Time-Harmonic Acoustic Wave Scattering in an Ocean with Depth-Dependent Sound Speed

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Abstract

Time-harmonic acoustic wave propagation in an inhomogeneous ocean with depth-dependent sound speed can be modeled by the Helmholtz equation in an infinite, three-dimensional waveguide of finite height. Using variational theory in Sobolev spaces we prove well-posedness of the corresponding scattering problem from a bounded inhomogeneity inside such an ocean. To this end, we introduce an exterior Dirichlet-to-Neumann operator for depth-dependent sound speed and prove boundedness, coercivity, and holomorphic dependence of this operator in function spaces adapted to our weak solution theory. Analytic Fredholm theory then yields existence and uniqueness of solution for the scattering problem for all but a countable sequence of frequencies. The latter result generalizes corresponding theory for waveguide scattering with constant sound speed and easily extends to various related scattering problems, e.g., to scattering from impenetrable obstacles.

1 Introduction

Propagation of sound waves inside an ocean is an active research area in applied mathematics and engineering at least since the mid-20th century for its crucial importance for techniques like SONAR or for oil exploration (see, e.g., the introduction of [BGWX04] or [Buc92]). After the millenium change, precise models for sound propagation became even more important due to the observation that man-made ocean noise pollution endangers marine mammals and legal thresholds for emitted sound energies were set up. Checking these thresholds, e.g., for acoustic pulses produced by an air guns, requires sufficiently accurate models yielding quantitatively exact simulations of sonic intensities. One approach satisfying this requirement is to model scattering of time-harmonic acoustic waves in the ocean using the Helmholtz equation and to discretize this equation using established approximation technique as, e.g., finite elements or boundary elements.

It is well-known that a sound knowledge on variation theory of weak solutions in Sobolev spaces, in particular the existence of Gårding inequalities, is crucial for proving convergence of such numerical approximations, see [SS11]. However, to the best of our knowledge, weak solution theory for ocean scattering problems has up to now merely been set up for the case of a constant sound speed, restricting the applicability of this approach to shallow seas. For this reason, it is our aim in this paper is to provide rigorous theory for weak solutions via a variational approach for wave scattering in a flat ocean with variable, depth-dependent refractive index. The crucial and non-trivial difficulty compared to known results for constant sound speed, see [AGL08, AGL11], is that the eigenmodes of the ocean are not known explicitly for such a setting. In consequence, we exploit on the one hand via estimates for these modes and their eigenvalues and, on the other hand, obtain holomorphic dependence of the eigenvalues on the frequency from abstract perturbation theory.

Let us emphasize here that various other models for sound propagation as well as various other approaches for existence of solutions to the Helmholtz equation in an ocean geometry exist. While the most simple techniques are based on ideas from ray propagation (e.g., the Lloyd mirror), more advanced methods rely on approximations to the wave or Helmholtz equation (e.g., the parabolic approximation). The monograph [Jen11] and the survey papers [Buc92, AK77, AL87] review these and similar methods from an engineering and applied mathematics perspective. Existence of classical (i.e., twice differentiable) solutions to the Helmholtz equation for constant and depth-dependent sound speed has been shown via integral equation techniques by Gilbert and Xu in a series of papers, partly focussing on inverse scattering problems, see [Xu90, GX90, Xu90, Xu92, GX96, BGWX04]. Finally note that [BGT85] considers finite element methods for ocean scattering with constant sound speed.

In this paper we use the following Helmholtz equation in the waveguide $\Omega := \mathbb{R}^2 \times (0, H)$ with constant depth $H > 0$,

$$\Delta u(x) + \frac{\omega^2}{c^2(x_3)} u(x) = 0 \qquad \text{for } x = (\tilde{x}, x_3)^\top \in \Omega, \tag{1}$$

to model scattering of time-harmonic acoustic waves with time-dependence $\exp(-i\omega t)$, angular frequency $\omega > 0$, and small amplitude. The sound speed $c : (0, H) \to \mathbb{R}_{>}0$ depends on various environmental parameters and is usually parametrized in terms of water temperature, salinity and pressure, see, e.g., [DWCH93]. At a mid-latitude location it decreases from about 1505 m/s at the sea surface to about 1485 m/s at the SOFAR channel in 600 meters depth and increases again to about 1515 m/s in 3000 meters depth. Accurate models for sound propagation over large distances imperatively need to reflect this depth-dependence, otherwise restricting their validity to shallow oceans.

When the sound speed is further locally perturbations inside the inhomogeneous waveguide Ω we model such local perturbations by a refractive index function $n : \Omega \to \mathbb{C}$ such that the support of the contrast $q = n^2 - 1$ is a bounded set $D \subset \overline{\Omega}$, i.e., $\operatorname{supp}(q) = \overline{D}$. Time-harmonic sound waves propagating in the perturbed ocean thus satisfy

$$\Delta u(x) + \frac{\omega^2}{c^2(x_3)} (1 + q(x)) u(x) = 0 \qquad \text{for } x \in \Omega. \tag{2}$$

We assume in the following that the background sound speed $c \in L^\infty(0, H)$ satisfies $0 < c_- \le c(x_3) \le c_+$ for almost all $x_3 \in (0, H)$, that is,

$$0 < \frac{\omega}{c_+} \le \frac{\omega}{c(x_3)} \le \frac{\omega}{c_-} \qquad \text{for almost all } x_3 \in (0, H). \tag{3}$$

Further, we model the free surface and the seabed of the ocean by a sound-soft and a sound-hard boundary, respectively,

$$u = 0 \text{ on } \Gamma_0 := \{x \in \mathbb{R}^3 : x_3 = 0\} \qquad \text{and} \qquad \frac{\partial u}{\partial x_3} = 0 \text{ on } \Gamma_H := \{x \in \mathbb{R}^3 : x_3 = H\}. \tag{4}$$

This setting yields a sufficiently accurate to model acoustic waves with small amplitude in an ocean with negligible seabed variation or a depth so large that little wave energy propagates to the seabed.

More flexible boundary models for, e.g., the ocean-seabed interface exist, see, e.g., [GL97], but for simplicity we restrict ourselves to the simpler condition Neumann condition from (4) describing a perfectly reflecting bottom.

When an incident sound field u^i that satisfies the unperturbed Helmholtz equation (1) subject to the waveguide boundary conditions (4) is scattered from the inhomogeneous medium described by q, then a scattered field u^s arises such that the total field $u = u^i + u^s$ solves the perturbed Helmholtz equation (2) with contrast q, subject to $u = 0$ on Γ_0 and $\partial u/\partial \nu = 0$ on Γ_H. On interfaces where q jumps we prescribe that both the trace and the normal derivative of u are continuous across the interface. To ensure uniqueness of solution we further need to impose a radiation condition on u. This condition will be constructed with the help of a modal analysis in Section 3 below. Note that we seek for weak solutions to the scattering problem, i.e., for a function u that is locally in H^1 and satisfies

$$\int_\Omega \left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_3)}(1 + q(x))u\overline{v} \right) dx = 0 \qquad \text{for all } v \in C_0^\infty(\Omega).$$

Let us finally indicate that all our results can be extended to dimension two; for the corresponding proofs we refer to the preprint [LR15]. The latter reference also contains full proofs of two technical lemmas from this paper that we omitted as they are rather similar to already published results. Moreover, we remark that Theorem 5.2 provides a proof concerning the extension of a solution on a bounded domain to all the waveguide that is missing in [AGL08].

The subsequent sections are organized as follows: Seeking for solutions to (1) by separation of variables, a Liouville eigenvalue problem turns up that we investigate in Section 2. After showing holomorphic dependence of the eigenvalues on the frequency, we use these eigenvalues in Section 3 rigorously set up the scattering problem we investigate, and in Section 4 to prove spectral characterizations of Sobolev-type function spaces. Those are exploited in Section 5 for analyzing the exterior Dirichlet-to-Neumann operator for the waveguide scattering problem. Finally, Section 6 contains and proves the main existence and uniqueness result of the paper via a Gårding inequality and analytic Fredholm theory.

Notation: We define, for $\rho > 0$, domains $\Omega_\rho = \{x \in \Omega : |\tilde{x}| < \rho\}$ and for arbitrary Lipschitz domains $U \subset \Omega$ the Sobolev spaces $H^1_W(U) = \{v \in H^1(U) : v|_{U \cap \{x_3 = 0\}} = 0\}$. This space is well-defined due to the well-known trace theorem in H^1. Moreover, $H^1_{W,\text{loc}}(\Omega) = \{v : \Omega \to \mathbb{C}, v \in H^1_W(\Omega_\rho) \text{ for all } \rho > 0\}$ and $H^2_{\text{loc}}(\Omega) = \{v : \Omega \to \mathbb{C} : v \in H^2(\Omega_\rho) \text{ for all } \rho > 0\}$. Whenever two real-valued expressions A, B satisfy that there is $c > 0$ such that $c^{-1}A \leq B \leq cA$ we write shorthand $A \simeq B$ to indicate this relation.

2 A Liouville Eigenvalue Problem

Our first aim is to derive a radiation condition characterizing outgoing time-harmonic waves in Ω. To this end, we start by seeking solutions to the Helmholtz equation (1) with boundary conditions (4) by separation of variables, i.e., of the form $u(\tilde{x}, x_3) = w(\tilde{x})\phi(x_3)$. One finds that w and ϕ need to solve

$$\frac{\Delta_{\tilde{x}} w(\tilde{x})}{w(\tilde{x})} = -\frac{\phi''(x_3)}{\phi(x_3)} - \frac{\omega^2}{c^2(x_3)} =: \lambda^2 \qquad \text{in } \Omega \tag{5}$$

for some $\lambda \in \mathbb{C}$. ($\Delta_{\tilde{x}}$ is the 2-dimensional Laplacian in the variables $\tilde{x}$.) Note that we choose λ^2 to be the eigenvalue since the square root of the eigenvalue is a crucial quantity for the scattering problem later on; the sign of λ^2 is chosen such that the eigenvalues found below as usual form a sequence tending to $+\infty$. We first consider the Liouville-type eigenvalue problem

$$\phi''(x_3) + \left[\frac{\omega^2}{c^2(x_3)} + \lambda^2\right]\phi(x_3) = 0 \quad \text{in } (0,H), \quad \phi(0) = 0, \quad \text{and} \quad \phi'(H) = 0. \tag{6}$$

The latter boundary conditions correspond to (4). Variational theory for weak solutions to the self-adjoint eigenvalue problem in the space $H^1_W(0,H) := \{\psi \in H^1(0,H) : \psi(0) = 0\}$ is well-known. (The latter space is well-defined by the well-known continuous embedding of $H^1(0,H)$ in the Hölder space $C^{0,1/2}(0,H)$.) Multiplying (6) with a test function $\psi \in H^1_W([0,H])$, formally integrating by parts and plugging in the boundary conditions for ϕ shows that an eigenpair $(\lambda^2, \phi) \in \mathbb{C} \times H^1_W([0,H])$ needs to satisfy

$$a(\phi,\varphi) := \int_0^H \left(\phi'\,\overline{\psi}' - \frac{\omega^2}{c^2(x_3)}\phi\,\overline{\psi}\right) dx_3 \stackrel{!}{=} \lambda^2 \int_0^H \phi\,\overline{\psi}\, dx_3 \quad \text{for all } \psi \in H^1_W([0,H]). \tag{7}$$

