Scattering of Herglotz waves from periodic structures and mapping properties of the Bloch transform

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October 10, 2014

Abstract

When an incident Herglotz wave function scatteres from a periodic Lipschitz continuous surface with Dirichlet boundary condition, then the classical (quasi-)periodic solution theory for scattering from periodic structures does not apply since the incident field lacks periodicity. Relying on the Bloch transform, we provide a solution theory in H^1 for this scattering problem: We first prove conditions guaranteeing that incident Herglotz wave functions propagating towards the periodic structure have traces in $H^{1/2}$ on the periodic surface. Second, we show that the solution to the scattering problem can be decomposed by the Bloch transform into its periodic components that solve a periodic scattering problem. Third, these periodic solutions yield an equivalent characterization of the solution to the original non-periodic scattering problem, which allows, for instance, to prove new characterizations of the Rayleigh coefficients of each of the periodic components. A corollary of our results is that under the conditions mentioned above the operator mapping densities to the restriction of their Herglotz wave function on the periodic surface is always injective; this result generally fails for bounded surfaces.

1 Introduction

We consider time-harmonic wave propagation modeled by the Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{1}$$

for a positive, constant wave number $k > 0$. Well-known entire solutions to this equations are plane waves, defined by

$$(x_1, x_2)^\top \mapsto \exp\left[\mathrm{i}k(\sin\theta\, x_1 - \cos\theta\, x_2)\right], \quad x = (x_1, x_2)^\top \in \mathbb{R}^2. \tag{2}$$

The direction of these plane waves obviously equals $(\sin(\theta), -\cos(\theta))^\top$ and hence these plane waves propagate downwards for $\theta \in (-\pi/2, \pi/2)$ and upwards for $\theta \in (\pi/2, 3\pi/2)$. The exceptional cases $\theta = \pm\pi/2$ correspond to plane waves either propagating to the left or to the right. Plane waves are quasiperiodic in x_1 with quasiperiodicity $k\sin(\theta)$: For any period $L > 0$ it holds that

$$e^{\mathrm{i}k(\sin\theta\,(x_1+L) - \cos\theta\, x_2)} = e^{\mathrm{i}L\,k\sin(\theta)}\ e^{\mathrm{i}k(\sin\theta\, x_1 - \cos\theta\, x_2)} \qquad \text{for } x = (x_1, x_2)^\top \in \mathbb{R}^2.$$

When a plane wave scatters from a periodic structure with period L then it is well-known that this scattering problem can be formulated in a framework of quasiperiodic waves. Such a setting has been studied in many papers, see, e.g., [1, 5, 6, 8, 10, 20, 25]. An important application of this quasiperiodic scattering theory is the development of non-destructive testing procedures for periodic structures, where one usually *fixes* the quasiperiodicity of the incident fields. Several methods to tackle such inverse problems have been investigated in quite some detail, see, e.g., [3, 13, 17–19, 22, 26]. Since there exists, however, only a finite number of *propagating* quasiperiodic incident plane waves, most of the above-mentioned papers rely either on quasiperiodic point sources or on evanescent incident wave fields. From an experimentalists point of view it would be much easier and in some sense also more natural to consider scattering from periodic

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structures using many plane waves with different directions. This observation directly leads to consider incident waves in the form of Herglotz wave functions,

$$v_\phi(x) = \int_{-\pi}^{\pi} e^{\mathrm{i}k(\sin\theta\, x_1 - \cos\theta\, x_2)} \phi(\theta)\,\mathrm{d}\theta\,, \quad x \in \mathbb{R}^2, \tag{3}$$

for suitable densities ϕ defined on (parts of) the unit circle. The importance of such incident fields from a practical point of view is one of our motivations to study scattering of Herglotz wave functions v_ϕ from periodic surfaces Γ given as graph of a Lipschitz continuous function.

It is well-known that incident Herglotz waves of the from given in (3) belong, e.g., to $C^2_{\mathrm{loc}}(\Gamma)$, see, e.g., [9]. However, a framework yielding weak (or variational) solutions requires, e.g., that the incident fields belong to Sobolev spaces $H^s(\Gamma)$ on the unbounded periodic surfaces. We are not aware of such results in the literature – indeed, there are counterexamples that show that in general the restriction of v_ϕ to a periodic surface does not even belong to, e.g., $L^2(\Gamma)$ (see Example 9 below). The lack of such results might be surprising, since Herglotz wave functions are among the most popular solutions to the Helmholtz equation. Several characterization of these functions exist, see, e.g., [15, 16, 23]; the most familiar one probably is that a solution u to the Helmholtz equation (1) in all of $\mathbb{R}^2$ is a Herglotz wave function if and only if

$$\sup_{r>0} \frac{1}{r} \int_{|x|<r} |u|^2 \,\mathrm{d}x \ < \infty.$$

We also note from [4] that Herglotz wave functions can be characterized using expansions in cylindrical Bessel functions J_n (see, e.g., [2]): An function u is a Herglotz wave function if and only if u can be written as $u(r\exp(\mathrm{i}\varphi)) = \sum_{n\in\mathbb{Z}} a_n J_n(r)\exp(\mathrm{i}n\varphi)$ with coefficients $(a_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. This yields hence a Hilbert space structure for Herglotz wave functions.

In this paper we will rely on the Bloch transform to give sharp conditions that guarantee that the restriction of v_ϕ to a periodic surface belongs to the above mentioned Sobolev spaces $H^s(\Gamma)$ (with $|s| \le 1$ for Lipschitz continuous surfaces). These conditions will be formulated in terms of either the support or else the behaviour at $\pm\pi/2$ of the density ϕ. Under these conditions we even show that the function v_ϕ belongs to H^1 on any horizontal strip of finite height (and hence even to, e.g., $H^{1/2}$ on any Lipschitz continuous surface inside such a strip; such surfaces might fail to be periodic or graph of a function). Our results do not depend on dimension and could be shown analogously in three dimensions.

The Bloch transform also allows to give equivalences between the solution to a scattering problem in the domain "above" the surface Γ (which is, in principle, a special case of a rough surface scattering problem) and a continuum of periodic scattering problems for the periodic components of the solution. Denoting the half-space–like domain above Γ by Ω, this equivalence yields, amongst others, new expressions for the Rayleigh coefficients of the periodic components of the non-periodic solution in Ω.

The mathematical tool we use to solve Dirichlet scattering problems in Ω is the variational solution theory for weak solutions to the Helmholtz equation in H^1 developed in [7]. These results could also be used to solve rough surface scattering problems for incident Herglotz wave functions on non-periodic surfaces (that still would be required to be graph of a function). Of course, due to the lack of geometric periodicity, the Bloch transform can in this case no longer be used to analyze this solution by decomposing it into periodic components.

The structure of this paper is as follows: In Section 2 we introduce the Bloch transform on a line and use it to define a Bloch transform on a periodic surface Γ. In Section 3 we show that the above-mentioned Herglotz wave functions are bounded in Sobolev spaces H^s on the surface Γ; this result is generalized in Section 4 to Sobolev spaces on horizontal strips. In Section 5 we provide an equivalent characterization of the non-periodic scattered field for arbitrary Dirichlet boundary conditions using a continuum of periodic scattering problems. The Appendix A contains a technical and in principle well-known result on isomorphisms between Sobolev spaces.

2 Bloch Transform for Sobolev Spaces on Periodic Surfaces

In the remainder of this paper we always denote the quasiperiodicity of a function by α; recall that a function u defined on some set $\Omega \subset \mathbb{R}^2$ is called α-quasiperiodic with period L if

$$u(x_1 + L, x_2) := e^{\mathrm{i}L\,\alpha} u(x), \quad x = (x_1, x_2)^\top \in \Omega. \tag{4}$$

(Of course, we implicitly assume here that for all $x \in \Omega$ the point $(x_1 + L, x_2)^\top$ belongs to Ω, too.) Obviously, if $L\alpha$ is a multiple of 2π, then this simply means that u is periodic. Thus, given a period $L > 0$, it is sufficient to consider quasiperiodicities

$$\alpha \in (-\pi/L, \pi/L].$$

Indeed, following definition (4), α-quasiperiodicity is precisely the same as $(\alpha + 2\pi/L)$-quasiperiodicity.

A main tool in our analysis will be the Bloch transform $\mathcal{J}_{\mathbb{R}}$, defined by

$$\mathcal{J}_{\mathbb{R}}\phi(\alpha; x_1) := \sqrt{\frac{L}{2\pi}} \sum_{j\in\mathbb{Z}} \phi(x_1 + Lj) e^{-\mathrm{i}\alpha(x_1+Lj)}, \quad x_1 \in (-L/2, L/2],\ \alpha \in (-\pi/L, \pi/L] \tag{5}$$

for $\phi \in C_0^\infty(\mathbb{R})$. (This transform is also known as Floquet- or Floquet–Bloch-transform.) Note that $\mathcal{J}_{\mathbb{R}}\phi$ is well-defined because ϕ has compact support. We restrict the argument x_1 of $\mathcal{J}_{\mathbb{R}}\phi$ to $(-L/2, L/2]$ since, by definition, this function would otherwise automatically be L-periodic in x_1. Further, the Bloch transform obviously commutes with L-periodic functions on the real line: If $w : \mathbb{R} \to \mathbb{C}$ is L-periodic, then

$$[\mathcal{J}_{\mathbb{R}}(w\phi)](\alpha; x_1) = \sqrt{\frac{L}{2\pi}} w(x_1 + Lj) \sum_{j\in\mathbb{Z}} \phi(x_1 + Lj) e^{-\mathrm{i}\alpha(x_1+Lj)} = w(x_1)\, \mathcal{J}_{\mathbb{R}}\phi(\alpha; x_1)$$

for $x_1 \in (-L/2, L/2]$ and $\alpha \in (-\pi/L, \pi/L]$. Even if we could define $\mathcal{J}_{\mathbb{R}}$ for all $\alpha \in \mathbb{R}$ by the same formula, for the reasons discussed below (4) we restrict α in the following to $(-\pi/L, \pi/L]$.

Obviously, $\mathcal{J}_{\mathbb{R}}$ is a classical one-dimensional Bloch transform and several mapping properties of this transform are of course well-known. A standard reference on this topic is [21]. Theorem 4 stated below is taken from [14, Annexe B], where a detailed proof can be found. Admittedly, the Bloch transform in [14] contains no phase shift in its definition; the resulting functions are hence not periodic in their second variable but α-quasiperiodic. This does, however, not affect the results below in any way, as can be seen by multiplying the Bloch transform $\mathcal{J}_{\mathbb{R}}$ by $x_1 \mapsto \exp(\mathrm{i}\alpha x_1)$.

To state the mapping properties of $\mathcal{J}_{\mathbb{R}}$, we need to introduce the L-periodic Sobolev spaces $H_{\mathrm{p}}^s(\mathbb{R})$ of L-periodic functions on $\mathbb{R}$. (From now on, the index p will always mean that functions in this space are L-periodic with respect to x_1.) For an L-periodic distribution $\phi \in \mathcal{D}'_{\mathrm{p}}(\mathbb{R})$ we define its Fourier coefficients $\hat{\phi}(j)$ for $j \in \mathbb{Z}$ by

$$\hat{\phi}(j) = \frac{1}{L}\phi\big(x_1 \mapsto \exp(-(2\pi\mathrm{i}/L)jx_1)\big) \quad \left[= \int_{-L/2}^{L/2} \phi(x_1) e^{-\frac{2\pi\mathrm{i}}{L}jx_1} \frac{\mathrm{d}x_1}{L} \text{ if } \phi \in L^2_{\mathrm{loc}}(\mathbb{R})\right]. \tag{6}$$

Further, we define the well-known Hilbert spaces $H_{\mathrm{p}}^s(\mathbb{R})$ by completion of smooth L-periodic functions in the norm

$$\|\phi\|_{H_{\mathrm{p}}^s(\mathbb{R})} := \sum_{j\in\mathbb{Z}} (1 + |j|^2)^s\, |\hat{\phi}(j)|^2, \quad s \in \mathbb{R}. \tag{7}$$

Later on in Lemma 5 and Theorem 6 we will have to deal with functions and distributions with period $2\pi/L$ (the dual period to L). Of course, these functions and distributions can again be development into Fourier series. The only difference in the definition of the corresponding basis functions and Fourier coefficients is that the period L in, e.g., (6) has to be replaced by $2\pi/L$. With these changes, we will again denote the Fourier coefficients as, e.g., $\hat{\phi}(j)$. The notation $H_{\mathrm{p}}^s(\mathbb{R})$ is, however, exclusively reserved for L-periodic functions.

