Homogenization of a moving boundary problem with prescribed normal velocity

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Abstract

The analysis and homogenization of a heat conduction problem with moving boundary for a highly heterogeneous, periodic two-phase medium is considered. In this context, the normal velocity governing the motion of the interface separating the two competing phases is assumed to be prescribed. Parametrizing the boundary motion via a height function, the so-called *Direct Mapping Method* is employed to construct a coordinate transform characterizing the changes with respect to the initial setup of the geometry. Using this transform, well-posedness of the problem is established. After characterizing the limit behavior (with respect to the heterogeneity parameter $\varepsilon \to 0$) of the functions related to the transformation, the corresponding homogenized problem is deduced.

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1 Introduction

We consider the analysis and the homogenization of a moving boundary problem that describes phase transitions occurring in highly heterogeneous two-phase media. Here, the two phases in question are separated via a sharp interface whose exact evolution is not known at the outset.

To me more specific, let $\Omega \subset \mathbb{R}^3$ be a bounded domain and let $\Omega_\varepsilon^{(1)}$, $\Omega_\varepsilon^{(2)} \subset \Omega$ be ε-periodic subdomains representing the initial set-up of the two-phases occupying Ω. Here, the small parameter ε represents the ratio of the characteristic lengths of the microscale size of the inhomogeneities of the medium) and the macroscale (overall size of the domain). The interface between the competing phases will be denoted by Γ_ε. Due to phase transitions, this geometrical setup might change with time leading to domains $\Omega_\varepsilon^{(i)}(t)$ $(i = 1, 2)$ and interface $\Gamma_\varepsilon(t)$ at time t which, in general, are not necessarily periodic anymore. With n_{Γ_ε} and V_{Γ_ε}, we denote normal vector pointing outwards $\Omega_\varepsilon^{(2)}$ and the normal velocity of $\Gamma_\varepsilon(t)$ in normal direction, respectively.

Now, let $\theta_\varepsilon^{(i)} = \theta_\varepsilon^{(i)}(t,x)$ denote the temperature in the respective domains. In this work, we consider a two-phase heat problem accounting for latent heat and phase transitions given by

$$\partial_t \theta_\varepsilon^{(i)} - \kappa_\varepsilon^{(i)} \Delta \theta_\varepsilon^{(i)} = f_\varepsilon^{(i)} \qquad \text{in } \Omega_\varepsilon^{(i)}(t), \tag{1.1a}$$

$$[\![\theta_\varepsilon]\!] = 0 \qquad \text{on } \Gamma_\varepsilon(t), \tag{1.1b}$$

$$-[\![\kappa_\varepsilon \nabla \theta_\varepsilon]\!] \cdot n_\varepsilon = L V_{\Gamma_\varepsilon} \qquad \text{on } \Gamma_\varepsilon(t), \tag{1.1c}$$

$$V_{\Gamma_\varepsilon} = \varepsilon v_\varepsilon \qquad \text{on } \Gamma_\varepsilon(t) \tag{1.1d}$$

complemented with appropriate boundary and initial conditions. The aim of this paper is twofold: (i) show that this two-phase problem admits a unique local-in-time solution where the interval of existence is independent of the parameter ε and (ii) investigate the limit behavior $\varepsilon \to 0$ thereby establishing an homogenized limit problem approximating (in some sense) the above system.

For the existence part, we rely on a particularly useful approach, which was originally introduced in [12], which is sometimes called *Direct Mapping Method* or *Hanzawa transformation*, and where a specific coordinate transformation is constructed. Please note that using this method it is not possible to consider any type of topological changes. Regarding the limit process in the context of mathematical homogenization, we employ the notion of (strong) two-scale convergence as introduced in [1, 15].

Combining the analysis of moving boundary problems with the mathematical homogenization leads to significant mathematical and technical challenges. First, the motion of the interface has to satisfy certain estimates uniformly with respect to the scale parameter ε. This means that the influence of ε has to be accounted for very carefully. Second, we have to show strong two-scale convergence of some functions related to the transformation as the usual two-scale convergence is not sufficient to pass to the limit (due to the coordinate transform).

Similar moving boundary problems to the system given by equations (1.1a) to (1.1d) without the heterogeneity parameter ε were considered in, e.g., [2, 3, 18]. The heterogeneous case might arise in situations where the spatial scale at which we can observe such transformations is several orders of magnitude below the size of the materials itself are; typical examples would be phase transformations in porous media or in steel. Such heterogeneous problems were considered in, e.g., [7, 8, 13].

For the more general setting of a fully coupled version of System 1.1 where the normal velocity is not prescribed but rather given as a function of the temperature and the geometry of the interface, typical choices would be $v_\varepsilon = \theta_\varepsilon - \theta_{crit}$ (the law of *kinetic undercooling*) or $v_\varepsilon = -H_{\Gamma_\varepsilon} + \theta_\varepsilon - \theta_{crit}$ (*Gibbs-Thomson undercooling*). Here, θ_{crit} denotes the critical temperature of the phase transition in question and $H_{\Gamma_\varepsilon}(t)$ the mean curvature function of the interface $\Gamma_\varepsilon(t)$. One possible way to tackle such fully coupled problems is in the context of maximal parabolic regularity, see, e.g., [19, 21]. This, however, runs into additional troubles in the heterogeneous case due to the extensive ε-independent estimates that would need to be established; e.g., $\theta_\varepsilon(t)$ would have to be uniformly bounded in $W^{2,\infty}(\Gamma_\varepsilon(t))$.

This work can therefore be seen as an important intermediate step in the analysis of the fully coupled case. In the existing literature regarding the homogenization of evolving microstructures, the changes in the geometry are usually assumed to be a priori known (the case of prescribed coordinate transform), see [6, 10, 17, 22]; a scenario which is easier to tackle.

This work is organized as follows: In Section 2, we introduce the ε-periodic geometry, the moving boundary problem with prescribed normal velocity as well as the level set equation associated with the normal velocity. The main results regarding the moving boundary problem, Theorems 3.1 to 3.4, are then given in Section 3. Finally, Sections 4 and 5 are dedicated to the detailed proofs of Theorem 3.1 and Theorem 3.2, respectively.

2 Setting and problem statement

2.1 Geometrical setup

Let $S = (0,T)$, $T > 0$, represent the time interval of interest and let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain whose outer normal vector we denote with $\nu = \nu(x)$. In addition, let $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ be a monotonically decreasing sequence of positive numbers converging to zero.

Now, take open and disjoint sets $Y^{(1)}, Y^{(2)} \subset (0,1)^3 =: Y$ such that $Y^{(1)}$ is connected, $\overline{Y^{(2)}} \subset Y$, and $Y = Y^{(1)} \cup \overline{Y^{(2)}}$. Moreover, let $\Gamma := \partial Y^{(2)}$ be a C^3-hypersurface. By $n_\Gamma = n_\Gamma(\gamma)$, $\gamma \in \Gamma$, we denote the normal vector of Γ pointing outwards of $Y^{(2)}$.

In order to circumvent problems due to complex structures at the boundary, we remove the boundary layer of thickness ε via

$$\widetilde{\Omega}_\varepsilon = \Omega \cap \left(\bigcup_{k\in Z_\varepsilon} \varepsilon(Y+k) \right), \quad \text{where } Z_\varepsilon = \{k \in \mathbb{Z}^3 \ : \ \varepsilon(Y+k) \subset \Omega\}.$$

Then, we introduce the εY-periodic domains $\Omega_\varepsilon^{(i)}$ $(i = 1,2)$ and the interface Γ_ε representing the two phases and the phase boundary, respectively, via

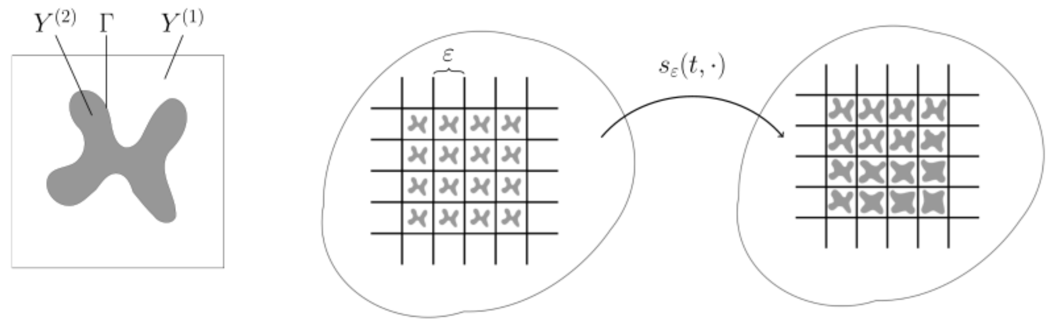
$$\Omega_\varepsilon^{(2)} = \widetilde{\Omega}_\varepsilon \cap \left(\bigcup_{k\in\mathbb{Z}^3} \varepsilon(Y+k) \right), \quad \Omega_\varepsilon^{(1)} = \Omega \setminus \overline{\Omega_\varepsilon^{(2)}}, \quad \Gamma_\varepsilon = \partial\Omega_\varepsilon^{(2)}.$$

Note that, by design, $\partial\Omega_\varepsilon^{(1)} = \partial\Omega$ and $\operatorname{dist}(\partial\Omega, \Gamma_\varepsilon) \geq \varepsilon$.

With $t \mapsto \Gamma_\varepsilon(t)$ and $t \mapsto \Omega_\varepsilon^{(i)}(t)$ for $t \in S$, we denote the evolution of the interface and the domains, respectively. We set

$$Q_\varepsilon^{(i)} := \bigcup_{t\in S} \{t\} \times \Omega_\varepsilon^{(i)}(t), \qquad \Xi_\varepsilon := \bigcup_{t\in S} \{t\} \times \Gamma_\varepsilon(t).$$

Finally, we assume the overall domain Ω to be time-independent; that is $\Omega = \Omega_\varepsilon^{(1)}(t) \cup \Omega_\varepsilon^{(2)}(t) \cup \Gamma_\varepsilon(t)$ for all $t \in S$. An illustration of the general geometrical setup is given via Figure 1.



(a) The unit cell Y (b) The periodic reference geometry (c) The actual geometry at time t

Figure 1: Illustration of the geometrical setup. Here, the motion function $s_\varepsilon(t,\cdot)$ characterizes the changes in geometry.

As a C^3-hypersurface, Γ admits a tubular neighborhood U_Γ of width $a>0$. Moreover, the function

$$\Lambda\colon \Gamma\times(-a,a)\to U_\Gamma,\quad \Lambda(\gamma,s):=\gamma+sn_\Gamma(\gamma)$$

is a C^2-diffeomorphism satisfying $\Lambda(\Gamma\times(-a,a))\subset Y$; we refer to [19, Section 3.1, p.65]. Similarly, we introduce the ε-scaled C^2-diffeomorphism

$$\Lambda_\varepsilon\colon \Gamma_\varepsilon\times(-\varepsilon a,\varepsilon a)\to U_{\Gamma_\varepsilon},\quad \Lambda_\varepsilon(\gamma,r)=\gamma+rn_{\Gamma_\varepsilon}(\gamma).$$

the family of interfaces

$$\Gamma_\varepsilon^{(l)}:=\{\Lambda_\varepsilon(\gamma,l)\ :\ \gamma\in\Gamma_\varepsilon\}\quad \text{for }\ l\in[-\varepsilon a,\varepsilon a]\,, \tag{2.1}$$

and the family of tubes around Γ_ε

$$U_{\Gamma_\varepsilon}(r):=\bigcup_{l\in(-\varepsilon ra,\varepsilon ra)}\Gamma_\varepsilon^{(l)}\qquad (r\in(0,1])\,.$$

We set $U_{\Gamma_\varepsilon}=U_{\Gamma_\varepsilon}(1)$. For $\gamma\in\Gamma_\varepsilon$, let $L_{\Gamma_\varepsilon}(\gamma)=-\nabla_{\Gamma_\varepsilon}n_{\Gamma_\varepsilon}(\gamma)$ denote the *Weingarten map*, where we have ([19, Section 2.1])

$$\sup_{\gamma\in\Gamma_\varepsilon}|L_{\Gamma_\varepsilon}(\gamma)|\le\frac{1}{2\varepsilon a}. \tag{2.2}$$

For $l\in[-\varepsilon a,\varepsilon a]$ and $\gamma\in\Gamma_\varepsilon^{(l)}$, the normal vector of the interface $\Gamma_\varepsilon^{(l)}$ in γ is given as $n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(\gamma))$, where $P_{\Gamma_\varepsilon}\colon U_{\Gamma_\varepsilon}\to\Gamma_\varepsilon$ denotes the projection operator. The inverse of Λ_ε is given via

$$\Lambda_\varepsilon^{-1}\colon U_{\Gamma_\varepsilon}\to\Gamma_\varepsilon\times[-\varepsilon a,\varepsilon a],\quad \Lambda_\varepsilon^{-1}(x)=(P_{\Gamma_\varepsilon}(x),d_{\Gamma_\varepsilon}(x))^T\,.$$

Here, $d_{\Gamma_\varepsilon}\colon U_\Gamma\to\mathbb{R}$ is the signed distance function for Γ_ε, i.e.,

$$d_{\Gamma_\varepsilon}(x)=\begin{cases}\operatorname{dist}(x,\Gamma_\varepsilon), & x\in U_{\Gamma_\varepsilon}\setminus\Omega_\varepsilon^{(2)}\\ -\operatorname{dist}(x,\Gamma_\varepsilon), & x\in U_{\Gamma_\varepsilon}\cap\Omega_\varepsilon^{(2)}\end{cases}.$$

2.2 Problem statement

For $k, l \in \mathbb{N}$, we introduce the Sobolev space

$$W^{(k,l),\infty}(S \times \Omega) = \left\{u \in L^\infty(S \times \Omega) : \partial_t^i u, D_x^j u \in L^\infty(S \times \Omega)\ (1 \le i \le k,\ 1 \le j \le l)\right\}$$

and note that $W^{(k,k),\infty}(S \times \Omega) = W^{k,\infty}(S \times \Omega)$.

