Analytical Characterization and Numerical Approximation of Interior Eigenvalues for Impenetrable Scatterers from Far Fields

Armin Lechleiter* Stefan Peters†

February 4, 2014

Abstract

We characterize the interior eigenvalues of a class of impenetrable, non-absorbing scattering objects from the spectra of the corresponding far field operators for a continuum of wave numbers. Our proof simplifies arguments from the original proof for Dirichlet scattering objects given in [Eckmann and Pillet, *Commun. Math. Phys.*, 1995:283–313] and furthermore extends to the cases of Neumann and Robin scattering objects. Further, the analytical characterization of interior eigenvalues of a scatterer can be exploited numerically: We present an algorithm that approximates interior eigenvalues from far field data without knowing the scattering object, we give several numerical examples for different scatterers and sound-hard as well as sound-soft boundary conditions, and we finally show through numerical examples that this algorithm remains stable under noise.

1 Introduction

It is well-known that direct and inverse exterior scattering problems from impenetrable scatterers are connected to the interior eigenvalues of the scattering object. Integral equation methods for the solution of exterior scattering problems might for instance fail at interior eigenvalues, see, e.g., [6, 15]. Further, several solution techniques for inverse shape identification problems as the linear sampling method, the factorization method, or the method of singular sources might as well fail at interior eigenvalues [11] (methods working at interior eigenvalues include, e.g., [5, 13, 9]). To indicate a third connection, it is well-known that the far field operator for impenetrable scattering problems with wave number k possesses an eigenvalue zero if and only if there is an interior eigenfunction of the Laplacian for the eigenvalue $-k^2$ that can be represented as a Herglotz wave function. Several methods tried to exploit this relationship to find interior eigenvalues from the knowledge of far field operators for many wave numbers, see, e.g. [2], [1], or [3, Chapter 4]. Exploiting this link is, however, subtle for at least two reasons: First, interior eigenfunctions can in general not be represented as Herglotz wave functions. Second, the far field operator is a compact operator. Hence, zero always belongs (and is in fact equal) to the essential spectrum of the far field operator. Thus, it is numerically difficult to decide whether zero merely belongs to the essential spectrum or even to the point spectrum.

The aim of this paper is to give a precise mathematically characterization of interior eigenvalues for Neumann- and Robin-type obstacles using multi-frequency data consisting of a continuum of far field patterns for positive wave numbers. This relation is called an inside-outside duality, following the terminology of Eckmann and Pillet in [7, 8], where this duality of the exterior scattering problem and the interior eigenvalue problem has been found for Dirichlet and Neumann obstacles. Our proof

*Center for Industrial Mathematics, University of Bremen, Bremen, Germany
†Center for Industrial Mathematics, University of Bremen, Bremen, Germany

to some extent simplifies the arguments from [7]; in particular, the proof avoids the Cayley transform as well as the scattering operator and is entirely based on the far field operator. We will, however, still rely on the fact that the scatterer is non-absorbing to profit from the normality of the far field operator. Let us note that the recent paper [12] already extended Eckmann's and Pillet's proof to penetrable scattering problems, proving an inside-outside duality for transmission eigenvalues under rather strong assumptions on the contrast. Further, we show in several numerical examples that the characterization provided by the inside-outside duality can be exploited numerically to detect interior eigenvalues from far field data for Dirichlet and Neumann obstacles. Finally, several numerical examples show that the algorithm we give to detect interior eigenvalues from far field data is able to handle noisy data. A detailed regularization analysis for this nonlinear algorithm goes beyond the scope of this paper and is postponed to a future paper.

Let us briefly indicate the results presented in the sequel of the paper. We consider a bounded Lipschitz domain $D \subset \mathbb{R}^3$ with connected complement representing the scattering object and a positive wave number $k > 0$. We consider an exterior time-harmonic scattering problem for the Helmholtz equation together with a non-absorbing boundary condition implemented in a boundary operator $\mathcal{B}$ representing either Dirichlet, Neumann, or Robin boundary conditions,

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad \mathcal{B}u = 0 \quad \text{on } \partial D.$$

The total wave field u can be split into a sum of an incident incoming plane wave $u^i(x,\theta) = \exp(\mathrm{i}k\,\theta \cdot x)$ with direction $\theta \in \mathbb{S}^2 = \{x \in \mathbb{R}^3,\, |x| = 1\}$ and a scattered field $u^s(\cdot,\theta)$ that satisfies Sommerfeld's radiation condition

$$\left(\frac{\partial u^s}{\partial |x|} - \mathrm{i}k u^s\right) = \mathcal{O}\left(\frac{1}{|x|^2}\right) \text{ as } |x| \to \infty, \qquad \text{uniformly in } \hat{x} = \frac{x}{|x|} \in \mathbb{S}^2. \tag{1}$$

In the following, we call solutions to the Helmholtz equation that satisfy (1) radiating solutions. As a consequence of this radiation condition, the scattered wave $u^s(\cdot,\theta)$ behaves like an outgoing spherical wave,

$$u^s(x,\theta) = \frac{\exp(\mathrm{i}k|x|)}{4\pi|x|}\left(u^\infty(\hat{x},\theta) + \mathcal{O}(1/|x|)\right) \text{ as } |x| \to \infty,$$

with a far field pattern $u^\infty(\cdot,\theta) \in L^2(\mathbb{S}^2)$. The far field operator is defined by

$$F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2), \qquad Fg(\hat{x}) := \int_{\mathbb{S}^2} u^\infty(\hat{x},\theta) g(\theta)\,\mathrm{d}S(\theta), \qquad \hat{x} \in \mathbb{S}^2, \tag{2}$$

and it is well-known that for the scattering problem introduced above this operator is compact and normal, that is, there exists a complete orthonormal eigensystem $(\lambda_j, g_j)_{j\in\mathbb{N}}$ such that $Fg = \sum_{j\in\mathbb{N}} \lambda_j (g, g_j) g_j$ for all $g \in L^2(\mathbb{S}^2)$. It is also well-known that all eigenvalues λ_j lie on a circle of radius $8\pi^2/k$ with center $8\pi^2\mathrm{i}/k$ in the complex plane, see, e.g., [11]. For all three boundary conditions considered here, the far field operator satisfies a factorization of the form $F = GTG^*$ with a solution operator G mapping boundary data ψ on ∂D to the far field of the radiating solution of the following scattering problem,

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad \mathcal{B}(v) = \psi \quad \text{on } \partial D. \tag{3}$$

The precise form of T, in particular the correct space for ψ, depends on the boundary conditions implemented in $\mathcal{B}$. All three operators $F = F(k)$, $G = G(k)$ and $T = T(k)$ obviously depend on the wave number. In all cases under investigation, two important properties hold: First, the eigenvalues $\lambda_j = \lambda_j(k)$ converge to zero either from the left or from the right as $j \to \infty$, that is, $\operatorname{Re}\lambda_j \gtrless 0$ for j large enough. This allows to order the phases of the λ_j and to speak of a smallest or a largest phase – if, for instance, $\operatorname{Re}\lambda_j < 0$ for large $j \in \mathbb{N}$, then the smallest phase $\vartheta_* = \min_{j\in\mathbb{N}} \vartheta_j$ of the eigenvalues $\lambda_j = r_j \exp(\mathrm{i}\vartheta_j)$, $\vartheta_j \in [0,\pi]$, is well-defined. Second, one can show that k^2 is an interior eigenvalue of the Laplacian in D,

$$\Delta u + k^2 u = 0 \quad \text{in } D, \qquad \mathcal{B}(u) = 0 \quad \text{on } \partial D, \tag{4}$$

if and only if there exists $\varphi \neq 0$ such that $\operatorname{Im}(T(k)\varphi,\, \varphi) = 0$. Moreover, $\operatorname{Im}(T(k)\varphi,\, \varphi)$ is always non-negative and the dimension of the eigenspace to (4) equals the dimension of the space of all φ such that $\operatorname{Im}(T(k)\varphi, \varphi) = 0$. Together, these two properties allow to show that a number $k_0^2 > 0$ is an interior eigenvalue if and only if the smallest phase ϑ_* tends to 0 if $k > 0$ tends to k_0 from below (if the largest phase is well-defined, it tends to π as k tends to k_0 from above).

Crucial tools in our analysis will be single- and double layer operators: Using the radiating fundamental solution Φ to the Helmholtz equation, these potentials are defined via

$$\operatorname{SL}\varphi(x) := \int_{\partial D} \Phi(x,y)\varphi(y)\,\mathrm{d}S(y), \quad x \in \mathbb{R}^3 \setminus \partial D, \qquad \Phi(x,y) := \frac{\exp(\mathrm{i}k|x-y|)}{4\pi|x-y|}, \quad x \neq y, \tag{5}$$

$$\operatorname{DL}\psi(x) := \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\psi(y)\,\mathrm{d}S(y), \quad x \in \mathbb{R}^3 \setminus \partial D. \tag{6}$$

Here and later on, ν denotes the outwards pointing unit normal vector field to D. It is well-known [14] that SL and DL are bounded from $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$ into $H^1(B_R)$ and $H^1(B_R \setminus \partial D)$ for any ball B_R centered in the origin with radius $R > 0$, respectively. Both potentials are smooth solutions to the Helmholtz equation in $\mathbb{R}^3 \setminus \partial D$ and radiating in $\mathbb{R}^3 \setminus \overline{D}$. Let us denote the exterior and interior trace operator on ∂D by $[\cdot]|^+$ and $[\cdot]|^-$, respectively. Then it is also well-known that the traces $\operatorname{SL}\varphi\big|^\pm, \partial\operatorname{SL}\varphi/\partial\nu\big|^\pm$ and $\operatorname{DL}\varphi\big|^\pm, \partial\operatorname{DL}\varphi/\partial\nu\big|^\pm$ are given by

$$\operatorname{SL}\varphi\Big|^\pm = S\varphi \qquad \text{in } H^{1/2}(\partial D), \tag{7}$$

$$\operatorname{DL}\psi\Big|^\pm = \pm\frac{1}{2}\psi + K\psi \qquad \text{in } H^{1/2}(\partial D), \tag{8}$$

$$\frac{\partial\operatorname{SL}\varphi}{\partial\nu}\Big|^\pm = \mp\frac{1}{2}\varphi + K'\varphi \qquad \text{in } H^{-1/2}(\partial D), \tag{9}$$

$$\frac{\partial\operatorname{DL}\psi}{\partial\nu}\Big|^\pm = N\psi \qquad \text{in } H^{-1/2}(\partial D), \tag{10}$$

where the boundary integral operator S is bounded from $H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$, K is bounded on $H^{1/2}(\partial D)$, K' is bounded on $H^{-1/2}(\partial D)$ and N in bounded from $H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$.