As we will see in the next theorem, the image spaces of the Bloch transform are the spaces $L^2((-\pi/L,\pi/L);H^s_{\mathrm{p}}(\mathbb{R}))$ of vector-valued, measurable functions from $(-\pi/L,\pi/L)$ into $H^s_{\mathrm{p}}(\mathbb{R})$. Those are defined, for all $s\in\mathbb{R}$, by

$$L^2((-\pi/L,\pi/L);H^s_{\mathrm{p}}(\mathbb{R}))=\left\{\tilde\phi:\,(-\pi/L,\pi/L)\to H^s_{\mathrm{p}}(\mathbb{R})\text{ is measurable, }\int_{-\pi/L}^{\pi/L}\|\tilde\phi(\alpha;\cdot)\|^2_{H^s_{\mathrm{p}}(\mathbb{R})}\,\mathrm{d}\alpha<\infty\right\}$$

with squared norm $\|\tilde\phi\|^2_{L^2((-\pi/L,\pi/L);H^s_{\mathrm{p}}(\mathbb{R}))}:=\int_{-\pi/L}^{\pi/L}\|\tilde\phi(\alpha;\cdot)\|^2_{H^s_{\mathrm{p}}(\mathbb{R})}\,\mathrm{d}\alpha\,.$

Theorem 1. *For $s\in\mathbb{R}$ the Bloch transform $\mathcal{J}_{\mathbb{R}}$ extends to an isometry between $H^s(\mathbb{R})$ and $L^2((-\pi/L,\pi/L);H^s_{\mathrm{p}}(\mathbb{R}))$. The inverse transform is given by*

$$\left(\mathcal{J}_{\mathbb{R}}^{-1}\tilde\phi\right)(x_1)=\sqrt{\frac{L}{2\pi}}\int_{-\pi/L}^{\pi/L}\tilde\phi(\alpha;x_1)\exp(\mathrm{i}\alpha x_1)\,\mathrm{d}\alpha\,,\qquad x_1\in\mathbb{R}.\tag{8}$$

Remark 2. *We will almost exclusively work with function spaces containing elements that are merely defined almost everywhere; to simplify notation we always neglect to write this down explicitly; in* (8) *we did for instance not note that the equality holds merely for almost every $x_1\in\mathbb{R}$.*

A convenient tool to analyze the Bloch transform is the usual continuous Fourier transform, defined by

$$\hat\phi(\xi)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp(-\mathrm{i}\xi x_1)\phi(x_1)\,\mathrm{d}x_1\,,\quad \xi\in\mathbb{R}.\tag{9}$$

This transform is an isomorphism from $H^s(\mathbb{R})$ into a weighted L^2-space on $\mathbb{R}$, defined via the Bessel potentials, yielding norms

$$\|f\|_{H^s(\mathbb{R})}=\|\hat f\|_{L^2_s(\mathbb{R})}:=\|\xi\mapsto(1+|\xi|^2)^{s/2}\hat f(\xi)\|_{L^2(\mathbb{R})},\quad s\in\mathbb{R}.$$

Since the Bloch transform $\mathcal{J}_{\mathbb{R}}\phi$ of $\phi\in C_0^\infty(\mathbb{R})$ is L-periodic for fixed α, we can develop it into a Fourier series with coefficients

$$\begin{aligned}
c(j,\alpha)&=\sqrt{\frac{L}{2\pi}}\frac{1}{L}\int_{-L/2}^{L/2}\sum_{j\in\mathbb{Z}}\phi(x_1+Lj)\exp\big(-\mathrm{i}\alpha(x_1+Lj)\big)\exp\left(-\frac{2\pi\mathrm{i}}{L}jx_1\right)\,\mathrm{d}x_1\\
&\overset{x_1+Lj=y_1}{=}\frac{1}{\sqrt{2\pi L}}\int_{\mathbb{R}}\phi(y_1)\exp\big(-\mathrm{i}\alpha y_1\big)\exp\left(-\frac{2\pi\mathrm{i}}{L}j(y_1-Lj)\right)\,\mathrm{d}y_1\\
&=\frac{1}{\sqrt{2\pi L}}\int_{\mathbb{R}}\phi(y_1)\exp\left(-\mathrm{i}\left(\alpha+\frac{2\pi}{L}j\right)y_1\right)\,\mathrm{d}y_1\;=\frac{1}{\sqrt{L}}\hat\phi\left(\alpha+\frac{2\pi}{L}j\right),\qquad j\in\mathbb{Z},\ \alpha\in(-\pi/L,\pi/L].
\end{aligned}$$

In consequence,

$$\mathcal{J}_{\mathbb{R}}\phi(\alpha;x_1)=\frac{1}{\sqrt{L}}\sum_{j\in\mathbb{Z}}\hat\phi\left(\alpha+\frac{2\pi}{L}j\right)\exp\left(\frac{2\pi\mathrm{i}}{L}jx_1\right),$$

and, for all $x_1\in\mathbb{R}$,

$$\begin{aligned}
\mathcal{J}_{\mathbb{R}}^{-1}\big(\mathcal{J}_{\mathbb{R}}\phi(\alpha;x_1)\big)&=\frac{1}{\sqrt{2\pi}}\int_{-\pi/L}^{\pi/L}\sum_{j\in\mathbb{Z}}\hat\phi\left(\alpha+\frac{2\pi}{L}j\right)\exp\left(\frac{2\pi\mathrm{i}}{L}jx_1\right)\exp(\mathrm{i}\alpha x_1)\,\mathrm{d}\alpha\\
&\overset{\alpha+\frac{2\pi}{L}j=\xi}{=}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\hat\phi(\xi)\exp\left(\mathrm{i}\xi x_1\right)\,\mathrm{d}\xi\;=\phi(x_1).
\end{aligned}$$

The latter relation extends by a standard density argument from $C_0^\infty(\mathbb{R})$ to all Sobolev spaces $H^s(\mathbb{R})$ with $s\in\mathbb{R}$.

As for the usual continuous Fourier transform, the adjoint of $\mathcal{J}_\mathbb{R}$ equals its inverse: For $\phi \in C_0^\infty(\mathbb{R})$ and $\tilde{v} \in C_0^\infty((-\pi/L, \pi/L); C_\mathrm{p}^\infty(\mathbb{R}))$ it holds that

$$\begin{aligned}
\langle \mathcal{J}_\mathbb{R}\phi, \, \tilde{v} \rangle_{L^2((-\pi/L,\pi/L);L^2_\mathrm{p}(\mathbb{R}))} &= \int_{-\pi/L}^{\pi/L} \sqrt{\frac{L}{2\pi}} \int_{-L/2}^{L/2} \sum_{j\in\mathbb{Z}} \phi(x_1 + Lj) e^{-\mathrm{i}\alpha(x_1+Lj)} \, \overline{\tilde{v}(\alpha, x_1)} \, \mathrm{d}x_1 \, \mathrm{d}\alpha \\
&= \sum_{j\in\mathbb{Z}} \int_{-L/2}^{L/2} \phi(x_1 + Lj) \sqrt{\frac{L}{2\pi}} \int_{-\pi/L}^{\pi/L} \overline{\tilde{v}(\alpha, x_1)} e^{-\mathrm{i}\alpha(x_1+Lj)} \, \mathrm{d}\alpha \, \mathrm{d}x_1 \\
&\overset{x_1+Lj=y_1}{=} \int_\mathbb{R} \phi(x_1) \, \overline{\sqrt{\frac{L}{2\pi}} \int_{-\pi/L}^{\pi/L} \tilde{v}(\alpha, x_1) \exp(\mathrm{i}\alpha x_1) \, \mathrm{d}\alpha} \, \mathrm{d}y_1 \; = \langle \phi, \, \mathcal{J}_\mathbb{R}^{-1}(\tilde{v}) \rangle_{L^2(\mathbb{R})}.
\end{aligned}$$

Since the Bloch transform $\mathcal{J}_\mathbb{R}$ is an isomorphism between $H^s(\mathbb{R})$ and $L^2((-\pi/L, \pi/L); H^s_\mathrm{p}(\mathbb{R}))$ for $s \in \mathbb{R}$, its inverse is an isomorphism between $L^2((-\pi/L, \pi/L); H^s_\mathrm{p}(\mathbb{R}))$ and $H^s(\mathbb{R})$. Of course, the adjoint $\mathcal{J}_\mathbb{R}^*$ with respect to the above L^2-inner product is naturally an isomorphism between $H^{-s}(\mathbb{R})$ and $L^2((-\pi/L, \pi/L); H^s_\mathrm{p}(\mathbb{R}))$, too. Of course, the equality $\mathcal{J}_\mathbb{R}^{-1} = \mathcal{J}^*$ on all spaces $L^2((-\pi/L, \pi/L); H^s_\mathrm{p}(\mathbb{R}))$ that we showed above is due to the fact that $\mathcal{J}_\mathbb{R}$, its adjoint, and its inverse do not depend on s.

Theorem 3. *(1) For $s \in \mathbb{R}$ the adjoint $\mathcal{J}_\mathbb{R}^*$ of the Bloch transform $\mathcal{J}_\mathbb{R} : H^s(\mathbb{R}) \to L^2((-\pi/L, \pi/L); H^s_\mathrm{p}(\mathbb{R}))$ with respect to the inner product of $L^2((-\pi/L, \pi/L); L^2_\mathrm{p}(\mathbb{R}))$ equals the inverse $\mathcal{J}_\mathbb{R}^{-1} : L^2((-\pi/L, \pi/L); H^{-s}_\mathrm{p}(\mathbb{R})) \to H^{-s}(\mathbb{R})$ and both are isomorphisms between their pre-image and image spaces. The inverse of $\mathcal{J}_\mathbb{R}$ equals its adjoint operator.*

(2) The Bloch transform can equivalently be represented as

$$\mathcal{J}_\mathbb{R}\phi(\alpha; x_1) = \frac{1}{\sqrt{L}} \sum_{j\in\mathbb{Z}} \hat{\phi}\left(\alpha + \frac{2\pi}{L} j\right) \exp\left(\frac{2\pi \mathrm{i}}{L} j x_1\right), \quad x_1 \in (-L/2, L/2], \; \alpha \in (-\pi/L, \pi/L].$$

for $\phi \in H^s(\mathbb{R})$, where $\hat{\phi}$ is the Fourier transform defined in (9).

(3) The Bloch transform commutes with L-periodic functions on the real line: If, e.g., $w : \mathbb{R} \to \mathbb{C}$ is a bounded and measurable function, then $(\mathcal{J}_\mathbb{R} w\phi)(\alpha, x_1) = w(x_1) \, \mathcal{J}_\mathbb{R}\phi(\alpha, x_1)$ for $x_1 \in (-L/2, L/2]$ and $\alpha \in (-\pi/L, \pi/L]$.

Next we will define an analogous Bloch transform $\mathcal{J}_\Gamma$ on Sobolev spaces $H^s(\Gamma)$ and periodic spaces $H^s_\mathrm{p}(\Gamma)$ for $-1 \le s \le 1$. To this end, assume now that Γ is the boundary of a periodic, impenetrable structure that is L-periodic in x_1 and given as the graph of a Lipschitz continuous function $\zeta : \mathbb{R} \to \mathbb{R}$,

$$\Gamma = \{(x_1, \zeta(x_1))^\top, x_1 \in \mathbb{R}\}.$$

Without loss of generality we can assume that there exists $\zeta_- > 0$ such that

$$0 < \zeta_- := \operatorname*{ess\,inf}_\mathbb{R}(\zeta) \le \zeta(x_1) \le \|\zeta\|_{L^\infty(\mathbb{R})} =: \zeta_+.$$

Following [24] we introduce Sobolev spaces on the L-periodic surface Γ. For $\varphi : \Gamma \to \mathbb{C}$ we introduce $\varphi_\zeta : \mathbb{R} \to \mathbb{C}$ by $\varphi_\zeta(x_1) = \varphi(x_1, \zeta(x_1))$ for $x_1 \in \mathbb{R}$, for $\varphi : \Gamma \to \mathbb{C}$, and for the above-mentioned L-periodic Lipschitz continuous function ζ defining Γ. Then

$$H^s(\Gamma) = \left\{\varphi : \Gamma \to \mathbb{C} \text{ such that } \varphi_\zeta \in H^s(\mathbb{R})\right\}, \qquad 0 \le s \le 1, \tag{10}$$

with norm $\|\varphi\|_{H^s(\Gamma)} := \|\varphi_\zeta\|_{H^s(\mathbb{R})}$. The spaces $H^s(\Gamma)$ for $-1 \le s < 0$ are then defined by duality with respect to the inner product

$$(\varphi, \psi)_\Gamma = \int_\Gamma \varphi \, \overline{\psi} \, \mathrm{d}S \; = \int_\mathbb{R} \varphi_\zeta(x_1) \overline{\psi}_\zeta(x_1) \sqrt{1 + |\zeta'(x_1)|^2} \, \mathrm{d}x_1 \, .$$

(The range of $s \in [-1, 1]$ is limited since the surface is merely assumed to be Lipschitz continuous.) We also introduce periodic spaces $H^s_\mathrm{p}(\Gamma)$, for $-1 \le s \le 1$, by lifting $H^s_\mathrm{p}(\mathbb{R})$ to Γ via

$$H^s_\mathrm{p}(\Gamma) = \left\{\varphi : \Gamma \to \mathbb{C} \text{ such that } x_1 \mapsto \varphi_\zeta(x_1) = \varphi(x_1, \zeta(x_1)) \in H^s_\mathrm{p}(\mathbb{R})\right\}. \tag{11}$$

The norm in $H^s_{\mathrm{p}}(\Gamma)$ is again defined be lifting the norm in $H^s_{\mathrm{p}}(\mathbb{R})$ to Γ, that is, $\|\varphi\|_{H^s_{\mathrm{p}}(\Gamma)} := \|\varphi_\zeta\|_{H^s_{\mathrm{p}}(\mathbb{R})}$. As an analogue to the spaces $L^2((-\pi/L,\pi/L);H^s_{\mathrm{p}}(\mathbb{R}))$, we define

$$L^2\big((-\pi/L,\pi/L);H^s_{\mathrm{p}}(\Gamma)\big) = \left\{\tilde{\phi}:\ (-\pi/L,\pi/L) \to H^s_{\mathrm{p}}(\Gamma) \text{ is measurable, } \int_{-\pi/L}^{\pi/L} \|\tilde{\varphi}(\alpha,\cdot)\|^2_{H^s_{\mathrm{p}}(\Gamma)} \,\mathrm{d}\alpha < \infty\right\}$$

with squared norm $\|\tilde{\varphi}\|^2_{L^2\left((-\pi/L,\pi/L);H^s_{\mathrm{p}}(\Gamma)\right)} := \int_{-\pi/L}^{\pi/L} \|\tilde{\varphi}(\alpha,\cdot)\|^2_{H^s_{\mathrm{p}}(\Gamma)} \,\mathrm{d}\alpha$. The Bloch transform $\mathcal{J}_\Gamma$ of $\varphi \in H^1(\Gamma)$ with compact support is then defined by

$$\mathcal{J}_\Gamma\varphi\big(\alpha;(x_1,\zeta(x_1))^\top\big) := \sqrt{\frac{L}{2\pi}} \sum_{j\in\mathbb{Z}} \varphi(x_1+Lj,\zeta(x_1+Lj))e^{-\mathrm{i}\alpha(x_1+Lj)}, \tag{12}$$

for $\alpha \in (-\pi/L,\pi/L]$, $x_1 \in (-L/2,L/2]$, and for any $\varphi \in H^1(\Gamma)$ with compact support. Since functions in $H^1(\mathbb{R})$ are continuous, functions in $H^1(\Gamma)$ are continuous, too; since φ in the last equation has, by assumption, compact support, it is also clear that $\mathcal{J}_\Gamma\varphi$ is well-defined. Obviously,

$$\mathcal{J}_\Gamma\varphi\big(\alpha;(x_1,\zeta(x_1))^\top\big) = \mathcal{J}_\mathbb{R}\varphi_\zeta(\alpha;x_1), \qquad \alpha \in (-\pi/L,\pi/L],\ x_1 \in (-L/2,L/2],$$

and hence the properties of $\mathcal{J}_\Gamma$ can be directly derived from those of $\mathcal{J}_\mathbb{R}$. Indeed, the spaces $H^s(\Gamma)$ and $H^s_{\mathrm{p}}(\Gamma)$ are defined by transporting $H^s(\mathbb{R})$ and $H^s_{\mathrm{p}}(\mathbb{R})$ to Γ, see (10) and (11). This means that the next result is a simple corollary of Theorem 4.