Now, take $\theta_\varepsilon^{(i)} = \theta_\varepsilon^{(i)}(t, x)$ $(i = 1, 2)$ to represent the temperature in the respective domains $Q_\varepsilon^{(i)}$. In the following, we consider the moving boundary problem given by:

Moving boundary problem with prescribed normal velocity

$$\partial_t\theta_\varepsilon^{(1)} - \kappa^{(1)}\Delta\theta_\varepsilon^{(1)} = f_\varepsilon^{(1)} \quad \text{in } Q_\varepsilon^{(1)}, \tag{2.3a}$$

$$\partial_t\theta_\varepsilon^{(2)} - \varepsilon^2\kappa^{(2)}\Delta\theta_\varepsilon^{(2)} = f_\varepsilon^{(2)} \quad \text{in } Q_\varepsilon^{(2)}, \tag{2.3b}$$

$$\theta_\varepsilon^{(1)} = \theta_\varepsilon^{(2)} \quad \text{on } \Xi_\varepsilon, \tag{2.3c}$$

$$-\left(\kappa^{(1)}\nabla\theta_\varepsilon^{(1)} - \varepsilon^2\kappa^{(2)}\nabla\theta_\varepsilon^{(2)}\right) \cdot n_\varepsilon = LV_{\Gamma_\varepsilon} \quad \text{on } \Xi_\varepsilon, \tag{2.3d}$$

$$V_{\Gamma_\varepsilon} = \varepsilon v_\varepsilon \quad \text{on } \Xi_\varepsilon, \tag{2.3e}$$

$$-\kappa^{(1)}\nabla\theta_\varepsilon^{(1)} \cdot \nu = 0 \quad \text{on } S \times \partial\Omega, \tag{2.3f}$$

$$\theta_\varepsilon^{(1)}(0) = \vartheta_\varepsilon^{(1)} \quad \text{in } \Omega_\varepsilon^{(1)}, \tag{2.3g}$$

$$\theta_\varepsilon^{(2)}(0) = \vartheta_\varepsilon^{(2)} \quad \text{in } \Omega_\varepsilon^{(2)}. \tag{2.3h}$$

Here, the positive constants $\kappa^{(i)}$ denote the heat conductivity coefficients and L denotes the constant of latent heat. The actual mathematical problem connected to this system is as follows: Given volume heat source densities $f_\varepsilon^{(i)} \colon Q_\varepsilon^{(i)} \to \mathbb{R}$, a function $v_\varepsilon \colon \Xi_\varepsilon \to \mathbb{R}$ governing the movement of the interface, and initial values $\vartheta_\varepsilon^{(i)} \colon \Omega_\varepsilon^{(i)} \to \mathbb{R}$, find the corresponding evolution of the domains, i.e., find $\Omega_\varepsilon^{(i)}(t)$ and $\Gamma_\varepsilon(t)$ for all $t \in S$, and the temperature functions $\theta_\varepsilon^{(i)} \colon Q_\varepsilon^{(i)} \to (0, \infty)$ such that all equations of the above system are satisfied.

Now, let $v_\varepsilon \in W^{(1,2),\infty}(S \times \Omega)$ be the outward normal velocity of the moving interface $\Gamma_\varepsilon(t)$. Let us assume that the corresponding motion of Γ_ε can be described via a regular C^1-motion. Then, there exists a level set function $\varphi_\varepsilon \colon S \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned} \Gamma_\varepsilon(t) &= \{x \in \Omega \ : \ \varphi_\varepsilon(t, x) = 0\}, \\ |\nabla\varphi_\varepsilon(t, x)| &> 0 \quad \text{on } \Xi_\varepsilon, \\ \varphi_\varepsilon(t, x) &< 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The normal velocity $\varepsilon v_\varepsilon$ and the level set function φ_ε are connected via ([16, Section 4.1])

$$\partial_t\varphi_\varepsilon = \varepsilon|\nabla\varphi_\varepsilon|v_\varepsilon \quad \text{on } \Xi_\varepsilon.$$

Based on these geometric considerations, we formulate the motion problem as a level set problem:

Motion problem via level set equation

Find $\varphi_\varepsilon \in C^1(S \times \Omega)$ such that

$$\partial_t \varphi_\varepsilon = \varepsilon |\nabla \varphi_\varepsilon| v_\varepsilon \quad \text{on } \Xi_\varepsilon, \tag{2.4a}$$

$$|\nabla \varphi_\varepsilon(t,x)| > 0 \quad \text{on } \Xi_\varepsilon, \tag{2.4b}$$

$$\frac{\partial_t \varphi_\varepsilon - \varepsilon |\nabla \varphi_\varepsilon| v_\varepsilon}{\varphi_\varepsilon} \in W^{(0,1),\infty}(S \times \Omega), \tag{2.4c}$$

$$\Gamma_\varepsilon = \{x \in \Omega \,:\, \varphi_\varepsilon(0,x) = 0\}, \tag{2.4d}$$

$$\Omega_\varepsilon^{(1)} = \{x \in \Omega \,:\, \varphi_\varepsilon(0,x) < 0\}. \tag{2.4e}$$

The family of sets $(\Gamma_\varepsilon(t))_{t\in S}$ defined via

$$\Gamma_\varepsilon(t) = \{x \in \Omega \,:\, \varphi_\varepsilon(t,x) = 0\}$$

is called the solution of the motion problem. The condition (2.4c) is a shorthand for: the function $\frac{\partial_t \varphi_\varepsilon - \varepsilon |\nabla \varphi_\varepsilon| v_\varepsilon}{\varphi_\varepsilon} \colon (S \times \Omega) \setminus \Xi_\varepsilon \to \mathbb{R}$ can be extended to a function in $W^{(0,1),\infty}(S \times \Omega)$. Note that this condition is merely technical in that it is not needed for the level set function φ_ε to correspond to the motion of the interface; it is, however, needed in Lemma 4.4.

We also point out that uniqueness of a solution of the motion problem only asserts uniqueness of the the family of hypersurfaces $(\Gamma_\varepsilon(t))_{t\in S}$ but not uniqueness of the level set function φ_ε. Indeed, for every $\alpha > 0$, $\alpha\varphi_\varepsilon$ corresponds to the same motion problem.

3 Main results

In this section, we present the main results. As some of the proofs are fairly long and technical, they are postponed to subsequent chapters: Section 4 and Section 5 are devoted to the proofs of Theorem 3.1 and Theorem 3.2, respectively.

We start by formulating the requirements for the data (normal velocity, source densities, and initial values) that are needed to ensure the well-posedness of the microscopic problems as well as to facilitate the passage $\varepsilon \to 0$.

(A1) Let $v_\varepsilon \in W^{(1,3),\infty}(S \times \Omega)$ with $\operatorname{supp}(v_\varepsilon) \subset U_{\Gamma_\varepsilon}$ and

$$l_v := \sup_{\varepsilon>0} \left(\|v_\varepsilon\|_{W^{1,\infty}(S\times\Omega)} + \varepsilon \|D_x^2 v_\varepsilon\|_\infty + \varepsilon^2 \|D_x^3 v_\varepsilon\|_\infty \right) < \infty.$$

(A2) For $i = 1, 2$, let $f_\varepsilon^{(i)} \in L^2(Q_\varepsilon^{(i)})$ and $\vartheta_\varepsilon^{(i)} \in L^2(\Omega_\varepsilon^{(i)})$ such that

$$\sup_{\varepsilon>0} \left(\|f_\varepsilon^{(i)}\|_{L^2(Q_\varepsilon^{(i)})} + \|\vartheta_\varepsilon^{(i)}\|_{L^2(\Omega_\varepsilon^{(i)})} \right) < \infty.$$

(A3) There is a function $v \in L^2(S \times \Omega; W^{1,2}_\#(Y))^3$ satisfying

$$[v_\varepsilon]^\varepsilon \to v, \;\; [Dv_\varepsilon]^\varepsilon \to D_y v, \;\; \varepsilon \left[D^2 v_\varepsilon\right]^\varepsilon \to D_y^2 v \quad \text{in } L^2(S \times \Omega \times Y)^3.$$

Here, $[v_\varepsilon]^\varepsilon : S \times \Omega \times Y \to \mathbb{R}$ is the periodic unfolding of $v_\varepsilon\colon S \times \Omega \to \mathbb{R}$ defined via $[v_\varepsilon]^\varepsilon(t,x,y) = v\left(t, \varepsilon y + \varepsilon\left[\frac{x}{\varepsilon}\right]\right)$ where $[x]$ denotes the unique $k \in \mathbb{Z}^3$ for which $x - k \in [0,1)^3$; for details, we refer to [4]. Furthermore, the number sign subscript # indicates spaces of periodic functions:

$$W^{1,2}_\#(Y) = \{u \in W^{1,2}_{\text{loc}}(\mathbb{R}^3) \ : \ u_{|Y} \in W^{1,2}(Y),\ u(y) = u(e_j + y) \text{ for a.a. } y \in Y\ (j = 1,2,3)\}.$$

If $[v_\varepsilon]^\varepsilon \to v$ in $L^2(S \times \Omega \times Y)$, we say that v_ε strongly two-scale converges to v ($v_\varepsilon \overset{2-str.}{\longrightarrow} v$); if $[v_\varepsilon]^\varepsilon \rightharpoonup v$, we say that v_ε two-scale converges to v ($v_\varepsilon \overset{2}{\rightharpoonup} v$). The correspondence of this notion to the usual definition of two-scale convergence (see [1]) can be found, e.g., in [5].

The regularity and the estimates postulated via Assumption (A1) ensure well-posedness of the motion problem given by equations (2.4a) to (2.4e) and the validity of corresponding a priori estimates. With Assumption (A2), these results can be used to tackle the heat problem given by equations (2.3a) to (2.3h)). Finally, Assumption (A3) is necessary for the homogenization process.

The following two results, namely, Theorem 3.1 and Theorem 3.2, are the cornerstones of this work; their proofs are given in Section 4 and Section 5, respectively.

Theorem 3.1. Under Assumption (A1), there is $T_v = T(l_v) \in S$, which is independent of $\varepsilon > 0$, and a function $h_\varepsilon\colon [0,T_v] \times \Gamma_\varepsilon \to (-\varepsilon a, \varepsilon a)$ such that

$$\Gamma_\varepsilon(t) = \{\gamma + h_\varepsilon(t,\gamma) n_{\Gamma_\varepsilon}(\gamma) \ : \ \gamma \in \Gamma_\varepsilon\} \quad (t \in [0,T_v]).$$

The time T_v is increasing for decreasing values of l_v and we have $(0,T_v) = S$ for sufficiently small $l_v > 0$. Also, there is a corresponding, regular C^1-motion $s_\varepsilon\colon [0,T_v] \times \overline{\Omega} \to \overline{\Omega}$ satisfying $s_\varepsilon(0) = \text{id}$, $s_\varepsilon(t, \Omega^{(i)}_\varepsilon) = \Omega^{(i)}_\varepsilon(t)$ $(i = 1,2)$, and

$$\|Ds_\varepsilon\|_\infty \le 2, \quad \|\left(Ds_\varepsilon\right)^{-1}\|_\infty \le 2.$$

Proof. This follows via Theorem 4.7 and Lemma 4.8. The statements and proof of these results are given in Section 4. □

In the following, we set $S_v = (0, T_v)$.

Theorem 3.2. Under Assumptions (A1) and (A2), there is $s \in L^\infty(S_v \times \Omega \times Y)$ with $\partial_t s$, $D_y s \in L^\infty(S_v \times \Omega \times Y)$ such that $Ds_\varepsilon \overset{2-str.}{\longrightarrow} D_y s$.

Proof. The proof of this theorem is given in Section 5, see Lemma 5.8. □

Using the results given in Theorems 3.1 and 3.2, it is then possible to investigate the associated heat conduction problem:

Theorem 3.3. Under Assumptions (A1) and (A2), there is a unique solution of the mathematical problem corresponding to the system given via equations (2.3a) to (2.3h). In addition, we find that

$$\sup_{\varepsilon>0}\left(\|\theta_\varepsilon\|^2_{L^\infty(S_v;L^2(\Omega))} + \|\nabla\theta^{(1)}_\varepsilon\|^2_{L^2(S_v\times\Omega^{(1)}_\varepsilon)} + \varepsilon^2\|\nabla\theta^{(2)}_\varepsilon\|^2_{L^2(S_v\times\Omega^{(2)}_\varepsilon)}\right) < \infty$$

Proof. Using the transformation function s_ε (given via Theorem 3.1) to arrive at a fixed-domain formulation of the problem, we are almost exactly in the situation described in [10] (without the mechanical part). □

We set $Q_Y = \bigcup_{(t,x)\in S_v\times\Omega}\{(t,x)\}\times Y^{(2)}(t,x)$. With $\mathbb{1}_E$, we denote the indicator function of a set E.

Theorem 3.4. Let Assumptions (A1)–(A3) hold. There are functions $\theta \in L^2(S_v; W^{1,2}(\Omega))$ and $\theta^{(2)} \in L^2(Q_Y)$, where $\theta^{(2)}(t,x,\cdot) \in W^{1,2}(Y^{(2)}(t,x))$ for almost all $(t,x) \in S_v \times \Omega$, such that

$$\mathbb{1}_{\Omega_\varepsilon^{(1)}}\theta_\varepsilon^{(1)} \rightharpoonup |Y^{(1)}(t,x)|\theta,\;\; \mathbb{1}_{\Omega_\varepsilon^{(2)}}\theta_\varepsilon^{(2)} \rightharpoonup \int_{Y^{(2)}(t,x)}\theta^{(2)}\,\mathrm{d}y \quad \text{in } L^2(S\times\Omega).$$

Moreover, they solve the following homogenized distributed microstructure problem: The macroscopic temperature θ is governed by an effective heat conduction problem given via

$$\partial_t\theta - \operatorname{div}(\kappa^h\nabla\theta) = f^h + f_\Gamma^h \quad \text{in } S_v\times\Omega, \tag{3.1a}$$
$$-\kappa^h\nabla\theta\cdot\nu = 0 \quad \text{on } S_v\times\partial\Omega, \tag{3.1b}$$
$$\theta(0) = \vartheta^h \quad \text{in } \Omega, \tag{3.1c}$$

which is coupled, via the Dirichlet boundary condition (3.1e), to a micro heat problem with time dependent microstructures for $\theta^{(2)}$ in the form of

$$\partial_t\theta^{(2)} - \kappa^{(2)}\Delta_y\theta^{(2)} = f^{(2)} \quad \text{in } Y^{(2)}(t,x),\ t\in S_v,\ x\in\Omega, \tag{3.1d}$$
$$\theta^{(2)} = \theta \quad \text{on } \Gamma(t,x),\ t\in S_v,\ x\in\Omega, \tag{3.1e}$$
$$\theta^{(2)}(0) = \vartheta^{(2)} \quad \text{in } \Omega\times Y^{(2)}. \tag{3.1f}$$

Finally, the motion of the interface $\Gamma(t,x)$ in normal direction is governed by

$$V_\Gamma = v \quad \text{on } \Gamma(t,x),\ t\in S_v,\ x\in\Omega. \tag{3.1g}$$

Here, the effective coefficients are given as

$$f^h = \int_{Y^{(1)}(t,x)} f^{(1)}\,\mathrm{d}y, \qquad f_\Gamma = \int_{\Gamma(t,x)} Lv + \kappa^{(2)}\nabla_y\theta^{(2)}\cdot n\,\mathrm{d}\sigma,$$
$$\vartheta^h = \int_{Y^{(1)}(t,x)}\vartheta^{(1)}\,\mathrm{d}y, \quad \left(\kappa^h\right)_{ij} = \kappa^{(1)}\min_{\tau\in W^{1,2}(Y^{(1)}(t,x))}\int_{Y^{(1)}(t,x)}(\nabla_y\tau + e_j)\cdot e_i\,\mathrm{d}y,$$

and $f^{(i)}$, $\vartheta^{(i)}$ ($i = 1, 2$), and v are the two-scale limits of their corresponding ε-counterparts.