To simplify notation, let us in the sequel denote both the duality pairing between $H^{\pm 1/2}(\partial D)$ that extends the $L^2(\partial D)$-inner product and the inner product itself by $(\cdot,\cdot)$ or $(\cdot,\cdot)_{L^2(\partial D)}$. The inner product on $L^2(\mathbb{S}^2)$ is denoted by $(\cdot,\cdot)_{L^2(\mathbb{S}^2)}$ or by $(\cdot,\cdot)$ if there is no danger of confusion. As mentioned above, the open ball of radius R centered in the origin is denoted by B_R.

This paper is structured as follows: In the next Section 2 we briefly prove a characterization for Dirichlet eigenvalues of the Laplacian from far field data, simplifying the original proof from [7]. In Section 3 we prove a similar characterization for Robin Neumann eigenvalues of the Laplacian form the corresponding far field data. In Section 4 we show how to exploit these algorithms numerically to obtain estimates for the interior eigenvalues from discrete far field data without knowing the scattering object or the boundary condition.

2 Characterizing Dirichlet Eigenvalues from Far Field Data

In this section, we want to briefly present the proof of the inside-outside duality for Dirichlet scattering objects. Despite we follow in principle the arguments from [7], we believe that there is an interest in their simplification (the Cayley transform used in [7] can for instance be avoided). Additionally, we rewrite these arguments using the common notation of the inverse scattering community. Apart from giving an easier access to the proof, we also prepare notation for the numerical examples on the detection of eigenvalues from far field data for the Dirichlet problem.

As noted in the introduction, the scatterer $D \subset \mathbb{R}^3$ is a bounded Lipschitz domain with connected complement and we consider an exterior Dirichlet scattering problem

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad u = 0 \quad \text{on } \partial D,$$

that is, $\mathcal{B}(u) = u$. We denote again by $u^s(\cdot, \theta)$ the radiating scattered field for an incident plane wave with direction θ, by $u^\infty(\cdot, \theta) \in L^2(\mathbb{S}^2)$ its far field pattern, and by F the far field operator, see (2). Recall the single-layer operator S on ∂D from (7). It is well-known that F can be factorized as $F = -GS^*G^*$ where $G : H^{1/2}(\partial D) \to L^2(\mathbb{S}^2)$ is a solution operator mapping ψ to the far field pattern v^∞ of the unique radiating solution $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D})$ to

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad v = \psi \quad \text{on } \partial D.$$

Lemma 1. *For all $k > 0$ and all $\varphi \in H^{-1/2}(\partial D)$ it holds that*

$$\mathrm{Im}\,(\varphi, S\varphi)_{L^2(\partial D)} \leq 0.$$

The function $\varphi \mapsto \mathrm{Im}\,(\varphi, S\varphi)$ vanishes at $\varphi \neq 0$ if and only if $S\varphi = 0$, that is, if and only if k^2 is an interior Dirichlet eigenvalue of the Laplacian in D. If $\mathrm{Im}\,(\varphi, S\varphi) = 0$ for some $0 \neq \varphi \in H^{-1/2}(\partial D)$, then the restriction of $w = \mathrm{SL}\,\varphi$ to D is an eigenfunction of the Dirichlet-Laplacian, while for any eigenfunction $w \in H^1_0(D)$ it holds that $\varphi = \partial w/\partial\nu|^- \in H^{-1/2}(\partial D) \neq 0$ satisfies $\mathrm{Im}\,(\varphi, S\varphi) = 0$.

Proof. It is well-known that $\mathrm{Im}\,(\varphi, S\varphi) \leq 0$ for all $\varphi \in H^{-1/2}(\partial D)$ due to the Sommerfeld radiation condition, while $\mathrm{Im}\,(\varphi, S\varphi) = 0$ for some $\varphi \neq 0 \in H^{-1/2}(\partial D)$ if and only if k^2 is a Dirichlet eigenvalue of the Laplacian in D, see, e.g., [11, Lemma 1.14], [15, Theorem 3.9.1] or [14, Chapter 9]. Further, the representation theorem implies that any such φ yields rise to an interior Dirichlet eigenfunction via the single layer operator restricted to D, $w = \mathrm{SL}\,\varphi|_D$. Vice versa, if $w \in H^1_0(D)$ is an eigenfunction, then $\varphi = \partial w/\partial\nu \in H^{-1/2}(\partial D) \neq 0$ satisfies, again due to the representation formula, $S\varphi = 0$ and therefore $\mathrm{Im}\,(\varphi, S\varphi) = 0$. □

Remark 2. *Lemma 1 in particular implies that the dimension of the eigenspace of the negative Dirichlet-Laplacian in D for the eigenvalue k^2 equals the dimension of the kernel of $\varphi \mapsto \mathrm{Im}\,(\varphi, S\varphi)$ and that the latter kernel is a linear space.*

Recall from the introduction that the eigenvalues λ_j of F all lie on the circle $\{z \in \mathbb{C} : |z - 8\pi^2\mathrm{i}/k| = 8\pi^2/k\}$ and they converge to 0 as $j \to \infty$ since F is compact. The fact that $\mathrm{Im}\,(\varphi, S\varphi) \leq 0$ allows to show that if k^2 is not a Dirichlet eigenvalue of D, then there is $N = N(k)$ such that $\mathrm{Re}\,\lambda_j < 0$ for $j > N$, see [11, Theorem 1.23] for a proof. Roughly speaking, the eigenvalues λ_j hence converge to zero from the left. We represent these eigenvalues in polar coordinates, such that

$$\lambda_j = r_j \exp(\mathrm{i}\vartheta_j), \qquad r_j \geq 0, \quad \vartheta_j \in [0, \pi].$$

For completeness, we define $\vartheta_j = \pi$ whenever $r_j = 0$ although this case will not be of interest in the sequel. If k^2 is not an interior Dirichlet eigenvalue, then all λ_j are different from zero and the phases ϑ_j are all included in the open interval $(0, \pi)$. Moreover, since $\mathrm{Re}\,\lambda_j < 0$ for large $j \in \mathbb{N}$ these phases converge to π as $j \to \infty$ and there is hence a smallest phase

$$\vartheta_* = \vartheta_{j_*} = \min_{j \in \mathbb{N}} \vartheta_j$$

among all phases ϑ_j. The eigenvalue λ_{j_*} with smallest phase is from now on denoted by λ_*.

Theorem 3. *If k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D, then*

$$\cot\vartheta_* = \max_{g\in L^2(\mathbb{S}^2)} \frac{\mathrm{Re}\,(Fg,\, g)_{L^2(\mathbb{S}^2)}}{\mathrm{Im}\,(Fg,\, g)_{L^2(\mathbb{S}^2)}}. \tag{11}$$

The maximum is attained at any eigenvector to the eigenvalue λ_* of F with smallest phase.

By abuse of notation we did not explicitly exclude the zero vector from the maximum in (11). Note also that the denominator $\mathrm{Im}\,(Fg,\, g)_{L^2(\mathbb{S}^2)}$ is positive if k^2 is not a transmission eigenvalue due to Lemma 1 and the factorization $F = -GS^*G^*$: $\mathrm{Im}\,(Fg,\, g) = -\mathrm{Im}\,(G^*g,\, SG^*g) > 0$ for $g \neq 0$ since G^* is injective. The proof of the last theorem relies on the following lemma.

Lemma 4. *Assume that f, g are continuous functions on $I := (0, \beta) \subset \mathbb{R}$ such that g takes positive values and that $\alpha \mapsto f(\alpha)/g(\alpha)$ is strictly monotonically decreasing on I. Assume further that $(\alpha_j)_{j\in\mathbb{N}} \subset I$ is a sequence such that $\alpha_j \geq \alpha_* > 0$ for all $j \in \mathbb{N}$. Further let $(c_j)_{j\in\mathbb{N}}$ be a sequence of non-negative numbers. If both series $\sum_{j\in\mathbb{N}} c_j f(\alpha_j)$ and $\sum_{j\in\mathbb{N}} c_j g(\alpha_j)$ are unconditionally convergent, then*

$$\frac{\sum_{j\in\mathbb{N}} c_j f(\alpha_j)}{\sum_{j\in\mathbb{N}} c_j g(\alpha_j)} \leq \frac{f(\alpha_*)}{g(\alpha_*)}.$$

Equality holds if and only if $c_j = 0$ whenever $\alpha_j \neq \alpha_*$ and if there is at least one α_j that equals α_*.