Theorem 4. *For $s \in [-1,1]$ the Bloch transform $\mathcal{J}_\Gamma$ can be extended to an isomorphism between $H^s(\Gamma)$ and $L^2\big((-\pi/L,\pi/L);H^s_{\mathrm{p}}(\Gamma)\big)$. The inverse transform is given by*

$$\left(\mathcal{J}_\Gamma^{-1}\tilde{\varphi}\right)(x) = \sqrt{\frac{L}{2\pi}} \int_{-\pi/L}^{\pi/L} \tilde{\varphi}(\alpha;x)\,\exp(\mathrm{i}\alpha x_1)\,\mathrm{d}\alpha\,, \qquad x = \big(x_1,\zeta(x_1)\big)^\top \in \Gamma,$$

and this inverse transform equals the adjoint of $\mathcal{J}_\Gamma$.

3 Herglotz Wave Functions

We turn again to the Herglotz wave functions from (13) and use them to define the Herglotz operator H by

$$H\phi(x) = \int_{-\pi}^{\pi} e^{\mathrm{i}k(\sin\theta\, x_1 - \cos\theta\,\zeta(x_1))}\phi(\theta)\,\mathrm{d}\theta\,, \quad x = \big(x_1,\zeta(x_1)\big)^\top \in \Gamma. \tag{13}$$

We will analyze mapping properties of H with respect to the Sobolev spaces $H^s(\Gamma)$ using the Bloch transform $\mathcal{J}_\Gamma$. To this end, we assume for the moment that ϕ belongs to $C_0^\infty(-\pi/2,\pi/2)$ and note that the change of variables $\ell = k\sin(\theta)$ implies that

$$H\phi(x) = \int_{-k}^{k} e^{\mathrm{i}\ell x_1 - \mathrm{i}\sqrt{k^2-\ell^2}\,\zeta(x_1)}\phi(\arcsin(k/\ell))\frac{\mathrm{d}\ell}{\sqrt{k^2-\ell^2}}.$$

Since ϕ is smooth and vanishes in a neighborhood of the endpoints $\pm\pi/2$, the singularities at $\ell = \pm k$ of $1/\sqrt{k^2-\ell^2}$ in the last expression do not cause problems. Formally computing the Bloch transform of $H\phi$ then yields that

$$\mathcal{J}_\Gamma(H\phi)\,(\alpha;x) = \sqrt{\frac{L}{2\pi}} \sum_{j\in\mathbb{Z}} \int_{-k}^{k} e^{\mathrm{i}\ell(x_1+Lj)-\mathrm{i}\sqrt{k^2-\ell^2}x_2}\frac{\phi(\arcsin(\ell/k))}{\sqrt{k^2-\ell^2}}\,\mathrm{d}\ell\ e^{-\mathrm{i}\alpha(x_1+Lj)} \tag{14}$$

$$= \sqrt{\frac{L}{2\pi}} \sum_{j\in\mathbb{Z}} \int_{-k}^{k} e^{\mathrm{i}\ell(x_1+Lj)-\mathrm{i}\sqrt{k^2-\ell^2}\,\zeta(x_1)}\frac{\phi(\arcsin(\ell/k))}{\sqrt{k^2-\ell^2}}\,\mathrm{d}\ell\ e^{-\mathrm{i}\alpha(x_1+Lj)} \tag{15}$$

for $x = (x_1, \zeta(x_1))^\top \in \Gamma$, $\alpha \in (-\pi/L, \pi/L]$, and $x_1 \in (-L/2, L/2]$. The convergence of the latter series in j is not clear without further arguments. To this end, we will first consider the truncated series

$$
\begin{aligned}
S_N &= \sum_{j=-N}^{N} \int_{-k}^{k} e^{\mathrm{i}\ell(x_1+Lj)-\mathrm{i}\sqrt{k^2-\ell^2}\,\zeta(x_1)} \frac{\phi(\arcsin(\ell/k))}{\sqrt{k^2-\ell^2}} \,\mathrm{d}\ell \;\, e^{-\mathrm{i}\alpha(x_1+Lj)} \\
&= \int_{-k}^{k} \left(\sum_{j=-N}^{N} e^{\mathrm{i}L(\ell-\alpha)j} \right) e^{\mathrm{i}\ell x_1-\mathrm{i}\sqrt{k^2-\ell^2}\,\zeta(x_1)} \frac{\phi(\arcsin(\ell/k))}{\sqrt{k^2-\ell^2}} \,\mathrm{d}\ell \;\, e^{-\mathrm{i}\alpha x_1},
\end{aligned}
$$

and investigate the behavior of this expression as $N \to \infty$. Let us, to this end, first define a smooth and compactly supported function $\psi : \mathbb{R} \to \mathbb{C}$ by

$$
\psi(\ell) = \begin{cases} \exp\left(\mathrm{i}\ell x_1 - \mathrm{i}\sqrt{k^2-\ell^2}\,\zeta(x_1)\right) \frac{\phi(\arcsin(\ell/k))}{\sqrt{k^2-\ell^2}} & |\ell| < k, \\ 0 & |\ell| \geq k. \end{cases} \tag{16}
$$

Since, by assumption, $\phi \in C_0^\infty(-\pi/2, \pi/2)$, the function $\ell \mapsto \phi(\arcsin(\ell/k))/\sqrt{k^2-\ell^2}$ belongs to $C_0^\infty(-k,k)$ and its extension by zero belongs to $C_0^\infty(\mathbb{R})$. The same holds for

$$
\ell \mapsto e^{\mathrm{i}\ell x_1-\mathrm{i}\sqrt{k^2-\ell^2}\,\zeta(x_1)} \frac{\phi(\arcsin(\ell/k))}{\sqrt{k^2-\ell^2}}
$$

and implies that ψ defined in (16) indeed belongs to $C_0^\infty(\mathbb{R})$. Using the notation $\mathcal{D}(\mathbb{R}) = C_0^\infty(\mathbb{R})$ for test functions and $\mathcal{D}'(\mathbb{R})$ for distributions on the real line, we can hence rewrite S_N as duality product

$$
\begin{aligned}
S_N &= \int_{-k}^{k} \left(\sum_{j=-N}^{N} e^{\mathrm{i}L(\ell-\alpha)j} \right) e^{\mathrm{i}\ell x_1-\mathrm{i}\sqrt{k^2-\ell^2}\,\zeta(x_1)} \frac{\phi(\arcsin(\ell/k))}{\sqrt{k^2-\ell^2}} \,\mathrm{d}\ell \;\, \exp(-\mathrm{i}\alpha x_1) \\
&= \int_{\mathbb{R}} \left(\sum_{j=-N}^{N} e^{\mathrm{i}L(\ell-\alpha)j} \right) \psi(\ell) \,\mathrm{d}\ell \;\, \exp(-\mathrm{i}\alpha x_1) = \Big\langle \sum_{j=-N}^{N} e^{\mathrm{i}L(\cdot-\alpha)j},\, \psi \Big\rangle_{\mathcal{D}'(\mathbb{R})\times\mathcal{D}(\mathbb{R})} \exp(-\mathrm{i}\alpha x_1).
\end{aligned} \tag{17}
$$

The distribution

$$
\delta_{\alpha,N}^{2\pi/L} := \frac{L}{2\pi} \sum_{j=-N}^{N} e^{\mathrm{i}L(\cdot-\alpha)j} \in \mathcal{D}'(\mathbb{R})
$$

is, for finite $N \in \mathbb{N}$, a $2\pi/L$-periodic, smooth function. It can hence also be interpreted as a $2\pi/L$-periodic distribution in $\mathcal{D}'_{2\pi/L}(\mathbb{R})$: By definition (see [27, Section 5.2]), a distribution v in $\mathcal{D}'(\mathbb{R})$ is $2\pi/L$-periodic (that is, it belongs to $\mathcal{D}'_{2\pi/L}(\mathbb{R})$), if v is shift-invariant with respect to shifts $\tau_{2\pi j/L}$ of length $2\pi j/L$,

$$
\langle v,\, \tau_{2\pi j/L}\varphi \rangle_{\mathcal{D}'(\mathbb{R})\times\mathcal{D}(\mathbb{R})} = \langle v,\, \varphi \rangle_{\mathcal{D}'(\mathbb{R})\times\mathcal{D}(\mathbb{R})} \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}), j \in \mathbb{Z}.
$$

Here, $\tau_{2\pi j/L}\varphi(\ell) := \varphi(\ell - 2\pi j/L)$. As shown in detail in [27, Section 5.2], the dual space of $\mathcal{D}'_{2\pi/L}(\mathbb{R})$ can be identified with the space $\mathcal{D}_{2\pi/L}(\mathbb{R})$ of smooth, $2\pi/L$-periodic functions.

Lemma 5. *For $N \to \infty$, the distribution $\delta_{\alpha,N}^{2\pi/L}$ converges in $\mathcal{D}'(\mathbb{R})$ to the $2\pi/L$-periodic Dirac distribution $\delta_\alpha^{2\pi/L}$ at $\alpha \in (-\pi/L, \pi/L]$, defined by*

$$
\delta_\alpha^{2\pi/L} = \lim_{M\to\infty} \sum_{j=-M}^{M} \delta_{\alpha+2\pi j/L}
$$

via the usual Dirac distribution $\delta_t \in \mathcal{D}'(\mathbb{R})$ at $t \in \mathbb{R}$, that is, for all $\varphi \in \mathcal{D}(\mathbb{R}) = C_0^\infty(\mathbb{R})$ it holds that

$$
\langle \delta_{\alpha,N}^{2\pi/L}, \varphi \rangle_{\mathcal{D}'(\mathbb{R})\times\mathcal{D}(\mathbb{R})} \to \langle \delta_\alpha^{2\pi/L}, \varphi \rangle_{\mathcal{D}'(\mathbb{R})\times\mathcal{D}(\mathbb{R})} = \sum_{j\in\mathbb{Z}} \varphi(\alpha + 2\pi j/L) \quad \text{as } N \to \infty.
$$

Proof. All limits in the claim of the lemma have to be understood in the sense of the convergence of distributions in $\mathcal{D}'(\mathbb{R})$, see, e.g., [27, Section 5.2]. To prove the convergence of $\delta_{\alpha,N}^{2\pi/L}$ to $\delta_{-\alpha}^{2\pi/L}$ in the sense of $\mathcal{D}'(\mathbb{R})$ as $N \to \infty$, let us first note from Lemma 5.2.1 in [27] that for periodic distributions, convergence in $\mathcal{D}'(\mathbb{R})$ is equivalent to convergence in $\mathcal{D}'_{2\pi/L}(\mathbb{R})$, where the duality pairing can, e.g., be defined via the Fourier coefficients of the distribution and the test function,

$$\langle u, \varphi \rangle_{\mathcal{D}'_{2\pi/L}(\mathbb{R}) \times \mathcal{D}_{2\pi/L}(\mathbb{R})} = \frac{2\pi}{L} \sum_{j \in \mathbb{Z}} \hat{u}(j) \hat{\varphi}(j),$$

see [27, Theorem 5.2.1]. Second, convergence in $\mathcal{D}'_{2\pi/L}(\mathbb{R})$ follows from the boundedness of the Fourier coefficients

$$\hat{\delta}_\alpha^{2\pi/L}(j) = \frac{2\pi}{L} \exp(-\mathrm{i} L \alpha j) \qquad \text{of} \qquad \delta_\alpha^{2\pi/L} \in \mathcal{D}'_{2\pi/L}(\mathbb{R})$$

by one, compare [27, Example 5.2.4]. Indeed, Theorem 5.2.1 in [27] shows that for all $s > 0$ there exists a constant $C(\varphi, s)$ such that $|\hat{\varphi}(j)| \leq C(\varphi, s)(1 + |s|^2)^{-s}$; hence

$$\begin{aligned} \big| \langle \delta_{\alpha,N}^{2\pi/L} - \delta_\alpha^{2\pi/L}, \varphi \rangle_{\mathcal{D}'_{2\pi/L}(\mathbb{R}) \times \mathcal{D}_{2\pi/L}(\mathbb{R})} \big| &\leq \frac{2\pi}{L} \sum_{j \in \mathbb{Z}, |j| > N} |\hat{\delta}_\alpha^{2\pi/L}(j)| \, |\hat{\varphi}(j)| \\ &\leq C(\varphi, s, L) \sum_{j \in \mathbb{Z}, |j| > N} (1 + |j|^2)^{-s} \overset{N \to \infty}{\to} 0 \qquad \text{if } s > 1/2. \end{aligned}$$

Hence, $\delta_{-\alpha,N}^{2\pi/L} \to \delta_{-\alpha}^{2\pi/L}$ in the sense of $\mathcal{D}'_{2\pi/L}(\mathbb{R})$ as $N \to \infty$. $\square$

Theorem 6. *If $\phi \in C_0^\infty(-\pi/2, \pi/2)$, then*

$$\mathcal{J}_\Gamma(H\phi)(\alpha; x) = \sqrt{\frac{2\pi}{L}} \sum_{j \in \mathbb{Z}} e^{\frac{2\pi \mathrm{i} j}{L} x_1 - \mathrm{i} \sqrt{k^2 - \left(\alpha + \frac{2\pi j}{L}\right)^2} \zeta(x_1)} \, \frac{\phi\big(\arcsin\big(\alpha/k + \frac{2\pi j}{kL}\big)\big)}{\sqrt{k^2 - \left(\alpha + \frac{2\pi j}{L}\right)^2}} \, \mathbb{1}_{\left\{\left|\alpha + \frac{2\pi j}{L}\right| < k\right\}} \tag{18}$$

for $\alpha \in (-\pi/L, \pi/L]$, $x_1 \in (-L/2, L/2]$, and $x = (x_1, \zeta(x_1))^\top \in \Gamma$.

