S'(q,u^i)[h]\|_{L^2(\Gamma_\mathrm{m})} \leq C\|h\|^2_{L^p(B_R)}\|u^i\|_{L^t(B_R)}$ for all $h \in L^p(B_R)$. The multi-static measurement operator $\mathcal{N} : \mathcal{D}_p \subset L^p(B_R) \to \mathrm{HS}\,(L^2(\Gamma_\mathrm{i}), L^2(\Gamma_\mathrm{m}))$, $\mathcal{N}(q)[\varphi] = k^2 V_{B_R\to\Gamma_\mathrm{m}}[qT_q(\mathrm{SL}_{\Gamma_\mathrm{i}}\varphi)]$, is a composition of the Hilbert-Schmidt embedding from $H^2(\Gamma_\mathrm{m})$ into $L^2(\Gamma_\mathrm{m})$ with the bounded operator $V_{B_R\to\Gamma_\mathrm{m}} : L^{tp/(t+p)}(B_R) \to H^2(\Gamma_\mathrm{m})$ and with the Fréchet differentiable operator $q \mapsto qT_q$. Hence, the chain rule [33, Prop. 4.10] in Banach spaces implies that $\mathcal{N}$ is Fréchet differentiable, too. Explicitly computing this derivative shows that

$$\mathcal{N}'(q)[h] : \varphi \mapsto k^2 V_{B_R\to\Gamma_\mathrm{m}} \circ \left[I + k^2\,(q\,\cdot) \circ T_q \circ V_{B_R\to B_R}\right] \circ (hT_q(\mathrm{SL}_{\Gamma_\mathrm{i}}\varphi)).$$

Lemma 17. *Assume that $p > d/2$. Then $\mathcal{N}$ is Fréchet differentiable and the derivative $\mathcal{N}'(q)[h] : \varphi \mapsto S'(q,\mathrm{SL}_{\Gamma_\mathrm{i}}\varphi)$ satisfies $\|\mathcal{N}(q+h) - \mathcal{N}(q) - \mathcal{N}'(q)[h]\|_{\mathrm{HS}(L^2(\Gamma_\mathrm{i}),L^2(\Gamma_\mathrm{m}))} \leq C\|h\|^2_{L^p(B_R)}$.*

For Newton-like schemes one often needs to evaluate the adjoint $[\mathcal{N}'(q)]^*$ of the Fréchet derivative. Assume that $\{\varphi_j\}_{j\in\mathbb{N}}$ is an arbitrary orthonormal basis of $L^2(\Gamma_\mathrm{m})$, abbreviate $\overline{V}(\varphi) := \overline{V(\overline{\varphi})}$ for any linear operator V, and choose an arbitrary $G \in \mathrm{HS}\,(L^2(\Gamma_\mathrm{i}), L^2(\Gamma_\mathrm{m}))$. Explicitly exploiting the inner product in $\mathrm{HS}\,(L^2(\Gamma_\mathrm{i}), L^2(\Gamma_\mathrm{m}))$, see Appendix C, shows that

$$[\mathcal{N}'(q)]^*[G] = k^2 \sum_{j=1}^{\infty} \overline{T_q[\mathrm{SL}_{\Gamma_\mathrm{i}}\varphi_j]} \cdot \Big([I + k^2 \overline{V}_{B_R\to B_R} \circ T_q^*(\overline{q}\ \cdot)] \circ [\overline{\mathrm{SL}}_{\Gamma_\mathrm{m}\to B_R}(G\varphi_j)] \Big). \quad \text{(E.3)}$$

Independently of the Lebesgue index $p > d/2$ of $q \in L^p(B_R)$ one can show that the latter expression defines a bounded, linear operator from $\mathrm{HS}\,(L^2(\Gamma_\mathrm{i}), L^2(\Gamma_\mathrm{m}))$ into $L^r(B_R)$ for all $r \in [1,\infty]$. The proof mainly exploits that $\mathrm{SL}_{\Gamma_\mathrm{i}}$ belongs to $\mathrm{HS}\,(L^2(\Gamma_\mathrm{i}), H^m(B_R))$ for all $m \in \mathbb{N}_0$, since, by assumption, the distance between Γ_i and B_R is strictly positive.

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