Proof. Due to the monotonicity of $\alpha \mapsto f(\alpha)/g(\alpha)$,

$$\frac{f(\alpha_j)}{g(\alpha_j)} \leq \frac{f(\alpha_*)}{g(\alpha_*)} \tag{12}$$

for all $j \in \mathbb{N}$. In particular, since $g(\alpha_j)$ is positive, $f(\alpha_j) \leq f(\alpha_*)\, g(\alpha_j)/g(\alpha_*)$ for all $j \in \mathbb{N}$, that is,

$$\sum_{j\in\mathbb{N}} c_j f(\alpha_j) \leq \sum_{j\in\mathbb{N}} c_j \frac{f(\alpha_*)}{g(\alpha_*)} g(\alpha_j) = \frac{f(\alpha_*)}{g(\alpha_*)} \sum_{j\in\mathbb{N}} c_j g(\alpha_j).$$

Since $\sum_{j\in\mathbb{N}} c_j g(\alpha_j)$ is a positive number, the latter inequality implies that

$$\frac{\sum_{j\in\mathbb{N}} c_j f(\alpha_j)}{\sum_{j\in\mathbb{N}} c_j g(\alpha_j)} \leq \frac{f(\alpha_*)}{g(\alpha_*)}. \tag{13}$$

The strict monotonicity of $\alpha \mapsto f(\alpha)/g(\alpha)$ yields that equality in (12) holds if and only if $\alpha_j = \alpha_*$. Thus, equality in (13) holds if and only if $c_j = 0$ whenever $\alpha_j \neq \alpha_*$ and if there is at least one α_j that equals $\alpha_* > 0$. $\square$

Proof of Theorem 3. We exploit that the eigenvectors $g_j \in L^2(\mathbb{S}^2)$ form a complete orthonormal basis of $L^2(\mathbb{S}^2)$ to represent $g \in L^2(\mathbb{S}^2)$ as $g = \sum_{j\in\mathbb{N}} (g, g_j) g_j$. Since $Fg = \sum_{j\in\mathbb{N}} \lambda_j (g, g_j) g_j$ this shows in particular that

$$(Fg, g) = \sum_{j\in\mathbb{N}} \lambda_j |(g, g_j)|^2. \tag{14}$$

Since $\mathrm{Re}\,(\lambda_j) = r_j \cos(\vartheta_j)$ and $\mathrm{Im}\,(\lambda_j) = r_j \sin(\vartheta_j)$ we want to apply Lemma 4 to $f(\alpha) = \cos(\alpha)$ and $g(\alpha) = \sin(\alpha)$ on $(0, \pi)$ and need to check the monotonicity of $h(\alpha) := f(\alpha)/g(\alpha) = \cot(\alpha)$. We find that $h'(\alpha) = 2/(\cos(2\alpha) - 1) < 0$ in $(0, \pi)$, that is, h is strictly monotonically decreasing. Setting $\alpha_j = \vartheta_j$, $\alpha_* = \vartheta_* \leq \vartheta_j$ and $c_j = r_j |(g, g_j)|^2$ for arbitrary $g \in L^2(\mathbb{S}^2)$, Lemma 4 implies that

$$\frac{\sum_{j\in\mathbb{N}} \mathrm{Re}\,(\lambda_j) |(g, g_j)|^2}{\sum_{j\in\mathbb{N}} \mathrm{Im}\,(\lambda_j) |(g, g_j)|^2} = \frac{\sum_{j\in\mathbb{N}} \cos(\vartheta_j) r_j |(g, g_j)|^2}{\sum_{j\in\mathbb{N}} \sin(\vartheta_j) r_j |(g, g_j)|^2} \leq \frac{\cos(\vartheta_*)}{\sin(\vartheta_*)} = \cot(\vartheta_*).$$

Due to the orthonormality of the eigenfunctions g_j and since $r_j > 0$ for all $j \in \mathbb{N}$ since k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D, equality holds if and only if g is chosen as an eigenfunction for the eigenvalue $\lambda_* = \lambda_{j_*}$ with the smallest phase among all eigenvalues of F. $\square$

Remark 5. *Due to the factorization $F = -GS^*G^*$ and the denseness of the range of G^* in $H^{-1/2}(\partial D)$, (11) can also be expressed using the single-layer operator S: Indeed, $(Fg,\, g)_{L^2(\mathbb{S}^2)} = -(S^*G^*g,\, G^*g)_{L^2(\partial D)} = -(\varphi,\, S\varphi)_{L^2(\partial D)}$ for $\varphi = G^*g \in H^{-1/2}(\partial D)$; in particular,*

$$\cot \vartheta_* = \max_{\varphi \in H^{-1/2}(\partial D)} \frac{\mathrm{Re}\,(\varphi,\, S\varphi)_{L^2(\partial D)}}{\mathrm{Im}\,(\varphi,\, S\varphi)_{L^2(\partial D)}}.$$

At this point it becomes crucial to consider the dependence of all the involved quantities on the wave number $k > 0$: We write $\vartheta_* = \vartheta_*(k)$, $S = S(k)$ and $\mathrm{SL} = \mathrm{SL}(k)$ to indicate this dependence. Further, we write $k \nearrow k_0$ to indicate that the positive number k tends to $k_0 > 0$ from below, i.e., $k_0 > k \to k_0$. We start with a crucial auxiliary result that has, in our opinion, an interest in its own: The derivative of $S(k)$ with respect to k is positive – and hence selfadjoint – when it is restricted to the kernel of $S(k)$.

Lemma 6. *Assume that k_0^2 is a Dirichlet eigenvalue of $-\Delta$ in D. Then $S(k_0)$ has a non-trivial kernel and for all elements φ_0 in this kernel it holds that $(\varphi_0,\, S(k_0)\varphi_0)_{L^2(\partial D)} = 0$. The mapping $k \mapsto (\varphi_0,\, S(k)\varphi_0)_{L^2(\partial D)}$ is differentiable at k_0 and*

$$\alpha := \frac{\mathrm{d}}{\mathrm{d}k}(\varphi_0,\, S(k)\varphi_0)_{L^2(\partial D)}\bigg|_{k=k_0} = 2k_0 \int_D |u_{k_0}|^2 \,\mathrm{d}x, \qquad \text{where } u_{k_0} = \mathrm{SL}(k_0)\varphi_0.$$

Proof. We already saw in Lemma 1 that $\mathrm{Im}\,(\varphi,\, S(k)\varphi)_{L^2(\partial D)}$ vanishes for a non-zero φ if and only if $S(k)\varphi = 0$, that is, if and only if k^2 is a Dirichlet eigenvalue of $-\Delta$ in D. Set $u_k = \mathrm{SL}(k)\varphi_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$, in particular, $u_{k_0} = \mathrm{SL}(k_0)\varphi_0$. Since the fundamental solution Φ is weakly singular, we compute that

$$\frac{\mathrm{d}}{\mathrm{d}k} u_k(x) = \frac{\mathrm{d}}{\mathrm{d}k} \int_{\partial D} \Phi(x,y)\varphi_0(y)\,\mathrm{d}S(y) = \int_{\partial D} \frac{\mathrm{d}}{\mathrm{d}k}\Phi(x,y)\varphi_0(y)\,\mathrm{d}S(y) = \int_{\partial D} \frac{\mathrm{i}}{4\pi} e^{\mathrm{i}k|x-y|}\varphi_0(y)\,\mathrm{d}S(y),$$

for $x \in \mathbb{R}$. The derivative of u_k with respect to k is hence well-defined in, e.g., $H^1_{\mathrm{loc}}(\mathbb{R}^3)$. In particular, the chain rule implies that

$$\Delta u_k' + k^2 u_k' + 2k u_k = 0, \qquad \text{where } u_k' := \frac{\mathrm{d}}{\mathrm{d}k} u_k \in H^1_{\mathrm{loc}}(\mathbb{R}^3). \tag{15}$$

Now we compute the derivative with respect to k of $k \mapsto (\varphi_0,\, S(k)\varphi_0)_{L^2(\partial D)}$,

$$\frac{\mathrm{d}}{\mathrm{d}k}(\varphi_0,\, S(k)\varphi_0)_{L^2(\partial D)} = \left(\varphi_0,\, \frac{\mathrm{d}}{\mathrm{d}k}S(k)\varphi_0\right) = \left(\varphi_0,\, \frac{\mathrm{d}}{\mathrm{d}k}u_k\right) = \left(\frac{\partial u_k}{\partial \nu}\bigg|^- - \frac{\partial u_k}{\partial \nu}\bigg|^+,\, \frac{\mathrm{d}}{\mathrm{d}k}u_k\right)_{L^2(\partial D)}.$$

Note that the normal derivative $(\partial u_{k_0}/\partial \nu)|^+$ taken from the exterior vanishes, since the radiating solution $u_{k_0} = \mathrm{SL}(k_0)\varphi_0$ to the Helmholtz equation vanishes by construction on ∂D and hence by the unique solvability of the exterior Dirichlet scattering problem everywhere in $\mathbb{R}^3 \setminus \overline{D}$. Now we use Green's first identity for $u_{k_0} \in H^1_0(D)$ and u_{k_0}' and exploit (15) to get that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}k}(\varphi_0,\, S(k_0)\varphi_0)_{L^2(\partial D)} &= \left(\frac{\partial u_{k_0}}{\partial \nu}\bigg|^-,\, \frac{\mathrm{d}u_{k_0}}{\mathrm{d}k}\right)_{L^2(\partial D)} = \int_D \left[\Delta u_{k_0}\overline{u_{k_0}'} + \nabla u_{k_0}\nabla\overline{u_{k_0}'}\right] \mathrm{d}x \\
&= \int_D \left[-k_0^2 u_{k_0}\overline{u_{k_0}'} - u_{k_0}\Delta\overline{u_{k_0}'}\right] \mathrm{d}x \\
&= \int_D \left[-k_0^2 u_{k_0}\overline{u_{k_0}'} + k_0^2\overline{u_{k_0}'}u_{k_0} + 2k_0\,|u_{k_0}|^2\right] \mathrm{d}x = 2k_0 \int_D |u_{k_0}|^2 \,\mathrm{d}x.
\end{aligned}$$