Proof. Following the above computations and the definition of the auxiliary function ψ in (16), we find the value of $\mathcal{J}_\Gamma(H\phi)(\alpha; x)$ as the limit of the terms S_N defined in (17),

$$\begin{aligned} \mathcal{J}_\Gamma(H\phi)(\alpha; x) &= \sqrt{\frac{L}{2\pi}} \lim_{N \to \infty} S_N = \sqrt{\frac{L}{2\pi}} \lim_{N \to \infty} \Big\langle \sum_{j=-N}^{N} e^{\mathrm{i} L(\cdot - \alpha) j}, \psi \Big\rangle_{\mathcal{D}'(\mathbb{R}) \times \mathcal{D}(\mathbb{R})} \exp(-\mathrm{i} \alpha x_1) \\ &= \sqrt{\frac{2\pi}{L}} \langle \delta_\alpha^{2\pi/L}, \psi \rangle_{\mathcal{D}'(\mathbb{R}) \times \mathcal{D}(\mathbb{R})} \exp(-\mathrm{i} \alpha x_1) = \sqrt{\frac{2\pi}{L}} \sum_{j \in \mathbb{Z}} \psi(\alpha + 2\pi j/L) e^{-\mathrm{i} \alpha x_1} \\ &= \sqrt{\frac{2\pi}{L}} \sum_{j \in \mathbb{Z}} e^{\frac{2\pi \mathrm{i} j}{L} x_1 - \mathrm{i} \sqrt{k^2 - \left(\alpha + \frac{2\pi j}{L}\right)^2} \zeta(x_1)} \frac{\phi\big(\arcsin\big(\alpha/k + \frac{2\pi j}{kL}\big)\big)}{\sqrt{k^2 - (\alpha + 2\pi j/L)^2}} \mathbb{1}_{\left\{\left|\alpha + \frac{2\pi j}{L}\right| < k\right\}} \end{aligned}$$

for $\alpha \in (-\pi/L, \pi/L]$, $x = (x_1, \zeta(x_1))^\top \in \Gamma$, and $x_1 \in (-L/2, L/2]$. Note that the last series contains at most a finite number of non-zero terms due to the condition $\left|\alpha + 2\pi j/L\right| < k$ for $k > 0$ and $\alpha \in (-\pi/L, \pi/L]$. Together with this condition, the root-like singularities of $(k^2 - (\alpha + 2\pi j/L)^2)^{-1/2}$ are canceled by the function ϕ which is, by assumption, compactly supported in $(-\pi/2, \pi/2)$. $\square$

Up to now it is not clear whether bounds for the Bloch transform of $\mathcal{J}_\Gamma(H\phi)$ in the spaces $L^2\big((-\pi/L, \pi/L); H_\mathrm{p}^s(\Gamma)\big)$ can be shown. An affirmative answer is given in Theorem 7 below. Roughly speaking, we are going to show that $H\phi$ is well-defined in $H^1(\Gamma)$ and that H is bounded from L^2 into $H^1(\Gamma)$, if the support of the density ϕ stays away from the Rayleigh frequencies at $\pm\pi/2$ that correspond to horizontally propagating waves.

Theorem 7. *For any number $\delta \in (0, \pi/2)$, then the Herglotz operator is bounded from $L^2(-\pi/2+\delta, \pi/2-\delta)$ into $H^s(\Gamma)$ for $|s| \le 1$. The operator norm of H depends on Γ merely through the number $\|\zeta'\|_{L^\infty(\mathbb{R})}$ and hence remains constant if one translates the surface Γ.*

Proof. We need to show that, under the stated assumptions, the Bloch transform $\mathcal{J}_\Gamma(H\phi)$ possesses a finite norm in $L^2\big((-\pi/L, \pi/L); H^s_{\mathrm{p}}(\Gamma)\big)$ for $|s| < 1$. Choose a function $\phi \in C_0^\infty(-\pi/2+\delta, \pi/2-\delta)$ and let us first note that

$$x \mapsto \psi_j(\alpha; x) =: \exp\left(\tfrac{2\pi \mathrm{i}}{L} j\, x_1 - \mathrm{i}\sqrt{k^2 - \left(\alpha + \tfrac{2\pi j}{L}\right)^2}\, x_2\right), \quad j \in \mathbb{Z},\ x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_1 \\ \zeta(x_1) \end{smallmatrix}\right) \in \Gamma, \tag{19}$$

is an L-periodic function on Γ, since ζ is L-periodic and $\exp(2\pi \mathrm{i} j/L\,(x_1+L)) = \exp(2\pi \mathrm{i} j/L\, x_1)$ because $j \in \mathbb{Z}$. Moreover, due to the indicator function in (18) we merely need to consider the case $(\alpha + 2\pi j/L)^2 < k^2$, such that the square root in (19) is real-valued and the absolute value of the exponential in (19) equals one. Hence, the definition of the norms $\|\cdot\|_{H^s_{\mathrm{p}}(\Gamma)}$ implies that $\|x \mapsto \psi_j(\alpha;x)\|_{H^0_{\mathrm{p}}(\Gamma)} = 2\pi$; further,

$$\begin{aligned} \big\|x \mapsto \psi_j(\alpha;x)\big\|_{H^1_{\mathrm{p}}(\Gamma)} &= \big\|x_1 \mapsto \psi_j(\alpha; (x_1, \zeta(x_1)^\top))\big\|_{H^1_{\mathrm{p}}(\mathbb{R})} \\ &\le \big\|x_1 \mapsto \exp\left(2\pi \mathrm{i}\, j x_1/L\right)\big\|_{H^1_{\mathrm{p}}(\mathbb{R})} \big\|x_1 \mapsto \exp\left(\mathrm{i}\sqrt{k^2 - (\alpha + 2\pi j/L)^2}\zeta(x_1)\right)\big\|_{H^1_{\mathrm{p}}(\mathbb{R})} \end{aligned}$$

since $H^1_{\mathrm{p}}(\mathbb{R})$ is a Banach algebra, see [27]. One first computes explicitly that

$$\big\|x_1 \mapsto \exp\left(2\pi \mathrm{i}\, j x_1/L\right)\big\|_{H^1_{\mathrm{p}}(\mathbb{R})} = 2\pi\left(1 + j^2/L^2\right)^{1/2},$$

second that

$$\|x_1 \mapsto \exp\left(\mathrm{i}\sqrt{k^2 - (\alpha + 2\pi j/L)^2}\,\zeta(x_1)\right)\|^2_{L^2_{\mathrm{p}}(\mathbb{R})} = 4\pi^2,$$

and third that

$$\begin{aligned} &\|x_1 \mapsto \exp\left(\mathrm{i}\sqrt{k^2 - (\alpha + 2\pi j/L)^2}\,\zeta(x_1)\right)\|^2_{H^1_{\mathrm{p}}(\mathbb{R})} \\ &\qquad = 4\pi^2 + \|x_1 \mapsto \sqrt{k^2 - (\alpha + 2\pi j/L)^2}\,\zeta'(x_1)\,\exp\left(\mathrm{i}\sqrt{k^2 - (\alpha + 2\pi j/L)^2}\zeta(x_1)\right)\|^2_{L^2_{\mathrm{p}}(\mathbb{R})}, \end{aligned}$$

that is,

$$\|x_1 \mapsto \exp\left(\mathrm{i}\sqrt{k^2 - (\alpha + 2\pi j/L)^2}\,\zeta(x_1)\right)\|^2_{H^1_{\mathrm{p}}(\mathbb{R})} \le 4\pi^2\left[1 + |k^2 - (\alpha + 2\pi j/L)^2|\,\|\zeta'\|^2_{L^\infty(\mathbb{R})}\right].$$

Since $(\alpha + 2\pi j/L)^2 < k^2$, the last inequality in particular implies that

$$\|\psi_j(\alpha;x)\|_{H^s_{\mathrm{p}}(\Gamma)} = \big\|x \mapsto \exp\left(\tfrac{2\pi \mathrm{i}}{L} j x_1 - \mathrm{i}\sqrt{k^2 - (\alpha + 2\pi j/L)^2} x_2\right)\big\|_{H^s_{\mathrm{p}}(\Gamma)} \le C(\zeta') \tag{20}$$

uniformly in $s \le 1$ and $\alpha \in (-\pi/L, \pi/L]$. Note that the constant $C(\zeta')$ merely depends on $\|\zeta'\|_{L^\infty(\mathbb{R})}$. To prove mapping properties of H we finally need to investigate the (squared) norm of $\mathcal{J}_\Gamma(H\phi)$ in $L^2\big((-\pi/L, \pi/L); H^s_{\mathrm{p}}(\Gamma)\big)$, that is, the expression

$$\int_{-\pi/L}^{\pi/L} \|\mathcal{J}_\Gamma(H\phi)(\alpha;\cdot)\|^2_{H^s_{\mathrm{p}}(\Gamma)}\,\mathrm{d}\alpha, \qquad s \le 1.$$

must be bounded in terms of the (squared) norm of ϕ in $L^2(-\pi/2+\delta, \pi/2-\delta)$. We compute that

$$\int_{-\pi/L}^{\pi/L} \|\mathcal{J}_\Gamma(H\phi)(\alpha;\cdot)\|^2_{H^s_{\mathrm{p}}(\Gamma)}\,\mathrm{d}\alpha \le \frac{C(\zeta)L}{2\pi} \sum_{j \in \mathbb{Z}} \int_{-\pi/L}^{\pi/L} \mathbb{1}_{\{|\alpha + \frac{2\pi j}{L}| < k\}} \left| \frac{\phi\big(\arcsin\big(\frac{\alpha L + 2\pi j}{Lk}\big)\big)}{\sqrt{k^2 - \big(\alpha + 2\pi j/L\big)^2}} \right|^2 \mathrm{d}\alpha \tag{21}$$

and exploit now that the support of ϕ is, by assumption, restricted to $[-\pi/2+\delta, \pi/2-\delta]$. We hence do not change the value of the last integral on the right if we merely integrate over α such that $-\pi/2+\delta < \arcsin\left(\frac{\alpha L+2\pi j}{Lk}\right) < \pi/2-\delta$, that is, if we merely consider α such that

$$\sin(-\pi/2+\delta) < \frac{\alpha L+2\pi j}{Lk} < \sin(\pi/2-\delta).$$

The latter condition is equivalent to $k\sin(-\pi/2+\delta) < \alpha + 2\pi j/L < k\sin(\pi/2-\delta)$, or

$$(\alpha+2\pi j/L)^2 < k^2(1-\cos^2(\pi/2-\delta)).$$

Consequently, $1/|k^2-(\alpha+2\pi j/L)^2| \le 1/(k^2\cos^2(\pi/2-\delta)) < \infty$. Moreover, the condition $|\alpha+2\pi j/L| < k$ is equivalent to

$$-k < \alpha + \frac{2\pi j}{L} < k \qquad \Longleftrightarrow \qquad -\frac{L}{2\pi}(k+\alpha) < j < \frac{L}{2\pi}(k-\alpha).$$

Since $\alpha \in [-\pi/L, \pi/L]$, this yields a-priori bounds $J_* \le j \le J^*$ for $j \in \mathbb{N}$,

$$J_* := \left\lceil -\frac{Lk}{2\pi} - \frac{1}{2} \right\rceil < j < \left\lfloor \frac{Lk}{2\pi} + \frac{1}{2} \right\rfloor := J^*.$$

Note that $J_* \le 0 \le J^*$, that is $j=0$ is always included in the sums below. Plugging the different estimates of the proof together, we find that

$$\begin{aligned}
&\int_{-\pi/L}^{\pi/L} \|\mathcal{J}_\Gamma(H\phi)(\alpha;\cdot)\|^2_{H^s_{\mathrm{p}}(\Gamma)}\,\mathrm{d}\alpha \\
&\quad\le \frac{C(\zeta)L}{2\pi k^2\cos^2(\pi/2-\delta)} \sum_{j=J_*}^{J^*} \int_{-\pi/L}^{\pi/L} \mathbb{1}_{\left\{\left|\frac{\alpha L+2\pi j}{Lk}\right|<\sin(\pi/2-\delta)\right\}} \left|\phi\big(\arcsin\left(\tfrac{\alpha L+2\pi j}{Lk}\right)\big)\right|^2 \mathrm{d}\alpha \\
&\quad\le \frac{C(\zeta)L}{2\pi k^2\cos^2(\pi/2-\delta)} \sum_{j=J_*}^{J^*} \int_{-\min(\pi/L,\pi/2-\delta)}^{\min(\pi/L,\pi/2-\delta)} \left|\phi\big(\arcsin\left(\tfrac{\alpha L+2\pi j}{Lk}\right)\big)\right|^2 \mathrm{d}\alpha \\
&\quad\overset{(*)}{\le} \frac{C(\zeta)L}{2\pi k\cos^2(\pi/2-\delta)} \sum_{j=J_*}^{J^*} \int_{-\min(\pi/L,\pi/2-\delta)}^{\min(\pi/L,\pi/2-\delta)} |\phi(t)|^2\cos(t)\,\mathrm{d}t \;\le\; \underbrace{\frac{C(\zeta)L(J^*-J_*+1)}{2\pi k\cos^2(\pi/2-\delta)}}_{C^*(\zeta,\delta)^2} \|\phi\|^2_{L^2(\pi/2+\delta,\pi/2-\delta)},
\end{aligned}$$

where we employed in $(*)$ the change of variables $\arcsin\left(\frac{\alpha L+2\pi j}{Lk}\right) = t$. Due to Theorem 4, this implies that $\|H\phi\|_{H^s(\Gamma)} \le C^*(\zeta,\delta)\|\phi\|_{L^2(\pi/2+\delta,\pi/2-\delta)}$ for all $\phi \in C_0^\infty(-\pi/2+\delta, \pi/2-\delta)$ and $s \le 1$. Since smooth functions with compact support in $(-\pi/2+\delta, \pi/2-\delta)$ are dense in $L^2(-\pi/2+\delta, \pi/2-\delta)$ we obtain the claimed norm bound actually for all $\phi \in L^2(-\pi/2+\delta, \pi/2-\delta)$ by a standard density argument. □

Higher-order regularity results for $H\phi$ can of course be shown if the surface, that is, its generating function ζ, is smoother; in this case, higher order derivatives in x_1 of $\mathcal{J}_\Gamma H\phi$ remain bounded. Since the proof is essentially the same as the one for H^1 we do not detail this point. However, in Theorem 12 in the next section we will show that the Herglotz wave function first defined in (3) are bounded in the Sobolev spaces H^s on any horizontal strip $\{0 < x_2 < h\}$ of finite height (the lower bound $x_2 = 0$ is of course not essential).