The sesquilinear form a is bounded in $H^1_W([0,H])$ and Poincaré's inequality together with the compact embedding of $H^1_W([0,H])$ in $L^2(0,H)$ shows that a is coercive in $H^1_W([0,H])$ up to a compact perturbation. Since ω/c^2 is real-valued, a is moreover symmetric, i.e., $a(\varphi,\psi) = a(\psi,\varphi)$ for all $\varphi, \psi \in H^1_W([0,H])$. Thus, standard eigenvalue theory, see, e.g., [McL00, Theorem 2.7], shows existence of a sequence of eigenvalues $\{\lambda_j^2\}_{j\in\mathbb{N}} \subset \mathbb{R}$ such that $\lambda_j^2 \to +\infty$ as $j \to \infty$ and associated real-valued eigenfunctions $\phi_j \in H^1_W([0,H])$ that are orthonormal in $L^2(0,H)$. We order the eigenvalues in increasing order, i.e., $-\infty < \lambda_1^2 \le \lambda_2^2 \le \lambda_3^2 \le \dots$ and define their square roots by

$$\lambda_j = \begin{cases} \sqrt{\lambda_j^2} & \text{if } \lambda_j^2 \ge 0, \\ -i\sqrt{|\lambda_j^2|} & \text{if } \lambda_j^2 < 0. \end{cases} \tag{8}$$

(The sign of λ_j is chosen to simplify expressions later on.) The square root function is then extended from the real axis to a holomorphic function in the slit complex plane with branch cut along the positive imaginary axis. The definition of a weak derivative in one dimension shows that ϕ_j' belongs to $H^1([0,H])$, such that $\phi \in H^2([0,H])$ satisfies $\phi_j''(x_3) + [\omega^2/c^2(x_3) + \lambda_j^2]\phi_j(x_3) = 0$ with equality in $L^2(0,H)$ and in particular almost everywhere in $(0,H)$. As $H^2([0,H]) \subset C^{1,1/2}([0,H])$ all eigenfunctions $\phi_j \in C^{1,1/2}([0,H])$ satisfy the boundary conditions $\phi_j(0) = 0$ and $\phi'(H) = 0$ in the classical sense of a point evaluation.

Remark 2.1. *For constant background sound speed $c_\pm$ it is well-known that*

$$\lambda_j^2 = \left(\frac{\pi}{2H}(2j-1)\right)^2 - \frac{\omega^2}{c_\pm^2} \quad \textit{and} \quad \phi_j(x_3) = \sin\left(\frac{\pi}{2H}(2j-1)x_3\right), \quad x_3 \in [0,H].$$

Lemma 2.2. *(a) For $j \in \mathbb{N}$ it holds that*

$$\left[\frac{\pi}{2H}(2j-1)\right]^2 - \frac{\omega^2}{c_-^2} \le \lambda_j^2 \le \left[\frac{\pi}{2H}(2j-1)\right]^2 - \frac{\omega^2}{c_+^2}. \tag{9}$$

Further, there are constants $0 < c < C$ such that $cj^2 \le |\lambda_j^2| \le Cj^2$ for j large enough and $cj \le \|\phi_j'\|_{L^2(0,H)} \le Cj$ as well as $\|\phi_j'\|_{L^2(0,H)} \le C(1+|\lambda_j|^2)^{1/2}$ for all $j \in \mathbb{N}$.

Proof. (a) The min-max theorem implies for all $j \in \mathbb{N}$ that

$$\begin{aligned}\lambda_j^2 &= \min_{V_j \subset H_W^1([0,H]),\dim(V_j)=j} \ \max_{\phi_j \in V_j, \|\phi_j\|=1} a(\phi_j,\phi_j) \\ &\leqslant \min_{V_j \subset H_W^1([0,H]),\dim(V_j)=j} \ \max_{\phi_j \in V_j, \|\phi_j\|=1} \int_0^H \left(|\phi_j'|^2 - \frac{\omega^2}{c_\pm^2}|\phi_j^+|^2\right) dx_3 = \left(\frac{\pi}{2H}(2j-1)\right)^2 - \frac{\omega^2}{c_\pm^2}.\end{aligned}$$

which shows (9) and the quadratic growth of λ_j^2. The remaining estimates follow from

$$\int_0^H |\phi_j'|^2 dx_3 = \int_0^H \left[\frac{\omega^2}{c^2(x_3)} + \lambda_j^2\right] |\phi_j|^2 dx_3 \leqslant \frac{\pi^2(2j-1)^2}{4H^2} + \omega^2 \frac{c_\pm^2 - c_\mp^2}{c_+^2 c_-^2}, \tag{10}$$

together with the fact that $\|\phi_j'\|_{L^2(0,H)} > 0$ since $\|\phi_j'\|_{L^2(0,H)}^2 = 0$ implies that $\phi_j \equiv 0$ as $\phi_j(0) = 0$. $\square$

The eigenvalues λ_j^2 obviously depend on the frequency $\omega > 0$. Writing $\lambda_j^2 = \lambda_j^2(\omega)$, the function $\omega \mapsto \lambda_j^2(\omega)$ can be extended holomorphically into a complex neighborhood of $\mathbb{R}_{>0}$.

Lemma 2.3. *For all $\omega_* > 0$ there exists an open neighborhood $U(\omega_*) \subset \mathbb{C}$ and index functions $\ell_j : U(\omega_*) \to \mathbb{N}$ for all $j \in \mathbb{N}$ that satisfy $\cup_{j\in\mathbb{N}} \ell_j(\omega) = \mathbb{N}$ and $\ell_j(\omega) \neq \ell_j'(\omega)$ for $j \neq j' \in \mathbb{N}$ and all $\omega \in U(\omega_*)$, such that the eigenvalue curves $\omega \mapsto \lambda_{\ell_j(\omega)}^2(\omega)$ are real-analytic functions in $U(\omega_*) \cap \mathbb{R}_{>0}$ and extend to holomorphic functions in $U(\omega_*)$.*

Proof. We exploit results on holomorphic families of operators from [Kat95, Chapter VII, §2 and §4]. The differential operators $L(\omega)\, u = u'' + ((\omega)^2/c^2)u$ on $(0,H)$ with boundary conditions $u(0) = 0$ and $u'(H) = 0$ yield a selfadjoint holomorphic family of type (A) since $u \mapsto (\omega^2/c^2)u$ is bounded on $L^2(0,H)$, $\omega \mapsto (\omega^2/c^2)u$ is holomorphic in the complex parameter ω, and the domain $\{u \in H^2(0,H),\, v(0) = 0\}$ of $L(\omega)$ is independent of ω, compare [Kat95, Ch. VII, §1.1, §2.1, Th. 2.6]. These differential operators also form of a holomorphic family of type (B) since the associated sesquilinear form a from (7) is bounded.

Now, choose some $\omega_* > 0$. From [Kat95, Ch. VII, §3.1, Example 4.23] it follows that for each eigenvalue $\lambda_j^2(\omega_*)$ with multiplicity one that there is a complex neighborhood U_j of ω_* such that $\omega \mapsto \lambda_j^2(\omega)$ can be extended from $U_j \cap \mathbb{R}$ as a holomorphic function of ω into U_j. If $\lambda_j^2(\omega_*)$ is a multiple eigenvalue, then $\omega \mapsto \lambda_j^2(\omega)$ is in general not differentiable at ω^*, such that the eigenvalue index needs to be re-ordered to obtain smooth eigenvalue curves, compare [Kat95, Ch. VII, §3.1, Ch. 2, Th. 6.1]. Indeed, the latter reference shows that if $\lambda_j^2(\omega_*)$ is a multiple eigenvalue, then it has finite multiplicity and there exists a complex neighborhood U_j of ω_* and an index function $\ell_j : U_j \cap \mathbb{R} \to \mathbb{N}$ such that $\omega \mapsto \lambda_{\ell_j(\omega)}^2(\omega)$ can be extended holomorphically from $U_j \cap \mathbb{R}$ into U_j.

It remains to show that the intersection $\cap_{j\in\mathbb{N}} U_j$ is non-empty. This is obviously true for any finite union $\cup_{|j|\le j_0} U_j$ with $j_0 \in \mathbb{N}$. Note that (9) implies that the eigenvalues $\lambda_j^2(\omega_*)$ of $L(\omega_*)$ are simple if $j > j_*$ for j_* large enough, e.g., for

$$j_* := \left\lceil \frac{\omega_*^2 H^2}{2\pi^2}\left(\frac{1}{c_-^2} - \frac{1}{c_+^2}\right) + \frac{H^2}{2\pi^2} \right\rceil.$$

Estimate (9) further implies that whenever $j > j_*$, the distance of $\lambda_j^2(\omega_*)$ to the rest of the spectrum of $L(\omega_*)$ is bounded from below by one. Thus, Theorem 4.8 in [Kat95, Ch. VII], compare also (4.45) in the same chapter, implies for all $j > j_*$ that the holomorphic extension of $\lambda_j^2(\omega_*)$ has a convergence radius of at least $(1 + 1/c_-^2)^{-1}$. (Set $\varepsilon = 1$, $a = 1$, $b = 0$, and $c = 1/c_-^2 \geq \|1/c^2\|_{L^\infty(0,H)}$ in (4.45).) In particular, all eigenvalues $\lambda_j^2(\omega_*)$, $j > j_*$, extend to holomorphic functions in $B(\omega_*, 1)$ and the lemma holds with $U(\omega_*) := \cap_{j=1}^{j_*} U_j \cap B(\omega_*, 1)$. □

Theorem 2.4. *There exists an complex neighborhood U of $\mathbb{R}_{>0}$ and index functions $\ell_j : U \to \mathbb{N}$ for $j \in \mathbb{N}$ such that the eigenvalue curves $\lambda_{\ell_j(\omega)}^2(\omega)$ are real-analytic curves that extend to holomorphic functions in U for all $j \in \mathbb{N}$. For each compact subset W of U, the set $K_0 = \{\omega \in W :$ there is $j \in \mathbb{N}$ such that $\lambda_j^2(\omega) = 0\}$ is finite.*

Proof. We cover the positive reals $(0, \infty)$ with the neighborhoods $U(\omega)$ of $\omega > 0$ constructed in Lemma 2.3. For each compact interval $[1/\ell, \ell]$ with $\ell \in \mathbb{N}$ there exists a finite sub cover, which allows to holomorphically continue all eigenvalue functions $\omega \mapsto \lambda_{\ell_j(\omega)}^2(\omega)$, $j \in \mathbb{N}$, into a complex neighborhood U_ℓ of $[1/\ell, \ell]$. As $\ell \in \mathbb{N}$ is arbitrary, this yields the claimed open set $U = \cup_{\ell \in \mathbb{N}} U_\ell$ in $\mathbb{C}$ containing $\mathbb{R}_{>0}$. Finally, Theorems 1.9 and 1.10 in [Kat95, Ch. VII, §1.3] state that on compact subsets W of U either for each $\omega \in W$ there is $j = j(\omega) \in \mathbb{N}$ such that $\lambda_{j(\omega)}^2 = 0$ or that the number of such ω in W is finite. As (9) excludes the first alternative the set K_0 from the claim is finite. □