Proof. Due to the strong convergence result of Lemma 5.8, this homogenization results follows via a standard two-scale limit procedure and is a special case of the homogenization of the thermoelasticity problem performed in [10]. □

4 Interface motion (proof of Theorem 3.1)

This section is devoted to the proof of Theorem 3.1. As a short guideline, this proof follows the following strategy:

(i) We investigate a nonlinear, parametrized ODE-system – given by equations (4.1a) to (4.1d) – tracking the interface motion. This is done via Lemmas 4.2 and 4.3.

(ii) We then show that the motion problem given via conditions (2.4a)-(2.4e) has a unique solution; see Lemma 4.4.

(iii) In Theorem 4.7, the local-in-time existence of the height function h_ε is then deduced via the implicit function theorem.

(iv) Finally, we construct a family of C^1-diffeomorphisms $s_\varepsilon(t,\cdot)\colon \overline{\Omega} \to \overline{\Omega}$ and investigate its properties; see Lemma 4.8.

The first two steps can be found in Section 4.1, and steps (iii) and (iv) are the topic of Section 4.2. In the following, we take $C > 0$ to denote any generic constant that is independent of both l_v and ε (but may depend on the interface $\Gamma = \partial Y^{(2)}$ as well as the overall domain Ω). In addition, we take $C(l_v)$ (sometimes with a subscript, e.g., $C_w(l_v)$) to denote the value at l_v of any monotonically increasing, continuous, and ε-independent function $C\colon [0,\infty) \to (0,\infty)$.

Note that this section is structurally similar to [2, Section 3], where the main substantial differences are due to the parameter ε and its role in the context of homogenization.

4.1 Interface motion problem

We consider the following nonlinear ODE system:

ODE system describing the interface motion

Find $y_\varepsilon, z_\varepsilon\colon S \times U_{\Gamma_\varepsilon} \to \mathbb{R}^3$ such that

$$\partial_t y_\varepsilon(t,x) = -\varepsilon \frac{z_\varepsilon(t,x)}{|z_\varepsilon(t,x)|} v_\varepsilon(t, y_\varepsilon(t,x)) \quad \text{in } S \times U_{\Gamma_\varepsilon}, \tag{4.1a}$$

$$\partial_t z_\varepsilon(t,x) = \varepsilon |z_\varepsilon(t,x)| \nabla v_\varepsilon(t, y_\varepsilon(t,x)) \quad \text{in } S \times U_{\Gamma_\varepsilon}, \tag{4.1b}$$

$$y_\varepsilon(0,x) = x \quad \text{in } U_{\Gamma_\varepsilon}, \tag{4.1c}$$

$$z_\varepsilon(0,x) = -n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon} x) \quad \text{in } U_{\Gamma_\varepsilon}. \tag{4.1d}$$

We extend every solution y_ε to all of Ω by setting $y_\varepsilon(t,x) = x$. Due to $\operatorname{supp} v_\varepsilon \subset U_{\Gamma_\varepsilon}$, y_ε is then continuous across $\partial U_{\Gamma_\varepsilon}$.

Remark 4.1. In Lemma 4.4, we show that the function y_ε characterizes the interface motion in the sense that $\Gamma_\varepsilon(t) = y_\varepsilon(t, \Gamma_\varepsilon)$. The function z_ε describes the direction of the motion. This is illsutrated in Figure 2. Note that, if $\nabla v_\varepsilon \equiv 0$, the solution satisfies $y_\varepsilon(t,\gamma) = \gamma + d_{\Gamma_\varepsilon}(y_\varepsilon(t,\gamma))n_{\Gamma_\varepsilon}(\gamma)$ for all $\gamma \in \Gamma$.

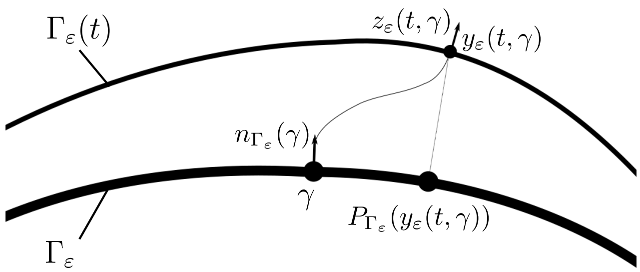


Figure 2: Part of the surface Γ_ε and its position at time t, $\Gamma_\varepsilon(t)$. The function y_ε characterizes the motion by tracking the paths of the material points. As an example, we see the path of y_ε for $\gamma = y_\varepsilon(0,\gamma)$ over the interval $(0,t)$. In addition, we see the change in the normal vector from $n_{\Gamma_e}(\gamma) = z_\varepsilon(0,\gamma)$ to $z_\varepsilon(t,\gamma)$. The goal is to find the corresponding height function h_ε that satisfies $h_\varepsilon(P_{\Gamma_\varepsilon}(y_\varepsilon(t,\gamma))) = d_{\Gamma_\varepsilon}(y_\varepsilon(t,\gamma))$.

We introduce functions

$$f_\varepsilon\colon \overline{S} \times \left(\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{0\}\right) \to \mathbb{R}^3 \times \mathbb{R}^3, \quad f_\varepsilon(t,(y,z)) = \left(\frac{z}{|z|}v_\varepsilon(t,y), |z|\,\nabla v_\varepsilon(t,y)\right)^T,$$
$$g_\varepsilon\colon \Omega \to \mathbb{R}^3 \times \mathbb{R}^3, \quad g_\varepsilon(x) = \left(x, -n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x))\right)^T.$$

Setting $w_\varepsilon = (y_\varepsilon, z_\varepsilon)^T$, equations (4.1a) to (4.1d) then become

$$\partial_t w_\varepsilon(t,x) = \varepsilon f_\varepsilon\left(t, w_\varepsilon(t,x)\right) \quad \text{in } S \times U_{\Gamma_\varepsilon}, \tag{4.2a}$$
$$w_\varepsilon(0,x) = g_\varepsilon(x) \quad \text{in } U_{\Gamma_\varepsilon}. \tag{4.2b}$$

Lemma 4.2. Let Assumption (A1) hold. The ODE system given via equations (4.1a) to (4.1d) admits a unique solution $(y_\varepsilon, z_\varepsilon) \in W^{(1,2),\infty}(S \times U_{\Gamma_\varepsilon})^6$. Additionally, there exists a monotonically increasing, continuous function $C_w\colon [0,\infty) \to (0,\infty)$, which is independent of the parameter ε, such that

$$\|D_x y_\varepsilon - \mathbb{I}\|_\infty + \|\partial_t D_x y_\varepsilon\|_\infty + \varepsilon\|D_x^2 y_\varepsilon\|_\infty \le l_v C_w(l_v),$$
$$\varepsilon\|D_x z_\varepsilon\|_\infty + \varepsilon^2\|D_x^2 z_\varepsilon\|_\infty \le C_w(l_v).$$

Proof. (i) Existence and Uniqueness. Due to the embedding $W^{k,\infty}(U_{\Gamma_\varepsilon}) = C^{k-1,1}(U_{\Gamma_\varepsilon})$ $(k \ge 1)$ (we refer to [11, Theorem 7]) we have $v_\varepsilon, \partial_j v_\varepsilon \in C^{1,1}(S \times U_{\Gamma_\varepsilon})$ $(j = 1,2,3)$ which, in turn, implies

$f_\varepsilon \in C^{1,1}\left(S \times (\mathbb{R}^3 \times K)\right)$ for every compact set $K \subset \mathbb{R}^3 \setminus \{0\}$. Therefore, for every $x \in \Omega$, *Picard-Lindeloef's existence theorem* ([23, Proposition 1.8]) guarantees the existence of a time $t_\varepsilon(x) \in S$ and a unique solution $w_\varepsilon(\cdot, x) = (y_\varepsilon(\cdot, x), z_\varepsilon(\cdot, x))^T \in C^{1,1}([0, t_\varepsilon(x)])^6$. Note that $|z_\varepsilon(0, x)| = 1$ independently of $x \in U_{\Gamma_\varepsilon}$. Taking a look at equation (4.1b), we see that

$$-\varepsilon t l_v \leq \int_0^t \frac{\partial_t (z_\varepsilon \cdot e_j)}{|z_\varepsilon|} \,\mathrm{d}\tau \leq \varepsilon t l_v \quad (j = 1, 2, 3).$$

The norm of every solution z_ε is therefore bounded from below and above via

$$e^{-\varepsilon l_v t} \leq |z_\varepsilon(t, x)| \leq e^{\varepsilon l_v t}. \tag{4.3}$$

As a consequence, a blow up due to $|z_\varepsilon| \to 0$ is not possible in finite time and we can extend to $w_\varepsilon(\cdot, x) \in C^{1,1}(\overline{S})^6$ for $x \in U_{\Gamma_\varepsilon}$.

(ii) Regularity and Estimates. For any $x_1, x_2 \in U_{\Gamma_\varepsilon}$, we find that

$$w_\varepsilon(t, x_1) - w_\varepsilon(t, x_2) = g_\varepsilon(x_1) - g_\varepsilon(x_2) + \int_0^t f_\varepsilon(\tau, w_\varepsilon(\tau, x_1)) - f_\varepsilon(\tau, w_\varepsilon(\tau, x_1)) \,\mathrm{d}\tau.$$

From $g_\varepsilon \in C^2(U_{\Gamma_\varepsilon})$, the Lipschitz continuity of f_ε as well as Df_ε, and Gronwall's inequality, we can infer $w_\varepsilon(t, \cdot) \in W^{(1,2),\infty}(S \times U_{\Gamma_\varepsilon})^6$.

In the following, let $\varepsilon > 0$ be sufficiently small such that ${}^1\!/\!\sqrt{2} \leq \|z_\varepsilon\|_\infty \leq \sqrt{2}$ (cf. inequality (4.3)). Differentiating the ODE with respect to $x \in U_{\Gamma_\varepsilon}$, we get

$$\partial_t D w_\varepsilon(t, x) = \varepsilon D_x \left(f_\varepsilon\left(t, w_\varepsilon(t, x)\right)\right). \tag{4.4}$$

We define $A_\varepsilon \colon S \times (\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{0\}) \to \mathbb{R}^{6\times 6}$ via

$$A_\varepsilon(t, (y, z)) := D_{(y,z)} f_\varepsilon\left(t, (y, z)\right) = \begin{pmatrix} \frac{z}{|z|} \otimes \nabla v_\varepsilon(t, y) & v_\varepsilon B(z) \\ |z| D^2 v_\varepsilon(t, y) & \nabla v_\varepsilon(t, y) \otimes \frac{z}{|z|} \end{pmatrix},$$

where $B \colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^{3\times 3}$ is given via

$$B(z) = D\left(z \mapsto \frac{z}{|z|}\right) = \frac{1}{|z|^3} \begin{pmatrix} z_2^2 + z_3^2 & -z_1 z_2 & -z_1 z_3 \\ -z_1 z_2 & z_1^2 + z_3^2 & -z_2 z_3 \\ -z_1 z_3 & -z_2 z_3 & z_1^2 + z_2^2 \end{pmatrix}. \tag{4.5}$$

Equation (4.4) can be rewritten into

$$\partial_t D w_\varepsilon(t, x) = \varepsilon A_\varepsilon(t, w_\varepsilon(t, x)) D w_\varepsilon(t, x). \tag{4.6}$$

With the estimate $\|B(z)\| \leq {}^{\sqrt{2}}\!/\!{}_{|z|}$ (Frobenius-Norm), the estimate for z_ε given by inequality (4.3), and Assumption (A1), we get (for sufficiently small ε)

$$\varepsilon |A_\varepsilon(t, (y_\varepsilon, z_\varepsilon)| \leq l_v(3\varepsilon + \sqrt{2}) \leq 2 l_v. \tag{4.7}$$

For the initial values of the *Jacobian* matrices, we have (for the derivative of $n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x))$, we refer to [19, Chapter 2, Section 3.1])

$$\begin{aligned} &Dy_\varepsilon(0,x) = \mathbb{I}_3, \\ &Dz_\varepsilon(0,x) = D\left(n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x))\right) = -L_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x))\left(\mathbb{I} - d_{\Gamma_\varepsilon}(x)L_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x))\right)^{-1}. \end{aligned}$$

As $|Dz_\varepsilon(0,x)| \leq {}^{C}\!/\!_{\varepsilon}$ for some $C > 0$, we can deduce estimate via *Gronwall's inequality* that

$$\varepsilon\,|Dw_\varepsilon(t,x)| \leq C\exp(2Tl_v) =: C_1(l_v). \tag{4.8}$$

For y_ε, we have

$$Dy_\varepsilon(t,x) = \mathbb{I}_3 + \varepsilon\int_0^t \left(A_\varepsilon^{(11)}(t,w_\varepsilon(\tau,x))Dy_\varepsilon(\tau,x) + A_\varepsilon^{(12)}(t,w_\varepsilon(\tau,x))Dz_\varepsilon(\tau,x)\right)\,\mathrm{d}\tau. \tag{4.9}$$

Inserting the estimate given in inequality (4.8) into equation (4.9), we see that

$$|Dy_\varepsilon(t,x)| \leq 1 + 3TC_1(l_v)l_v. \tag{4.10}$$

Looking at equation (4.9) and using the estimate for A_ε (cf. inequality (4.7)), we get (for small ε)

$$|\partial_t Dy_\varepsilon(t,x)| \leq 3C_1(l_v)l_v. \tag{4.11}$$

Similarly, differentiating Dw_ε with respect to x_j ($j = 1,2,3$) and estimating the different terms accordingly, we also get

$$\varepsilon^2|\partial_j Dw_\varepsilon(t,x)| \leq C_2(l_v). \tag{4.12}$$

With this estimate, we can further bound $|\partial_{x_i}Dy_\varepsilon(t,x)|$ via

$$|\partial_{x_i}Dy_\varepsilon(t,x)| \leq l_vC_3(l_v). \tag{4.13}$$

The details regarding these calculations are given in [9, Lemma 6.6]. Now, combining inequalities (4.8) and (4.10) to (4.13), the function C_w can be directly constructed via $C_j(l_v)$ ($j = 1,2,3$). □

Note that $l_vC_w(l_v) \to 0$ for $l_v \to 0$. In the following lemma, we show that $y_\varepsilon(t,\cdot)$ is a homeomorphism (a minimal requirement for it to correspond to a meaningful transformation) for $t \in \overline{S}$ small enough. Moreover, for small l_v, this holds for all $t \in \overline{S}$.