$\square$

Lemma 7. *Let $k_0 > 0$ and $0 \neq \varphi_0 \in H^{-1/2}(\partial D)$ such that $(\varphi_0,\, S(k_0)\varphi_0)_{L^2(\partial D)} = 0$. Then it holds that $\lim_{k \nearrow k_0} \vartheta_*(k) = 0$.*

Proof. We know from Lemma 1 that $(\varphi_0,\, S(k_0)\varphi_0)_{L^2(\partial D)} = 0$ implies that k_0^2 is a Dirichlet eigenvalue. Assume that $I = (k_0 - \varepsilon, k_0 + \varepsilon)$ is an interval that does not contain other Dirichlet eigenvalues. We showed in Lemma 4 (see also Remark 5) that

$$\cot \vartheta_*(k) = \max_{g \in L^2(\mathbb{S}^2)} \frac{\mathrm{Re}\,(Fg,\, g)_{L^2(\mathbb{S}^2)}}{\mathrm{Im}\,(Fg,\, g)_{L^2(\mathbb{S}^2)}} = \max_{\varphi \in H^{-1/2}(\partial D)} \frac{\mathrm{Re}\,(\varphi,\, S(k)\varphi)_{L^2(\mathbb{S}^2)}}{\mathrm{Im}\,(\varphi,\, S(k)\varphi)_{L^2(\mathbb{S}^2)}} \qquad \text{for } k \in I \setminus \{k_0\}.$$

Define $f(k) = (\varphi_0,\, S(k)\varphi_0)_{L^2(\partial D)}$ for $k \in I$ and note that the last Lemma 6 states that this function is differentiable at k_0. Taylor's theorem states that

$$f(k) = f(k_0) + \alpha(k - k_0) + r(k),$$

where $f(k_0) = 0$ by construction and the remainder $r(k)$ satisfies $r(k) = o(|k - k_0|)$ as $k \to k_0$. Further, note that $\mathrm{Im}\,(r(k)) \leq 0$ due to Lemma 1, because the derivative $\alpha = \mathrm{d}f/\,\mathrm{d}k$ at k_0 is real-valued and $\mathrm{Im}\, f(k) \leq 0$. Hence,

$$\cot \vartheta_*(k) = \max_{\varphi \in H^{-1/2}(\partial D)} \frac{\mathrm{Re}\,(\varphi,\, S(k)\varphi)_{L^2(\mathbb{S}^2)}}{\mathrm{Im}\,(\varphi,\, S(k)\varphi)_{L^2(\mathbb{S}^2)}} \overset{\varphi=\varphi_0}{\geq} \frac{\alpha(k - k_0) + \mathrm{Re}\,(r(k))}{\mathrm{Im}\,(r(k))} \to \infty \quad \text{as } k \nearrow k_0. \tag{16}$$

Indeed, since α is positive, $k \nearrow k_0$ implies that $\alpha(k - k_0) \leq 0$ tends slower to zero than $0 < \mathrm{Im}\,(r(k)) = o(|k - k_0|)$, that is, $[\alpha(k - k_0) + \mathrm{Re}\,(r(k))]/\mathrm{Im}\,(r(k)) \to \infty$. Obviously, $\cot \vartheta_*(k) \to \infty$ for $\vartheta_*(k) \in (0, \pi)$ implies that $\vartheta_*(k) \to 0$. $\square$

Our final result in this section is the following characterization of Dirichlet eigenvalues of $-\Delta$ in D. Roughly speaking, this characterization states that interior eigenvalues k_0^2 are characterized by the fact that the eigenvalue $\lambda_* = \lambda_{j_*}(k)$ of $F(k)$ with the smallest phase tends to 0 from the right as $k \nearrow k_0$. More precisely, the phase $\vartheta_*(k) \in (0, \pi)$ of $\lambda_*(k)$ tends to 0 as $k \nearrow k_0$ – this behavior is exceptional since the eigenvalues λ_j usually accumulate from the left at zero, that is, $\vartheta_j(k) \to \pi$ as $j \to \infty$ for all $k > 0$.

Theorem 8. *Assume that $k_0 > 0$ and that $I = (k_0 - \varepsilon, k_0)$ contains no k such that k^2 is a Dirichlet eigenvalue of $-\Delta$ in D. As above, we denote the phases of the eigenvalues $\lambda_j(k)$ of $F(k)$ by $\vartheta_j(k) \in (0, \pi)$ and set $\vartheta_*(k) = \min_{j \in \mathbb{N}} \vartheta_j(k)$. Then*

$$k_0^2 \textit{ is a Dirichlet eigenvalue of } -\Delta \textit{ in } D \textit{ if and only if } \lim_{k \nearrow k_0} \vartheta_*(k) = 0. \tag{17}$$

Proof. If k_0^2 is a Dirichlet eigenvalue, then $\lim_{k \nearrow k_0} \vartheta_*(k) = 0$ follows directly from Lemma 7.

To prove that $\lim_{k \nearrow k_0} \vartheta_*(k) = 0$ implies that k_0^2 is a Dirichlet eigenvalue we argue by contradiction: Assume that this limit relation holds but that k_0^2 is not a Dirichlet eigenvalue. Due to Lemma 7, $\vartheta_*(k) \to 0$ as $k \nearrow k_0$ implies that

$$\max_{\varphi \in H^{-1/2}(\partial D)} \frac{\mathrm{Re}\,(\varphi,\, S(k)\varphi)_{L^2(\mathbb{S}^2)}}{\mathrm{Im}\,(\varphi,\, S(k)\varphi)_{L^2(\mathbb{S}^2)}} \to \infty \qquad \text{as } k \nearrow k_0.$$

Hence, there exist sequences $k_j \in I$ such that $k_j \nearrow k_0$ and $\varphi_j \in H^{-1/2}(\partial D)$ with $\|\varphi_j\|_{H^{-1/2}(\partial D)} = 1$ such that $0 > \mathrm{Im}\,(\varphi_j,\, S(k_j)\varphi_j)_{L^2(\partial D)} \to 0$ as $j \to \infty$ and $\mathrm{Re}\,(\varphi_j,\, S(k_j)\varphi_j)_{L^2(\partial D)} < 0$ for $j \in \mathbb{N}$ large enough. Since the sequence φ_j is bounded, there exists a weakly convergent subsequence that we also denote by φ_j, such that $\varphi_j \rightharpoonup \varphi_0$ for some $\varphi_0 \in H^{-1/2}(\partial D)$. Define $v_j = \mathrm{SL}(k_j)\varphi_j$. Note that Green's first identity, the jump relation (9), and the Sommerfeld radiation condition imply that

$$(\varphi_j,\, S(k_j)\varphi_j)_{L^2(\partial D)} = \int_{\partial D} \left[\frac{\partial v_j}{\partial \nu}\bigg|^- - \frac{\partial v_j}{\partial \nu}\bigg|^+ \right] \overline{v_j}\, \mathrm{d}S = \int_{B_R \setminus \partial D} [|\nabla v_j|^2 - k_j^2 |v_j|^2]\, \mathrm{d}x - \int_{\partial B_R} \frac{\partial v_j}{\partial \nu} \overline{v_j}\, \mathrm{d}S$$
$$= \int_{B_R \setminus \partial D} [|\nabla v_j|^2 - k_j^2 |v_j|^2]\, \mathrm{d}x - \mathrm{i} k_j \int_{\partial B_R} |v_j|^2\, \mathrm{d}S + \mathcal{O}(1/R) \quad \text{as } R \to \infty, \quad (18)$$

such that the far field v_j^∞ of v_j satisfies

$$\mathrm{Im}\,(\varphi_j,\, S(k_j)\varphi_j)_{L^2(\partial D)} = -\frac{k_j}{4\pi^2} \|v_j^\infty\|^2_{L^2(\mathbb{S}^2)}, \qquad j \in \mathbb{N}. \quad (19)$$