3 The Radiation Condition and the Scattering Problem

Now we are in a position the rigorously define a radiation conditon for solution to the Helmholtz equation (1) and to subsequently formulate the scattering problem targeting weak radiating solutions. Going back to the construction of solutions to (1) by separation of variables in (5) we note that $u(\tilde{x}, x_3) = \sum_{j \in \mathbb{N}} c(j)\, w_j(\tilde{x}) \phi_j(x_3)$ with coefficients $c(j) \in \mathbb{C}$ formally solves (1) whenever

$$\Delta_{\tilde{x}} w_j - \lambda_j^2 w_j = 0 \qquad \text{in } \mathbb{R}^2. \tag{11}$$

Thus, the eigenfunctions ϕ_j to (7) give rise to plane wave-like solutions $x \mapsto \exp(-\lambda_j\, \theta \cdot \tilde{x}) \phi_j(x_3)$ of the Helmholtz equation in Ω with direction $\theta \in \mathbb{R}^2$, $|\theta|_2 = 1$. These so-called waveguide modes are propagating if $i\lambda_j$ is positive (i.e., $\lambda_j^2 < 0$) and evanescent if $i\lambda_j \in i\mathbb{R}$ (i.e., $\lambda_j^2 > 0$). The number of such propagating modes is (up to rotation) determined by the largest integer

$$J_* = J_*(\omega, c, H) \text{ such that } \lambda_{J_*}^2 < 0. \tag{12}$$

In analogy to the case of a constant sound speed, see [AGL08], solutions to the Helmholtz equation are called radiating if, roughly speaking, the above series representation holds for $|\tilde{x}|$ sufficiently large with modes w_j that are required to satisfy Sommerfeld's radiation condition if $i\lambda_j \in \mathbb{R}_{>0}$ is positive and to be bounded if $\lambda_j \in \mathbb{R}_{>0}$ is positive.

The case $\lambda_j^2 = 0$ is somewhat exceptional as the mode u_j is constant in $\tilde{x}$; note that the number of propagating modes changes at the corresponding frequency.

Assumption 3.1. *We assume that the frequency $\omega > 0$ is such that $\lambda_j^2 \neq 0$ for all $j \in \mathbb{N}$.*

Note that Theorem 2.4 states that there exists at most a countable set of exceptional frequencies without accumulation point in $(0, \infty)$. As for sufficiently small $\omega > 0$ all λ_j^2 are positive due to (9), the only possible accumulation point of the exceptional frequencies is $+\infty$.

Now we can rigorously formulate the scattering problem tackled in the sequel: Consider $c \in L^\infty(0, H)$ such that $0 < c_- \leq c(x_3) \leq c_+$ for $x_3 \in (0, H)$, a contrast $q \in L^\infty(\Omega)$ such that $\mathrm{Im}(q) \geq 0$ and $\mathrm{supp}(q) \subset \Omega_\rho$, and an incident field $u^i \in H^1_{W,\mathrm{loc}}(\Omega)$ that satisfies the Helmholtz equation (1) weakly in Ω,

$$\int_\Omega \left(\nabla u^i \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_3)} u^i \overline{v} \right) dx = 0 \qquad \text{for all } v \in H^1_{W,\mathrm{loc}}(\Omega) \text{ with compact support.} \tag{13}$$

We seek for a total field $u \in H^1_{W,\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega \left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_3)} (1+q) u \overline{v} \right) dx = 0 \qquad \text{for all } v \in H^1_{W,\mathrm{loc}}(\Omega) \text{ with compact support.} \tag{14}$$

Additionally, we require that

$$u^s(x) = u(x) - u^i(x) = \sum_{j \in \mathbb{N}} w_j(\tilde{x}) \phi_j(x_3) \qquad \text{for all } |\tilde{x}| > \rho, \tag{15}$$

with solutions $w_j \in C^\infty(|\tilde{x}| > \rho)$ to $(\Delta_{\tilde{x}} - \lambda_j^2) w_j = 0$ in $\{|\tilde{x}| > \rho\}$ that satisfy

$$\begin{cases} \lim_{|\tilde{x}| \to \infty} \sqrt{|\tilde{x}|} \left(\frac{\partial w_j}{\partial |\tilde{x}|} - i |\lambda_j| w_j \right) = 0 \text{ uniformly in } \tilde{x}/|\tilde{x}| & \text{if } \lambda_j^2 < 0, \\ w_j(\tilde{x}) \text{ is uniformly bounded in } |\tilde{x}| > \rho & \text{if } \lambda_j^2 > 0, \end{cases} \qquad \text{for all } j \in \mathbb{N}. \tag{16}$$

The series in (15) is required to converge in $H^1(\Omega_R \setminus \Omega_\rho)$ for all $R > \rho$.

In the sequel, any solution to the Helmholtz equation outside Ω_ρ that satisfies (16) for all $j \in \mathbb{N}$ is called a radiating solution.

Remark 3.2. *(1) Any solution solving* (14) *can be represented in series form as in* (15) *since the eigenfunctions* $\{\phi_j\}_{j \in \mathbb{N}} \subset H^1_W(0, H)$ *are an orthonormal basis of* $L^2(0, H)$*. Thus, the above assumption on the scattered field* u^s *merely requires the conditions* (16) *to be satisfied.*
(2) The Neumann boundary conditions from (4) *are implicitly included in* (13) *and* (14)*.*

4 Spectral Characterization of Function Spaces

To analyze the scattering problem defined in the last section we will ultimately transform it into a variational problem on the bounded domain Ω_ρ for $\rho > 0$ such that $\mathrm{supp}(q) \subset \Omega_\rho$. To this end, we will define and analyze exterior Dirichlet-to-Neumann operators in the next section. As a technical tool, we next prove a spectral characterization of the H^1-Norm and a related trace estimate. Note that all results in this section hold true irrespective of whether Assumption 3.1 holds or not.

Standard theory on orthogonal bases in Hilbert spaces allows to expand $u \in L^2(\Omega_\rho)$ into its Fourier series with respect to the basis $\{\phi_j\}_{j \in \mathbb{N}}$,

$$u(x) = \sum_{j \in \mathbb{N}} u_j(\tilde{x}) \phi_j(x_3) \qquad \text{where } u_j(\tilde{x}) = \int_0^H u(\tilde{x}, x_3) \overline{\phi_j}(x_3) \, dx_3. \tag{17}$$

This series converges in $L^2(\Omega_\rho)$ and $\|u\|^2_{L^2(\Omega_\rho)} = \sum_{j\in\mathbb{N}} \|u_j\|^2_{L^2(\{|\tilde{x}|<\rho\})}$ holds due to Parseval's identity.

Lemma 4.1. *(a) The coefficients u_j of $u \in H^1_W(\Omega_\rho)$ from* (17) *belong to $H^1(\{|\tilde{x}| < \rho\})$ and*

$$\|\nabla_{\tilde{x}} u\|^2_{L^2(\Omega_\rho)} = \sum_{j\in\mathbb{N}} \|\nabla_{\tilde{x}} u_j(\tilde{x})\|^2_{L^2(\{|\tilde{x}|<\rho\})}.$$

(b) If $u \in C^2(\overline{\Omega_\rho})$, then the series expansion (17) *converges absolutely and uniformly. Additionally, this expansion can be differentiated term by term with respect to x_3 and the series converges absolutely and uniformly,*

$$\frac{\partial u}{\partial x_3}(x) = \sum_{j\in\mathbb{N}} u_j(\tilde{x})\phi_j'(x_3) \qquad \text{for } x \in \overline{\Omega_\rho}. \tag{18}$$

Proof. (a) The function $u = \sum_{j\in\mathbb{N}} u_j(\tilde{x})\phi_j(x_3) \in L^2(\Omega_\rho)$ belongs to $H^1(\Omega_\rho)$ if and only if its first-order distributional derivatives all belong to $L^2(\Omega_\rho)$. As $u \mapsto \partial u/\partial x_i$ is continuous from $H^1(\Omega_\rho)$ into $L^2(\Omega_\rho)$, we can exchange this partial derivative for $i = 1, \dots, m-1$ with the inner product of $L^2(0, H)$,

$$\int_0^H \frac{\partial u}{\partial x_i}(x)\overline{\phi_l}(x_3)\, dx_3 = \frac{\partial}{\partial x_i}\int_0^H \sum_{j\in\mathbb{N}} u_j(\tilde{x})\phi_j(x_3)\overline{\phi_l}(x_3)\, dx_3 = \frac{\partial}{\partial x_i} u_\ell(\tilde{x}). \tag{19}$$

The right-hand side is square-integrable, since the left-hand side can be estimated in $L^2(\{|\tilde{x}| < \rho\})$ by $\|\partial u/\partial x_i\|_{L^2(\Omega_\rho)}$. Thus, $u_j(\tilde{x}) \in H^1(\{|\tilde{x}| < \rho\})$. Parseval's identity states that $\|\partial u/\partial x_i\|^2_{L^2(\Omega_\rho)} = \sum_{j\in\mathbb{N}} \|\partial u_j/\partial x_i\|^2_{L^2(\{|\tilde{x}|<\rho\})}$ for $i = 1, \dots, m-1$.

(b) This follows from results on function expansions in terms of the Sturm-Liouville eigenfunctions $\{\phi_j\}_{j\in\mathbb{N}}$, compare, e.g., [LS60, Chapter 3, §3-§5]. □

To state a spectral characterization of the H^1-norm on Ω_ρ we introduce

$$\Sigma_\rho := \{x \in \Omega : |\tilde{x}| = \rho\} = \left\{x = (\rho\cos\varphi, \rho\sin\varphi, x_3)^\top : \varphi \in (0,\pi),\, x_3 \in (0,H)\right\}, \tag{20}$$

the cylindrical part of the boundary of Ω_ρ. As $\{\exp(in\varphi)\phi_j\}_{n\in\mathbb{Z},j\in\mathbb{N}}$ is a (non-normalized) orthogonal basis of $L^2(\Sigma_\rho)$ we can further expand $u \in L^2(\Omega_\rho)$ in cylindrical coordinates as

$$\begin{aligned} u(x) = \sum_{j\in\mathbb{Z}} u_j(\tilde{x})\phi_j(x_3) &= \sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{Z}} \hat{u}(j,n,r)e^{in\varphi}\phi_j(x_3), \qquad x = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ x_3 \end{pmatrix} \in \Omega_\rho, \\ \text{where } \hat{u}(j,n,r) &:= \frac{1}{2\pi}\int_0^{2\pi}\int_0^H u(r,\varphi,x_3)e^{-in\varphi}\phi_j(x_3)\, dx_3,\, d\varphi, \quad n \in \mathbb{Z},\, j \in \mathbb{N}. \end{aligned} \tag{21}$$

Note that whenever we differentiate the Fourier coefficient $\hat{u}(j,n,r)$ with respect to the radial variable we write $\hat{u}'(j,n,r)$ instead of $\partial\hat{u}(j,n,r)/\partial r$.