Lemma 4.3. There is a monotonically decreasing and continuous function $\delta\colon (0,\infty) \to (0,\infty)$ (we set $t_v = \min\{\delta(l_v), T\}$) such that:

(i) The function $y_\varepsilon(\tau,\cdot)\colon U_{\Gamma_\varepsilon} \to y_\varepsilon\left(\tau, U_{\Gamma_\varepsilon}\right)$ is a Lipschitz homeomorphism for all $\tau \in [0,t_v]$.

(ii) For $t \in [0, t_v]$, let

$$y_{\varepsilon,t}^{-1} \colon y_\varepsilon(t, U_{\Gamma_\varepsilon}) \to U_{\Gamma_\varepsilon}$$

be the unique function that satisfies $y_{\varepsilon,t}^{-1}(y_\varepsilon(t,x)) = x$ for all $x \in U_{\Gamma_\varepsilon}$. The function

$$y_\varepsilon^{-1} \colon \bigcup_{t\in[0,t_v]} \big(\{t\} \times y_\varepsilon(t, U_{\Gamma_\varepsilon})\big) \to U_{\Gamma_\varepsilon}, \quad y_\varepsilon^{-1}(t,w) := y_{\varepsilon,t}^{-1}(w)$$

is Lipschitz continuous with respect to $t \in [0, t_v]$.

Proof. (i). We recall the characterization of Dy_ε established in the proof of the preceding lemma, i.e., equation (4.9):

$$Dy_\varepsilon(t,x) = \mathbb{I}_3 + \varepsilon \int_0^t \left(A_\varepsilon^{(11)}(t, w_\varepsilon(\tau,x)) Dy_\varepsilon(\tau,x) + A_\varepsilon^{(12)}(t, w_\varepsilon(\tau,x)) Dz_\varepsilon(\tau,x)\right) \,\mathrm{d}\tau.$$

From here, we conclude that

$$\|Dy_\varepsilon(t,\cdot) - \mathbb{I}_3\|_\infty \leq 3tl_v C_1(l_v) \quad \text{for all } t \in \overline{S}.$$

This shows (employing the *Neumann* series) that $y_\varepsilon(t,\cdot) \colon U_{\Gamma_\varepsilon} \to y_\varepsilon(t, U_{\Gamma_\varepsilon})$ is a Lipschitz homeomorphism for all $t \in [0, t_v]$ where $t_v = \min\{(4l_v C_1(l_v))^{-1}, T\}$. Here, the function δ is given via $(4l_v C_1(l_v))^{-1}$.

(ii). It holds $y_\varepsilon(t, y_\varepsilon^{-1}(t,x)) = x$ for all $(t,x) \in \bigcup_{t\in[0,t_v]} (\{t\} \times y_\varepsilon(t, U_{\Gamma_\varepsilon}))$. Implicit differentiation leads to

$$\partial_t \left(y_\varepsilon(t, y_\varepsilon^{-1}(t,x))\right) = \partial_t y_\varepsilon(t, y_\varepsilon^{-1}(t,x)) + Dy_\varepsilon(t, y_\varepsilon^{-1}(t,x)) \partial_t y_\varepsilon^{-1}(t,x) = 0$$

and, therefore,

$$\begin{aligned} \partial_t y_\varepsilon^{-1}(t,x) &= -\left(Dy_\varepsilon(t, y_\varepsilon^{-1}(t,x))\right)^{-1} \partial_t y_\varepsilon(t, y_\varepsilon^{-1}(t,x)) \\ &= \varepsilon \left(Dy_\varepsilon(t, y_\varepsilon^{-1}(t,x))\right)^{-1} \frac{z_\varepsilon(t, y_\varepsilon^{-1}(t,x))}{|z_\varepsilon(t, y_\varepsilon^{-1}(t,x))|} v_\varepsilon(t, y_\varepsilon(t, y_\varepsilon^{-1}(t,x))). \end{aligned} \tag{4.14}$$

As the right hand side is bounded by virtue of the estimates provided in Lemma 4.2, this implies Lipschitz continuity of y_ε^{-1} with respect to $t \in [0, t_v]$. □

With the following lemma, we show that any solution of the motion problem given by equations (2.4a) to (2.4e) can be characterized via y_ε and that, indeed, there is a unique solution to the motion problem.

Lemma 4.4. (i) Let $\{\Gamma_\varepsilon(t)\}_{t\in[0,t_v]}$ be a solution of the free boundary problem given by equations (2.4a) to (2.4e). Then, for all $t \in [0, t_v]$, $\Gamma_\varepsilon(t) = y_\varepsilon(t, \Gamma_\varepsilon)$.

(ii) There is a unique solution to the motion problem posed in the time interval $[0, t_v]$.

Proof. (i). This is shown in [2, Lemma 3.2] using the method of characteristics.

(ii). This proof follows closely along the lines of [2, Theorem 3.1] adapting the ideas to our setting. We introduce a Lipschitz continuous function $\widetilde{\varphi}_\varepsilon\colon [0,t_v]\times\Omega \to [-\varepsilon a, \varepsilon a]$ via (as a reminder: $\Gamma_\varepsilon^{(l)} = \{\Lambda_\varepsilon(\gamma, l) \ : \ \gamma \in \Gamma_\varepsilon\}$, see equation (2.1))

$$\widetilde{\varphi}_\varepsilon(t,x) = \begin{cases} -\varepsilon a, & x \in \Omega_\varepsilon^{(1)} \setminus y_\varepsilon(t, U_{\Gamma_\varepsilon}) \\ -l, & x \in y_\varepsilon(t, \Gamma_\varepsilon^{(l)}) \text{ for some } l \in (-\varepsilon a, \varepsilon a) \\ \varepsilon a, & y \in \Omega_\varepsilon^{(2)} \setminus y_\varepsilon(t, U_{\Gamma_\varepsilon}) \end{cases}.$$

In the same way as in [2, Theorem 3.1], it can be shown that

$$\nabla\widetilde{\varphi}_\varepsilon(t,x) = z_\varepsilon(t, y_\varepsilon^{-1}(t,x)) \quad \text{for all } x \in y_\varepsilon(t, U_{\Gamma_\varepsilon}),\ t \in [0,t_v]$$

and, therefore,

$$e^{\varepsilon l_v t} \geq |\nabla\widetilde{\varphi}_\varepsilon(t,x)| \geq e^{-\varepsilon l_v t} \quad \text{for all } x \in y_\varepsilon(t, U_{\Gamma_\varepsilon}),\ t \in [0,t_v] \tag{4.15}$$

as well as

$$\partial_t\widetilde{\varphi}_\varepsilon(t,y) = \varepsilon|\nabla\widetilde{\varphi}_\varepsilon(t,y)| v_\varepsilon(t,y) \quad \text{in } \bigcup_{t\in[0,t_v]} (\{t\}\times y_\varepsilon(t, U_{\Gamma_\varepsilon}))\,. \tag{4.16}$$

Due to the Lipschitz continuity of the involved derivatives, we get

$$\widetilde{\varphi}_\varepsilon \in W^{(2,2),\infty}\left(\bigcup_{t\in[0,t_v]} (\{t\}\times y_\varepsilon(t, U_{\Gamma_\varepsilon}))\right).$$

Now, let $g\colon \mathbb{R}\to[0,1]$ be a C^2-function such that $g(0)=0$, $g'(0)=1$, $g'(r)=0$ if $r\notin(-a/2, a/2)$, and $|g''|\leq 3/a$. We introduce $\varphi_\varepsilon = \varepsilon g\circ(\varepsilon^{-1}\widetilde{\varphi}_\varepsilon) \in W^{(2,2),\infty}([0,t_v]\times\Omega)$. Then, $\varphi_\varepsilon = 0$ if and only if $\widetilde{\varphi}_\varepsilon = 0$ which implies

$$\Gamma_\varepsilon = \{x\in\Omega \ : \ \varphi_\varepsilon(0,x) = 0\}.$$

and

$$\{x\in\Omega \ : \ \varphi_\varepsilon(t,x)=0\} = y_\varepsilon(t,\Gamma_\varepsilon) \quad \text{for all } t\in[0,t_v].$$

It then can easily be checked that φ_ε satisfies the conditions of the motion problem given by equations (2.4a) to (2.4e). □

Lemma 4.5. There is a continuous function $C_\varphi\colon [0,\infty)\to(0,\infty)$ such that

$$\varepsilon^{-1}\|\partial_t\widetilde{\varphi}_\varepsilon\|_\infty + \|\partial_t\nabla\widetilde{\varphi}_\varepsilon\|_\infty \leq l_v C_\varphi(l_v),$$
$$\|\nabla\widetilde{\varphi}_\varepsilon\|_\infty + \varepsilon\|D^2\widetilde{\varphi}_\varepsilon\|_\infty \leq C_\varphi(l_v).$$

Proof. In this proof, we rely on the estimates provided in Lemma 4.2. Let $t \in [0, t_v]$ and $x \in y_\varepsilon(t, U_{\Gamma_\varepsilon})$. The second spatial derivative is given as

$$D^2\widetilde{\varphi}_\varepsilon(t,x) = (Dy_\varepsilon(t, y_\varepsilon^{-1}(t,x)))^{-1} Dz_\varepsilon(t, y_\varepsilon^{-1}(t,x))$$

and can therefore be estimated via

$$\left|D^2\widetilde{\varphi}_\varepsilon(t,x)\right| \leq \frac{4}{\varepsilon} C_w(l_v)$$

where C_w is the function given by Lemma 4.2. Furthermore, as $\widetilde{\varphi}_\varepsilon$ satisfies inequality (4.15) and equation (4.16), we can estimate

$$|\partial_t \widetilde{\varphi}_\varepsilon(t,x)| \leq \varepsilon |\nabla\widetilde{\varphi}_\varepsilon(t,x)||v_\varepsilon(t,x)| \leq \varepsilon e^{\varepsilon l_v t} l_v.$$

Taking the derivative with respect to $x \in y_\varepsilon(t, U_{\Gamma_\varepsilon})$ in equation (4.16), we get

$$\partial_t \nabla\widetilde{\varphi}_\varepsilon(t,x) = \varepsilon|\nabla\widetilde{\varphi}_\varepsilon(t,x)|\nabla v_\varepsilon(t,x) + \varepsilon D^2\widetilde{\varphi}_\varepsilon(t,x)\frac{\nabla\widetilde{\varphi}_\varepsilon(t,x)}{|\nabla\widetilde{\varphi}_\varepsilon(t,x)|}|v_\varepsilon(t,x)|$$

and find the upper bound

$$|\partial_t \nabla\widetilde{\varphi}_\varepsilon(t,x)| \leq l_v \left(\varepsilon_0 e^{\varepsilon_0 l_v t_v} + 4C_w(l_v)\right).$$

□

4.2 Motion function

For $\varepsilon > 0$ and $\gamma \in \Gamma_\varepsilon$, we introduce the function $F_{\varepsilon,\gamma} \colon [0, t_v] \times (-\varepsilon a, \varepsilon a) \to \mathbb{R}$ via $F_{\varepsilon,\gamma}(t,r) = \varphi_\varepsilon(t, \Lambda_\varepsilon(\gamma, r))$. Then, $F_{\varepsilon,\gamma}(0,0) = \varphi_\varepsilon(0, \Lambda_\varepsilon(\gamma, 0)) = 0$ for all $\gamma \in \Gamma_\varepsilon$.

Lemma 4.6. For all $\varepsilon > 0$ and $\gamma \in \Gamma_\varepsilon$, it holds $\partial_2 F_{\varepsilon,\gamma}(0,0) = -1$. Furthermore, there are $\widetilde{t}_v \in [0, t_v]$ and $0 < R_v < a$ such that $\partial_2 F_{\varepsilon,\gamma}(t,r) \leq -1/3$ for all $t \in [0, \widetilde{t}_v]$ and $r \in [-\varepsilon R_v, \varepsilon R_v]$.

Proof. We calculate

$$\partial_2 F_{\varepsilon,\gamma}(t,r) = g'(\varepsilon^{-1}\widetilde{\varphi}_\varepsilon(t, \Lambda_\varepsilon(\gamma,r)))\nabla\widetilde{\varphi}_\varepsilon(t, \Lambda_\varepsilon(\gamma,r)) \cdot n_{\Gamma_\varepsilon}(\gamma) \tag{4.17}$$

and see that

$$\partial_2 F_{\varepsilon,\gamma}(0,0) = -1 < 0.$$

For any $t \in [0, t_v]$ and $r \in (-\varepsilon a, \varepsilon a)$, we have

$$\partial_2 F_{\varepsilon,\gamma}(t,r) = -1 + \int_0^r \partial_2^2 F_{\varepsilon,\gamma}(0,s)\,\mathrm{d}s + \int_0^t \partial_t\partial_2 F_{\varepsilon,\gamma}(\tau,r)\,\mathrm{d}\tau.$$

Starting off with the first integrand, $\partial_2^2 F_{\varepsilon,\gamma}$, we get

$$\partial_2^2 F_{\varepsilon,\gamma}(t,r) = \varepsilon^{-1} g''(\varepsilon^{-1}\widetilde{\varphi}_\varepsilon(t,\Lambda_\varepsilon(\gamma,r)))\left(\nabla\widetilde{\varphi}_\varepsilon(t,\Lambda_\varepsilon(\gamma,r))\cdot n_{\Gamma_\varepsilon}(\gamma)\right)^2 + D^2\widetilde{\varphi}_\varepsilon(t,\Lambda_\varepsilon(\gamma,r))n_{\Gamma_\varepsilon}(\gamma)\cdot n_{\Gamma_\varepsilon}(\gamma).$$

Using the estimates collected in Lemma 4.5, we can conclude that

$$\varepsilon\left|\partial_2^2 F_{\varepsilon,\gamma}(t,r)\right| \le \frac{3}{a}e^{2\varepsilon l_v t} + C_\varphi(l_v).$$

For the second integrand, $\partial_t\partial_2 F_{\varepsilon,\gamma}$, we calculate

$$\partial_t\partial_2 F_{\varepsilon,\gamma}(t,r) = \varepsilon^{-1} g''(\varepsilon^{-1}\widetilde{\varphi}_\varepsilon(t,\Lambda_\varepsilon(\gamma,r)))\partial_t\widetilde{\varphi}_\varepsilon(t,\Lambda_\varepsilon(\gamma,r))\nabla\widetilde{\varphi}_\varepsilon(t,\Lambda_\varepsilon(\gamma,r))\cdot n_{\Gamma_\varepsilon}(\gamma) + g'(\varepsilon^{-1}\widetilde{\varphi}_\varepsilon(t,\Lambda_\varepsilon(\gamma,r)))\partial_t\nabla\widetilde{\varphi}_\varepsilon(t,\Lambda_\varepsilon(\gamma,r))\cdot n_{\Gamma_\varepsilon}(\gamma)$$

and estimate

$$|\partial_t\partial_2 F_{\varepsilon,\gamma}(t,r)| \le \frac{3}{a} l_v C_\varphi(l_v)\left(C_\varphi(l_v)+1\right)$$

and finally arrive at

$$\partial_2 F_{\varepsilon,\gamma}(t,r) \le -1 + \frac{r}{\varepsilon}\left(\frac{3}{a}e^{2\varepsilon_0 l_v t} + C_\varphi(l_v)\right) + t l_v\left(\frac{3}{a}C_\varphi(l_v)\left(C_\varphi(l_v)+1\right)\right).$$

□

Theorem 4.7 (Height function)**.** There is a time $T_v \in (0,\tau_v]$ monotonically decreasing with respect to l_v and such that $T_v = T$ for l_v sufficiently small such that:

(i) There is a height function $h_\varepsilon\colon \Gamma_\varepsilon \times [0,T_v] \to (-\varepsilon a, \varepsilon a)$ satisfying

$$\Gamma_\varepsilon(t) = \{\Lambda_\varepsilon(\gamma, h_\varepsilon(t,\gamma)) \ : \ \gamma\in\Gamma_\varepsilon\} \quad \text{for all } t\in[0,T_v]$$

(ii) It holds the estimate

$$\frac{5}{\varepsilon a}\|h_\varepsilon\|_{L^\infty((0,T_v)\times\Gamma_\varepsilon)} + 2\|\nabla_{\Gamma_\varepsilon} h_\varepsilon\|_{L^\infty((0,T_v)\times\Gamma_\varepsilon)} \le \frac{1}{2}.$$

Moreover, $\|\partial_t h_\varepsilon\|_\infty \le 3\varepsilon l_v C_\varphi(l_v)$.