The Herglotz operator is also bounded from a weighted L^2-space into $H^s(\Gamma)$. To state this result, define $L^2_{\cos}(-\pi/2, \pi/2)$ as the closure of $C_0^\infty(-\pi/2, \pi/2)$ in the norm

$$\|\phi\|^2_{L^2_{\cos}(-\pi/2,\pi/2)} := \left[\int_{-\pi/2}^{\pi/2} |\phi(\theta)|^2/\cos\theta\;\mathrm{d}\theta\right]^{1/2}.$$

Theorem 8. *The Herglotz operator H is bounded from $L^2_{\cos}(-\pi/2, \pi/2)$ into $H^s(\Gamma)$ for $|s| \le 1$.*

Proof. We use of course (21) to conclude by a change of variables $\alpha + 2\pi j/L = k\sin\theta$ that

$$\int_{-\pi/L}^{\pi/L} \|\mathcal{J}_\Gamma(H\phi)(\alpha;\cdot)\|^2_{H^s_{\mathrm{p}}(\Gamma)}\,\mathrm{d}\alpha \le \frac{C(\zeta)L}{2\pi}\sum_{j\in\mathbb{Z}}\int_{-\pi/L}^{\pi/L} \mathbb{1}_{\{|\alpha+\frac{2\pi j}{L}|<k\}} \left|\frac{\phi\big(\arcsin\big(\frac{\alpha L+2\pi j}{Lk}\big)\big)}{\sqrt{k^2-\big(\alpha+2\pi j/L\big)^2}}\right|^2 \mathrm{d}\alpha$$

$$\le \frac{C(\zeta)L}{2\pi}\sum_{j=J_*}^{J^*}\int_{-\pi/2}^{\pi/2}\frac{|\phi(\theta)|^2}{\cos(\theta)}\,\mathrm{d}\theta \le \frac{C(\zeta)L(J^*-J_*+1)}{2\pi}\|\phi\|^2_{L^2_{\cos}(-\pi/2,\pi/2)}.$$

□

Since the spaces $L^2(-\pi/2+\delta,\pi/2-\delta)$ can via extension by zero obviously be considered as subspaces of $L^2_{\cos}(\pi/2,\pi/2)$ we will in the sequel always work with $L^2_{\cos}(-\pi/2,\pi/2)$. We close this section by showing via an explicit example that the statements of Theorem 7 and of Theorem 8 are sharp.

Example 9. *The assumptions of Theorems 7 and 8 are sharp in the following sense: If ϕ is constant on $(-\pi/2,\pi/2)$, then ϕ belongs to $L^2(-\pi/2,\pi/2)$, but $H\phi$ does* not *belong to $L^2(\Gamma)$ for $\Gamma=\{x_2=0\}$. To show that $H\phi$ fails be belong to $L^2(\{x_2=0\})$, recall that the Bessel function J_0 can be represented as*

$$J_0(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}\sin\theta\, t}\,\mathrm{d}\theta = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{\mathrm{i}\sin\theta\, t}\,\mathrm{d}\theta\,, \quad t\in\mathbb{R},$$

see e.g. [2, (9.1.21)]. Consider now $\phi\equiv 1/\pi$ in $(-\pi/2,\pi/2)$ and the flat surface $\Gamma=\{x_2=0\}$. Then

$$H\phi(x) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{\mathrm{i}k\sin\theta\, x_1}\,\mathrm{d}\theta = J_0(kx_1).$$

The Bessel function $x_1\mapsto J_0(kx_1)$, however, does not belong to $L^2(\mathbb{R})$ since, e.g., its Fourier transform is not square integrable.

4 Scattering Problems and a Volumetric Bloch Transform

In this section we introduce a precise mathematical formulation of the scattering problem that we use to model scattering of the Herglotz wave $H\phi$ from the periodic surface Γ. As above, $\phi\in L^2_{\cos}(-\pi/2,\pi/2)$ and the surface $\Gamma=\{(x_1,\zeta(x_1))^\top, x_1\in\mathbb{R}\}$ is given as the graph of an L-periodic and Lipschitz continuous function $\zeta:\mathbb{R}\to\mathbb{R}$. We denote the domain above Γ by

$$\Omega = \Big\{x=(x_1,x_2)^\top\in\mathbb{R}^2,\ \zeta(x_1)<x_2\Big\}$$

and set

$$\Omega_h = \Big\{x=(x_1,x_2)^\top\in\mathbb{R}^2,\ \zeta(x_1)<x_2<h\Big\} \quad\text{and}\quad \Gamma_h = \Big\{x=(x_1,x_2)^\top\in\mathbb{R}^2,\ x_2=h\Big\}$$

for $h>\zeta_+=\|\zeta\|_{L^\infty(\mathbb{R})}$. We will frequently identify Γ_h with the real line, writing $\hat{f}(\xi)$ or $\hat{f}(\xi,h)$ for the Fourier transform in x_1 of a function $f:\Gamma_h\to\mathbb{C}$.

The Dirichlet scattering problem we consider is to find a weak solution

$$u\in\mathcal{H}(\Omega) =: \big\{u\in H^1_{\mathrm{loc}}(\Omega) \text{ and } u\in H^1(\Omega_h) \text{ for all } h>\zeta_+\big\}$$

that satisfies the following Dirichlet problem for the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad\text{in } L^2(\Omega), \qquad u|_\Gamma = H\phi \quad\text{in } H^{1/2}(\Gamma). \tag{22}$$

Moreover, u is required to satisfy a radiation condition in form of the angular spectrum representation,

$$u(x_1,x_2) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\big(\mathrm{i}\xi x_1 + \mathrm{i}\sqrt{k^2-\xi^2}(x_2-h)\big)\widehat{(u|_{\Gamma_h})}(\xi)\,\mathrm{d}\xi\,, \qquad x_2\ge h. \tag{23}$$

Due to [7] we know that the latter problem possesses a unique solution.

Theorem 10. *For any $\phi \in L^2_{\cos}(-\pi/2,\pi/2)$ there exists a variational solution $u \in \mathcal{H}(\Omega)$ to (22–23). Further, for any $h > \zeta_+ = \|\zeta\|_{L^\infty(\mathbb{R})}$ there is $C = C(h) > 0$ such that*

$$\|u\|_{H^1(\Omega_h)} \leq C\|\phi\|_{L^2_{\cos}(-\pi/2,\pi/2)}.$$

Proof. Setting $f = H\phi \in H^{1/2}(\Gamma)$, the above scattering problem can be reformulated in the subspace $V_0 := \{u \in H^1(\Omega_h),\, u|_\Gamma = 0\}$ of $H^1(\Omega_h)$, $h > \zeta_+ = \|\zeta\|_{L^\infty(\mathbb{R})}$. To this end, we first use an extension F of f such that $F \in H^1(\Omega_{\zeta_+})$, $F|_\Gamma = f$, and such that the trace of F on Γ_{ζ_+} vanishes. This extension exists due to, e.g., [24, Th. 3.37]. We extend F by zero into $\Omega \setminus \Omega_{\zeta_+}$. Obviously, the restriction of this extension to Ω_h belongs to $H^1(\Omega_h)$ for all $h \geq \zeta_+$. Second, we employ the exterior Dirichlet-to-Neumann operator $T_h : H^{1/2}(\Gamma_h) \to H^{-1/2}(\Gamma_h)$ on Γ_h defined by

$$(T_h\phi)(x_1) = \frac{\mathrm{i}}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{k^2 - \xi^2}\, \exp(\mathrm{i}\xi x_1)\widehat{(u|_{\Gamma_h})}(\xi)\, \mathrm{d}\xi\,, \quad x_1 \in \mathbb{R}.$$

Then (22–23) can be formulated variationally as follows: We seek u in $H^1(\Omega_h)$ for $h > \zeta_+$ in the form $u = w + F$ where $w \in V_0$ is afterwards extended via the angular spectrum representation (23) to all of Ω in order to obtain a solution in $\mathcal{H}(\Omega)$. Since u must solve (22–23), the variational problem for $w \in V_0$ reads

$$a(w,v) := \int_{\Omega_h} \left(\nabla w \cdot \nabla \overline{v} - k^2 w\overline{v}\right) \mathrm{d}x - \int_{\Gamma_h} \overline{v} T_h(w)\, \mathrm{d}S \overset{!}{=} \int_{\Omega_h} \left(\nabla F \cdot \nabla \overline{v} - k^2 F\overline{v}\right) \mathrm{d}x \quad \text{for all } v \in V_0. \tag{24}$$

From [7] we know that the sesquilinear form a satisfies an inf-sup condition. Hence, a solution w to the latter variational problem exists and satisfies $\|w\|_{H^1(\Omega_h)} \leq C\|F\|_{H^1(\Omega_h)} \leq C\|f\|_{H^{1/2}(\Gamma)}$. Note that [7, Remark 2.1] ensures that after extension to Ω by (23) the solution u is independent of the value h chosen to define V_0. Theorem 7 now implies that $\|f\|_{H^{1/2}(\Gamma)} = \|H\phi\|_{H^{1/2}(\Gamma)} \leq C\|\phi\|_{L^2_{\cos}(-\pi/2,\pi/2)}$. □

As for functions defined on the surface Γ, we analyze the solution u to (22–23) using a Bloch transform. This transform is denoted by $\mathcal{J}_\Omega$ and defined by

$$\mathcal{J}_\Omega\phi(\alpha;x) := \sqrt{\frac{L}{2\pi}} \sum_{j\in\mathbb{Z}} \phi(x_1 + Lj, x_2) e^{-\mathrm{i}\alpha(x_1+Lj)}, \qquad x = (x_1,x_2)^\top \in \Omega,\ -\pi/L < \alpha \leq \pi/L, \tag{25}$$

for smooth functions $\phi : \Omega \to \mathbb{C}$ with compact support in $\overline{\Omega}$. Note that this implies in particular that

$$\mathcal{J}_\Omega u(\alpha;x) = \mathcal{J}_\Gamma\left(u|_\Gamma\right)(\alpha;x) \qquad \text{for } x \in \Gamma. \tag{26}$$

In the next theorem, we show that the Bloch transform $\mathcal{J}_\Omega$ extends to a transform between certain Sobolev spaces, in the same way as the transforms $\mathcal{J}_\mathbb{R}$ and $\mathcal{J}_\Gamma$ in Section 2. This result will then be used to analyze the solution u to (22–23). Before stating the mapping properties of $\mathcal{J}_\Omega$, we introduce the domain

$$\Omega_h^{\mathrm{p}} = \{x \in \Omega,\, x_1 \in (-L/2,L/2),\, x_2 < h\}\,, \qquad \text{for } h > \zeta_+ = \|\zeta\|_{L^\infty(\mathbb{R})}. \tag{27}$$

The boundary part of Ω_h^{p} that intersects Γ is denoted by

$$\Gamma_{\mathrm{p}} = \{x \in \Gamma,\, x_1 \in (-L/2,L/2)\}\,. \tag{28}$$

We define the periodic Sobolev spaces $H^n_{\mathrm{p}}(\Omega_h)$ of L-periodic functions in x_1 in the usual way:

$$H^n_{\mathrm{p}}(\Omega_h) = \{u \in H^n_{\mathrm{loc}}(\Omega_h),\ u \text{ is } L\text{-periodic}\} \qquad \text{for } n \in \mathbb{N}_0, \tag{29}$$

and equip this space with the norm $u \mapsto \|u\|_{H^n(\Omega_h^{\mathrm{p}})}$, that is, with the usual H^n-norm over one period in x_1. (E.g., $\|u\|^2_{H^1(\Omega_h^{\mathrm{p}})} = \int_{\Omega_h^{\mathrm{p}}} \left(|\nabla u|^2 + |u|^2\right) \mathrm{d}x$ for $n = 1$). An L-periodic function hence belongs to $H^n_{\mathrm{p}}(\Omega_h)$ if its $H^n(\Omega_h^p)$-norm is finite. For $s > 0$, the intermediate spaces $H^s_{\mathrm{p}}(\Omega_h)$ are then defined by interpolation, see [24, Ch. 3 & App. B]. Note that this definition is consistent with the one of $H^s_{\mathrm{p}}(\mathbb{R})$.

Theorem 11. *The Bloch transform $\mathcal{J}_\Omega$ extends to an isomorphism between $H^s(\Omega_h)$ and $L^2((-\pi/L,\pi/L);H^s_{\mathrm{p}}(\Omega_h))$ for all $s\in[0,1]$ and $h>h_0:=\zeta_++3\zeta_-$.*
(b) The inverse transform to $\mathcal{J}_\Omega$ is given by

$$\left(\mathcal{J}_\Omega^{-1}\tilde{\phi}\right)(x)=\sqrt{\frac{L}{2\pi}}\int_{-\pi/L}^{\pi/L}\tilde{\phi}(\alpha;x)\exp(\mathrm{i}\alpha x_1)\,\mathrm{d}\alpha\,,\qquad x=(x_1,x_2)^\top\in\Omega, \tag{30}$$

and defines an isomorphism between $L^2((-\pi/L,\pi/L);H^s_{\mathrm{p}}(\Omega_h))$ and $H^s(\Omega_h)$ for all $s\in[0,1]$ and $h>h_0$.