Lemma 4.2. *For $u \in H^1_W(\Omega_\rho)$ it holds that*

$$\|u\|^2_{H^1(\Omega_\rho)} \simeq \sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{Z}} \int_0^\rho \left[(1+|\lambda_j|^2)|\hat{u}(j,n,r)|^2 + \frac{n^2}{r^2}|\hat{u}(j,n,r)|^2 + |\hat{u}'(j,n,r)|^2\right] r\, dr. \tag{22}$$

Proof. It is sufficient to show the claim for $u = \sum_{j\in\mathbb{N}} u_j(\tilde{x})\phi_j(x_3)$ in the dense subset $H^1_W(\Omega_\rho)\cap C^2(\overline{\Omega_\rho})$ of $H^1_W(\Omega_\rho)$. In this situation, Lemma 4.1 states that $(\partial u/\partial x_3)(x) = \sum_{j\in\mathbb{N}} u_j(\tilde{x})\phi_j'(x_3)$ is an absolutely and uniformly converging series representation in Ω_ρ.

Parseval's identity applied to the orthogonal eigenfunctions $\{\phi_j\}_{j\in\mathbb{N}}$ and the trigonometric monomials, together with the transformation formula, shows that

$$\|u\|^2_{L^2(\Omega_\rho)} = \sum_{j\in\mathbb{N}} \|u_j\|^2_{L^2(\{|\tilde{x}|<\rho\})} = \sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{Z}} \int_0^\rho |\hat{u}(j,n,r)|^2\, r\, dr.$$

Recall the representation of the gradient in cylinder coordinates,

$$\nabla u = \frac{\partial u}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial u}{\partial \varphi}\boldsymbol{e}_\varphi + \frac{\partial u}{\partial x_3}\boldsymbol{e}_{x_3}, \quad \text{with } \boldsymbol{e}_r = \begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix},\ \boldsymbol{e}_\varphi = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix},\ \text{and } \boldsymbol{e}_{x_3} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Lemma 4.1 shows that $\nabla_{\tilde{x}} u \in H^1(\{|\tilde{x}| < \rho\})$, that $\|\nabla_{\tilde{x}} u\|^2_{L^2(\Omega_\rho)} = \sum_{j\in\mathbb{N}} \|\nabla_{\tilde{x}} u_j\|^2_{L^2(\{|\tilde{x}|<\rho\})}$, and the transformation theorem together with the orthogonality of the trigonometric polynomials implies that (see, e.g., (A.35) in [Kir11])

$$\|\nabla_{\tilde{x}} u_j\|^2_{L^2(\{|\tilde{x}|<\rho\})} = 2\pi \sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{Z}} \int_0^\rho \left[\left(1+\frac{n^2}{r^2}\right)|\hat{u}(j,n,r)|^2 + |\hat{u}'(j,n,r)|^2\right] r\, dr. \tag{23}$$

Finally, the variational formulation of the eigenvalue problem (6) for (λ_j, ϕ_j) allows to show that

$$\begin{aligned}
\Big\|\frac{\partial u}{\partial x_3}\Big\|^2_{L^2(\Omega_\rho)} &= \int_{\{|\tilde{x}|<\rho\}} \sum_{j,j'\in\mathbb{N}} u_j(\tilde{x})\overline{u_{j'}(\tilde{x})}\, d\tilde{x} \int_0^H \Big(\frac{\omega^2}{c^2(x_3)} + \lambda_j^2\Big)\phi_j(x_3)\phi_{j'}(x_3)\, dx_3 \\
&= \sum_{j,j'\in\mathbb{N}} \int_{\{|\tilde{x}|<\rho\}} u_j(\tilde{x})\overline{u_{j'}(\tilde{x})}\, d\tilde{x} \left[\lambda_j^2 \int_0^H \phi_j\overline{\phi_{j'}}\, dx_3 + \int_0^H \frac{\omega^2}{c^2(x_3)}\phi_j(x_3)\phi_{j'}(x_3)\, dx_3\right] \\
&= \sum_{j\in\mathbb{N}} \lambda_j^2 \|u_j\|^2_{L^2(\{|\tilde{x}|<\rho\})} + \int_{\{|\tilde{x}|<\rho\}} \int_0^H \frac{\omega^2}{c^2(x_3)}\Big|\sum_{j\in\mathbb{N}} u_j(\tilde{x})\phi_j(x_3)\Big|^2 dx_3\, d\tilde{x} \\
&\leqslant \sum_{j\in\mathbb{N}} \lambda_j^2 \|u_j\|^2_{L^2(\{|\tilde{x}|<\rho\})} + \frac{\omega^2}{c_\mp^2}\|u\|^2_{L^2(\Omega_\rho)} = \sum_{j\in\mathbb{N}} \left[\lambda_j^2 + \frac{\omega^2}{c_\mp^2}\right] \|u_j\|^2_{L^2(\{|\tilde{x}|<\rho\})} \\
&= \sum_{j\in\mathbb{N}} \left[\lambda_j^2 + \frac{\omega^2}{c_\mp^2}\right] \sum_{n\in\mathbb{Z}} \int_0^\rho |\hat{u}(j,n,r)|^2\, r\, dr.
\end{aligned}$$

Since merely the first eigenvalues λ_j^2, $1 \le j \le J_* \in \mathbb{N}$ are negative, the corresponding finitely many terms can be estimated by the L^2-Norm of u, such that the last inequality together with Lemma 2.2(a) and (23) shows the claimed norm equivalence. Note that the equivalence constants depend on ω and J_*; however, they can be chosen uniformly for frequencies ω in any compact subset of $\mathbb{R}_{>0}$. □

It is well-known that the trace operator T, first defined for continuous functions $u \in C(\overline{\Sigma_\rho})$ by $u \mapsto u|_{\Sigma_\rho}$, can be extended to a bounded linear operator from $H^1(\Omega_\rho)$ into $H^{1/2}(\Sigma_\rho) \subset L^2(\Sigma_\rho)$, see, e.g., [McL00]. We introduce special subspaces of this trace space adapted to $H^1_W(\Omega_\rho)$,

$$V := \Big\{\psi \in L^2(\Sigma_\rho) : \psi = \sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{Z}} \hat\psi(j,n)e^{in\cdot}\phi_j, \sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{Z}}(1+|n|^2+|\lambda_j|^2)^{1/2}|\hat\psi(j,n)|^2 < \infty\Big\},$$

a Hilbert space with inner product $(\psi,\theta)_V = 2\pi\rho\sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{Z}}(1+|n|^2+|\lambda_j|^2)^{1/2}\hat\psi(j,n)\overline{\hat\theta(j,n)}$ for $\psi,\theta\in V$. The dual space V' with respect to $L^2(\Sigma_\rho)$ of V is equipped with inner product

$$(\psi,\theta)_{V'} = 2\pi\rho\sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{Z}}(1+|n|^2+|\lambda_j|^2)^{-1/2}\hat\psi(j,n)\overline{\hat\theta(j,n)} \qquad \text{for } \psi,\theta\in V',$$

and the dual evaluation $\langle\cdot,\cdot\rangle_{V'\times V}$ is simply abbreviated as $\langle\cdot,\cdot\rangle$.

Theorem 4.3. *The trace operator T is bounded and onto from $H^1(\Omega_\rho)$ into V.*

Proof. The Cauchy-Schwarz inequality applied to the representation of $u\in H^1_W(\Omega_\rho)$ from (21) shows that

$$\begin{aligned}\rho^2|\hat u(j,n,\rho)|^2 &= \int_0^\rho \frac{d}{dr}(r^2|\hat u(j,n,r)|^2)\,dr = 2\int_0^\rho r|\hat u(j,n,r)|^2\,dr + 2\,\mathrm{Re}\int_0^\rho \hat u(j,n,r)\overline{\hat u'(j,n,r)}r^2dr\\ &\le 2\int_0^\rho |\hat u(j,n,r)|^2 r\,dr + 2\rho\Big(\int_0^\rho |\hat u(j,n,r)|^2 r\,dr\Big)^{1/2}\Big(\int_0^\rho |\hat u'(j,n,r)|^2 r\,dr\Big)^{1/2}.\end{aligned}$$

Thus,

$$\begin{aligned}&(1+|n|^2+|\lambda_j|^2)^{1/2}|\hat u(j,n,\rho)|^2 \le C\int_0^\rho \left[(1+|n|^2+|\lambda_j|^2)|\hat u(j,n,r)|^2 + |\hat u'(j,n,r)|^2\right] r\,dr\\ &\le C\int_0^\rho \left[\left(1+\frac{|n|^2}{r^2}\right)|\hat u(j,n,r)|^2 + (1+|\lambda_j|^2)|\hat u(j,n,r)|^2 + |\hat u'(j,n,r)|^2\right] r\,dr.\end{aligned}$$

Due to (22), summation over $n\in\mathbb{Z}$ and $j\in\mathbb{N}$ shows that $\|Tu\|_V \le C\|u\|_{H^1(\Omega_\rho)}$. The definition of V together with Lemma 4.2 shows that T is onto. □

As usual, we restrict functions in $H^1_W(\Omega_\rho)$ to Σ_ρ without explicitly relying on the trace operator.

5 The Exterior Dirichlet-to-Neumann Operator

We will now construct and analyze an exterior Dirichlet-to-Neumann map on the surface Σ_ρ, mapping Dirichlet data on Σ_ρ to the Neumann data on Σ_ρ of the (unique) radiating solution in $\Omega\setminus\overline{\Omega_\rho}$ to the Helmholtz equation (1). To this end, we assume that Assumption 3.1 holds, i.e., no eigenvalue $\lambda_j^2\in\mathbb{R}$ vanishes, such that

$$u(x) = \sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{Z}} \hat u(j,n)\,H^{(1)}_n(i\lambda_j r)e^{in\varphi}\phi_j(x_3), \qquad x = \begin{pmatrix} r\cos\varphi\\ r\sin\varphi\\ x_3\end{pmatrix}\in\Omega\setminus\overline{\Omega_\rho}, \tag{24}$$

defines a formal solution to the Helmholtz equation (1) in $\Omega \setminus \overline{\Omega_\rho}$ that satisfies the boundary conditions (4). Indeed, the Hankel function $H_n^{(1)}$ of the first kind and order n satisfies Bessel's differential equation, such that

$$\tilde{x} \mapsto v_{j,n}(\tilde{x}) = H_n^{(1)}(i\lambda_j r) e^{in\varphi}, \qquad \tilde{x} = r \left(\begin{smallmatrix} \cos\varphi \\ \sin\varphi \end{smallmatrix} \right), \, r > 0,$$

satisfies the two-dimensional Helmholtz equation $(\Delta_{\tilde{x}} - \lambda_j^2) v_{n,j} = 0$ in $\mathbb{R}^2 \setminus \{0\}$, see [CK12]. The asymptotic expansion of the Hankel function for large arguments moreover shows that each term of u satisfies the radiation condition (16) such that (24) defines a radiating solution. Formally computing the normal derivative of u on Σ_ρ motivates the following definition.