Proof. (i). Note that $F_{\varepsilon,\gamma}(0,0) = 0$ and $\partial_2 F_{\varepsilon,\gamma}(0,0) = -1$. By the *Implicit Function Theorem*, we infer that, for every $\varepsilon > 0$ and for every $\gamma\in\Gamma_\varepsilon$, there is a time $\tau_{\varepsilon,\gamma} > 0$ and a differentiable function $h_{\varepsilon,\gamma}\colon [0,\tau_{\varepsilon,\gamma}] \to (-\varepsilon a, \varepsilon a)$ such that $F_{\varepsilon,\gamma}(t, h_{\varepsilon,\gamma}(t)) = 0$ for all $t\in[0,\tau_{\varepsilon,\gamma}]$. Let $\tau_{\varepsilon,\gamma}\in\overline{S}$ always be the maximal possible point in time for this to be true. It holds that

$$\sup\{|h_{\varepsilon,\gamma}(t)| \ : \ \gamma\in\Gamma_\varepsilon\} = \sup\{|d_{\Gamma_\varepsilon}(y_\varepsilon(t))| \ : \ \gamma\in\Gamma_\varepsilon\} \le \varepsilon t l_v \quad \text{for all } t\in[0,\tau_{\varepsilon,\gamma}],$$

Here, the equality holds due to

$$\Gamma_\varepsilon(t) = \{\Lambda_\varepsilon(\gamma, h_{\varepsilon,\gamma}(t)) \ : \ \gamma \in \Gamma_\varepsilon\} = \{y_\varepsilon(t,\gamma) \ : \ \gamma \in \Gamma_\varepsilon\}.$$

And, for the inequality, we observe that $y_\varepsilon(0,\gamma) \in \Gamma_\varepsilon$ and that y_ε satisfies equation (4.1a). Now, take $\tau_v = \min\{t_v, l_v^{-1} R_v\}$. We claim that

$$\inf\{\tau_{\varepsilon,\gamma} \ : \varepsilon > 0, \ \gamma \in \Gamma_\varepsilon\} \geq \tau_v.$$

Let us assume this is not the case, i.e., there are $\varepsilon > 0$ and $\gamma \in \Gamma_\varepsilon$ such that $\tau_{\varepsilon,\gamma} < \tau_v$. Since

$$(i) \ F_{\varepsilon,\gamma}(\tau_{\varepsilon,\gamma}, h_{\varepsilon,\gamma}(\tau_{\varepsilon,\gamma})) = 0,$$
$$(ii) \ \partial_2 F_{\varepsilon,\gamma}(\tau_{\varepsilon,\gamma}, h_{\varepsilon,\gamma}(\tau_{\varepsilon,\gamma})) < -\frac{1}{3},$$

we can apply the *Implicit Function Theorem* again which contradicts the assumption that $\tau_{\varepsilon,\gamma}$ is maximal. Here, (ii) holds true by virtue of Lemma 4.6. As a consequence, we are able to define $h_\varepsilon \colon [0,\tau_v] \times \Gamma_\varepsilon \to (-\varepsilon a, \varepsilon a)$ via $h_\varepsilon(t,\gamma) := h_{\varepsilon,\gamma}(t)$.

(ii). Owing to the regularity of Λ_ε and φ_ε, we have $h_\varepsilon \in W^{2,\infty}((0,T) \times \Gamma_\varepsilon)$. For all $t \in [0,\tau_v]$ and $\gamma \in \Gamma_\varepsilon$, we have $F_{\varepsilon,\gamma}(t, h_\varepsilon(t,\gamma)) = 0$ implying vanishing derivatives with respect to time and space. Implicit differentiation with respect to time yields

$$\partial_t h_\varepsilon(t,\gamma) = -\frac{\partial_t F_{\varepsilon,\gamma}(t, h_\varepsilon(t,\gamma))}{\partial_2 F_{\varepsilon,\gamma}(t, h_\varepsilon(t,\gamma))}. \tag{4.18}$$

Considering that $\|g'\|_\infty \leq 1$, we are therefore led to

$$|\partial_t h_\varepsilon(t,\gamma)| \leq 3 \, |\partial_t \widetilde{\varphi}_\varepsilon(t, \Lambda_\varepsilon(\gamma, h_\varepsilon(t,\gamma)))| \leq 3\varepsilon l_v C_\varphi(l_v).$$

Let us first observe that $\nabla_{\Gamma_\varepsilon} h_\varepsilon(t,\gamma) = 0$ if and only if

$$n_{\Gamma_\varepsilon}(t, \Lambda_\varepsilon(\gamma, h_\varepsilon(t,\gamma)) = n_{\Gamma_\varepsilon}(\gamma).$$

The normal vector at $\gamma \in \Gamma_\varepsilon(t)$ is given as

$$n_{\Gamma_\varepsilon}(t,\gamma) = \frac{\nabla \varphi_\varepsilon(t,\gamma)}{|\nabla \varphi_\varepsilon(t,\gamma)|} = \frac{\nabla \widetilde{\varphi}_\varepsilon(t,\gamma)}{|\nabla \widetilde{\varphi}_\varepsilon(t,\gamma)|}.$$

For the surface gradient of h_ε, we can find the representation (we point to [19, Section 2.5])

$$\nabla_{\Gamma_\varepsilon} h_\varepsilon(t,\gamma) = (\mathbb{I}_3 - h_\varepsilon(t,\gamma) L_{\Gamma_\varepsilon}(\gamma)) \left(n_{\Gamma_\varepsilon}(\gamma) - \frac{1}{n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t) \cdot n_{\Gamma_\varepsilon}(\gamma)} n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t) \right), \tag{4.19}$$

where we have set $\overline{\gamma}_t = y_\varepsilon(t,\gamma)$. Due to

$$n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t) = n_{\Gamma_\varepsilon}(\gamma) + \int_0^t \underbrace{\frac{\partial_t \nabla \widetilde{\varphi}_\varepsilon(t,\overline{\gamma}_t) |\nabla \widetilde{\varphi}_\varepsilon(t,\overline{\gamma}_t)| - \nabla \widetilde{\varphi}_\varepsilon(t,\overline{\gamma}_t) \partial_t |\nabla \widetilde{\varphi}_\varepsilon(t,\overline{\gamma}_t)|}{|\nabla \widetilde{\varphi}_\varepsilon(t,\overline{\gamma}_t)|^2}}_{=:\Phi_\varepsilon(\tau,\overline{\gamma}_t)} \, \mathrm{d}\tau,$$

we estimate

$$|n_{\Gamma_\varepsilon}(\gamma) - n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t)| \leq \int_0^t |\Phi_\varepsilon(\tau,\overline{\gamma}_t)| \,\mathrm{d}\tau \leq 2tl_v e^{3\varepsilon_0 l_v t} C_\varphi(l_v),$$

and (for t small enough, but independent of ε and decreasing with increasing l_v)

$$0 < 1 - 2te^{3\varepsilon l_v t} l_v C_\varphi(l_v) \leq n_{\Gamma_\varepsilon}(\gamma)\cdot n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t) \leq 1.$$

Combining these estimates to bound the difference

$$n_{\Gamma_\varepsilon}(\gamma) - \frac{1}{n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t)\cdot n_{\Gamma_\varepsilon}(\gamma)} n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t) = n_{\Gamma_\varepsilon}(\gamma) - n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t) + \frac{n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t)\cdot n_{\Gamma_\varepsilon}(\gamma) - 1}{n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t)\cdot n_{\Gamma_\varepsilon}(\gamma)} n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t),$$

we are led to

$$\left| n_{\Gamma_\varepsilon}(\gamma) - \frac{1}{n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t)\cdot n_{\Gamma_\varepsilon}(\gamma)} n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t)\right| \leq 2tl_v e^{3\varepsilon_0 l_v} C_\varphi(l_v)\left(1 + \sum_{k=0}^\infty \left(2tl_v e^{3\varepsilon_0 l_v t} C_\varphi(l_v)\right)^k\right).$$

In summary, estimating equation (4.19) leads us to

$$|\nabla_{\Gamma_\varepsilon} h_\varepsilon(t,\gamma)| \leq \left(1+\frac{tl_v}{2a}\right)\left(2te^{3\varepsilon_0 l_v t} l_v C_\varphi(l_v)\left(1 + \sum_{k=0}^\infty \left(2tl_v e^{3\varepsilon_0 l_v t} C_\varphi(l_v)\right)^k\right)\right).$$

□

Let $\chi \in \mathcal{D}(\mathbb{R}_{\geq}0)$ be a cut-off function that satisfies

$$0 \leq \chi \leq 1, \qquad \chi(r) = 1 \text{ if } r < \frac{1}{3}, \qquad \chi(r) = 0 \text{ if } r > \frac{2}{3}.$$

In addition, let $\chi'(r) < 0$ if $1/3 < r < 2/3$ as well as $\|\chi'\|_\infty \leq 4$.

We introduce the function $s_\varepsilon\colon [0,T_v]\times\overline{\Omega} \to \overline{\Omega}$ via

$$s_\varepsilon(t,x) = \begin{cases} x + h_\varepsilon(t, P_{\Gamma_\varepsilon}(x)) n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x))\chi\left(\frac{\operatorname{dist}(x,\Gamma_\varepsilon)}{\varepsilon a}\right), & x \in U_{\Gamma_\varepsilon} \\ x, & x \notin U_{\Gamma_\varepsilon} \end{cases}. \tag{4.20}$$

Lemma 4.8. The function $s_\varepsilon\colon [0,T_v]\times\overline{\Omega} \to \overline{\Omega}$ is a regular C^1-motion with $\Gamma_\varepsilon(t) = s_\varepsilon(t,\Gamma_\varepsilon)$ for all $t \in [0,T_v]$.

Proof. With the estimates provided in Theorem 4.7, we can conclude that $s_\varepsilon(t,\cdot)\colon \overline{\Omega} \to \overline{\Omega}$ is a regular C^1-deformation with $\Gamma_\varepsilon(t) = s_\varepsilon(t,\Gamma_\varepsilon)$ for all $t \in [0,T_v]$. For details, we refer to [9, Lemma 2.9]. The regularity with respect to time follows via $h_\varepsilon \in C^{1,1}([0,T_v]\times\overline{\Omega})$. □

5 Limit behavior (proof of Theorem 3.2)

In this section, the limit behavior of the functions related to the Hanzawa transformation s_ε as given by Lemma 4.8, in particular $F_\varepsilon = Ds_\varepsilon$ and $J_\varepsilon = \det F_\varepsilon$, are investigated. To be able to pass to the limit $\varepsilon \to 0$, strong two-scale convergence of these quantities has to be established. We start by introducing the folding and unfolding operators, similar (in spirit) considerations can be found, e.g., in [14], and by formulating a few technical lemmas.

In an effort to keep the notations for the estimations shorter, we introduce functions

$$q_\varepsilon \colon S_v \times U_{\Gamma_\varepsilon} \to \mathbb{R}^3, \quad q_\varepsilon(t,x) := z_\varepsilon(t, y_\varepsilon^{-1}(t,x)), \tag{5.1a}$$

$$\eta_\varepsilon \colon S_v \times \Gamma_\varepsilon \to \Omega, \qquad \eta_\varepsilon(t,\gamma) := \Lambda_\varepsilon(\gamma, h_\varepsilon(t,\gamma)). \tag{5.1b}$$

5.1 Preliminaries and auxiliary lemmas

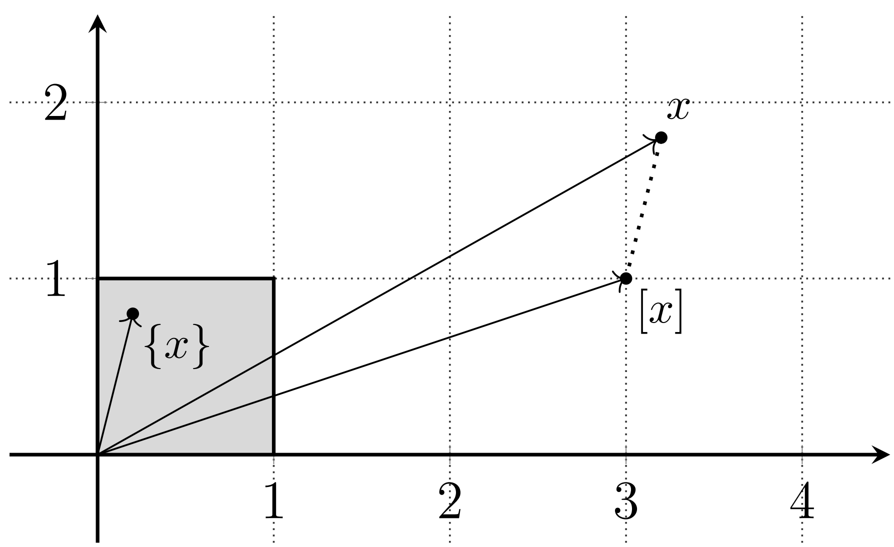


Figure 3: Simple example demonstrating the construction of $[x]$ and $\{x\}$.