The operator mapping φ_j to v_j^∞ is compact and hence the far fields v_j^∞ converge strongly in $L^2(\mathbb{S}^2)$. This strong limit equals the weak limit which is $v_0^\infty \in L^2(\mathbb{S}^2)$, the far field of $v_0 := \mathrm{SL}(k_0)\varphi_0$. Note now that the right-hand side in (19) tends to zero, that is, v_0^∞ must vanish. Rellich's lemma then implies that v_0 vanishes in the exterior of D. However, since we assumed that k_0^2 is no interior Dirichlet eigenvalue, v_0 must vanish inside of D, too, and the jump relations for the single-layer potential imply that φ_0 must also vanish, that is, $\varphi_j \rightharpoonup 0$. Since the single-layer operator SL is bounded from $H^{-1/2}(\partial D)$ into $H^1(B_R)$ for all $R > 0$ it is also a compact operator into $L^2(B_R)$. Hence, $v_j \to 0$ strongly in $L^2(B_R)$. Due to elliptic regularity results, SL is also bounded from $H^{-1/2}(\partial D)$ into $H^2(B_{2R} \setminus B_{R/2})$ for $R > 0$ large enough. Since $\varphi_j \rightharpoonup 0$ this mapping property implies that $\int_{\partial B_R} (\partial v_j / \partial \nu) \overline{v_j}\, \mathrm{d}S$ tends strongly to zero as $j \to \infty$. Note that we already found above that $\mathrm{Re}\,(\varphi_j,\, S(k_j)\varphi_j)_{L^2(\partial D)} \leq 0$. This motivates to take the real part of (18),

$$0 \geq \mathrm{Re}\,(\varphi_j,\, S(k_j)\varphi_j)_{L^2(\partial D)} = \int_{B_R} [|\nabla v_j|^2 - k^2 |v|^2]\, \mathrm{d}x - \int_{\partial B_R} \frac{\partial v_j}{\partial \nu} \overline{v_j}\, \mathrm{d}S,$$

to obtain that

$$\int_{B_R} |\nabla v_j|^2\, \mathrm{d}x \leq \int_{B_R} |v_j|^2\, \mathrm{d}x + \int_{\partial B_R} \frac{\partial v_j}{\partial \nu} \overline{v_j}\, \mathrm{d}S \to 0 \qquad \text{as } j \to \infty.$$

In particular, v_j converges strongly to zero in $H^1(B_R)$, as well as its trace $v_j|_{\partial D} = S(k_j)\varphi_j$ tends strongly to zero in $H^{1/2}(\partial D)$. Since, by assumption k_0^2 is not a Dirichlet eigenvalue, the single-layer operator $S(k_0)$ is an isomorphism. This allows to conclude that $\varphi_j \to 0$ strongly in $H^{-1/2}(\partial D)$, which contradicts our initial assumption that $\|\varphi_j\|_{H^{-1/2}(\partial D)} = 1$ for all $j \in \mathbb{N}$. □

Remark 9. *One can also prove that the number M of eigenvalue curves $k \mapsto \lambda_j(k)$ that tend to 0 from the right as $k \nearrow k_0$ equals the dimension N of the eigenspace of the interior Dirichlet eigenvalue k_0^2. The proof of Lemma 7 together with Lemma 1 implies that N linear independent eigenfunctions create N eigenvalue curves that tend to 0 from the right, that is, $N \leq M$. On the other hand, the contradiction argument in the proof of Theorem 8 shows that each eigenvalue curve corresponds to an interior Dirichlet eigenvalue, that is, $M \leq N$. This dimensional correspondence also holds for the Robin scattering problem in Section 3 and is visible in the numerical examples in Section 4, too.*

3 Characterizing Robin and Neumann Eigenvalues from Far Field Data

In this section, we want to apply a technique similar to the one from the last section to extend the inside-outside duality between the interior eigenvalues and the spectrum of the far field operator to the case of Robin obstacles. As we will see, the above arguments require adaptions or extensions at several points. Note that the important case of (sound-hard) Neumann boundary conditions will be included in the theory outlined below. The case of absorbing impedance boundary conditions is not included since we will rely on the fact that non-absorbing boundary conditions yield a normal far field operator that possesses in particular an eigenvalue decomposition.

Once again let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected complement and let the boundary operator $\mathcal{B}$ take the form $\mathcal{B}(u) = \partial u/\partial\nu + \tau u$ on ∂D for a real-valued function $\tau \in L^\infty(\partial D)$. This choice yields the exterior Robin scattering problem

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad \frac{\partial u}{\partial \nu} + \tau u = 0 \quad \text{on } \partial D. \tag{20}$$

Since we do not exclude the special case $\tau = 0$, all succeeding arguments also hold true for the Neumann case $B(u) = \partial u/\partial\nu$. Our goal in this section is to provide a characterization of the interior Robin eigenvalues $k^2 > 0$ corresponding to this scattering problem, e.g. of those wave numbers $k > 0$ for which there is a non-trivial solution to

$$\Delta u + k^2 = 0 \quad \text{in } D, \qquad \frac{\partial u}{\partial \nu} + \tau u = 0 \quad \text{on } \partial D.$$

Since τ is real-valued, the far field operator F from (2) is a compact and normal operator [4]. We denote its eigensystem again as $(\lambda_j, g_j)_{j\in\mathbb{N}}$, that is, $Fg = \sum_{j\in\mathbb{N}} \lambda_j (g, g_j) g_j$. Due to [4] we know that the λ_j again lie on the circle $\{z \in \mathbb{C},\ |z - 8\pi^2 \mathrm{i}/k| = 8\pi^2/k\}$.

As mentioned in the introduction, there is a factorization of the far field operator F corresponding to the above-introduced Robin boundary conditions,

$$F = -GT^*G^*. \tag{21}$$

Here, $G : H^{-1/2}(\partial D) \to L^2(\mathbb{S}^2)$ is the compact and injective solution operator, defined in (22), mapping a Robin boundary datum ψ to the far field v^∞ of the unique radiating solution to the exterior Robin boundary value problem,

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad \frac{\partial v}{\partial \nu} + \tau v = \psi \quad \text{on } \partial D. \tag{22}$$

Moreover, the operator $T :\, H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is given by

$$T\psi = N\psi + K'(\tau\psi) + \tau K\psi + \tau S(\tau\psi), \tag{23}$$

where N, K', K and S are the boundary integral operators defined in (7)–(10). For the proof of this factorization we refer to [11, Theorem 2.6]. Before we proceed to exploit this factorization to describe the behavior of the eigenvalues λ_j of F, we note that for a fixed wave number k the imaginary part $\operatorname{Im} F$ is positive, since

$$\operatorname{Im}(Fg,\, g) = \frac{k}{16\pi^2}\|Fg\|^2_{L^2(\mathbb{S}^2)} = \frac{k}{16\pi^2}\|F^*g\|^2_{L^2(\mathbb{S}^2)} \geq 0 \qquad \text{for all } g \in L^2(\mathbb{S}^2). \tag{24}$$

The equalities in the equation above are a direct consequence of [11, Theorem 2.5].

Lemma 10. *If D is a Lipschitz domain, then $T :\, H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is Fredholm of index zero. Moreover, T can be represented as $T = N(0) + C$ where $N(0)$ is the hypersingular boundary integral operator N from (10) for wave number $k = 0$ and C is a compact operator. The operator $-N(0)$ is strictly positive and self-adjoint,*

$$-\big(N(0)\psi,\, \psi\big) \geq c_0 \|\psi\|^2_{H^{1/2}(\partial D)} \qquad \text{for all } \psi \in H^{1/2}(\partial D). \tag{25}$$

Proof. The mapping properties of the boundary integral operators S, K and K' on the Lipschitz boundary ∂D from (7)–(10), the boundedness of the multiplication by τ on $L^2(\partial D)$, and the compact embeddings $H^{1/2}(\partial D) \hookrightarrow L^2(\partial D) \hookrightarrow H^{-1/2}(\partial D)$ imply that $\psi \mapsto K'(\tau\psi) + \tau K\psi + \tau S(\tau\psi)$ is compact from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$. Further, $N = N(k)$ is a Fredholm operator due to [15, Prop. 3.5.5] and Lemma 3.9.8 in [15] shows that the difference $N(k) - N(0)$ is compact, too. Finally, [15, Theorem 3.5.4] shows that $-N(0)$ is strictly positive and hence also self-adjoint. □

The next lemma is the corresponding result to Lemma 1 for Robin boundary conditions.

Lemma 11. *For all $k > 0$ and all $\psi \in H^{1/2}(\partial D)$ it holds that*

$$\mathrm{Im}\,(T\psi,\, \psi)_{L^2(\partial D)} \geq 0. \tag{26}$$

The function $\psi \mapsto \mathrm{Im}\,(T\psi,\, \psi)$ vanishes at $\psi \neq 0$ if and only if $T\psi = 0$. Further, T fails to be an isomorphism if and only if k^2 is an interior Robin eigenvalue of $-\Delta$ in D.

Proof. Inequality (26) follows from (24), the factorization of F and the dense range of G^*,

$$0 \leq \frac{k}{16\pi^2}\|Fg\|^2_{L^2(\mathbb{S}^2)} = \mathrm{Im}\,(Fg,\, g)_{L^2(\mathbb{S}^2)} = -\mathrm{Im}\,(T^*G^*g,\, G^*g)_{L^2(\partial D)} = \mathrm{Im}\,(TG^*g,\, G^*g)_{L^2(\partial D)} \tag{27}$$

for $g \in L^2(\mathbb{S}^2)$. Assume now that $\mathrm{Im}\,(T\psi,\, \psi) = 0$ for a $0 \neq \psi \in H^{1/2}(\partial D)$. Since the range of G^* is dense in $H^{1/2}(\partial D)$, there exists $\{g_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{S}^2)$ such that $G^*g_j \to \psi$ as $j \to \infty$. Due to (27),

$$0 \leq \frac{k}{16\pi^2}\|Fg_j\|^2_{L^2(\mathbb{S}^2)} = \mathrm{Im}\,(TG^*g_j,\, G^*g_j)_{L^2(\mathbb{S}^2)} \to \mathrm{Im}\,(T\psi,\, \psi)_{L^2(\mathbb{S}^2)} = 0 \quad \text{as } j \to \infty.$$

We conclude that $Fg_j \to 0$ as $j \to \infty$ and (24) shows that $F^*g_j \to 0$ as well. For arbitrary $g \in L^2(\mathbb{S}^2)$ this implies that $-(G^*g,\, TG^*g_j)_{L^2(\partial D)} = (g,\, F^*g_j)_{L^2(\mathbb{S}^2)} \to 0$ as $j \to \infty$. Since $G^*g_j \to \psi$ as $j \to \infty$, it follows that $(G^*g, T\psi) = 0$ for all $g \in L^2(\mathbb{S}^2)$ and the denseness of the range of G^* shows that $T\psi = 0$. The other direction is trivial: If $T\psi = 0$, then $\mathrm{Im}\,(T\psi,\, \psi) = 0$.