Definition 5.1. *For $\psi \in V$ with Fourier coefficients $(\hat{\psi}(j,n))_{j\in\mathbb{N}, n\in\mathbb{Z}}$, the Dirichlet-to-Neumann operator Λ is defined by*

$$\Lambda : V \to V', \quad \psi \mapsto i \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{Z}} \lambda_j \frac{H_n^{(1)'}(i\lambda_j \rho)}{H_n^{(1)}(i\lambda_j \rho)} \hat{\psi}(j,n) e^{in\varphi} \phi_j(x_3), \qquad x = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ x_3 \end{pmatrix} \in \Sigma_\rho. \tag{25}$$

Theorem 5.2. *The operator Λ from (25) is well-defined and bounded from V into V'. For $\psi \in V$,*

$$u(x) = \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{Z}} \hat{\psi}(j,n) \frac{H_n^{(1)}(i\lambda_j r)}{H_n^{(1)}(i\lambda_j \rho)} e^{in\varphi} \phi_j(x_3), \qquad x = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ x_3 \end{pmatrix} \in \Omega \setminus \overline{\Omega_\rho}, \tag{26}$$

belongs to $H^1_{\mathrm{loc}}(\Omega \setminus \overline{\Omega_\rho})$ and there is $C = C(\rho) > 0$ independent of ψ such that $\|u\|_{H^1_{\mathrm{loc}}(\Omega\setminus\overline{\Omega_\rho})} \leq C \|\psi\|_V$. Further, u is a the unique weak solution to the Helmholtz equation in $\Omega \setminus \overline{\Omega_\rho}$ with boundary values $u|_{\Sigma_\rho} = \psi$ that satisfies both the waveguide boundary conditions (4) and the radiation condition (16).

Proof. (1) The boundedness of Λ follows almost literally as the proof of Lemma 2.1 in [AGL08] and will be omitted. The only additional fact required here is the estimate $|\lambda_j^2| \leq C j^2$ for $j \in \mathbb{N}$ from (9).

(2) In this part we abbreviate $\Omega_R \setminus \overline{\Omega_\rho}$ for $R > \rho$ by $\Omega_{\rho,R}$ and $\{\tilde{x} \in \mathbb{R}^2, \, \rho < |\tilde{x}| < R\}$ by $\tilde{\Omega}_{\rho,R}$. For $|\tilde{x}| > \rho$, the function u from (26) can be written as

$$u(x) = \sum_{j\in\mathbb{N}} u_j(\tilde{x}) \phi_j(x_3) \quad \text{with } u_j(\tilde{x}) = \sum_{n\in\mathbb{Z}} \hat{\psi}(j,n) \frac{H_n^{(1)}(i\lambda_j r)}{H_n^{(1)}(i\lambda_j \rho)} e^{in\varphi}, \quad \tilde{x} = \left(\begin{smallmatrix} r\cos\varphi \\ r\sin\varphi \end{smallmatrix} \right).$$

We first show that the latter series converges in $H^1(\Omega_{\rho,R})$ for arbitrary $R > \rho$, such that $u \in H^1_{\mathrm{loc}}(\Omega \setminus \overline{\Omega_\rho})$. As in the proof of Theorem 4.2 one shows that

$$\|u\|^2_{H^1(\Omega_{\rho,R})} \leq C \sum_{j\in\mathbb{N}} \left[\|u_j\|^2_{H^1(\tilde{\Omega}_{\rho,R})} + (1 + |\lambda_j|^2) \|u_j\|^2_{L^2(\tilde{\Omega}_{\rho,R})} \right]. \tag{27}$$

For $\xi \in \mathbb{R}$ and a parameter $c_0 = 1 + \max_{j\in\mathbb{N}}(-\lambda_j^2) < \infty$ we set

$$\alpha(\xi) = \begin{cases} \sqrt{c_0 - \xi^2} & \text{if } c_0 \geq \xi^2, \\ i\sqrt{\xi^2 - c_0} & \text{if } c_0 \geq \xi^2. \end{cases}$$

The latter function is then used to define

$$\tilde{v}_{n,\xi}(\tilde{x}) := \frac{H_n^{(1)}(r\alpha(\xi))}{H_n^{(1)}(\rho\alpha(\xi))} e^{in\varphi} \quad \text{for } \tilde{x} = r \left(\begin{smallmatrix} \cos\varphi \\ \sin\varphi \end{smallmatrix}\right) \in \tilde{\Omega}_{\rho,R},\ n \in \mathbb{Z} \text{ and } n \in \mathbb{Z}.$$

It is not difficult to see that the smooth function $\tilde{v}_{n,\xi}$ belongs to $H^1(\tilde{\Omega}_{\rho,R})$. Moreover, Lemma A6 in [CH07] states that there exists $C > 0$ independent of $\xi \in \mathbb{R}$ and $n \in \mathbb{Z}$ such that

$$\|\tilde{v}_{n,\xi}\|^2_{H^1(\tilde{\Omega}_{\rho,R})} \leq C(\rho, R)\,(1 + n^2 + \xi^2)^{1/2}.$$

Since $c_0 = 1 + \max_{j\in\mathbb{N}}(-\lambda_j^2)$ it holds $c_0 + \lambda_j^2 > 1$ for all $j \in \mathbb{N}$ such that there exists a unique positive solution $\xi_j > 0$ to $\xi_j^2 = c_0 + \lambda_j^2$. Thus, $\alpha(\xi_j) = i\lambda_j$ and $\xi_j^2 \leq C(k)(1 + |\lambda_j|^2)$, such that

$$\|\tilde{v}_{n,\xi_j}\|^2_{H^1(\tilde{\Omega}_{\rho,R})} \leq C\,(1 + n^2 + \xi_j^2)^{1/2} = C\,(1 + n^2 + c_0 + \lambda_j^2)^{1/2} \leq C\,(1 + n^2 + |\lambda_j|^2)^{1/2}$$

with $C = C(\rho, R, k)$ and

$$\|u_j\|^2_{H^1(\tilde{\Omega}_{\rho,R})} \leq \sum_{n\in\mathbb{Z}} |\hat{\psi}(j,n)|^2 \|\tilde{v}_{n,\xi_j}\|^2_{H^1(\tilde{\Omega}_{\rho,R})} \leq C(\rho, R, k) \sum_{n\in\mathbb{Z}} (1 + n^2 + |\lambda_j|^2)^{1/2} |\hat{\psi}(j,n)|^2.$$

Of course, the corresponding L^2-estimate $\|u_j\|^2_{L^2(\tilde{\Omega}_{\rho,R})} \leq \sum_{n\in\mathbb{Z}} |\hat{\psi}(j,n)|^2 \|\tilde{v}_{n,\xi_j}\|^2_{L^2(\tilde{\Omega}_{\rho,R})}$ holds as well and we conclude by (27) that

$$\|u\|^2_{H^1(\Omega_{\rho,R})} \leq C \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{Z}} \left[(1 + n^2 + |\lambda_j|^2)^{1/2} + (1 + |\lambda_j|^2)\|\tilde{v}_{n,\xi_j}\|^2_{L^2(\tilde{\Omega}_{\rho,R})}\right] |\hat{\psi}(j,n)|^2. \tag{28}$$

To estimate the L^2-norm of $\tilde{v}_{n,\xi_j}$ we note that

$$\|\tilde{v}_{n,\xi_j}\|^2_{L^2(\tilde{\Omega}_{\rho,R})} = 2\pi \int_\rho^R \left| \frac{H_n^{(1)}(i\lambda_j r)}{H_n^{(1)}(i\lambda_j \rho)} \right|^2 \frac{dr}{r} \leq \frac{2\pi}{\rho} \int_\rho^R \frac{\left|H_n^{(1)}(i\lambda_j r)\right|^2}{\left|H_n^{(1)}(i\lambda_j \rho)\right|^2}\, dr \leq 2\pi(R - \rho) \quad \text{for all } j \in \mathbb{N},$$

since $\left|H_n^{(1)}(i\lambda_j r)\right|^2 / \left|H_n^{(1)}(i\lambda_j \rho)\right| \leq 1$ for $r > 0$ by Lemma A2 in [CH07]. Moreover, if $j > J_*$ (the parameter J_* was defined in (12)), i.e., if $\lambda_j^2 > 0$, then $\left|H_n^{(1)}(i\lambda_j r)\right|^2 / \left|H_n^{(1)}(i\lambda_j \rho)\right|^2 \leq \exp(-(r-\rho)|\lambda_j|)$ for $r > \rho$ due to [CH07, Lem. A3] and

$$\|\tilde{v}_{n,\xi_j}\|^2_{L^2(\tilde{\Omega}_{\rho,R})} \leq \frac{2\pi}{\rho} \int_\rho^R e^{-(r-\rho)|\lambda_j|}\, dr \leq \frac{2\pi}{|\lambda_j|} \left[1 - e^{-(R-\rho)|\lambda_j|}\right] \leq \frac{4\pi}{|\lambda_j|} \leq \frac{C}{(1 + |\lambda_j|^2)^{1/2}}.$$

Together with (28), the last estimate shows that $\|u\|^2_{H^1(\Omega_{\rho,R})} \leq C\|\psi\|^2_V$.

The function u satisfies the Helmholtz equation in $\Omega_{\rho,R}$ in the classical sense, and for this reason also weakly, by construction of the eigenfunctions ψ_j to (7), since $\tilde{v}_{n,\xi_j}$ solves $(\Delta_{\tilde{x}} - \lambda_j^2)\tilde{v}_{n,\xi_j} = 0$ in $\{|\tilde{x}| > \rho\}$ and since the series in (26) was shown to converge in $H^1(\Omega_{\rho,R})$. The same argument shows that u satisfies the waveguide boundary conditions and it is obvious that $u|_{\Sigma_\rho} = \psi$. Well-known properties of Hankel and Kelvin functions show that $\tilde{v}_{n,\xi_j}$ is a radiating solution to the Helmholtz equation if $1 \leq j \leq J_*$, i.e., $\lambda_j^2 < 0$, whereas $\tilde{v}_{n,\xi_j}$ is bounded (and even exponentially decaying) if $j \geq J_*$, i.e., $\lambda_j^2 > 0$. This implies that u satisfies the radiation condition (16). To show uniqueness of u, consider the difference $w = \sum_{j\in\mathbb{N}} w_j \phi_j$ of u and a further radiating solution v to the same exterior boundary value problem. For arbitrary $j \in \mathbb{N}$, $w_j \in H^1_{\mathrm{loc}}(\{|\tilde{x}| > \rho\})$ satisfies $(\Delta_{\tilde{x}} - \lambda_j^2)\tilde{v}_{n,\xi_j} = 0$ in $\{|\tilde{x}| > \rho\}$ and vanishes on $\{|\tilde{x}| = \rho\}$. Moreover, if $\lambda_j^2 < 0$ the function w_j satisfies the two-dimensional Sommerfeld radiation condition (the first condition in (16)) and, for this reason, must vanish (see [CK12]). If $\lambda_j^2 > 0$ the function w_j is exponentially decaying in $\tilde{x}$ and belongs in particular to $H^1(\{|\tilde{x}| > \rho\})$. A partial integration shows that $\int_{\{|\tilde{x}|>\rho\}} (|\nabla_{\tilde{x}} w_j|^2 + \lambda_j^2 |w_j|^2)\, d\tilde{x} = 0$. In consequence, w vanishes as well, which implies uniqueness of the radiating solution. □

The next lemma formulates a weak coercivity result for Λ when applied to $\psi \in V$ with representation $\psi = \sum_{j,n} \hat{\psi}(j,n) \exp(in\,\cdot)\phi_j$.