For $x \in \mathbb{R}^3$, $[x]$ is defined to be the unique $k \in \mathbb{Z}^3$ such that $\{x\} := x - [x] \in [0,1)^3$ and, for functions $f \colon \Omega \to \mathbb{R}$ and $f_b \colon \Gamma_\varepsilon \to \mathbb{R}$, we denote the periodic unfolding via $[f]^\varepsilon : \Omega \times Y \to \mathbb{R}$ and $[f_b]^\varepsilon : \Omega \times \Gamma \to \mathbb{R}$ defined by

$$[f]^\varepsilon(x,y) = f\left(\varepsilon y + \left[\frac{x}{\varepsilon}\right]\right), \quad [f_b]^\varepsilon(x,\gamma) = f_b\left(\varepsilon\gamma + \left[\frac{x}{\varepsilon}\right]\right).$$

We get the integral identities (see [4])

$$\int_\Omega f(x)\,\mathrm{d}x = \int_{\Omega\times Y} [f]^\varepsilon(x,y)\,\mathrm{d}(x,y),$$
$$\int_{\Gamma_\varepsilon} f_b(x)\,\mathrm{d}x = \frac{1}{\varepsilon}\int_{\Omega\times\Gamma} [f_b]^\varepsilon(x,y)\,\mathrm{d}(x,y)$$

and, for $\mathrm{id}\colon \Omega \to \Omega$ and $n, m \in \mathbb{N}$, it holds

$$|[\mathrm{id}]^{\varepsilon_n} - [\mathrm{id}]^{\varepsilon_m}| \leq \sqrt{2}\,(\varepsilon_n + \varepsilon_m)\,. \tag{5.2}$$

In addition, for functions $g\colon \Omega \times Y \to \mathbb{R}$ and $g_b\colon \Omega \times \Gamma \to \mathbb{R}$, we set

$$\begin{aligned} [g]_\varepsilon\colon \Omega \to \mathbb{R}, &\quad [g]_\varepsilon(x) = g\left(x, \left\{\frac{x}{\varepsilon}\right\}\right), \\ [g_b]_\varepsilon\colon \Gamma_\varepsilon \to \mathbb{R}, &\quad [g_b]_\varepsilon(x) = g_b\left(x, \left\{\frac{x}{\varepsilon}\right\}\right). \end{aligned}$$

We find that, $f \in W^{1,2}(\Omega; W^{1,2}_{\#}(Y))$,

$$\left\| f - [[f]_\varepsilon]^\varepsilon \right\|^2_{L^2(\Omega\times Y)} \to 0 \tag{5.3}$$

as $\left(\varepsilon y + \varepsilon\left[\frac{x}{\varepsilon}\right], \left[y + \left[\frac{x}{\varepsilon}\right]\right]\right)$ converges uniformly to (x, y).

The following identities are a consequence of the periodicity of the initial configuration. For $x \in U_{\Gamma_\varepsilon}$, $y \in Y$, $\gamma \in \Gamma$, and $r \in (-\varepsilon a, \varepsilon a)$, it holds

$$[n_\varepsilon]^\varepsilon(x, \gamma) = n(\gamma), \tag{5.4a}$$

$$[\Lambda_\varepsilon]^\varepsilon(x, \gamma, r) = \varepsilon\Lambda\left(\gamma, \frac{r}{\varepsilon}\right) + \varepsilon\left[\frac{x}{\varepsilon}\right], \tag{5.4b}$$

$$[L_{\Gamma_\varepsilon}]^\varepsilon(x, \gamma) = \varepsilon^{-1}L_\Gamma(\gamma), \tag{5.4c}$$

$$[P_{\Gamma_\varepsilon}]^\varepsilon(x, y) = \varepsilon P_\Gamma(y) + \varepsilon\left[\frac{x}{\varepsilon}\right], \tag{5.4d}$$

$$[DP_{\Gamma_\varepsilon}]^\varepsilon(x, y) = \big(\mathbb{I} - d_\Gamma(y)L_\Gamma(P_\Gamma(y))\big)^{-1}\left(\mathbb{I} - n(P_\Gamma(y)) \otimes n(P_\Gamma(y))\right). \tag{5.4e}$$

With these relations in mind, we are able to connect the limit behavior of the auxiliary function η_ε and the height function h_ε.

Lemma 5.1. Let $n, m \in \mathbb{N}$. It holds

$$\left|\varepsilon_n^{-1}[\eta_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1}[\eta_{\varepsilon_m}]^{\varepsilon_m}\right| \leq \left|\varepsilon_n^{-1}[h_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1}[h_{\varepsilon_m}]^{\varepsilon_m}\right|$$

as well as

$$\left|[D\eta_{\varepsilon_n}]^{\varepsilon_n} - [D\eta_{\varepsilon_m}]^{\varepsilon_m}\right| \leq \frac{1}{2a}\left|\varepsilon_n^{-1}[h_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1}[h_{\varepsilon_n}]^{\varepsilon_n}\right| + \left|[\nabla h_{\varepsilon_n}]^{\varepsilon_n} - [\nabla h_{\varepsilon_m}]^{\varepsilon_m}\right|.$$

Proof. Since Λ is contractive and equations (5.4a) and (5.4b) hold, we conclude

$$\begin{aligned} &\left|\varepsilon_n^{-1}[\eta_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1}[\eta_{\varepsilon_m}]^{\varepsilon_m}\right| \\ &\qquad = \left|\Lambda\left(\gamma, \varepsilon_n^{-1}[h_{\varepsilon_n}]^{\varepsilon_n}\right) - \Lambda(\gamma, \varepsilon_m^{-1}[h_{\varepsilon_m}]^{\varepsilon_m})\right| \leq \left|\varepsilon_n^{-1}[h_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1}[h_{\varepsilon_m}]^{\varepsilon_m}\right|. \end{aligned}$$

The spatial derivative of η_ε is given as

$$D_{\Gamma_\varepsilon}\eta_\varepsilon = \mathrm{Id} + \nabla_{\Gamma_\varepsilon}h_\varepsilon \otimes n_\varepsilon - h_\varepsilon L_{\Gamma_\varepsilon}.$$

Using equations (5.4a) to (5.4c), we estimate

$$\left|[D_{\Gamma_\varepsilon}\eta_{\varepsilon_n}]^{\varepsilon_n} - [D_{\Gamma_\varepsilon}\eta_{\varepsilon_m}]^{\varepsilon_m}\right| \leq \left|\left[\nabla_{\Gamma_{\varepsilon_n}}h_{\varepsilon_n}\right]^{\varepsilon_n} - \left[\nabla_{\Gamma_{\varepsilon_m}}h_{\varepsilon_m}\right]^{\varepsilon_m}\right| + \frac{1}{2a}\left|\varepsilon_n^{-1}[h_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1}[h_{\varepsilon_n}]^{\varepsilon_n}\right|.$$

□

In the next few lemmas, we establish some technical results which are needed to show the strong two-scale convergence of F_ε and J_ε.

Lemma 5.2. (i) Let $u_\varepsilon \in W^{1,2}(\Omega)$ and $u \in L^2(\Omega; W^{1,2}_\#(Y))$ such that $[u_\varepsilon]^\varepsilon \to u$ and $\varepsilon\,[\nabla u_\varepsilon]^\varepsilon \to \nabla_y u$ strongly in $L^2(\Omega \times Y)$. Then, $[u_\varepsilon]^\varepsilon \to u$ strongly in $L^2(\Omega \times \Gamma)$.

(ii) For all $u \in W^{1,2}(\Omega)$, it holds that

$$\varepsilon \|u\|^2_{L^2(\Gamma_\varepsilon(t))} \leq 4C_{tr} \left(\|u\|^2_{L^2(\Omega)} + \varepsilon^2 \|\nabla u\|^2_{L^2(\Omega)} \right).$$

Proof. (i). This is due to the trace embedding operator $W^{1,2}(Y) \hookrightarrow L^2(\Gamma)$.

(ii). Let C_{tr} be the trace constant of the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Gamma_\varepsilon)$. For $u \in W^{1,2}(\Omega)$ and $t \in [0, T_v]$, we have

$$\begin{aligned}
\varepsilon \int_{\Gamma_\varepsilon(t)} |u(\gamma)|^2 \,\mathrm{d}\gamma &= \varepsilon \int_{\Gamma_\varepsilon} |u(y_\varepsilon(t,\gamma))|^2 |\det(D_{\Gamma_\varepsilon} y_\varepsilon(t,\gamma))| \,\mathrm{d}\gamma \\
&\leq 2C_{tr} \left(\int_\Omega |u \circ y_\varepsilon(x)|^2 \,\mathrm{d}x + \varepsilon^2 \int_\Omega |\nabla(u \circ y_\varepsilon)(x)|^2 \,\mathrm{d}x \right) \\
&\leq 2C_{tr} \left(\int_\Omega |u \circ y_\varepsilon(x)|^2 \,\mathrm{d}x + 2\varepsilon^2 \int_\Omega |\nabla u \circ y_\varepsilon(x)|^2 \,\mathrm{d}x \right).
\end{aligned}$$

The time parametrized coordinate transformation $x \mapsto y_\varepsilon^{-1}(t,x)$ (note that $y_\varepsilon^{-1}(t,\Omega) = \Omega$) then leads to

$$\varepsilon \|u\|^2_{L^2(\Gamma_\varepsilon(t))} \leq 4C_{tr} \left(\|u\|^2_{L^2(\Omega)} + \varepsilon^2 \|\nabla u\|^2_{L^2(\Omega)} \right)$$

□

Parts of the analysis rely on the ability to estimate certain differences of some composites of functions involving y_ε. In the following lemma, we collect some general results.

Lemma 5.3. Let $(f_\varepsilon) \subset W^{1,\infty}(\Omega)$ and $n, m \in \mathbb{N}$ $(n > m)$.

1. Let $\|\nabla f_{\varepsilon_m}\|_\infty$ be bounded independently of the parameter ε and $[f_\varepsilon]^\varepsilon$ be a Cauchy sequence. Then, there are $C, C_m > 0$ such that

$$\|f_{\varepsilon_n}([y_{\varepsilon_n}]^{\varepsilon_n}) - f_{\varepsilon_m}([y_{\varepsilon_m}]^{\varepsilon_m})\|_{L^2(\Omega\times Y)} \leq C_m + C\left\| \,[y_{\varepsilon_n}]^{\varepsilon_n} - [y_{\varepsilon_m}]^{\varepsilon_m} \right\|^2_{L^2(\Omega\times Y)}.$$

   and such that $\lim_{m\to\infty} C_m = 0$.

2. Let $f \in W^{1,\infty}(\Omega; W^{1,\infty}_\#(Y))$ such that $[f_\varepsilon]^\varepsilon \to f$. For $g_\varepsilon = y_\varepsilon$ or $g_\varepsilon = y_\varepsilon^{-1}$, we can estimate

$$\begin{aligned}
&\|f_{\varepsilon_n}([g_{\varepsilon_n}]^{\varepsilon_n}) - f_{\varepsilon_m}([g_{\varepsilon_m}]^{\varepsilon_m})\|^2_{L^2(\Omega\times Y)} \\
&\quad \leq C_m + C\left(\left\| \,[g_{\varepsilon_n}]^{\varepsilon_n} - [g_{\varepsilon_m}]^{\varepsilon_m} \right\|^2_{L^2(\Omega\times Y)} + \left\| \varepsilon_n^{-1}\,[g_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1}\,[g_{\varepsilon_m}]^{\varepsilon_m} \right\|^2_{L^2(\Omega\times Y)} \right)
\end{aligned}$$

   where $C, C_m > 0$ and $\lim_{m\to\infty} C_m = 0$.

3. Let $f \in W^{1,\infty}(\Omega; W^{1,\infty}_{\#}(Y))$ such that $[f_\varepsilon]^\varepsilon \to f$ and $\varepsilon [\nabla f_\varepsilon]^\varepsilon \to \nabla_y f$. Then, we estimate

$$\begin{aligned}&\|f_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n}) - f_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})\|^2_{L^2(\Omega\times\Gamma)} \\ &\qquad \le C_m + C\left(\| [h_{\varepsilon_n}]^{\varepsilon_n} - [h_{\varepsilon_m}]^{\varepsilon_m} \|^2_{L^2(\Omega\times\Gamma)} + \|\varepsilon_n^{-1} [h_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1} [h_{\varepsilon_m}]^{\varepsilon_m} \|^2_{L^2(\Omega\times\Gamma)}\right)\end{aligned}$$

where $C, C_m > 0$ and $\lim_{m\to\infty} C_m = 0$.

Proof. Proofs of these technical estimates are given in [9, Lemma 6.20]. □

5.2 Limit behavior

Based on the estimates established via Lemma 4.2, it is clear that y_ε converges strongly to the identity operator and that both Dy_ε and z_ε have two-scale converging subsequences. This in itself, however, is not enough to guarantee strong convergence of their unfolded counterparts, which in consequence may also impede strong convergence of $[F_\varepsilon]^\varepsilon$ and $[J_\varepsilon]^\varepsilon$ – a property that is needed to make sure that passing to the limit $\varepsilon \to 0$ is justified.

In the following lemma, we investigate the limit behavior of the dilated functions $\varepsilon^{-1} [y_\varepsilon - \mathrm{Id}]^\varepsilon$ and $[z_\varepsilon]^\varepsilon$.