Proof. (a) The proof relies on the diffeomorphisms Ψ and Θ between Ω_h and $U_h=\{y=(y_1,y_2)^\top\in\mathbb{R}^2,\ 0<y_2<h\}$ for $h>h_0=\zeta_++3\zeta_-$, constructed in Appendix A and their mapping properties, see Proposition 15. Recall that $u\mapsto u\circ\Psi$ and $v\mapsto v\circ\Theta$ are isomorphisms between $H^1(\Omega_h)$ and $H^1(U_h)$ for $h>h_0$. Hence, it is sufficient to show the mapping properties of the Bloch transform by transporting functions from Ω_h to U_h and vice versa, afterwards relying on mapping properties of the Bloch transform $\mathcal{J}_{\mathbb{R}^2_+}$, defined by

$$\mathcal{J}_{\mathbb{R}^2_+}\phi(\alpha;y):=\sqrt{\frac{L}{2\pi}}\sum_{j\in\mathbb{Z}}\phi(y_1+Lj,y_2)e^{-\mathrm{i}\alpha(y_1+Lj)},\qquad y=(y_1,y_2)^\top\in\mathbb{R}^2_+,\ -\pi/L<\alpha\le\pi/L, \tag{31}$$

for smooth functions ϕ with compact support in $\overline{\mathbb{R}^2_+}$. Note that $\mathcal{J}_{\mathbb{R}^2_+}\phi(\alpha;\cdot)$ is by definition L-periodic in y_1. Hence, knowing $\mathcal{J}_{\mathbb{R}^2_+}\phi(\alpha;y)$ for y in the closure of Ω_h^{p} is sufficient to know $\mathcal{J}_{\mathbb{R}^2_+}\phi(\alpha;y)$ for all $y\in\Omega_h$.

The Bloch transform $\mathcal{J}_{\mathbb{R}^2_+}$ extends to an isomorphism between $H^s(U_h)$ and $L^2((-\pi/L,\pi/L);H^s_{\mathrm{p}}(U^h_{\mathrm{p}}))$ for $s\in[0,1]$ and $h>h_0$. For $s=0$ this follows by interpreting $L^2(U_h)=L^2((0,h);L^2(\mathbb{R}))$ as an L^2-space on $(0,h)$ with values in $L^2(\mathbb{R})$: Since $\mathcal{J}_{\mathbb{R}^2_+}$ does not act on y_2, applying Theorem 1 with $s=0$ yields

$$\begin{aligned}\|\mathcal{J}_{\mathbb{R}^2_+}\phi(\alpha;y)\|^2_{L^2((-\pi/L,\pi/L);L^2_{\mathrm{p}}(U^h_{\mathrm{p}}))}&=\int_{-\pi/L}^{\pi/L}\int_0^h\|\mathcal{J}_{\mathbb{R}^2_+}\phi(\alpha;(\cdot,y_2))\|^2_{L^2(-L/2,L/2)}\,\mathrm{d}y_2\,\mathrm{d}\alpha\\&=\int_0^h\|\phi(\cdot,y_2)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}y_2\;=\|\phi\|^2_{L^2(U_h)}.\end{aligned}$$

The proof for $s=1$ is analogous, since derivatives in x_2 interchange with the Bloch transform (see also the proof of the subsequent Theorem 12). The case $s\in(0,1)$ can then be treated by interpolation, see [24, Ch. 3 & App. B].

Next, we consider $\phi\in H^s(\Omega_h)$ for $s\in[0,1]$ and note that

$$\mathcal{J}_\Omega\phi(\alpha,x)=\left[\mathcal{J}_{\mathbb{R}^2_+}(\phi\circ\Psi)\right](\alpha,\Theta(x))\qquad\text{for }x\in\Omega,\ \alpha\in(-L/2,L/2). \tag{32}$$

Since $\phi\circ\Psi\in H^1(U_h)$, the Bloch transform $\mathcal{J}_{\mathbb{R}^2_+}(\phi\circ\Psi)$ belongs to $L^2((-L/2,L/2);H^s_{\mathrm{p}}(U_h))$, that is, $(\alpha,x)\mapsto\left[\mathcal{J}_{\mathbb{R}^2_+}(\phi\circ\Psi)\right](\alpha,\Theta(x))$ belongs to $L^2((-L/2,L/2);H^s_{\mathrm{p}}(\Omega_h))$. This shows part (a).

(b) For part (b) we use again the inversion formula for the one-dimensional Bloch transform $\mathcal{J}_{\mathbb{R}}$ from Theorem 1. Assume that $\tilde{\psi}=\mathcal{J}_{\mathbb{R}^2_+}\psi$ for $\psi\in H^s(\Omega_h)$. Then $\psi(\cdot,x_2)=\mathcal{J}_{\mathbb{R}}^{-1}\psi(\cdot,x_2)$ for $x_2\in(0,h)$, that is,

$$\psi(x)=\sqrt{\frac{L}{2\pi}}\int_{-\pi/L}^{\pi/L}\tilde{\psi}(\alpha;x)\exp(\mathrm{i}\alpha x_1)\,\mathrm{d}\alpha\;=:\left(\mathcal{J}_{\mathbb{R}^2_+}\tilde{\psi}\right)(x))$$

for $x\in U_h$ and $\alpha\in(-L/2,L/2)$. Due to (32), the inverse transform to $\mathcal{J}_\Omega$ is hence given by (30): For $\phi\in H^s(\Omega_h)$ and $\tilde{\phi}=\mathcal{J}_\Omega\phi$ it holds that

$$\begin{aligned}\mathcal{J}_\Omega^{-1}\tilde{\phi}(\alpha,x)&=\left(\mathcal{J}_{\mathbb{R}^2_+}^{-1}(\tilde{\phi}(\alpha,\Theta(\cdot)))\right)(\Psi(x))=\sqrt{\frac{L}{2\pi}}\int_{-\pi/L}^{\pi/L}(\tilde{\phi}(\alpha,\Psi(\Theta(x))))\exp(\mathrm{i}\alpha x_1)\,\mathrm{d}\alpha\\&=\sqrt{\frac{L}{2\pi}}\int_{-\pi/L}^{\pi/L}\tilde{\phi}(\alpha,x)\exp(\mathrm{i}\alpha x_1)\,\mathrm{d}\alpha\qquad\text{for }x\in\Omega_h\text{ and }\alpha\in(-L/2,L/2).\end{aligned}$$

□

As announced above, we can now prove that the restriction of the Herglotz wave function

$$v_\phi(x) = \int_{-\pi}^{\pi} e^{\mathrm{i}k(\sin\theta\, x_1 - \cos\theta\, x_2)} \phi(\theta)\,\mathrm{d}\theta\,, \qquad x \in \mathbb{R}^2,$$

defines a bounded linear operator $H_U : \phi \mapsto v_\phi|_{U_h}$ from $L^2_{\cos}(-\pi/2, \pi/2)$ into $H^1(U_h)$ for any strip $U_h = \{y = (y_1, y_2)^\top \in \mathbb{R}^2,\, 0 < y_2 < h\}$.

Theorem 12. *The operator H_U is bounded from $L^2_{\cos}(-\pi/2, \pi/2)$ into $H^s(U_h)$ for any height $h > 0$ and any $s \in \mathbb{R}$.*

Proof. We abbreviate $\|\cdot\|_{L^2_{\cos}(-\pi/2,\pi/2)}$ in the entire proof by $\|\cdot\|$ and we first show the claimed result for $s = 0$. To this end, we use again that the L^2-norm of $H_U\phi = v_\phi|_{U_h}$ can be expressed as $\|v_\phi\|^2_{L^2(U_h)} = \int_0^h \|v_\phi(\cdot, t)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}t$. Theorem 7 applied to $\{x_2 = t\}$ instead of Γ states that $\|v_\phi(\cdot, t)\|_{L^2(\mathbb{R})} \le C\|\phi\|$ with C independent of t. Hence follows the claimed norm bound for $s = 0$.

Next we consider the case $s = 1$. Since we merely need to bound the first weak derivatives of v_ϕ, we use again Theorem 7 to obtain that

$$\left\| \frac{\partial v_\phi}{\partial x_1} \right\|^2_{L^2(U_h)} = \int_0^h \left\| \frac{\partial v_\phi}{\partial x_1}(\cdot, t) \right\|^2_{L^2(\mathbb{R})} \mathrm{d}x_2 \le \int_0^h \|v_\phi(\cdot, t)\|^2_{H^1(\mathbb{R})}\,\mathrm{d}x_2 \le C\|\phi\|^2.$$

To bound the partial derivative with respect to x_2 we rely on the Bloch transform $\mathcal{J}_{\mathbb{R}^2_+}$ defined in 31 (and, by abuse of notation, apply $\mathcal{J}_{\mathbb{R}^2_+}$ also to functions that are merely defined in U_h). Note that $\mathcal{J}_{\mathbb{R}^2_+}$ commutes by definition with derivatives with respect to x_2,

$$\mathcal{J}_{\mathbb{R}^2_+}\left(\frac{\partial u}{\partial x_2}\right)(\alpha; x) = \frac{\partial}{\partial x_2}\mathcal{J}_{\mathbb{R}^2_+} u(\alpha; x), \qquad x \in \mathbb{R}^2_+,\ \alpha \in (-\pi/L, \pi/L].$$

Since $\mathcal{J}_{\mathbb{R}^2_+}$ is for fixed x_2 simply a one-dimensional Bloch transform in x_1, Theorem 6 implies that

$$\mathcal{J}_{\mathbb{R}^2_+}(H_U\phi)(\alpha; x) = \sqrt{\frac{2\pi}{L}} \sum_{j\in\mathbb{Z}} e^{\frac{2\pi \mathrm{i} j}{L}x_1 - \mathrm{i}\sqrt{k^2 - \left(\alpha + \frac{2\pi j}{L}\right)^2}\, x_2}\ \frac{\phi\big(\arcsin\big(\alpha/k + \frac{2\pi j}{kL}\big)\big)}{\sqrt{k^2 - \left(\alpha + \frac{2\pi j}{L}\right)^2}}\ \mathbb{1}_{\left\{\left|\alpha + \frac{2\pi j}{L}\right| < k\right\}},$$

that is,

$$\frac{\partial}{\partial x_2}\mathcal{J}_{\mathbb{R}^2_+}(H_U\phi)(\alpha; x) = -\mathrm{i}\sqrt{\frac{2\pi}{L}} \sum_{j\in\mathbb{Z}} e^{\frac{2\pi \mathrm{i} j}{L}x_1 - \mathrm{i}\sqrt{k^2 - \left(\alpha + \frac{2\pi j}{L}\right)^2}\, x_2}\ \phi\big(\arcsin\big(\alpha/k + \tfrac{2\pi j}{kL}\big)\big)\mathbb{1}_{\left\{\left|\alpha + \frac{2\pi j}{L}\right| < k\right\}},$$

It is now obvious that the second derivative of $\mathcal{J}_{\mathbb{R}^2_+}(H_U\phi)$ is bounded in $L^2(-\pi/L, \pi/L; H^1_{\mathrm{p}}(\{x_2 = t\}))$, that is, $H_U\phi(\cdot, t)$ is bounded in $H^1(\mathbb{R})$ by $C\|\phi\|$ with a constant C that is uniform in t. This bound then yields that $\|v_\phi(\cdot, x_2)\|_{H^1(U_h)} \le C\|\phi\|$. The corresponding bound for $s \in \mathbb{N}$ follows analogously; intermediate values $s > 0$ are then treated using an interpolation argument. □

Note that the last Theorem 12 implies that for any Lipschitz continuous surface Γ contained in a strip U_h the mapping $\phi \mapsto v_\phi|_\Gamma$ is bounded from $L^2_{\cos}(-\pi/2, \pi/2)$ into $H^{1/2}(\Gamma)$. Of course, such surfaces are neither required to be the graph of a function not to be periodic.

5 Periodic Scattering Problems and Equivalences

Now we show that the Bloch transform $\mathcal{J}_\Omega$ of a solution to the Helmholtz equation (22) yields a periodic solution to a (shifted) Helmholtz equation in Ω. Vice versa, a family of periodic solutions to this (shifted) Helmholtz equation yields a solution to (22) via an inverse Bloch transform. To state this result, we first need to introduce tools for the variational formulation of the periodic Helmholtz equation in Ω^{p}_h.

Define, for $\alpha \in (-\pi/L, \pi/L]$, the shifted differential operators

$$\nabla_\alpha f = \nabla f + \begin{pmatrix} \mathrm{i}\alpha f \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathrm{div}_\alpha F = \mathrm{div}\, F + \mathrm{i}\alpha F^{(1)}$$

for scalar functions f and vector fields $F = (F^{(1)}, F^{(2)})^\top$ and consider some function $f_{\mathrm{p}} \in H^{1/2}_{\mathrm{p}}(\Gamma)$. Then the weak formulation of the shifted Helmholtz equation is to find a solution u in the periodic Fréchet space

$$\mathcal{H}_{\mathrm{p}}(\Omega) := \{u \in H^1_{\mathrm{loc}}(\Omega),\ u \text{ is } L\text{-periodic, and } u \in H^1_{\mathrm{p}}(\Omega_h) \text{ for all } h > \zeta_+\}$$

to the problem

$$\mathrm{div}_\alpha \nabla_\alpha u + k^2 u = \Delta u + 2\mathrm{i}\alpha \frac{\partial u}{\partial x_1} + (k^2 - \alpha^2) u = 0 \quad \text{in } L^2(\Omega), \qquad u|_\Gamma = f_{\mathrm{p}} \quad \text{in } H^{1/2}_{\mathrm{p}}(\Gamma), \tag{33}$$

such that u satisfies the radiation condition

$$u(x) = \sum_{j \in \mathbb{Z}} \hat{u}_j e^{\frac{2\pi \mathrm{i} j}{L} x_1 + \mathrm{i}\beta_j (x_2 - h)} \quad \text{for } x_2 > h, \text{ with } \beta_j = \begin{cases} \sqrt{k^2 - |2\pi j/L + \alpha|^2} & k^2 \geq |2\pi j/L + \alpha|^2, \\ \mathrm{i}\sqrt{|2\pi j/L + \alpha|^2 - k^2} & \text{else.} \end{cases} \tag{34}$$

The numbers $\hat{u}_j = \widehat{(u|_{\Gamma_h})}(j)$ are the so-called Rayleigh coefficients of u and given as the Fourier coefficients of the restriction $u|_{\Gamma_h}$. As for Γ_h we introduce the space $V_0^{\mathrm{p}} = \{u \in H^1_{\mathrm{p}}(\Omega_h),\ u|_\Gamma = 0\}$ and the operators

$$T^h_{\mathrm{p}} : H^{1/2}_{\mathrm{p}}(\Gamma_h) \to H^{-1/2}_{\mathrm{p}}(\Gamma_h), \qquad \phi \mapsto \mathrm{i} \sum_{j \in \mathbb{Z}} \beta_j \hat{\phi}(j) \exp(\frac{2\pi \mathrm{i}}{L} j x_1),$$

where $\hat{\phi}(j)$ are the Fourier coefficients of ϕ defined in (6). These definitions allow to derive the following variational formulation for (33–34),

$$a^\alpha_{\mathrm{p}}(u, v) := \int_{\Omega^{\mathrm{p}}_h} \left(\nabla_\alpha u \cdot \nabla_\alpha \overline{v} - k^2 u \overline{v}\right) \mathrm{d}x \; - \int_{-L/2}^{L/2} \overline{v}(\cdot, h) T^h_{\mathrm{p}}(u(\cdot, h))\, \mathrm{d}x_1 \; = 0 \qquad \text{for all } v \in V_0^{\mathrm{p}}. \tag{35}$$

If the surface Γ is given as the graph of a Lipschitz continuous function ζ, then it is well-known that existence and uniqueness of solution to this variational problem holds for all $k > 0$. Such results go back to [6, 11, 12, 19]. Moreover, for fixed k, the solution operator to the above variational problem is uniformly bounded for all α in $(-\pi/L, \pi/L]$.