Let now k^2 be an interior Robin eigenvalue of $-\Delta$ in D and $w \in H^1(D)$ a corresponding eigenfunction. Due to the representation theorem, w can be written as

$$w = \mathrm{SL}\left(\frac{\partial w}{\partial \nu}\bigg|^-\right) - \mathrm{DL}(w|^-) \qquad \text{in } H^1(D).$$

Since $\partial w/\partial\nu = -\tau w$ on ∂D, we find that $w = -\,\mathrm{SL}(\tau\, w|^-) - \mathrm{DL}(w|^-)$. Setting $\psi = w|^-$ and exploiting the jump relations (7)–(10) we obtain that

$$w|^- = -S(\tau\psi) + \frac{1}{2}\psi - K\psi \quad \text{in } H^{1/2}(\partial D), \qquad \frac{\partial w}{\partial \nu}\bigg|^- = -\frac{1}{2}\tau\psi - K'(\tau\psi) - Nw \quad \text{in } H^{-1/2}(\partial D).$$

Using these equations, we deduce that

$$\frac{\partial w}{\partial \nu}\bigg|^- + \tau w|^- = -\big[\tau S(\tau\psi) + \tau K\psi + K'(\tau\psi) + N\psi\big] = -T\psi.$$

Since w satisfies homogeneous Robin boundary conditions we obtain that $T\psi = 0$. The representation $w = -\,\mathrm{SL}(\tau\psi) - \mathrm{DL}\,\psi$ on the other hand implies that $\psi \neq 0$, since otherwise w would vanish in D, contradicting the assumption that w is an eigenfunction. Hence, the kernel of T is non-trivial. If we finally assume that $T\psi = 0$ in $H^{-1/2}(\partial D)$ for some $0 \neq \psi \in H^{1/2}(\partial D)$, then the same arguments show that $w = -\,\mathrm{SL}(\tau\psi) - \mathrm{DL}\,\psi$ defines a Robin eigenfunction of $-\Delta$ in D. □

Contrary to the Dirichlet case from Section 2, the eigenvalues λ_j now tend to zero from the right.

Lemma 12. *Assume that k^2 is no interior Robin eigenvalue of $-\Delta$ in D. Then the eigenvalues λ_j of F converge to zero from the right, i.e.,* $\operatorname{Re}\lambda_j > 0$ *for $j \in \mathbb{N}$ large enough.*

Proof. Recall that $g_j \in L^2(\mathbb{S}^2)$ is the eigenfunction corresponding to the eigenvalue λ_j and define $\psi_j = G^* g_j/\sqrt{|\lambda_j|}$. Then

$$\begin{aligned}(T\psi_j,\, \psi_\ell)_{L^2(\partial D)} &= \frac{1}{\sqrt{|\lambda_j|\,|\lambda_\ell|}}(TG^*g_j,\, G^*g_\ell)_{L^2(\partial D)} = \frac{1}{\sqrt{|\lambda_j|\,|\lambda_\ell|}}(GTG^*g_j,\, g_\ell)_{L^2(\mathbb{S}^2)} \\ &= -\frac{1}{\sqrt{|\lambda_j|\,|\lambda_\ell|}}(Fg_j,\, g_\ell)_{L^2(\mathbb{S}^2)} = -\frac{\lambda_j}{|\lambda_\ell|}\delta_{j,\ell} = -s_j\delta_{j,\ell}\end{aligned}$$

where $s_j := \lambda_j/|\lambda_j|$. By construction, $|s_j| = 1$ and $\operatorname{Im}(s_j) > 0$. Since λ_j converges to zero, the only possible accumulation point of s_j is either 1 oder -1. In the remainder of this proof we will show that the accumulation point is 1, which implies the statement of the lemma.

We exploit the splitting $T = N(0) + C$ from Lemma 10 where $-N(0)$ and C are self-adjoint, strictly positive and compact operators, respectively, to note that

$$s_j = (-N(0)\psi_j,\, \psi_j)_{L^2(\partial D)} - (C\psi_j,\, \psi_j)_{L^2(\partial D)}, \qquad j \in \mathbb{N}. \tag{28}$$

This implies in particular that $\operatorname{Re}(s_j) \geq c_0 \|\psi_j\|^2_{H^{1/2}(\partial D)} - \operatorname{Re}(C\psi_j,\, \psi_j)_{L^2(\partial D)}$. Next, we show that the sequence ψ_j is bounded using a contradiction argument: Assume that there is a subsequence, also denoted by ψ_j, such that $\|\psi_j\|_{H^{1/2}(\partial D)} \to \infty$ as $j \to \infty$. Then $\psi_j' := \psi_j/\|\psi_j\|_{H^{1/2}(\partial D)}$ satisfies

$$c_0 + \operatorname{Re}(C\psi_j',\, \psi_j')_{L^2(\partial D)} \leq -\frac{\operatorname{Re}(s_j)}{\|\psi_j\|^2_{H^{1/2}(\partial D)}} \to 0 \quad \text{as } j \to \infty. \tag{29}$$

Since the sequence ψ_j' is bounded, we can extract a weakly convergent subsequence, again denoted by ψ_j' such that $\psi_j' \rightharpoonup \psi'$ as $j \to \infty$. Since C is compact, the image sequence $C\psi_j'$ converges strongly in $H^{-1/2}(\partial D)$ and $(C\psi_j',\, \psi_j')_{L^2(\partial D)} \to (C\psi',\, \psi')_{L^2(\partial D)}$. Now, (29) allows to conclude that

$$c_0 + \lim_{j\to\infty} \operatorname{Re}(C\psi_j',\, \psi_j')_{L^2(\partial D)} = c_0 + \operatorname{Re}(C\psi',\, \psi')_{L^2(\partial D)} \leq 0. \tag{30}$$

Since $c_0 > 0$, this means that $\operatorname{Re}(C\psi',\, \psi')_{L^2(\partial D)} < 0$. Similar arguments applied to the imaginary part of (28) yield

$$0 = -\lim_{j\to\infty} \frac{\operatorname{Im}(s_j)}{\|\psi_j\|^2_{H^{1/2}(\partial D)}} = \lim_{j\to\infty} \operatorname{Im}(T\psi_j',\, \psi_j')_{L^2(\partial D)} = \operatorname{Im}(T\psi',\, \psi')_{L^2(\partial D)}.$$

Our assumption that k^2 is no interior eigenvalue together with Lemma 11 now implies that $\psi' = 0$. This contradicts the fact that $\operatorname{Re}(C\psi',\, \psi')_{L^2(\partial D)} < 0$ and finally shows that $\{\psi_j\}_{j\in\mathbb{N}}$ is bounded.

To conclude, consider again the imaginary part of (28) and exploit that $(-N(0)\psi_j,\, \psi_j)_{L^2(\partial D)}$ is real-valued together with $\operatorname{Im} s_j \to 0$ to deduce that $\operatorname{Im}(C\psi_j,\, \psi_j)_{L^2(\partial D)} \to \operatorname{Im}(C\psi,\, \psi) = 0$ as $j \to \infty$. This shows that $\operatorname{Im}(T\psi,\, \psi) = \operatorname{Im}(C\psi,\, \psi) = 0$. Since k^2 is no interior eigenvalue, Lemma 11 implies that $\psi = 0$. Hence, $(C\psi_j,\, \psi_j) \to 0$ and $\operatorname{Re}(s_j) \geq c_0\|\psi_j\|^2 \geq 0$ as $j \to \infty$, such that the accumulation point of s_j has to be 1. □

Let us again represent the eigenvalues λ_j of F in polar coordinates,

$$\lambda_j = r_j \exp(\mathrm{i}\vartheta_j), \qquad r_j \geq 0, \quad \vartheta_j \in (0, \pi),$$

assuming that k^2 is no interior Robin eigenvalue such that none of the eigenvalues λ_j vanish. Since $\operatorname{Re}\lambda_j > 0$ for large $j \in \mathbb{N}$, the phases ϑ_j converge to 0 as $j \to \infty$ and therefore we can define the largest phase

$$\vartheta^* = \vartheta_{j^*} = \max_{j \in \mathbb{N}} \vartheta_j$$

among all phases ϑ_j. As in the previous section we denote the eigenvalue corresponding to the largest phase ϑ^* as λ^*. Adapting the arguments of Theorem 3 and Lemma 4 to the different phase behavior for the Robin boundary conditions, we obtain the following characterization of the largest phase ϑ^*.

Theorem 13. *If k^2 is not a Robin eigenvalue of $-\Delta$ in D, then*

$$\cot \vartheta^* = \min_{g \in L^2(\mathbb{S}^2)} \frac{\operatorname{Re}(Fg,\, g)_{L^2(\mathbb{S}^2)}}{\operatorname{Im}(Fg,\, g)_{L^2(\mathbb{S}^2)}}, \tag{31}$$

where the minimum is attained at any eigenvector g^* corresponding to the eigenvalue λ^* of F with smallest phase.

Remark 14. *Inserting the factorization* (21) *of the far field operator and using the denseness of the range of G^*, the equality in* (31) *can equivalently be expressed as*

$$\cot \vartheta^* = \min_{\psi \in H^{1/2}(\partial D)} \frac{\operatorname{Re}(\psi,\, T\psi)_{L^2(\mathbb{S}^2)}}{\operatorname{Im}(\psi,\, T\psi)_{L^2(\partial D)}}. \tag{32}$$

where the minimum is attained at $\psi = G^* g^*$.