Lemma 5.4. There exist functions $y, z \in L^2(S \times \Omega; H^1_{\#}(Y))^3$ such that

$$\frac{1}{\varepsilon} [y_\varepsilon - \mathrm{Id}]^\varepsilon \to y - \mathrm{Id}, \quad [z_\varepsilon]^\varepsilon \to z, \quad [Dy_\varepsilon]^\varepsilon \to D_y y, \quad \varepsilon [Dz_\varepsilon]^\varepsilon \to D_y z.$$

Proof. Let $\delta > 0$ be given and let $n, m \in \mathbb{N}$, such that $n > m$ and such that $e^{\varepsilon_m l_v T_v} < 2$.[1] Taking a look at the ODE sytem given by equations (4.1a) to (4.1d) and its corresponding system that emerges by differentiation with respect to the spatial variable, we find that (in $S \times \Omega \times \Sigma$, $(i = n, m)$)

$$\varepsilon_i^{-1} \partial_t [y_{\varepsilon_i} - \mathrm{Id}]^{\varepsilon_i} = \frac{[z_{\varepsilon_i}]^{\varepsilon_i}}{|[z_{\varepsilon_i}]^{\varepsilon_i}|} v_{\varepsilon_i}([y_{\varepsilon_i}]^{\varepsilon_i}), \tag{5.5a}$$

$$\partial_t [z_{\varepsilon_i}]^{\varepsilon_i} = \varepsilon_i \left| [z_{\varepsilon_i}]^{\varepsilon_i} \right| \nabla v_{\varepsilon_i}([y_{\varepsilon_i}]^{\varepsilon_i}), \tag{5.5b}$$

$$\partial_t [Dy_{\varepsilon_i}]^{\varepsilon_i} = \varepsilon_i A^{(11)}_{\varepsilon_i}([w_{\varepsilon_i}]^{\varepsilon_i}) [Dy_{\varepsilon_i}]^{\varepsilon_i} + \varepsilon_i A^{(11)}_{\varepsilon_i}([w_{\varepsilon_i}]^{\varepsilon_i}) [Dz_{\varepsilon_i}]^{\varepsilon_i}, \tag{5.5c}$$

$$\varepsilon_i \partial_t [Dz_{\varepsilon_i}]^{\varepsilon_i} = \varepsilon_i^2 A^{(21)}_{\varepsilon_i}([w_{\varepsilon_i}]^{\varepsilon_i}) [Dy_{\varepsilon_i}]^{\varepsilon_i} + \varepsilon_i^2 A^{(22)}_{\varepsilon_i}([w_{\varepsilon_i}]^{\varepsilon_i}) [Dz_{\varepsilon_i}]^{\varepsilon_i}. \tag{5.5d}$$

Now, subtracting these equations for $i = n$ and $i = m$ from one another, multiplying with the corresponding differences, and integrating over $\Omega \times Y$, we are led to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varepsilon_n^{-1} [y_{\varepsilon_n} - \mathrm{Id}]^{\varepsilon_n} - \varepsilon_m^{-1} [y_{\varepsilon_m} - \mathrm{Id}]^{\varepsilon_m} \right\|^2_{L^2(\Omega\times Y)} \\ &\qquad \le 2\int_{\Omega\times Y} \left| \frac{[z_{\varepsilon_n}]^{\varepsilon_n}}{|[z_{\varepsilon_n}]^{\varepsilon_n}|} v_{\varepsilon_n}([y_{\varepsilon_n}]^{\varepsilon_n}) - \frac{[z_{\varepsilon_m}]^{\varepsilon_m}}{|[z_{\varepsilon_m}]^{\varepsilon_m}|} v_{\varepsilon_m}([y_{\varepsilon_m}]^{\varepsilon_m}) \right| \\ &\qquad\qquad \left| \varepsilon_n^{-1} [y_{\varepsilon_n} - \mathrm{Id}]^{\varepsilon_n} - \varepsilon_m^{-1} [y_{\varepsilon_m} - \mathrm{Id}]^{\varepsilon_m} \right| \mathrm{d}(x,y),\end{aligned} \tag{5.6a}$$

[1] This is a mere technicality to allow for a more compact notation of the estimates. Here, we do not care about the details of the specific estimates, we only want to ensure convergence.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\| \left[z_{\varepsilon_n}\right]^{\varepsilon_n} - \left[z_{\varepsilon_m}\right]^{\varepsilon_m} \right\|^2_{L^2(\Omega\times Y)} \leq 2\int_{\Omega\times Y} \left|\varepsilon_n\right| \left[z_{\varepsilon_n}\right]^{\varepsilon_n} \left|\nabla v_{\varepsilon_n}(\left[y_{\varepsilon_n}\right]^{\varepsilon_n}) - \varepsilon_m\right| \left[z_{\varepsilon_m}\right]^{\varepsilon_m} \left|\nabla v_{\varepsilon_m}(\left[y_{\varepsilon_m}\right]^{\varepsilon_m})\right| \left|\left[z_{\varepsilon_n}\right]^{\varepsilon_n} - \left[z_{\varepsilon_m}\right]^{\varepsilon_m}\right| \mathrm{d}(x,y). \tag{5.6b}$$

To proceed in showing that these sequences are Cauchy sequences, several independent estimates are needed to manage the right hand sides of inequalities (5.6a) and (5.6b). In the following, we heavily rely on the estimates established by Lemma 4.2. With the reverse triangle inequality, we get

$$\left|\left|\left[z_{\varepsilon_n}\right]^{\varepsilon_n}\right| - \left|\left[z_{\varepsilon_m}\right]^{\varepsilon_m}\right|\right| \leq \left|\left[z_{\varepsilon_n}\right]^{\varepsilon_n} - \left[z_{\varepsilon_m}\right]^{\varepsilon_m}\right|, \tag{5.7a}$$

Since $e^{\varepsilon_m l_v T_v} < 2$, we also see that

$$\left|\frac{\left[z_{\varepsilon_n}\right]^{\varepsilon_n}}{\left|\left[z_{\varepsilon_n}\right]^{\varepsilon_n}\right|} - \frac{\left[z_{\varepsilon_m}\right]^{\varepsilon_m}}{\left|\left[z_{\varepsilon_m}\right]^{\varepsilon_m}\right|}\right| \leq 10\left|\left[z_{\varepsilon_n}\right]^{\varepsilon_n} - \left[z_{\varepsilon_m}\right]^{\varepsilon_m}\right|. \tag{5.7b}$$

Moreover, for $f_\varepsilon = v_\varepsilon, \varepsilon\nabla v_\varepsilon$, we can apply Lemma 5.3 to get

$$\left\|f_{\varepsilon_n}(\left[y_{\varepsilon_n}\right]^{\varepsilon_n}) - f_{\varepsilon_m}(\left[y_{\varepsilon_n}\right]^{\varepsilon_m})\right\|^2_{L^2(\Omega\times Y)} \leq C_m + C\left(\left\|\left[f_{\varepsilon_n}\right]^{\varepsilon_n} - \left[f_{\varepsilon_m}\right]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times Y)} + \left\|\left[y_{\varepsilon_n}\right]^{\varepsilon_n} - \left[y_{\varepsilon_m}\right]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times Y)}\right), \tag{5.7c}$$

where $\lim C_m = 0$. As y_ε is a cauchy sequence (it converges strongly to the identity operator), it can also be estimated via a function C_m converging to 0. The matrix valued function B, which is defined via equation (4.5), is Lipschitz continuous with Lipschitz constant 2, i.e.,

$$\left|B(\left[z_{\varepsilon_n}\right]^{\varepsilon_n}) - B(\left[z_{\varepsilon_m}\right]^{\varepsilon_m})\right| \leq 2\left|\left[z_{\varepsilon_n}\right]^{\varepsilon_n} - \left[z_{\varepsilon_m}\right]^{\varepsilon_m}\right|. \tag{5.7d}$$

Adding inequalities (5.6a) and (5.6b), using the estimates given by inequalities (5.7a) to (5.7c) as well as Assumption (A3), and applying Gronwall's inequality, we infer

$$\left\|\varepsilon_n^{-1}\left[y_{\varepsilon_n} - \mathrm{Id}\right]^{\varepsilon_n} - \varepsilon_m^{-1}\left[y_{\varepsilon_m} - \mathrm{Id}\right]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times Y)} + \left\|\left[z_{\varepsilon_n}\right]^{\varepsilon_n} - \left[z_{\varepsilon_m}\right]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times Y)} \leq C_m + C\left(\left\|\left[v_{\varepsilon_n}\right]^{\varepsilon_n} - \left[v_{\varepsilon_m}\right]^{\varepsilon_m}\right\|^2 + \left\|\varepsilon_n\left[\nabla v_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m\left[\nabla v_{\varepsilon_m}\right]^{\varepsilon_m}\right\|^2\right) \tag{5.8}$$

for all $n, m \in \mathbb{N}$ such that $n, m > N$ for sufficiently large $N \in \mathbb{N}$ (which is independent of ε and t). This implies

$$\frac{1}{\varepsilon}\left[y_\varepsilon - \mathrm{Id}\right]^\varepsilon \to y - \mathrm{Id}, \quad \left[z_\varepsilon\right]^\varepsilon \to z \quad \text{in } L^2(S\times\Omega\times Y)^3.$$

Similarly, we also get (for more details, we refer to [9, Lemma 6.21])

$$\left[Dy_\varepsilon\right]^\varepsilon \to D_y y, \quad \varepsilon\left[Dz_\varepsilon\right]^\varepsilon \to D_y z \quad \text{in } L^2(S\times\Omega\times Y)^{3\times 3}.$$

□

Remark 5.5. As a consequence of Lemma 5.2, this implies

$$\frac{1}{\varepsilon}\left[y_\varepsilon - \mathrm{Id}\right]^\varepsilon \to y - \mathrm{Id}, \quad \left[z_\varepsilon\right]^\varepsilon \to z \quad \text{in } L^2(S\times\Omega\times\Gamma)^3.$$

Lemma 5.6. The following convergences hold:

$$\frac{1}{\varepsilon}\left[y_\varepsilon^{-1} - \mathrm{Id}\right]^\varepsilon \to y^{-1} - \mathrm{Id}, \quad \left[q_\varepsilon\right]^\varepsilon \to z(y^{-1}), \quad \varepsilon^{-1}\widetilde{\varphi}_\varepsilon \to \widetilde{\varphi}, \quad \varepsilon\nabla q_\varepsilon \to \nabla_y q \quad \text{in } L^2(S\times\Omega\times Y).$$

Proof. We recall that y_ε^{-1} can be characterized by equation (4.14). This leads us to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}&\left\|\varepsilon_n^{-1}\left[y_{\varepsilon_n}^{-1} - \mathrm{Id}\right]^{\varepsilon_n} - \varepsilon_m^{-1}\left[y_{\varepsilon_m}^{-1} - \mathrm{Id}\right]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times Y)} \\
&\leq \int_{\Omega\times Y}\Bigg|Dy_{\varepsilon_n}(\left[y_{\varepsilon_n}^{-1}\right]^{\varepsilon_n})^{-1}\frac{z_{\varepsilon_n}(\left[y_{\varepsilon_n}^{-1}\right]^{\varepsilon_n})}{|z_{\varepsilon_n}(\left[y_{\varepsilon_n}^{-1}\right]^{\varepsilon_n})|}v_{\varepsilon_n}(y_{\varepsilon_n}(\left[y_{\varepsilon_n}^{-1}\right]^{\varepsilon_n})) \\
&\qquad - Dy_{\varepsilon_m}(\left[y_{\varepsilon_m}^{-1}\right]^{\varepsilon_m})^{-1}\frac{z_{\varepsilon_m}(\left[y_{\varepsilon_m}^{-1}\right]^{\varepsilon_m})}{|z_{\varepsilon_m}(\left[y_{\varepsilon_m}^{-1}\right]^{\varepsilon_m})|}v_{\varepsilon_m}(y_{\varepsilon_m}(\left[y_{\varepsilon_m}^{-1}\right]^{\varepsilon_m}))\Bigg| \\
&\qquad\qquad \cdot \left|\varepsilon_n^{-1}\left[y_{\varepsilon_n}^{-1} - \mathrm{Id}\right]^{\varepsilon_n} - \varepsilon_n^{-1}\left[y_{\varepsilon_n}^{-1} - \mathrm{Id}\right]^{\varepsilon_m}\right|\mathrm{d}(x,y).
\end{aligned}$$

Taking into considerations the a-priori estimates available for the involved functions and the strong convergence results formulated in Lemma 5.4, as well as the estimates given in Lemma 5.3, it follows that $\varepsilon^{-1}\left[y_\varepsilon^{-1} - \mathrm{Id}\right]^\varepsilon$ is a Cauchy sequence. Similarly, $\left[q_\varepsilon\right]^\varepsilon = \left[z_\varepsilon(y_\varepsilon^{-1})\right]^\varepsilon$ is a Cauchy sequence due to Lemma 5.3 (2). Since $\partial_t\widetilde{\varphi}_\varepsilon$ is governed by equation (4.16) and since $\nabla\widetilde{\varphi}_\varepsilon = q_\varepsilon$, we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varepsilon_n^{-1}\widetilde{\varphi}_{\varepsilon_n} - \varepsilon_m^{-1}\widetilde{\varphi}_{\varepsilon_m}\right\|^2_{L^2(\Omega\times Y)} \leq \int_{\Omega\times Y}\left|\,|q_{\varepsilon_n}|\,v_{\varepsilon_n} - |q_{\varepsilon_m}|\,v_{\varepsilon_m}\right|\mathrm{d}(x,y)$$

which shows that $\varepsilon^{-1}\widetilde{\varphi}_\varepsilon$ also converges strongly. Finally, as

$$\varepsilon\nabla q_\varepsilon = \varepsilon D^2\widetilde{\varphi}_\varepsilon = \varepsilon\big(Dy_\varepsilon(y_\varepsilon^{-1})\big)^{-1}Dz_\varepsilon(y_\varepsilon^{-1}),$$

we also get the strong convergence of $\varepsilon\left[\nabla q_\varepsilon\right]^\varepsilon$. □

Since the quantity $\varepsilon\|h_\varepsilon\|_\infty + \|\nabla_{\Gamma_\varepsilon}h_\varepsilon\|_\infty$ is bounded indepedently of the parameter ε, we can find a constant $C_h > 0$ such that

$$\frac{1}{\sqrt{\varepsilon}}\|h_\varepsilon\|_{L^2(S\times\Gamma_\varepsilon)} + \sqrt{\varepsilon}\|\nabla_{\Gamma_\varepsilon}h_\varepsilon\|_{L^2(S\times\Gamma_\varepsilon)^3} \leq C_h.$$

As a result, we conclude the existence of a function $h \in L^2(S\times\Omega;H^1(\Gamma))$ such that, up to a subsequence,

$$\frac{1}{\varepsilon}h_\varepsilon \stackrel{2}{\to} h, \quad \nabla_{\Gamma_\varepsilon}h_\varepsilon \stackrel{2}{\to} \nabla_\Gamma h$$

Furthermore, it is clear that $h \in L^\infty(S\times\Omega\times\Gamma)$ and that $\left[h_\varepsilon\right]^\varepsilon \in L^\infty(S\times\Omega\times\Gamma)$ is bounded independently of ε. As a consequence, there is a function $\tilde{h} \in L^\infty(S\times\Omega\times\Gamma)$ such that $\left[h_\varepsilon\right]^\varepsilon \rightharpoonup \tilde{h}$ in $L^2(S\times\Omega\times Y)$. In the following, we are concerned with the limit behavior of h_ε.

Lemma 5.7. There is $h \in L^2(S\times\Omega; H^1_\#(\Gamma))$ such that $\varepsilon^{-1}\left[h_\varepsilon\right]^\varepsilon \to h$ and such that $\left[\nabla_{\Gamma_\varepsilon} h_\varepsilon\right]^\varepsilon \to \nabla_y h$ in $L^2(S\times\Omega\times\Gamma)$.