Theorem 13. *(a) The solution $u = u(\alpha) \in H^1_{\mathrm{p}}(\Omega_h)$ to (35) exists for all $k > 0$ and $\alpha \in (-\pi/L, \pi/L]$. For fixed k there is $C = C(k) > 0$ such that $\|u(\alpha)\|_{H^1_{\mathrm{p}}(\Omega_h)} \leq C(k) \|f_p\|_{H^{1/2}_{\mathrm{p}}(\Gamma)}$, uniformly in $\alpha \in (-\pi/L, \pi/L]$.*

(b) A function $u \in H^1(\Omega_h)$ is a variational solution to (22–23) for boundary data $f \in H^{1/2}(\Gamma)$ if and only if $\mathcal{J}_\Omega u(\alpha, \cdot)$ is for (almost) every $\alpha \in (-\pi/L, \pi/L)$ a variational solution in $H^1_{\mathrm{p}}(\Omega_h)$ to (35) for boundary data $\mathcal{J}_\Gamma f(\alpha; \cdot) \in H^{1/2}_{\mathrm{p}}(\Gamma)$. The Rayleigh coefficients of $\mathcal{J}_\Omega u(\alpha; \cdot)$ are given by the Fourier transform $\hat{u}(\xi, h)/\sqrt{L}$ of u, evaluated at the points $\alpha + 2\pi j/L$.

Proof. (a) It is well-known that the sesquilinear form a^α_{p} defines a Fredholm operator of index zero. Hence, for fixed $k > 0$ and $\alpha \in (-\pi/L, \pi/L)$ one merely needs to check uniqueness of solution to (35). Since the uniqueness result for Lipschitz surfaces is well-known due to [12], we only sketch a proof under the additional assumption that the surface Γ is of class $C^{1,1}$. Under this assumption, any solution u to the homogeneous problem belongs by elliptic regularity results to $H^2_{\mathrm{p}}(\Omega_h)$. We extend u by (34) to a function in $\mathcal{H}_{\mathrm{p}}(\Omega)$; this extension then also belongs to $H^2_{\mathrm{p}}(\Omega_h)$ for all $h > \zeta_+$.

By checking the imaginary part of $a^\alpha_{\mathrm{p}}(u, u)$ for a solution to the homogeneous problem with $f = 0$, one notes that all propagating modes of such a solution must vanish. Hence u and all of its first and second partial derivatives decay exponentially. Multiplying the shifted Helmholtz equation by $\partial\overline{u}/\partial x_2$ and integrating by parts, one finds that

$$\begin{aligned}
0 &= \int_U \left[\operatorname{div}_\alpha \nabla_\alpha u \frac{\partial \overline{u}}{\partial x_2} + k^2 u \frac{\partial \overline{u}}{\partial x_2}\right] \mathrm{d}x \\
&= -\frac{1}{2}\int_U \frac{\partial}{\partial x_2}\left[|\nabla_\alpha u|^2 - k^2|u|^2\right] \mathrm{d}x - \int_{\Gamma_\mathrm{p}} \left[\nu + i\alpha(1,0)^\top\right]\cdot \nabla_\alpha u \frac{\partial \overline{u}}{\partial x_2}\,\mathrm{d}S \\
&= -\frac{1}{2}\int_{\Gamma_\mathrm{p}} \nu_2 \left[|\nabla_\alpha u|^2 - k^2|u|^2\right] \mathrm{d}x - \int_{\Gamma_\mathrm{p}} \left[\nu\cdot\nabla u + \mathrm{i}\alpha\frac{\partial u}{\partial x_1} - \alpha^2 u + \mathrm{i}\nu_1\alpha u\right]\frac{\partial \overline{u}}{\partial x_2}\,\mathrm{d}S\,,
\end{aligned}$$

with $U = \{-L/2 < x_1 < L/2,\, x_2 > \zeta(x_1)\}$ and the upwards pointing unit normal ν on Γ. Since u vanishes on Γ due to the homogeneous Dirichlet boundary condition we find that

$$0 = \frac{1}{2}\int_{\Gamma_\mathrm{p}} \nu_2 |\nabla_\alpha u|^2\,\mathrm{d}x + \int_{\Gamma_\mathrm{p}} \left[(\nu_1 + \mathrm{i}\alpha)\frac{\partial u}{\partial x_1} + \nu_2\frac{\partial u}{\partial x_2}\right]\frac{\partial \overline{u}}{\partial x_2}\,\mathrm{d}S\,,$$

moreover, it holds that $\nabla u = \nu\cdot(\partial u/\partial\nu)$ on Γ and $\partial u/\partial x_j = e_j\cdot\nu(\partial u/\partial\nu) = \nu_j(\partial u/\partial\nu) \in H_\mathrm{p}^{1/2}(\Gamma)$. In particular, $0 = \int_{\Gamma_\mathrm{p}} \nu_2\left[|\nabla_\alpha u|^2 + 2\,|\partial u/\partial\nu)|^2\right]\mathrm{d}x$. Since $\nu_2 > 0$ on $\Gamma = \{(x_2, \zeta(x_2))^\top,\, x_2\in\mathbb{R}\} \supset \Gamma_\mathrm{p}$ this is only possible if $\partial u/\partial\nu$ vanishes. Hence, Holmgren's lemma implies that u vanishes entirely.

To show that the bounds $C(k\alpha)$ for the solution operators are uniform in α we show that the sesquilinear form a_p^α depends continuously on α. For $-\pi/L \le \alpha, \tilde\alpha \le \pi/L$ it holds that

$$\begin{aligned}
\left|a_\mathrm{p}^\alpha(u,v) - a_\mathrm{p}^{\tilde\alpha}(u,v)\right| &\le \frac{2\pi}{L}|\alpha-\tilde\alpha|\,\|u\|_{H_\mathrm{p}^1(\Omega_h)}\|v\|_{H_\mathrm{p}^1(\Omega_h)} \\
&\quad + L\underbrace{\left|\sum_{j\in\mathbb{Z}}\left(\sqrt{k^2-(\alpha+2\pi j/L)^2} - \sqrt{k^2-(\tilde\alpha+2\pi j/L)^2}\right)\widehat{(u|_{\Gamma_h^\mathrm{p}})}\widehat{(v|_{\Gamma_h^\mathrm{p}})}\right|}_{(*)}
\end{aligned}$$

and the first term on the right-hand side obviously tends to zero as $\tilde\alpha\to\alpha$. Further,

$$(*) \le C\left[\sup_{j\in\mathbb{Z}} \frac{\left|\sqrt{k^2-(\alpha+2\pi j/L)^2} - \sqrt{k^2-(\tilde\alpha+2\pi j/L)^2}\right|}{(1+j^2)^{1/2}}\right]\|u\|_{H_\mathrm{p}^1(\Omega_h}\|v\|_{H_\mathrm{p}^1(\Omega_h}.$$

For $j=0$ the term over which the supremum is taken equals $|(k^2-(\alpha+2\pi j/L)^2)^{1/2} - (k^2-(\tilde\alpha+2\pi j/L)^2)^{1/2}|$ and tends to zero as $\tilde\alpha\to\alpha$. Further, if for some $0\neq j^*\in\mathbb{Z}$ it holds that $k^2 = (\alpha+2\pi j^*/L)^2$, then $|(k^2-(\tilde\alpha+2\pi j^*/L)^2)^{1/2}|/j^* \to 0$ as $\tilde\alpha\to\alpha$. It is hence sufficient to show that $\sup_{j\notin\{0,j^*\}}|f(\alpha,\tilde\alpha)| \to 0$ as $\tilde\alpha\to\alpha$ where

$$f(\alpha,\tilde\alpha) = \frac{1}{j}\frac{|\tilde\alpha-\alpha|\,[|\tilde\alpha+\alpha| + 4\pi j/L]}{|\sqrt{k^2-(\alpha+2\pi j/L)^2} + \sqrt{k^2-(\tilde\alpha+2\pi j/L)^2}|}.$$

For $j\neq j^*$ there is $\delta>0$ such that all terms $(k^2-(\alpha+2\pi j/L)^2)^{1/2}$ have magnitude larger than δ (since none of these terms vanishes and since they grow as j for $|j|\to\infty$). Since $\sqrt{k^2-(\alpha+2\pi j/L)^2}$ is either real and positive, zero, or purely imaginary with positive imaginary part, it follows that the denominator in the definition of $f(\alpha,\tilde\alpha)$ is bounded below in magnitude by $\delta/\sqrt{2}$. We can hence estimate

$$\sup_{j\notin\{0,j^*\}} |f(\alpha,\tilde\alpha)| \le \frac{\sqrt{2}}{\delta}|\tilde\alpha-\alpha|\,[|\tilde\alpha+\alpha| + 4\pi/L] \to 0 \quad \text{as } \tilde\alpha\to\alpha.$$

Hence, $\left|a_\mathrm{p}^\alpha(u,v) - a_\mathrm{p}^{\tilde\alpha}(u,v)\right| \le C\,|\alpha-\tilde\alpha|\,\|u\|_{H_\mathrm{p}^1(\Omega_h)}\|v\|_{H_\mathrm{p}^1(\Omega_h)}$ for some constant C independent of α and $\tilde\alpha$. Now, Strang's lemma (see, e.g., [28, Th. 4.2.11]) implies that the solution operator to (35) is continuous in α. Since this operator is also pointwise bounded in α, the compactness of $[-\pi/L, \pi/L]$ then implies that $C(k,\alpha) \le C(k)$ for $\alpha\in[-\pi/L,\pi/L]$.

(b) Assume that $u \in H^1(\Omega_h)$ is a variational solution to (22–23) for boundary data $f \in H^{1/2}(\Gamma)$. It is obvious that for a smooth function $v \in C^\infty(\overline{\Omega_h})$ with compact support the relations

$$\mathcal{J}_\Omega \left(\frac{\partial v}{\partial x_1} \right) = \left(\frac{\partial}{\partial x_1} - \mathrm{i}\alpha \right) \mathcal{J}_\Omega v \qquad \text{and} \qquad \mathcal{J}_\Omega \left(\frac{\partial v}{\partial x_2} \right) = \mathcal{J}_\Omega v \tag{36}$$

hold. By density of such functions in $H^1(\Omega_h)$, these relations extend to all $v \in H^1(\Omega_h)$. Since u satisfies the Helmholtz equation in the weak sense it follows that the x_1-periodic function $x \mapsto w(\alpha; x) = \mathcal{J}_\Omega u(\alpha; \cdot)$ satisfies the shifted Helmholtz equation (33) in the weak sense. It is also clear that the boundary condition $u|_\Gamma = f$ satisfied in $H^{1/2}(\Gamma)$ transforms due to (26) into $w(\alpha; \cdot)|_\Gamma = \mathcal{J}_\Gamma f(\alpha; \cdot)$ in $H_\mathrm{p}^{1/2}(\Gamma)$. We finally need to check the radiation condition (34) satisfied by the periodic solutions to 33. To this end, we use the angular spectrum representation (23), yielding that

$$\begin{aligned}
\mathcal{J}_\Omega u \left(\alpha; (x_1, x_2)^\top \right) &= \frac{1}{\sqrt{2\pi L}} \sum_{j \in \mathbb{Z}} \mathcal{F} \left[x_1 \mapsto \int_{\Gamma_h} e^{\mathrm{i}\xi x_1 + \mathrm{i}\sqrt{k^2 - \xi^2}(x_2 - h)} \hat{u}(\xi, h) \,\mathrm{d}\xi \right] \left(\alpha + \tfrac{2\pi j}{L}, x_2 \right) \exp\left(\tfrac{2\pi \mathrm{i}}{L} j x_1 \right) \\
&= \frac{1}{\sqrt{L}} \sum_{j \in \mathbb{Z}} e^{\frac{2\pi \mathrm{i}}{L} j x_1 + \mathrm{i}\sqrt{k^2 - (\alpha + \frac{2\pi j}{L})^2}(x_2 - h)} \; \hat{u} \left(\alpha + \frac{2\pi j}{L}, x_2 \right) \\
&= \frac{1}{\sqrt{L}} \sum_{j \in \mathbb{Z}} \hat{u} \left(\alpha + \frac{2\pi j}{L}, h \right) e^{\frac{2\pi \mathrm{i}}{L} j x_1 + \mathrm{i}\beta_j (x_2 - h)} \qquad \text{for } x_2 > h.
\end{aligned}$$

The Rayleigh coefficients of $\mathcal{J}_\Omega u(\alpha; \cdot)$ can hence be expressed via $\hat{u}(\xi, h)$ evaluated at $\xi = \alpha + 2\pi j / L$.