To indicate the dependency of the relevant quantities on the wave number k, we write from now on again $\vartheta^* = \vartheta^*(k)$, $\mathrm{SL} = \mathrm{SL}(k)$, $\mathrm{DL} = \mathrm{DL}(k)$ as well as $T = T(k)$. Further, we write $k \searrow k_0$ to indicate that the positive wave number k tends to k_0 from above, that is, $k_0 < k \to k_0$.

Similar to the Dirichlet case – see Lemma 6 – one shows that the derivative of $T(k)$ with respect to k is positive when it is restricted to the kernel of $T(k)$.

Lemma 15. *Assume that k_0^2 is a Robin eigenvalue of $-\Delta$ in D. Then $T(k_0)$ has a non-trivial kernel and for all elements $\psi_0 \in H^{1/2}(\partial D)$ in this kernel it holds that $(\psi_0,\, T(k_0)\psi_0)_{L^2(\partial D)} = 0$. The mapping $k \mapsto (\psi_0,\, T(k)\psi_0)_{L^2(\partial D)}$ is differentiable at k_0 and*

$$\frac{\mathrm{d}}{\mathrm{d}k}(\psi_0,\, T(k)\psi_0)_{L^2(\partial D)}\bigg|_{k=k_0} = 2k_0 \int_D |u_{k_0}|^2 \,\mathrm{d}x, \qquad \textit{where } u_{k_0} = \mathrm{SL}(k_0)(\tau\psi_0) + \mathrm{DL}(k_0)\psi_0.$$

Proof. We have already proven in Lemma 11 that $\operatorname{Im}(\psi,\, T(k)\psi)_{L^2(\partial D)} = 0$ for a non-trivial $\psi \in L^2(\partial D)$ implies that k^2 is an interior Robin eigenvalue. Define $u_k := \mathrm{SL}(k)(\tau\psi_0) + \mathrm{DL}(k)\psi_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \partial D)$. In Lemma 6 we have shown that the single layer potential $\mathrm{SL}(k)$ is differentiable in k. A similar calculation for the double layer potential $\mathrm{DL}(k)$ shows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}\mathrm{DL}(k)(x) &= \frac{\mathrm{d}}{\mathrm{d}k}\int_{\partial D}\frac{\partial}{\partial\nu}\Phi(x,y)\psi_0(y)\,\mathrm{d}S(y) = \int_{\partial D}\frac{\partial}{\partial\nu}\frac{\mathrm{d}}{\mathrm{d}k}\Phi(x,y)\psi_0(y)\,\mathrm{d}S(y)\\ &= \int_{\partial D}\frac{\mathrm{i}}{4\pi}\frac{\partial}{\partial\nu}\exp(\mathrm{i}k|x-y|)\psi_0(y)\,\mathrm{d}S(y), \qquad x \in \mathbb{R}^3,\end{aligned}$$

implying that the derivative of u_k with respect to k is also well-defined in, e.g. $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \partial D)$. In particular, $u_k' := \mathrm{d}u_k/\mathrm{d}k \in H^1(D)$ and we can use the chain rule to obtain

$$\Delta u_k' + k^2 u_k' + 2k u_k = 0 \quad \text{in } D. \tag{33}$$

Since $u_k = \mathrm{SL}(k)(\tau\psi_0) + \mathrm{DL}(k)\psi_0$ one easily verifies the jump relation

$$u_k|^- - u_k|^+ = \psi_0. \tag{34}$$

Moreover, we have already computed in the proof of Lemma 11 that

$$T(k)\psi_0 = \frac{\partial u_k}{\partial \nu}\bigg|^- + \tau\, u_k|^- .$$

These two relations allow to compute the derivative with respect to k of $k \mapsto (\psi_0,\, T(k)\psi_0)_{L^2(\partial D)}$,

$$\frac{\mathrm{d}}{\mathrm{d}k}\big(\psi_0,\, T(k)\psi_0\big)_{L^2(\partial D)} = \left(\psi_0,\, \frac{\mathrm{d}}{\mathrm{d}k}T(k)\psi_0\right) = \left(u_k|^- - u_k|^+ \,,\, \frac{\mathrm{d}}{\mathrm{d}k}\,\frac{\partial u_k}{\partial \nu}\bigg|^- + \tau \frac{\mathrm{d}}{\mathrm{d}k}\, u_k|^- \right)_{L^2(\partial D)}.$$

For $k = k_0$ the trace $u_{k_0}|^+$ taken from the exterior of D vanishes because k_0^2 is an interior eigenvalue. Indeed, the radiating solution u_{k_0} to the homogeneous Robin boundary value problem (20) vanishes outside of D and hence its trace vanishes on ∂D. Now we can apply Green's first identity for $u_{k_0} \in H_0^1(D)$, use (33) and the boundary condition $\partial u_{k_0}/\partial\nu = -\tau u_{k_0}$ to compute that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}k}(\psi_0,\, T(k_0)\psi_0)_{L^2(\partial D)} &= \left(u_{k_0}|^- \,,\, \frac{\mathrm{d}}{\mathrm{d}k}\,\frac{\partial u_{k_0}}{\partial \nu}\bigg|^- + \tau \frac{\mathrm{d}}{\mathrm{d}k}\, u_{k_0}|^- \right)_{L^2(\partial D)} \\
&= -\int_D \left[\Delta \overline{u'_{k_0}} u_{k_0} + \nabla u_{k_0} \nabla \overline{u'_{k_0}}\right] \mathrm{d}x - \int_{\partial D} \tau \overline{u_{k_0}}' u_{k_0}|^- \,\mathrm{d}S \\
&= -\int_D \left[\Delta \overline{u'_{k_0}} u_{k_0} - \Delta u_{k_0} \overline{u'_{k_0}}\right] \mathrm{d}x - \int_{\partial D} \frac{\partial u_{k_0}}{\partial \nu}\bigg|^- \overline{u'_{k_0}}\,\mathrm{d}S + \int_{\partial D} \frac{\partial u_{k_0}}{\partial \nu}\bigg|^- \overline{u'_{k_0}}\,\mathrm{d}S \\
&= \int_D \left[2k_0 \overline{u_{k_0}} u_{k_0} + k^2 \overline{u'_{k_0}} u_{k_0} - k^2 \overline{u'_{k_0}} u_{k_0}\right] \mathrm{d}x = 2k_0 \int_D |u_{k_0}|^2 \mathrm{d}x.
\end{aligned}$$

□

Lemma 16. *Let $k_0 > 0$ and $0 \neq \psi \in H^{1/2}(\partial D)$ such that $(\psi_0,\, T(k_0)\psi_0)_{L^2(\partial D)} = 0$. Then it holds that $\lim_{k\searrow k_0} \vartheta^*(k) = \pi$.*

Proof. Using Lemma 15 one can easily adapt the proof of Lemma 7 to get the desired result. □

In the following Theorem 17 we obtain a similar characterization of interior Robin eigenvalues of $-\Delta$ in D as we have already shown for Dirichlet eigenvalues in Theorem 8. In the Dirichlet case we found that the Dirichlet eigenvalues can be characterized by the behavior of the smallest phase of the eigenvalues of the far field operator. In contrast the Robin eigenvalues k_0^2 (or Neumann eigenvalues for the special case $\tau = 0$) can be characterized by the fact that the largest phase ϑ^* converges to π as k approaches k_0 from above.

Theorem 17. *Assume that $k_0 > 0$ and that $I = (k_0, k_0 + \varepsilon)$ contains no k such that k^2 is a Robin eigenvalue of $-\Delta$ in D. As above, we denote the phases of the eigenvalues $\lambda_j(k)$ of $F(k)$ by $\vartheta_j(k) \in (0, \pi)$ and set $\vartheta^*(k) = \max_{j\in\mathbb{N}} \vartheta_j(k)$. Then*

$$k_0^2 \textit{ is a Robin eigenvalue of } -\Delta \textit{ in } D \textit{ if and only if } \lim_{k\searrow k_0} \vartheta^*(k) = \pi. \tag{35}$$

Proof. If k_0^2 is a Robin eigenvalue, $\lim_{k\searrow k_0} \vartheta^*(k) = \pi$ follows directly from Lemma 16.