Lemma 5.3. *There exist constants $C > 0$ and $c_0 > 0$ such that Λ is L^2-coercive at small frequencies: For $0 < \omega \leq C$ it holds that $-\langle \Lambda\psi, \psi\rangle \geq c_0 \omega \|\psi\|^2_{L^2(\Sigma_\rho)}$ for all $\psi \in V$.*

Proof. Due to (9) we can choose $C > 0$ so small such that the lower bound $(\pi(2j-1)/(2H))^2 - \omega^2/c_-^2$ of all eigenvalues $\lambda_j^2 = \lambda_j^2(\omega)$ is positive for $0 < \omega < C$. As in the proof of Lemma 2.2 in [AGL08] this implies that

$$-\langle \Lambda\psi, \psi\rangle \geq 2\pi\rho \sum_{j\in\mathbb{N}} \Big[\lambda_j |w(j,0)|^2 + c_1 \lambda_j |\hat{w}(j,1)|^2 + \sum_{n=2}^{\infty} \lambda_j \Big(\frac{\lambda_j \rho}{\lambda_j\rho + 2n} + \frac{n}{\lambda_j \rho}\Big)|\Big] \hat{w}(j,n)|^2,$$

with $\hat{w}(j,n) = \hat{\psi}(j,n) + \hat{\psi}(j,-n)$ for $n \neq 0$ and $\hat{w}(j,0) = \hat{\psi}(j,0)$, $j \in \mathbb{N}$. Due to the binomial formula,

$$\lambda_j^{1/2} \left(\frac{\lambda_j\rho}{\lambda_j\rho + 2n} + \frac{n}{\lambda_j\rho}\right) \geq 2\lambda_j^{1/2} \left(\frac{n}{\lambda_j\rho + 2n}\right)^{1/2} \geq \frac{2\lambda_j^{1/2}}{(\lambda_j\rho + 2)^{1/2}} \geq \frac{2\lambda_1^{1/2}}{(\lambda_1\rho + 2)^{1/2}} > 0, \quad n \in \mathbb{N},$$

because $n \mapsto n/(\lambda_j\rho + 2n)$ and $j \mapsto \lambda_j/(\lambda_j\rho + 2)$ increase in n and j, respectively. By Lemma 2.2(a) it holds for $j \in \mathbb{N}$ and $0 < \omega < \min\{(\pi c_-)/(4H), 1\}$ that

$$\frac{3}{c_-^2}\omega^2 \leq \omega^2 \left(\frac{\pi^2}{4\omega^2 H^2}(2j-1)^2 - \frac{1}{c_-^2}\right) \leq \lambda_j^2. \tag{29}$$

By possibly further reducing $0 < C < 1$, the latter estimate shows that $\lambda_j \geq c_* \omega$ holds for all $j \in \mathbb{N}$ and some $c_* > 0$. Further reducing $0 < C \leq \min(c_*^{-1/2}, (\pi c_-)/(2H))$ yields $\lambda_j^2 < 1$ due to (29), i.e., $\lambda_j \leq \sqrt{\lambda_j}$. In consequence, for some $c_0 > 0$ and all $\psi \in V$ there holds

$$-\langle \Lambda\psi, \psi\rangle \geq c_0 \sum_{j\in\mathbb{N}} \left[\lambda_j |w(j,0)|^2 + \lambda_j |\hat{w}(j,1)|^2 + \sqrt{\lambda_j} \sum_{n=2}^{\infty} |\hat{w}(j,n)|^2\right] \geq c_0\omega \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{Z}} |\hat{\psi}(j,n)|^2$$

and Plancherel's identity implies the claim. □

Lemma 5.4. *If $\omega > 0$ there exists $C = C(\omega) > 0$ such that there holds*

$$- \operatorname{Re} \langle \Lambda\psi, \psi \rangle \geq -2\pi\rho\, C \sum_{j=1}^{J_*} \sum_{n\in\mathbb{Z}} |\hat{\psi}(j,n)|^2 \geq -C\|\psi\|^2_{L^2(\Sigma_\rho)}$$

for all $\psi \in V$ with Fourier coefficients $\{\hat{\psi}(j,n)\}_{j\in\mathbb{N},\, n\in\mathbb{Z}}$ (see (12) for the definition of $J_* = J_*(\omega, c, H)$).

We omit the proof that precisely follows the arguments of the proof of Lemma 2.3 in [AGL08].

To be able to apply analytic Fredholm theory when establishing existence theory for the scattering problem (14–16) we finally show that $\Lambda = \Lambda_\omega$ depends holomorphically on ω.

Lemma 5.5. *For all $\omega_* > 0$ such that $\lambda_j(\omega_*) \neq 0$ for $j \in \mathbb{N}$ and all $\omega^* > 0$ small enough to satisfy the assumption of Lemma 5.3 there exists an open connected set $U \subset \mathbb{C}$ containing ω_* and ω^* such that $\omega \mapsto \Lambda_\omega$ is an analytic operator-valued function in U.*

Proof. Due to Theorem 8.12(b) in [Muj85] we merely need to show that

$$\begin{aligned}
\langle \Lambda\phi, \psi \rangle &= 2\pi i\rho \sum_{j\in\mathbb{N}} \lambda_j(\omega) \sum_{n\in\mathbb{Z}}^{\infty} \frac{H_n^{(1)'}(i\lambda_j(\omega)\rho)}{H_n^{(1)}(i\lambda_{\ell_j(\omega)}(\omega)\rho)} \hat{\phi}(j,n)\overline{\hat{\psi}(j,n)} \qquad (30)\\
&= 2\pi i\rho \sum_{j\in\mathbb{N}} \lambda_j(\omega) \sum_{n\in\mathbb{Z}}^{\infty} \Big[\frac{H_{n-1}^{(1)'}(i\lambda_j(\omega)\rho)}{H_n^{(1)}(i\lambda_j(\omega)\rho)} - \frac{n}{i\lambda_j(\omega)\rho}\Big] \hat{\phi}(j,n)\overline{\hat{\psi}(j,n)}, \quad \text{for } \phi,\, \psi \in V,
\end{aligned}$$

is a holomorphic function in an open connected set $U \subset \mathbb{C}$ that satisfies the properties claimed in the lemma. Relying on the index functions ℓ_j, $j \in \mathbb{N}$, Lemma 2.4 states that all eigenvalues $\omega \mapsto \lambda^2_{\ell_j(\omega)}(\omega)$ can be extended to holomorphic functions in some neighborhood U_0 of $\mathbb{R}_{>0}$. We choose $\delta_1 > 0$ such that $U_1 = \{z \in U,\, 0 \leq \operatorname{Re}(z) \leq \omega^* + 1,\, |\operatorname{Im}(z)| \leq \delta_1\} \subset U$ is connected, compact and contains ω_* and ω^*. Due to Theorem 2.4, the set $K_0 = \{\omega \in U_1 : \text{there is } j \in \mathbb{N} \text{ such that } \lambda_j^2(\omega) = 0\}$ is finite. Thus, by further reducing the parameter δ_1 we can assume without loss of generality that K_0 contains merely real numbers. Recall that the square root function $z \mapsto z^{1/2}$ that was defined for complex numbers via a branch cut at the positive real axis is holomorphic in the slit complex plane $\mathbb{C} \setminus i\mathbb{R}_{\geq 0}$. The roots $\omega \mapsto \lambda_{\ell_j(\omega)}(\omega)$ are hence holomorphic functions in the set $U_2 := \{z \in U_1 : \operatorname{Im} z < 0 \text{ if } \omega \in K_0\}$. Further restricting this set we define the open set $U_3 := \{z \in U_2,\, B(z, \delta_2) \subset U_2\}$ for a parameter $\delta_2 > 0$. For δ_2 small enough U_3 is open, connected and contains ω_* and ω^*. Recall that the Hankel function $z \mapsto H_n^{(1)}(z)$ and its derivative are holomorphic in the domain $\{z \in \mathbb{C},\, z \neq 0,\, -\pi/2 < \arg(z) < \pi\}$. The fraction $z \mapsto H_n^{(1)'}(z)/H_n^{(1)}(z)$ is holomorphic for $z \neq 0$ and $\arg(z) \in [0, \pi)$ since $z \mapsto H_n^{(1)}(z)$ does not possess zeros in this domain. Moreover, an infinite number of zeros of $z \mapsto H_n^{(1)}(z)$ in the lower complex half-plane is contained in the quadrant $-\pi < \arg(z) \leq -\pi/2$, while at most n zeros are contained in $-\pi/2 < \arg(z) \leq 0$, compare the paragraph on complex zeros of the Hankel function in [AS64, pg. 373–374]. If follows from [CS82, eq. (2.8)] or [AS64, pg. 374] that these finitely many zeros lie in the sector $-\pi/2 < \arg(z) \leq -\epsilon$ for some $\epsilon > 0$, independent of n, i.e., $z \mapsto H_n^{(1)'}(z)/H_n^{(1)}(z)$ is holomorphic in $\{z \neq 0,\, \arg(z) \in (-\epsilon, \pi + \epsilon)\}$. Since the numbers $i\lambda_j$ are either positive or purely imaginary with positive imaginary part we deduce that, upon reducing the parameter $\delta_1 > 0$ for the construction of $U_{1,2,3}$ a second time, the function $\omega \mapsto H_n^{(1)'}(i\lambda_{\ell_j(\omega)}(\omega)\rho)/H_n^{(1)}(i\lambda_{\ell_j(\omega)}(\omega)\rho)$ is holomorphic for $\omega \in U_3$.