If $w(\alpha, \cdot)$ is a function in $\mathcal{D}'((-\pi/L, \pi/L); H_\mathrm{p}^1(\Omega_h))$ that solves for (almost) every $\alpha \in (-\pi/L, \pi/L)$ the variational problem (35) for boundary data $\mathcal{J}_\Gamma f(\alpha; \cdot) \in H_\mathrm{p}^{1/2}(\Gamma)$, then the uniform boundedness of the solution operator to this problem established in part (a) of this theorem implies that

$$\|w\|^2_{L^2((-\pi/L, \pi/L); H_\mathrm{p}^1(\Omega_h))} = \int_{-\pi/L}^{\pi/L} \|w(\alpha; \cdot)\|^2_{H_\mathrm{p}^1(\Omega_h)} \,\mathrm{d}\alpha \le C \int_{-\pi/L}^{\pi/L} \|\mathcal{J}_\Gamma f(\alpha; \cdot)\|^2_{H_\mathrm{p}^{1/2}(\Gamma)} \,\mathrm{d}\alpha \le C \|f\|^2_{H^{1/2}(\Gamma)}.$$

Due to (36), we know that the inverse Bloch transform $u = \mathcal{J}_\Omega^{-1} w$ satisfies the Helmholtz equation in Ω_h. As above, the boundary condition $u|_\Gamma = f$ in $H^{1/2}(\Gamma)$ is clear because of (26). It remains to show that u can be extended to a solution to the Helmholtz equation in the form (23). The representations (34) of the periodic solutions $w(\alpha, \cdot)$,

$$w(\alpha, x) = \sum_{j \in \mathbb{Z}} \hat{w}_j(\alpha) e^{\frac{2\pi \mathrm{i} j}{L} x_1 + \mathrm{i}\beta_j (x_2 - h)}, \qquad x_2 > h,$$

imply for $x_2 > h$ that holds

$$\begin{aligned}
u(x) = \mathcal{J}_\Omega^{-1} w(x) &= \sqrt{\frac{L}{2\pi}} \int_{-\pi/L}^{\pi/L} w(\alpha; x) \exp(\mathrm{i}\alpha x_1) \,\mathrm{d}\alpha \\
&= \sqrt{\frac{L}{2\pi}} \sum_{j \in \mathbb{Z}} \int_{-\pi/L}^{\pi/L} \hat{w}_j(\alpha) e^{\mathrm{i}(\frac{2\pi j}{L} + \alpha) x_1 + \mathrm{i}\sqrt{k^2 - (\alpha + \frac{2\pi j}{L})^2}(x_2 - h)} \,\mathrm{d}\alpha \\
&= \sqrt{\frac{L}{2\pi}} \sum_{j \in \mathbb{Z}} \int_{2\pi(j-1/2)/L}^{2\pi(j+1/2)/L} \hat{w}_j(\xi) e^{\mathrm{i}\xi x_1 + \mathrm{i}\sqrt{k^2 - \xi^2}(x_2 - h)} \,\mathrm{d}\xi = \sqrt{\frac{L}{2\pi}} \int_{\mathbb{R}} \hat{w}_j(\xi) e^{\mathrm{i}\xi x_1 + \mathrm{i}\sqrt{k^2 - \xi^2}(x_2 - h)} \,\mathrm{d}\xi\,.
\end{aligned}$$

□

The correspondence between the solution to the non-periodic problem (22–23) and the continuum of quasiperiodic problems (33–34) has a couple of consequences.

Corollary 14. *The Herglotz operator H is injective from $L^2_{\cos}(-\pi/2, \pi/2)$ into $H^s(\Gamma)$ for $s \in [1/2, 1]$ and any surface $\Gamma = \{(x_1, \zeta(x_1))^\top,\, x_1 \in \mathbb{R}\}$ given by the graph of a Lipschitz continuous function ζ.*

Proof. The injectivity of H is equivalent to the injectivity of H^- defined by changing the sign of the second coordinate, that is, $H^-\phi = \int_{\pi/2}^{3\pi/2} \exp(\mathrm{i}k(\sin\theta\, x_1 + \cos\theta\, \zeta(x_1)))\, \phi(\theta)\, \mathrm{d}\theta$. Note that changing the sign of the second coordinate does not affect any of the bounds shown above and all results shown for H also hold for H^-.

If $H^-\phi$ vanishes on Γ, then the Herglotz wave function v_ϕ defines a solution to the Helmholtz equation that belongs to $H^1(U_h)$ for all $h > 0$ by Theorem 12 and hence belongs to $\mathcal{H}(\Omega)$. Moreover, it is obvious that the Bloch transform of v_ϕ equals

$$\mathcal{J}_\Omega v_\phi\,(\alpha; x) = \sqrt{\frac{2\pi}{L}} \sum_{j\in\mathbb{Z}} e^{\frac{2\pi \mathrm{i} j}{L} x_1 + \mathrm{i}\sqrt{k^2 - \left(\alpha + \frac{2\pi j}{L}\right)^2}\, x_2}\ \frac{\phi\big(\arcsin\big(\alpha/k + \frac{2\pi j}{kL}\big)\big)}{\sqrt{k^2 - \big(\alpha + \frac{2\pi j}{L}\big)^2}}\ \mathbb{1}_{\left\{\left|\alpha + \frac{2\pi j}{L}\right| < k\right\}}$$

for $\alpha \in (-\pi/L, \pi/L]$ and $x \in \Omega$. This implies that each periodic component $\mathcal{J}_\Omega v_\phi\,(\alpha; \cdot)$ is an upwards radiating function that satisfies (34). Theorem 13 implies that v_ϕ itself is also upwards radiating, i.e., v_ϕ satisfies (23). We have hence shown that v_ϕ is a solution to the homogeneous Dirichlet problem (22–23). Due to Theorem 10, such a solution must vanish. Since v_ϕ is analytic (as any strong solution to the Helmholtz equation) it must vanish in all of $\mathbb{R}^2$, which is only possible if $\phi = 0$, see [9, Th. 3.19]. □

A second consequence of Theorem 13 is the one-to-one correspondence between the propagating information of solutions to the quasiperiodic problems (33–34) and to problem (22–23): If $u \in \mathcal{H}(\Omega)$ solves (22–23), then the Rayleigh expansion of $u_\alpha = \mathcal{J}_\Omega u(\alpha; \cdot)$ consists of an infinite number of terms that are exponentially decaying and a finite number of terms that are propagating plane waves. The latter terms correspond to indices $j \in \mathbb{Z}$ such that $(\alpha + 2\pi j/L)^2 < k^2$, that is, such that the number $\beta_j = (k^2 - (\alpha + 2\pi j/L)^2)^{1/2}$ is real valued. The propagating part of u_α is hence

$$u_\alpha^{\mathrm{prop}}(x) = \frac{1}{\sqrt{L}} \sum_{j:\, |\alpha + 2\pi j/L| \le k} \hat{u}\left(\alpha + \frac{2\pi j}{L}, h\right) \exp\left(\frac{2\pi \mathrm{i} j}{L} x_1 + \mathrm{i}\beta_j (x_2 - h)\right), \qquad x_2 \ge h.$$

One can analogously define the propagating part of $u \in \mathcal{H}(\Omega)$ by neglecting all numbers ξ in the angular spectrum representation (23) such that $k^2 < \xi^2$, since their contribution for large x_2 will be exponentially small,

$$u^{\mathrm{prop}}(x) = \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} \exp\big(\mathrm{i}\xi x_1 + \mathrm{i}\sqrt{k^2 - \xi^2}(x_2 - h)\big) \hat{u}(\xi, h)\, \mathrm{d}\xi\,, \qquad x_2 \ge h.$$

Note on the inverse Bloch transformation applied to u_α^{prop} equals the propagating part u^{prop} of u,

$$\begin{aligned}
\mathcal{J}_\Omega^{-1} u_\alpha^{\mathrm{prop}}(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi/L}^{\pi/L} \sum_{j: |\alpha + 2\pi j/L| < k} \hat{u}\left(\alpha + \tfrac{2\pi j}{L}, h\right) e^{\mathrm{i}\left(\alpha + \frac{2\pi j}{L}\right) x_1 + \mathrm{i}\beta_j (x_2 - h)}\, \mathrm{d}\alpha \\
&= \frac{1}{\sqrt{2\pi}} \sum_{j\in\mathbb{Z}} \int_{-\pi/L}^{\pi/L} \mathbb{1}_{\{|\alpha + 2\pi j/L| < k\}} \hat{u}\left(\alpha + \tfrac{2\pi j}{L}, h\right) e^{\mathrm{i}\left(\alpha + \frac{2\pi j}{L}\right) x_1 + \mathrm{i}\sqrt{k^2 - (\alpha + 2\pi j/L)^2)}(x_2 - h)}\, \mathrm{d}\alpha \\
&= \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} \hat{u}(\xi, h) e^{\mathrm{i}\xi x_1 + \mathrm{i}\sqrt{k^2 - \xi^2)}(x_2 - h)}\, \mathrm{d}\xi\ = u^{\mathrm{prop}}(x).
\end{aligned}$$

This is in some sense bad news from the point of view of inverse problems: Even if one leaves the quasiperiodic solution framework to obtain an infinite-dimensional space of propagating solutions to the Helmholtz equation, there will always be some information lost in the evanescent fields radiating from the periodic structure.

A Diffeomorphisms and Isomorphisms Between Sobolev Spaces

As in the entire paper, we assume that Ω is the domain above the periodic surface Γ given as the graph of a periodic function ζ. Here, we explicitly construct a coordinate transform mapping Ω to the upper half-plane $\mathbb{R}^2_+ = \{y \in \mathbb{R}^2,\, y_2 > 0\}$. Moreover, this coordinate transform is invariant on points x with

$x_2 > h_0$ for $h > 0$ large enough (see the explicit bound below), and hence it does not perturb the radiation condition encoded in the angular spectrum representation (23) or in the Rayleigh series (34). In particular, for $h > h_0$, the coordinate transform maps Ω_h to

$$U_h = \{y = (y_1, y_2)^\top \in \mathbb{R}^2,\, 0 < y_2 < h\}. \tag{37}$$

To construct such a transform, let us first recall that $\Gamma = \{x = (x_1, x_2)^\top,\, x_2 = \zeta(x_1)\}$ for $\zeta \in C^{0,1}(\mathbb{R}) = W^{1,\infty}(\mathbb{R})$. Moreover, the numbers $\zeta_+ := \|\zeta\|_\infty$ and $\zeta_- := \operatorname{ess\,inf}_{\mathbb{R}}(\zeta) > 0$ are by assumption strictly positive. Choose a monotone function $\chi \in C^\infty(\mathbb{R} \to \mathbb{R})$ such that $\chi(t) = 1$ if $t < \zeta_+$, $\chi(t) = 0$ if $t > \zeta_+ + 3\zeta_-$, and such that $-1 < 2\zeta_- \chi'(t) \leq 0$ for $t \in \mathbb{R}$. Then we define

$$\Psi : \mathbb{R}^2_+ \to \Omega, \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} y_1 \\ y_2 + \zeta(y_1)\chi(y_2) \end{pmatrix},$$

and note that this Lipschitz continuous mapping is invertible and hence a Lipschitz continuous diffeomorphism. Indeed, $\det D\Psi(y) = 1 + \zeta(y_1)\chi'(y_2) \neq 0$, because $1 + \zeta(y_1)\chi'(y_2) > 1 - \zeta(y_1)/(2\zeta_-) > 1/2$. The inverse $\Theta = \Psi^{-1}$ is given by

$$\Theta : \Omega \to \mathbb{R}^2_+, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 \\ x_2 - \zeta(y_1)\chi(y_2) \end{pmatrix}.$$

If $h > h_0 := \zeta_+ + 3\zeta_-$, then Θ and Ψ are also Lipschitz continuous diffeomorphisms between Ω_h and U_h (see (37)) and between

$$\Omega_h^{\mathrm{p}} := \{x \in \Omega_h,\, x_1 \in (-L/2, L/2)\} \quad \text{and} \quad U_{\mathrm{p}}^h := \left\{ y = (y_1, y_2)^\top \in (-L/2, L/2) \times (0, h) \right\}.$$

Both mappings are obviously L-periodic in their first variable. Further, if ζ is more regular than merely Lipschitz continuous, then also the regularity of Ψ and Θ increases: If $\zeta \in W^{n,\infty}(\mathbb{R})$, then Ψ and Θ are both diffeomorphisms of class $W^{n,\infty}$.

Proposition 15. *If $\zeta \in W^{n,\infty}(\mathbb{R})$, $n \in \mathbb{N}$, and if $h > h_0 := \zeta_+ + 3\zeta_-$ is large enough, then Ψ and Θ are diffeomorphisms of class $W^{n,\infty}$ and the mappings $u \mapsto u \circ \Psi$ and $v \mapsto v \circ \Theta$ are isomorphisms from $H^s(\Omega_h)$ into $H^s(U_h)$ and from $H^s(U_h)$ and $H^s(\Omega_h)$ for $0 \leq s \leq n$, respectively:*

$$\frac{1}{C}\|u\|_{H^s(\Omega_h)} \leq \|u \circ \Psi\|_{H^s(U_h)} \leq C\|u\|_{H^1(\Omega_h)}, \qquad \frac{1}{C}\|v\|_{H^s(U_h)} \leq \|v \circ \Theta\|_{H^s(\Omega_h)} \leq C\|v\|_{H^1(U_h)},$$

for some $C > 0$ large enough. The same relations hold for the periodic Sobolev spaces $H^s_{\mathrm{p}}(\Omega_h)$, see (29), and $H^s_{\mathrm{p}}(U_h)$ on the bounded domains Ω_h^{p} and U_{p}^h instead of $H^s(\Omega_h)$ and $H^s(U_h)$, respectively.

Proof. We merely indicate a proof for Lipschitz continuous ζ, that is, for $n = 1$ (the case of larger n can be treated analogously). If $\zeta \in W^{1,\infty}(\mathbb{R})$, then both Θ and Ψ are Lipschitz continuous and the chain rule implies that these transforms transform $H^1(\Omega_h)$ into $H^1(U_h)$ and vice versa. The same holds for the L^2-spaces $L^2(\Omega_h)$ into $L^2(U_h)$. Since Ψ and Θ are inverses to each other, both $u \mapsto u(\Psi(\cdot))$ and $v \mapsto v(\Theta(\cdot))$ are continuous and continuously invertible from $H^s(\Omega_h)$ into $H^s(U_h)$ and from $H^s(U_h)$ and $H^s(\Omega_h)$ for $s = 0$ and $s = 1$, respectively. The general result then follows from interpolation theory, see, e.g., [24, App. B]. The proof for the Sobolev spaces $H^s_{\mathrm{p}}(\Omega_h)$ and $H^s_{\mathrm{p}}(U_h)$ of periodic functions are similar but additionally rely on the L-periodicity of Ψ and Θ. $\square$

Acknowledgements. AL was supported through an exploratory project granted by the University of Bremen as part of the excellence initiative funded by the federal and state governments of Germany. DLN was supported by the ANR project METAMATH, grant ANR-11-MONU-016 of the French Agence Nationale de la Recherche.

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