Proof. Let $\delta > 0$ and $n, m \in \mathbb{N}$, $n > m$. Using the representation of the height function h_ε in terms of $F_{\varepsilon,\gamma}$ as given by equation (4.18), we have

$$\partial_t h_\varepsilon(t,\gamma) = -\frac{\partial_t F_{\varepsilon,\gamma}(t, h_\varepsilon(t,\gamma))}{\partial_2 F_{\varepsilon,\gamma}(t, h_\varepsilon(t,\gamma))} \quad (t \in [0,T_v],\ \gamma \in \Gamma_\varepsilon). \tag{5.9}$$

Now, integrating over $\Omega\times\Gamma$ and testing with the difference $\varepsilon_n^{-1}\left[h_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^{-1}\left[h_{\varepsilon_n}\right]^{\varepsilon_m}$ leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left\|\varepsilon_n^{-1}\left[h_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^{-1}\left[h_{\varepsilon_n}\right]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times\Gamma)}\\ &\le 2\int_{\Omega\times\Gamma}\left|\varepsilon_n^{-1}\frac{\partial_t\left[F_{\varepsilon_n,\gamma}(h_{\varepsilon_n})\right]^{\varepsilon_n}}{\left[\partial_2 F_{\varepsilon_n,\gamma}(h_{\varepsilon_n})\right]^{\varepsilon_n}} - \varepsilon_m^{-1}\frac{\partial_t\left[F_{\varepsilon_m,\gamma}(h_{\varepsilon_m})\right]^{\varepsilon_m}}{\left[\partial_2 F_{\varepsilon_m,\gamma}(h_{\varepsilon_m})\right]^{\varepsilon_m}}\right|\\ &\qquad\qquad \left|\varepsilon_n^{-1}\left[h_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^{-1}\left[h_{\varepsilon_n}\right]^{\varepsilon_m}\right|\mathrm{d}(x,\gamma).\end{aligned}$$

Using that $\partial_t\widetilde{\varphi}_\varepsilon$ is governed by equation (4.16) and $q_\varepsilon = \nabla\widetilde{\varphi}_\varepsilon$, we get

$$\varepsilon^{-1}\partial_t\left[F_{\varepsilon,\gamma}(h_\varepsilon)\right]^\varepsilon = \left|q_\varepsilon(\left[\eta_\varepsilon\right]^\varepsilon))\right| v_\varepsilon(\left[\eta_\varepsilon\right]^\varepsilon). \tag{5.10}$$

Applying Lemma 5.3(3) to q_ε and v_ε, respectively, and using the strong convergence of $\left[v_\varepsilon\right]^\varepsilon$, $\left[\nabla v_\varepsilon\right]^\varepsilon$, $\left[q_\varepsilon\right]^\varepsilon$, and $\varepsilon\left[\nabla q_\varepsilon\right]^\varepsilon$, we are led to

$$\begin{aligned}&\left\|\varepsilon_n^{-1}\partial_t\left[F_{\varepsilon_n,\gamma}(h_{\varepsilon_n})\right]^{\varepsilon_n} - \varepsilon_m^{-1}\partial_t\left[F_{\varepsilon_m,\gamma}(h_{\varepsilon_m})\right]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times\Gamma)}\\ &\quad\le C(m) + C\left(\left\|\left[h_{\varepsilon_n}\right]^{\varepsilon_n} - \left[h_{\varepsilon_m}\right]^{\varepsilon_m})\right\|^2_{L^2(\Omega\times\Gamma)} + \left\|\varepsilon_n^{-1}\left[h_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^{-1}\left[h_{\varepsilon_n}\right]^{\varepsilon_n}\right\|^2_{L^2(\Omega\times\Gamma)}\right)\end{aligned} \tag{5.11}$$

where $\lim_{m\to\infty} C(m) = 0$. As a next step, we estimate the difference with respect to $\partial_2 F_{\varepsilon,\gamma}$. In view of equation (4.17), we have

$$\left[\partial_2 F_{\varepsilon,\gamma}(h_\varepsilon)\right]^\varepsilon = g'(\varepsilon^{-1}\widetilde{\varphi}_\varepsilon(\left[\eta_\varepsilon\right]^\varepsilon))q_\varepsilon(\left[\eta_\varepsilon\right]^\varepsilon)\cdot n \tag{5.12}$$

and, due to the strong convergence of $\varepsilon^{-1}\left[\widetilde{\varphi}_\varepsilon\right]^\varepsilon$, $\left[q_\varepsilon\right]^\varepsilon = \left[\nabla\widetilde{\varphi}_\varepsilon\right]^\varepsilon$, and $\varepsilon\left[\nabla q_\varepsilon\right]^\varepsilon$, we can infer (again applying Lemma 5.3(3))

$$\begin{aligned}&\left\|\varepsilon_n^{-1}\left[\partial_2 F_{\varepsilon_n,\gamma}(h_{\varepsilon_n})\right]^{\varepsilon_n} - \varepsilon_m^{-1}\left[\partial_2 F_{\varepsilon_m,\gamma}(h_{\varepsilon_m})\right]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times\Gamma)}\\ &\quad\le C_m + C\left(\left\|\left[h_{\varepsilon_n}\right]^{\varepsilon_n} - \left[h_{\varepsilon_m}\right]^{\varepsilon_m})\right\|^2_{L^2(\Omega\times\Gamma)} + \left\|\varepsilon_n^{-1}\left[h_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^{-1}\left[h_{\varepsilon_n}\right]^{\varepsilon_n}\right\|^2_{L^2(\Omega\times\Gamma)}\right)\end{aligned} \tag{5.13}$$

where $\lim_{m\to\infty} C_m \to 0$. Combining the estimates given by inequalities (5.11) and (5.13) and applying Gronewall's inequality, it is then easy to see that $\varepsilon^{-1}\left[h_\varepsilon\right]^\varepsilon$ is, in fact, Cauchy.

Using the representation of h_ε given in equation (4.19), we have

$$\left[\nabla_{\Gamma_\varepsilon} h_\varepsilon\right]^\varepsilon = \left(\mathbb{I}_3 - \varepsilon^{-1}\left[h_\varepsilon\right]^\varepsilon L_\Gamma\right)\left(n - \frac{1}{n_{\Gamma_\varepsilon}(\left[\eta_\varepsilon\right]^\varepsilon)\cdot n} n_{\Gamma_\varepsilon}(\left[\eta_\varepsilon\right]^\varepsilon)\right).$$

Consequently, since $n_{\Gamma_\varepsilon}(\eta_\varepsilon)\cdot n_\varepsilon > 1/2$ and $|\varepsilon^{-1}h_\varepsilon| \le a/10$ in $[0,T_v]\times\Gamma_\varepsilon$, we are led to

$$\begin{aligned}&\left\|\left[\nabla_{\Gamma_{\varepsilon_n}} h_{\varepsilon_n}\right]^{\varepsilon_n} - \left[\nabla_{\Gamma_{\varepsilon_m}} h_{\varepsilon_m}\right]^{\varepsilon_m}\right\|_{L^2(S\times\Gamma_\varepsilon)}\\ &\quad\le \frac{3}{2a}\left\|\varepsilon_n^{-1}\left[h_{\varepsilon_n}\right]^{\varepsilon} - \varepsilon_m^{-1}\left[h_{\varepsilon_m}\right]^{\varepsilon}\right\|_{L^2(S\times\Gamma_\varepsilon)} + 6\left\|\frac{n_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n})}{n_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n})\cdot n} - \frac{n_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})}{n_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})\cdot n}\right\|_{L^2(S\times\Gamma_\varepsilon)}\end{aligned}$$

Now, due to $n_{\Gamma_\varepsilon}(\eta_\varepsilon) = \frac{\nabla\widetilde{\varphi}_\varepsilon(\eta_\varepsilon)}{|\nabla\widetilde{\varphi}_\varepsilon(\eta_\varepsilon)|} = \frac{q_\varepsilon(\eta_\varepsilon)}{|q_\varepsilon(\eta_\varepsilon)|}$, we further estimate

$$\begin{aligned}&\left\|\frac{n_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n})}{n_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n})\cdot n} - \frac{n_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})}{n_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})\cdot n}\right\|_{L^2(S\times\Gamma_\varepsilon)}\\ &\qquad\le 6\left\|\frac{q_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n})}{|q_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n})|} - \frac{q_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})}{|q_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})|}\right\|_{L^2(S\times\Gamma_\varepsilon)}\\ &\qquad\le 36\left\|q_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n}) - q_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})\right\|_{L^2(S\times\Gamma_\varepsilon)}.\end{aligned}$$

As both $[q_\varepsilon]^\varepsilon$ and $\varepsilon\,[\nabla q_\varepsilon]^\varepsilon$ converge, we can apply Lemma 5.3(3) and conclude

$$\begin{aligned}&\left\|\left[\nabla_{\Gamma_{\varepsilon_n}} h_{\varepsilon_n}\right]^{\varepsilon_n} - \left[\nabla_{\Gamma_{\varepsilon_m}} h_{\varepsilon_m}\right]^{\varepsilon_m}\right\|_{L^2(S\times\Gamma_\varepsilon)}\\ &\qquad\le C_m + \frac{3}{2a}\left\|\varepsilon_n^{-1}\left[h_{\varepsilon_n}\right]^{\varepsilon} - \varepsilon_m^{-1}\left[h_{\varepsilon_m}\right]^{\varepsilon}\right\|_{L^2(S\times\Gamma_\varepsilon)}\\ &\qquad+ C\left(\left\|[h_{\varepsilon_n}]^{\varepsilon} - [h_{\varepsilon_m}]^{\varepsilon}\right\|_{L^2(S\times\Gamma_\varepsilon)} + \left\|\varepsilon_n^{-1}\left[h_{\varepsilon_n}\right]^{\varepsilon} - \varepsilon_m^{-1}\left[h_{\varepsilon_m}\right]^{\varepsilon}\right\|_{L^2(S\times\Gamma_\varepsilon)}\right),\end{aligned}$$

where, again, $\lim_{m\to\infty} C_m = 0$.

□

We introduce $\psi_\varepsilon = s_\varepsilon - \mathrm{Id}$ which implies (see equation (4.20)) $D\psi_\varepsilon = Ds_\varepsilon$.

Lemma 5.8. There is $\psi \in L^2(S\times\Omega; H^1_\#(Y))$ such that $\varepsilon^{-1}\,[\psi_\varepsilon]^\varepsilon \to \psi$ and such that $[\nabla\psi_\varepsilon]^\varepsilon \to \nabla_y\psi$ in $L^2(S\times\Omega\times Y)$.

Proof. Let $n, m \in \mathbb{N}$ such that $m > n$ and set $\mu_\varepsilon(t,x) = h_\varepsilon(t, P_{\Gamma_\varepsilon}(x))$ as well as $\mu(t,x,y) = h(t,x,P_\Gamma(y))$. We calculate

$$\varepsilon_n^{-1}\left[\psi_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^{-1}\left[\psi_{\varepsilon_m}\right]^{\varepsilon_m} = \left(\varepsilon_n^{-1}\left[\mu_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^{-1}\left[\mu_{\varepsilon_m}\right]^{\varepsilon_m}\right)\chi\left(a^{-1}d_\Gamma\right)n(P_\Gamma).$$

As a consequence,

$$\begin{aligned}&\int_{\Omega\times U_\Gamma}\left|\varepsilon_n^{-1}\left[\psi_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^{-1}\left[\psi_{\varepsilon_m}\right]^{\varepsilon_m}\right|^2 \mathrm{d}(x,y)\\ &\qquad\le \int_{\Omega\times U_\Gamma}\left|\varepsilon_n^{-1}\left[\mu_{\varepsilon_n}\right]^{\varepsilon_n} - \mu\right|^2 + \left|\varepsilon_m^{-1}\left[\mu_{\varepsilon_m}\right]^{\varepsilon_m} - \mu\right|^2 \mathrm{d}(x,y).\end{aligned}$$

Now, for fixed $x\in\Omega$, $[\mu_\varepsilon]^\varepsilon$ and μ are constant in the y variable in the direction of the normal vector. As a consequence,

$$\int_{\Omega\times U_\Gamma}\left|\varepsilon^{-1}\left[\mu_\varepsilon\right]^{\varepsilon} - \mu\right|^2 \mathrm{d}(x,y) = 2a\int_{\Omega\times\Gamma}\left|\varepsilon_n^{-1}\left[h_{\varepsilon_n}\right]^{\varepsilon_n} - h\right|^2 \mathrm{d}(x,y).$$

The unfolded deformation gradient is given via (we refer to [20, Section 2])

$$
\begin{aligned}
[\nabla\psi_\varepsilon]^\varepsilon = ([\nabla\mu_\varepsilon]^\varepsilon)^T n(P_\Gamma)\chi\left(a^{-1}d_\Gamma\right) \\
+ \varepsilon^{-1}[\mu_\varepsilon]^\varepsilon \Big(L_\Gamma(P_\Gamma)\left(\mathbb{I} - d_\Gamma L_\Gamma(P_\Gamma)\right)^{-1}(\mathbb{I} - n(P_\Gamma)\otimes n(P_\Gamma))\chi\left(a^{-1}d_\Gamma\right) \\
+ \chi'\left(a^{-1}d_\Gamma\right) n(P_\Gamma)\otimes n(P_\Gamma)\Big).
\end{aligned}
$$

which leads us to

$$
\begin{aligned}
&\int_{\Omega\times U_\Gamma} |[\nabla\psi_{\varepsilon_n}]^{\varepsilon_n} - [\nabla\psi_{\varepsilon_m}]^{\varepsilon_m}|^2 \,\mathrm{d}(x,y) \\
&\qquad \le C\int_{\Omega\times U_\Gamma} \left|\varepsilon_n^{-1}[\mu_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1}[\mu_{\varepsilon_m}]^{\varepsilon_m}\right|^2 + |[\nabla\mu_{\varepsilon_n}]^{\varepsilon_n} - [\nabla\mu_{\varepsilon_m}]^{\varepsilon_m}|^2 \,\mathrm{d}(x,y),
\end{aligned}
$$

where $C > 0$ is independent of ε. Since

$$
\nabla\mu_\varepsilon(t,x) = (DP_{\Gamma_\varepsilon}(x))^T \nabla_{\Gamma_\varepsilon} h_\varepsilon(t, P_{\Gamma_\varepsilon}(x))
$$

and

$$
\begin{aligned}
&\int_{\Omega\times U_\Gamma} |[\nabla_{\Gamma_\varepsilon} h_\varepsilon(P_{\Gamma_\varepsilon})]^\varepsilon(x,y) - \nabla_y h(t,x,P_\Gamma(y))|^2 \,\mathrm{d}(x,y) \\
&\qquad = 2a\int_{\Omega\times\Gamma} |[\nabla_{\Gamma_\varepsilon} h_{\varepsilon_n}]^{\varepsilon_n} - \nabla_y h|^2 \,\mathrm{d}(x,y),
\end{aligned}
$$

we can conclude $[\nabla\psi_\varepsilon]^\varepsilon \to \nabla_y\psi$. □

Remark 5.9. Looking at the Definition of s_ε given via equation (4.20), it is then clear that both Ds_ε and $\det Ds_\varepsilon$ are also strongly two-scale convergent. Due to the uniform boundedness in L^∞, it is also clear that the limit functions are essentially bounded.

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