Thus, each term in the series in (30) is holomorphic in U_3. Holomorphy of the entire series follows from the uniform and absolute convergence of this series: If we set

$$g_j(\omega) = \lambda_j(\omega) \sum_{n\in\mathbb{Z}} \Big[\frac{H_{n-1}^{(1)}(i\lambda_j(\omega)\rho)}{H_n^{(1)}(i\lambda_j(\omega)\rho)} - \frac{n}{i\lambda_j(\omega)\rho}\Big] \hat{\phi}(j,n)\overline{\hat{\psi}(j,n)} \tag{31}$$

$$= \lambda_j(\omega) \sum_{n\in\mathbb{Z}} \frac{H_{n-1}^{(1)}(i\lambda_j(\omega)\rho)}{H_n^{(1)}(i\lambda_j(\omega)\rho)} \hat{\phi}(j,n)\overline{\hat{\psi}(j,n)} - R_j(u,v), \quad R_j(u,v) := \sum_{n\in\mathbb{Z}} \frac{n}{i\rho} \hat{\phi}(j,n)\overline{\hat{\psi}(j,n)},$$

then $R_j(u,v)$ is a bounded sesquilinear form on V independent of ω. For all $j > J_*(\omega^*+1,c,H)$ and all $\omega \in U_3 \cap \mathbb{R}$ it holds that $i\lambda_j(\omega) \in i\mathbb{R}_{>0}$, such that

$$\left|\frac{H_{n-1}^{(1)}(i\lambda_j(\omega)\rho)}{H_n^{(1)}(i\lambda_j(\omega)\rho)}\right| = \left|\frac{K_{n-1}(\lambda_j(\omega)\rho)}{K_n(\lambda_j(\omega)\rho)}\right| \le C \qquad \text{for } \omega \in U_3,\ n \in \mathbb{Z},$$

due to [AGL08, Lemma A.2 & (A10)]. For $1 \le j \le J_*(\omega^*+1,c,H)$ the asymptotic expansion of the Hankel functions for large orders, see [AS64, (9.3.1)], implies that there is a constant $C > 0$ such that the last bound is uniformly valid for all $j \in \mathbb{N}$. We deduce that

$$|g_j(\omega)| \le \sum_{n\in\mathbb{Z}} (C|\lambda_j(\omega)| + n/\rho)\, |\hat{\phi}(j,n)\hat{\psi}(j,n)| \le C\|u\|_V\|v\|_V$$

since $\omega \mapsto \lambda_j(\omega)$ is holomorphic on U_0 and hence bounded on $U_3 \Subset U_0$. We deduce that the series in (31) converges absolutely and uniformly for each $\omega \in U_3$ and is hence a holomorphic function of ω.

As in the proof of Theorem 5.2 one shows that the series in (30) is uniformly convergent in $j \in \mathbb{N}$ as well, which finally implies the claim. □

6 Existence and Uniqueness of Solutions to the Scattering Problem

We have now prepared all tools to provide existence theory for weak solutions of the waveguide scattering problem (14–16). Plugging together these tools is rather standard and follows, e.g., the approach taken in [AGL08].

Assume that $u^i \in H^1_{W,\mathrm{loc}}(\Omega)$ is an incident field that solves the Helmholtz equation (13) weakly. Assume further that $u \in H^1_{W,\mathrm{loc}}(\Omega)$ solves (14) for all $v \in H^1_W(\Omega)$ with compact support such that $u^s = u - u^i$ satisfies the radiation conditions (16). Since (14) implies that $\Delta u = \operatorname{div} \nabla u \in L^2_{\mathrm{loc}}(\Omega)$ is locally square integrable, we can integrate by parts, to find that

$$0 = \int_\Omega \left(\Delta u + \frac{\omega^2}{c^2(x_m)}(1+q)u\right) \overline{v}\, dx \quad \text{holds for all } v \in H^1(\Omega) \text{ with compact support in } \Omega,$$

that is, $\Delta u + (\omega^2/c^2)(1+q)u = 0$ holds in $L^2(\Omega_R)$ for all $R > 0$. As the boundary of Ω is flat, elliptic regularity results, see, e.g., [McL00, Ch. 4], moreover imply that $u \in H^2_{\mathrm{loc}}(\Omega)$.

As $u^s = u - u^i$ is by assumption radiating, Theorem (5.2) implies that $[\partial u^s/\partial\nu]|_{\Sigma_\rho} = \Lambda(u^s|_{\Sigma_\rho})$ holds in V', where ν denotes the exterior unit normal vector to Ω_ρ. In consequence, the normal derivative of $u = u^i + u^s$ on Σ_ρ equals

$$\frac{\partial u}{\partial \nu} = \frac{\partial u^i}{\partial \nu} + \frac{\partial u^s}{\partial \nu} = \frac{\partial u^i}{\partial \nu} + \Lambda\left((u-u^i)\big|_{\Sigma_\rho}\right) \qquad \text{in } V'.$$

Multiplying $\Delta u + (\omega^2/c^2)(1+q)u = 0$ in Ω_ρ by $v \in H^1_W(\Omega_\rho)$ and integrating by parts in Ω_ρ we find

$$\begin{aligned} 0 &= \int_{\Omega_\rho} \left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_3)}(1+q)u\overline{v}\right) dx - \int_{\Sigma_\rho} \frac{\partial u}{\partial \nu}\overline{v}\, ds + \int_{\Gamma_{0,\rho}} \frac{\partial u}{\partial \nu}\overline{v}\, ds + \int_{\Gamma_{H,\rho}} \frac{\partial u}{\partial \nu}\overline{v}\, ds \\ &= \int_{\Omega_\rho} \left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_3)}(1+q)u\overline{v}\right) dx - \langle \Lambda(u), \overline{v}\rangle - \left\langle \left(\frac{\partial u^i}{\partial \nu} - \Lambda(u^i)\right), \overline{v} \right\rangle. \end{aligned}$$

Hence, a reformulation of the scattering problem (14–16) on Ω_ρ is to find $u \in H^1_W(\Omega_\rho)$ solving

$$B_\omega(u,v) := \int_{\Omega_\rho} \left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_3)}(1+q)u\overline{v}\right) dx - \langle \Lambda(u), \overline{v}\rangle \stackrel{!}{=} F(v) \qquad \text{for all } v \in H^1_W(\Omega_\rho), \quad (32)$$

where $F(v) = \langle (\partial u^i/\partial\nu - \Lambda(u^i)), \overline{v}\rangle$ for $v \in H^1_W(\Omega_\rho)$. Of course, (32) can also be considered for arbitrary continuous anti-linear forms $F \in H^1_W(\Omega_\rho)' := \mathcal{L}(H^1_W(\Omega_\rho), \mathbb{C})$ that can be used to tackle source problems instead of scattering problems.

Theorem 6.1 (Existence and uniqueness of solution)**.** *Assume that Assumption 3.1 holds.*

(1) The sesquilinear form B_ω and the anti-linear form F from (32) *are bounded on $H^1_W(\Omega_\rho)$ and B_ω satisfies a Gårding inequality. Thus, the Fredholm alternative holds: Whenever the variational problem* (32) *for $u^i = 0$ (i.e., for $F = 0$) possesses only the trivial solution, existence and uniqueness of solution holds for any $F \in H^1_W(\Omega_\rho)'$.*

(2) There exists $\omega_0 > 0$ such that the variational problem (32) *is uniquely solvable for all incident fields u^i for all frequencies $\omega \in (0, \omega_0)$.*

(3) The variational problem (32) *is uniquely solvable for all $F \in H^1_W(\Omega_\rho)'$ and all frequencies $\omega > 0$ except possibly for a discrete set of exceptional frequencies $\{\omega_\ell\}_{\ell=1}^{L_*} \subset \mathbb{R}_{>\omega_0}$, $L_* \in \mathbb{N} \cup \{+\infty\}$. If $L_* = \infty$, then $\omega_\ell \to \infty$ as $\ell \to \infty$.*

Proof. (1) Boundedness of Λ, see Lemma 5.2, and the trace estimate from Theorem 4.3, and the trace estimate $\|\partial u^i/\partial\nu\|_{H^{-1/2}(\Sigma_\rho)} \leq \|\mathrm{div}\nabla u^i\|_{L^2(\Omega_\rho)} \leq (\omega^2/c_-^2)\|u^i\|_{L^2(\Omega_\rho)}$ imply the boundedness of B_ω and F on $H^1_W(\Omega_\rho)$. The Gårding inequality for B_ω follows from the lower bound of Λ from Lemma 5.4,

$$\begin{aligned} \mathrm{Re}(B_\omega(u,u)) &\geq \|u\|^2_{H^1_W(\Omega_\rho)} - \left(\frac{\omega^2}{c_-^2}(1 + \|q\|_{L^\infty(\Omega_\rho)}) + 1 \right) \|u\|^2_{L^2(\Omega_\rho)} - \mathrm{Re}\left(\int_{\Sigma_\rho} \Lambda u \, \overline{v}\, ds \right) \\ &\geq \|u\|^2_{H^1_W(\Omega_\rho)} - \left(\frac{\omega^2}{c_-^2}(1 + \|q\|_{L^\infty(\Omega_\rho)}) + 1 \right) \|u\|^2_{L^2(\Omega_\rho)} - C\|u\|^2_{L^2(\Sigma_\rho)} \qquad \text{for } u \in H^1_W(\Omega_\rho). \end{aligned}$$

As the embedding of $H^1_W(\Omega_\rho)$ in $L^2(\Omega_\rho)$ is compact and since the trace operator from $H^1_W(\Omega_\rho)$ into $L^2(\Sigma_\rho)$ is compact due to the compact embedding of $H^{1/2}(\Sigma_\rho)$ in $L^2(\Sigma_\rho)$ the latter estimate provides indeed a Gårding inequality for B_ω. In consequence, the variational problem (32) is Fredholm of index zero and uniqueness of solution implies existence of solution together with continuous dependence of the solution on the right-hand side.

(2) L^2-coercivity of Λ for small frequencies (see Lemma 5.3) and Poincaré's inequality $\|u\|^2_{L^2(\Omega_\rho)} \leq (H^2/2)\|\nabla u\|^2_{L^2(\Omega_\rho)^m}$ for $u \in H^1_W(\Omega_\rho)$ imply that

$$\begin{aligned}\mathrm{Re}(B_\omega(u,u)) &\geq \|\nabla u\|^2_{L^2(\Omega_\rho)} - \frac{\omega^2}{c_-^2}(1+\|q\|_{L^\infty(\Omega_\rho)})\|u\|^2_{L^2(\Omega_\rho)} + c\omega\|u\|^2_{L^2(\Sigma_\rho)} \\ &\geq \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega_\rho)} + \frac{1}{H^2}\|u\|^2_{L^2(\Omega_\rho)} - \frac{\omega^2}{c_-^2}(1+\|q\|_{L^\infty(\Omega_\rho)})\|u\|^2_{L^2(\Omega_\rho)} \qquad \text{for } u \in H^1_W(\Omega_\rho).\end{aligned}$$

If $\omega > 0$ is small enough, the right-hand side of this estimate is strictly positive and B_ω is coercive on $H^1_W(\Omega_\rho)$ such that the Lax-Milgram Lemma implies the claim.

(3) Part (2) shows that (32) is uniquely solvable if $\omega > 0$ is less than some $\omega_0 > 0$. For larger ω we exploit holomorphy of $\omega \mapsto \Lambda_\omega$. More precisely, fix an arbitrary $\omega^* > 0$ such that $\lambda_j^2(\omega^*) \neq 0$ for all $j \in \mathbb{N}$ and some $\omega_* \in (0,\omega_0)$. Lemma 5.5 shows that there exists an open connected set $U \subset \mathbb{C}$ containing ω_* and ω^* such that $\omega \mapsto \Lambda_\omega$ is an analytic operator-valued function in U. In consequence, the entire sesquilinear form $B_\omega(u,v)$ depends analytically on ω in U. Moreover, (32) is uniquely solvable for frequency ω_*. Hence, analytic Fredholm theory implies that (32) is uniquely solvable for all $\omega \in U$ except possibly for a countable sequence of exceptional frequencies without accumulation point in U. In particular, there exists at most a countable set of real frequencies without finite accumulation point where uniqueness of solution fails. □

Theorem 6.2. *Assume that Assumption 3.1 holds.*

(1) If the variational problem (32) *is uniquely solvable for any* $F \in H^1_W(\Omega_\rho)'$*, then any solution* $u \in H^1_W(\Omega_\rho)$ *can be extended to the unique weak solution* $\tilde{u} \in H^1_{W,\mathrm{loc}}(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$ *of the waveguide scattering problem (14–16) by setting* $\tilde{u}|_{\Omega_\rho} = u|_{\Omega_\rho}$ *and*

$$\tilde{u}(x) = u^i(x) + \sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{Z}} \hat{u}(j,n)\frac{H_n^{(1)}(i\lambda_j r)}{H_n^{(1)}(i\lambda_j \rho)} e^{in\varphi}\phi_j(x_3) \quad \text{for } x = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ x_3\end{pmatrix} \text{ in } \Omega\setminus\overline{\Omega_\rho}, \tag{33}$$

with coefficients $\hat{u}(j,n)$ *defined by*

$$\hat{u}(j,n) = \int_0^H \int_0^{2\pi} (u-u^i)\begin{pmatrix}\rho\cos\varphi \\ \rho\sin\varphi \\ x_3\end{pmatrix} e^{-in\varphi}\phi_j \, d\varphi\, dx_3, \qquad j\in\mathbb{N},\ n\in\mathbb{Z}. \tag{34}$$

(2) If $\mathrm{Im}(q) > 0$ *on a non-empty open subset* D *of* Ω_ρ*, then* (32) *and the scattering problem (14–16) are both uniquely solvable for all incident fields.*

Proof. In this proof, we indicate by $(\cdot)|^{\pm}_{\Sigma_\rho}$ if a trace on Σ_ρ is taken from the inside (-) or from the outside (+) of Ω.

(1) Assume that $u \in H^1_W(\Omega_\rho)$ is the unique solution (32). As in the beginning of this section, we choose $v \in H^1_W(\Omega_\rho)$ such that $v|_{\Sigma_\rho} = 0$ and integrate by parts in (32) to find that

$$0 = -\int_{\Omega_\rho}\left(\Delta u + \frac{\omega^2}{c^2(x_3)}(1+q)u\right)\overline{v}\,dx + \int_{\Gamma_H\cap\{|\tilde{x}|<\rho\}} \frac{\partial u}{\partial x_3}\overline{v}\,ds.$$

Integrating now a second time by parts for a test function $v \in H^1_W(\Omega_\rho)$ and exploiting the definition of the right-hand side F in (32) then shows that

$$\left\langle \left(\frac{\partial u}{\partial \nu} - \Lambda(u) \right), \overline{v} \right\rangle = \left\langle \left(\frac{\partial u^i}{\partial \nu} - \Lambda(u^i) \right), \overline{v} \right\rangle \qquad \text{for all } v \in H^1_W(\Omega_\rho). \tag{35}$$

Now we define $u^s \in H^1_W(\Omega_\rho)$ by $u = u^i + u^s$ such that (35) and the surjectivity of the trace operator imply that $(\partial u^s/\partial\nu)|_{\Sigma_\rho} = \Lambda(u^s|_{\Sigma_\rho})$ holds in V'. In $\Omega \setminus \overline{\Omega}_\rho$ we define u^s by the series in (33), such that $\tilde{u} = u^i + u^s$ holds in $\Omega \setminus \overline{\Omega}_\rho$. By the radiation condition (16) for $(u - u^i)\big|_{\Omega_\rho}$, the trace trace estimate from Theorem 4.3, and the representation of functions in V we note that the coefficients $\hat{u}(j,n)$ in (34) are defined such that

$$u^s|^-_{\Sigma_\rho} = (u - u^i)\big|^-_{\Sigma_\rho} = \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{Z}} \hat{u}(j,n) e^{in\cdot} \phi_j = u^s|^+_{\Sigma_\rho} = (u - u^i)\big|^+_{\Sigma_\rho}, \qquad \text{holds in } V.$$

This implies that that $u|^-_{\Sigma_\rho}$ equals the restriction $\tilde{u}|^+_{\Sigma_\rho}$, i.e., the extension $\tilde{u}$ is continuous over Σ_ρ in the trace sense. By construction of Λ and $\tilde{u}$, it follows from Theorem (5.2) that $\tilde{u} \in H^1_{W,\mathrm{loc}}(\Omega)$ is a radiating solution to the Helmholtz equation in $\Omega \setminus \overline{\Omega}_\rho$ with normal derivative

$$\frac{\partial \tilde{u}}{\partial \nu}\bigg|^+_{\Sigma_\rho} = \left[\frac{\partial u^i}{\partial \nu} + \frac{\partial u^s}{\partial \nu}\right]\bigg|^+_{\Sigma_\rho} = \left[\frac{\partial u^i}{\partial \nu}\right]\bigg|^+_{\Sigma_\rho} + \Lambda(u^s|^+_{\Sigma_\rho}) = \left[\frac{\partial u^i}{\partial \nu}\right]\bigg|^-_{\Sigma_\rho} + \Lambda(u^s|^-_{\Sigma_\rho}) = \left[\frac{\partial \tilde{u}}{\partial \nu}\right]\bigg|^-_{\Sigma_\rho} \qquad \text{in } V'.$$

As the normal derivative of $\tilde{u}$ across Σ_ρ is hence also continuous in the trace sense, the latter function is a weak solution in $H^1_{W,\mathrm{loc}}(\Omega)$ to the Helmholtz equation that solves the waveguide scattering problem (14–16). Elliptic regularity results [McL00, Chapter 4] show that $\tilde{u} \in H^2_{\mathrm{loc}}(\Omega)$. Uniqueness of this solution follows from uniqueness of solution to (32).

(2) We merely need to show that $\mathrm{Im}(q) > 0$ on a non-empty open subset $D \subset \Omega_\rho$ implies that any solution $u \in H^1_W(\Omega_\rho)$ to (32) with $F = 0$ vanishes. Extend such a u by (33) to a solution in $H^1_{W,\mathrm{loc}}(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$ of the scattering problem with $u^i = 0$ and, by abuse of notation, call the extended function again u. Taking the imaginary part of (32) with $v = u$ and integrating by parts in $\Omega_R \setminus \overline{\Omega_\rho}$ yields

$$\begin{aligned} 0 = \mathrm{Im}\, B_\omega(u,u) &= -\int_{\Omega_\rho} \frac{\omega^2\,\mathrm{Im}(q)}{c^2(x_3)} |u|^2\,dx - \mathrm{Im} \int_{\Sigma_\rho} \Lambda(u)\,\overline{u}\,ds = \int_{\Omega_\rho} \frac{\omega^2\,\mathrm{Im}(q)}{c^2(x_3)} |u|^2\,dx - \mathrm{Im} \int_{\Sigma_\rho} \frac{\partial u}{\partial \nu}\,\overline{u}\,ds \\ &= \int_{\Omega_\rho} \frac{\omega^2\,\mathrm{Im}(q)}{c^2(x_3)} |u|^2\,dx + \mathrm{Im} \int_{\Omega_R\setminus\overline{\Omega_\rho}} \left[|\nabla u|^2 - \frac{\omega^2}{c^2(x_3)} |u|^2 \right] dx - \mathrm{Im} \int_{\Sigma_R} \frac{\partial u}{\partial \nu}\,\overline{u}\,ds, \qquad R > \rho. \end{aligned}$$

The expansion $u = \sum_{j\in\mathbb{N}} u_j \phi_j$ of $u \in H^1_{W,\mathrm{loc}}(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$ shows that

$$\int_{\Sigma_R} \frac{\partial u}{\partial \nu}\,\overline{u}\,ds = \sum_{j\in\mathbb{N}} \int_{|\tilde{x}|=R} \frac{\partial u_j}{\partial \nu}\,\overline{u_j}\,ds.$$

If $1 \le j \le J_*$, then u_j is a solution to a Helmholtz equation with positive wave number in $\mathbb{R}^2$ that satisfies the Sommerfeld radiation condition due to (16) and it is well-known (see, e.g.,[CK12]) that this implies that $\operatorname{Im} \int_{|\tilde{x}|=R} (\partial u_j / \partial \nu) \overline{u_j} \, ds \geq 0$ (the latter expression is a multiple of the L^2-norm of the far field pattern of u_j). For $j > J_*$, u_j is a bounded and hence exponentially decreasing solution to a Helmholtz equation with negative wave number (this follows, e.g., from the estimates of the Hankel functions in the proof of Theorem 5.2), such that $\operatorname{Im} \int_{|\tilde{x}|=R} (\partial u_j / \partial \nu) \overline{u_j} \, ds \to 0$ as $R \to \infty$. Choosing $R > 0$ large enough, we conclude that

$$0 = \operatorname{Im} B_\omega(u,u) = \int_{\Omega_\rho} \frac{\omega^2 \operatorname{Im}(q)}{c^2(x_3)} |u|^2 \, dx + \operatorname{Im} \int_{\Sigma_R} \frac{\partial u}{\partial \nu} \overline{u} \, ds \geq 0.$$

Thus, u vanishes on the open, nonempty set D. The unique continuation property for solutions to $\Delta u + (\omega^2 / c^2(x_3))(1+q)u = 0$, see [JK85, Th. 6.3, Rem. 6.7], implies that u vanishes in all of Ω. □

Remark 6.3. *(1) If the inhomogeneous medium described by the contrast q is replaced by an impenetrable obstacle $D \Subset \Omega_\rho$ with either Dirichlet, Neumann or impedance boundary conditions, the approach from the beginning of this section yields a variational problem for the total field restricted to $\Omega_\rho \setminus \overline{D}$. For a Neumann or impedance boundary condition this problem is posed in $H^1_W(\Omega_\rho \setminus \overline{D})$, whereas for a Dirichlet boundary condition, the variational space additionally needs to incorporate homogeneous Dirichlet boundary conditions on ∂D. The existence and uniqueness results of Theorem 6.1 and Theorem 6.2(1) hold for those scattering problems in an analogous way.*

(2) The Gårding inequality from Theorem 6.1(1) implies that a conforming Galerkin scheme applied to (32) *converges if the sequence of discrete variational spaces is dense in $H^1_W \Omega_\rho$.*

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