Assume now that $\lim_{k\searrow k_0} \vartheta^*(k) = \pi$ but that k_0^2 is no Robin eigenvalue. From Lemma 13 it follows that

$$\min_{\psi\in H^{1/2}(\partial D)} \frac{\mathrm{Re}\,(\psi,\, T(k)\psi)_{L^2(\partial D)}}{\mathrm{Im}\,(\psi,\, T(k)\psi)_{L^2(\partial D)}} \to -\infty \qquad \text{as } k \searrow k_0.$$

Hence, there is a sequence $\{k_j\}_{j\in\mathbb{N}} \subset I$ with $k_j \searrow k_0$ as $j \to \infty$ and functions $\psi_j \in H^{1/2}(\partial D)$ with $\|\psi_j\|_{H^{1/2}(\partial D)} = 1$ such that

$$0 > \mathrm{Im}\,(\psi_j,\, T(k_j)\psi_j)_{L^2(\partial D)} \to 0 \quad \text{as } j \to \infty, \tag{36}$$

and such that $\mathrm{Re}\,(\psi_j,\, T(k_j)\psi_j)_{L^2(\partial D)} > 0$ for j large enough. Since the range of G^* is dense in $H^{1/2}(\partial D)$, there exist sequences $\{g_{j,\ell}\}_{\ell\in\mathbb{N}} \subset L^2(\mathbb{S}^2)$ such that $\psi_j = \lim_{\ell\to\infty} G^*(k_j) g_{j,\ell}$. Since the sequence $\{\psi_j\}_{j\in\mathbb{N}}$ is bounded in $H^{1/2}(\partial D)$ we can extract a weakly convergent subsequence, still denoted by ψ_j, such that $\psi_j \rightharpoonup \psi_0 \in H^{1/2}(\partial D)$. Define

$$v_j = \mathrm{DL}(k_j)\psi_j + \mathrm{SL}(k_j)(\tau\psi_j), \qquad j \in \mathbb{N}_0. \tag{37}$$

Since $\mathrm{DL}(k_j)$ and $\mathrm{SL}(k_j)$ from sequences of uniformly bounded linear operators, v_j converges weakly in $H^1(B_R \setminus \partial D)$ to $v_0 = \mathrm{DL}(k_0)\psi_0 + \mathrm{SL}(k_0)(\tau\psi_0) \in H^1(B_R \setminus \partial D)$ for $R > 0$ large enough such that $\overline{D} \subset B_R$. Due to the jump relations (7)–(10) it holds that $\partial v_j/\partial\nu|^+ + \tau\, v_j|^+ = T(k_j)\psi_j$. Thus, the far fields of the radiating solutions v_j to the Helmholtz equation are given by

$$v_j^\infty = G(k_j)T(k_j)\psi_j = \lim_{\ell\to\infty} G(k_j)T(k_j)G^*(k_j)g_{j,\ell} = -\lim_{\ell\to\infty} F^*(k_j)g_{j,\ell}. \tag{38}$$

Since T is an isomorphism and G is compact, the mapping $\psi_j \mapsto v_j^\infty$ is compact and $v_j^\infty \to v_0^\infty \in L^2(\mathbb{S}^2)$ strongly in $L^2(\mathbb{S}^2)$. According to (24) and (26),

$$\begin{aligned} 0 < \frac{k_j}{16\pi^2}\|F^*(k_j)g_{j,\ell}\|^2_{L^2(\mathbb{S}^2)} &\overset{(24)}{=} \mathrm{Im}\,(F(k_j)g_{j,\ell},\, g_{j,\ell})_{L^2(\partial D)} \overset{\ell\to\infty}{\longrightarrow} -\mathrm{Im}\,(T^*(k_j)\psi_j,\, \psi_j)_{L^2(\partial D)} \\ &= -\mathrm{Im}\,(\psi_j,\, T^*(k_j)\psi_j)_{L^2(\partial D)} \to 0 \quad \text{as } j \to \infty \text{ due to (36).} \end{aligned}$$

Hence, $\lim_{\ell\to\infty} F^*(k_j)g_{j,\ell} = v_j^\infty$ tends to zero in $L^2(\mathbb{S}^2)$ as $j \to \infty$, that is, $v_0^\infty = 0$. Rellich's lemma implies that v_0 vanishes in $\mathbb{R}^3 \setminus \overline{D}$. Moreover, k_0^2 is no Robin eigenvalue, that is, v_0 vanishes everywhere. The jump relations (7)–(10) imply that $\psi_0 = 0$ must vanish, too, that is, $\psi_j \rightharpoonup 0$ in $H^{1/2}(\partial D)$.

We will now show that v_j converges strongly to zero in $H^1(B_R \setminus \partial D)$. First we note that, up to extraction of a subsequence, $\tau\psi_j$ converges weakly to zero in $L^2(\partial D)$ and therefore strongly to zero in $H^{-1/2}(\partial D)$. Thus, $\mathrm{SL}(k_j)(\tau\psi_j)$ also converges strongly to zero in $H^1(B_R \setminus \partial D)$. Second, we show that $\mathrm{DL}(k_j)w_j$ converges strongly to zero in $H^1(B_R \setminus \partial D)$, too (the weak convergence to zero is clear). To this end, let us recall from the proof of Lemma 10 that $T(k_j)$ can be written as $T(k_j) = N(k_j) + C(k_j)$ with a compact operator $C(k_j)$. Thus,

$$\mathrm{Re}\,(\psi_j,\, T(k_j)\psi_j)_{L^2(\partial D)} = \mathrm{Re}\,(\psi_j,\, N(k_j)\psi_j)_{L^2(\partial D)} + \mathrm{Re}\,(\psi_j,\, C(k_j)\psi_j)_{L^2(\partial D)}.$$

Since $\psi_j \rightharpoonup 0$ in $H^{1/2}(\partial D)$, the sequence $C(k_j)\psi_j$ converges strongly in $H^{-1/2}(\partial D)$ to $C(k_0)\psi_0 = 0$. Setting $v_j' = \mathrm{DL}(k_j)\psi_j$, Green's first identity shows that

$$\begin{aligned} \mathrm{Re}\,(\psi_j,\, T(k_j)\psi_j)_{L^2(\partial D)} = &-\int_{B_R\setminus\partial D} \left[|\nabla v_j'|^2 - k_j^2|v_j'|^2\right] \mathrm{d}x \\ &+ \mathrm{Re}\,(\psi_j,\, C(k_j)\psi_j)_{L^2(\partial D)} + \mathrm{Re}\int_{\partial B_R} \frac{\partial v_j'}{\partial\nu}\overline{v_j'}\,\mathrm{d}S. \end{aligned}$$

The last surface integral tends to zero as $j \to \infty$ since $\psi_j \rightharpoonup 0$ and since both mappings $\psi_j \mapsto v_j'|_{\partial B_R}$ and $\psi_j \mapsto \partial v_j'/\partial\nu|_{\partial B_R}$ are compact due to elliptic regularity results. Exploiting the positivity of $\mathrm{Re}\,(\psi_j,\, T(k_j)\psi_j)_{L^2(\partial D)} > 0$ for $j \in \mathbb{N}$ large enough yields that

$$\int_{B_R\setminus\partial D} |\nabla v_j'|^2\,\mathrm{d}x \le \int_{B_R\setminus\partial D} |v_j'|^2\,\mathrm{d}x \qquad \text{for } j \in \mathbb{N} \text{ large enough.}$$

Since $v_j' = \mathrm{DL}(k_j)\psi_j$ converges weakly to zero in $H^1(B_R \setminus \partial D)$, this series of functions converges strongly to zero in $L^2(B_R \setminus \partial D)$ and from the last inequality we get that $v_j' = \mathrm{DL}(k_j)\psi_j$ converges even strongly in $H^1(B_R \setminus \partial D)$. Now it follows that $v_j = \mathrm{DL}(k_j)\psi_j + \mathrm{SL}(k_j)(\tau\psi_j)$, defined in (37), converges strongly to $0 = v_0 = \mathrm{DL}(k_0)\psi_0 + \mathrm{SL}(k_0)(\tau\psi_0)$ in $H^1(B_R \setminus \partial D)$. The jump relation (34) for the combined single- and double-layer potential implies that $\psi_0 = v_0|^- - v_0|^+ = 0$. Hence, $\psi_j \to 0$ strongly in $H^{1/2}(\partial D)$ as $j \to \infty$. This, however, contradicts our assumption $\|\psi_j\|_{H^{1/2}(\partial D)} = 1$. $\square$

4 Numerically Detecting Interior Eigenvalues from Far Fields

In this section we provide numerical examples to verify the theoretical results from the previous sections. In particular, we show that it is possible to numerically compute the interior eigenvalues in a domain D of $-\Delta$ for Dirichlet and Neumann boundary conditions from far field operators for many wave numbers. We also show that the corresponding algorithm remains stable under perturbation of the data by synthetic noise.

To use the theory from the previous sections, we need to numerically approximate the radiating solution u^s to an exterior scattering problem with Dirichlet or Neumann boundary conditions,

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad \mathcal{B}(u^s) = -\mathcal{B}(u^i) \quad \text{on } \partial D.$$

Measurements of radiating waves (or, alternatively, numerical approximations to the solution of this problem) for several incident plane waves $u^i(\cdot, \theta_\ell)$ yield approximations $u^\infty_{\mathrm{appr}}(\theta_j, \theta_\ell)$ to the far field patterns $u^\infty(\cdot, \theta_\ell)$ that allow to approximate the far field operator F: Choose a regular, triangular surface mesh $\Gamma = \{\Gamma_j,\, j = 1, \dots, N\}$ of the unit sphere (see, e.g. [15, Ch. 4.1]) consisting of $N \in \mathbb{N}$ patches $\Gamma_j \subset \mathbb{S}^2$ and define P_N to be the $L^2(\mathbb{S}^2)$-orthogonal projection on the space of bounded functions on $\mathbb{S}^2$ that are constant on each surface patch Γ_j. Denote by $\mathbf{1}_{\Gamma_j} : \mathbb{S}^2 \to \mathbb{C}$ the indicator function of the jth surface patch Γ_j, by $P_N[g](j)$ the value of the projection $P_N[g]$ on the jth patch and define θ_j, $j = 1, \dots, N$ to be the midpoint of the jth surface patch Γ_j (defined as the image of the centroid of the reference triangle under the parametrization of the patch). Then

$$F_N g = \sum_{j=1}^N \mathbf{1}_{\Gamma_j} \sum_{\ell=1}^N u^\infty_{\mathrm{appr}}(\theta_j, \theta_\ell) P_N[g](\ell) \tag{39}$$

is a finite-dimensional approximation $F_N : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ to the far field operator F defined via an interpolation projection.

Assume for a moment that we deal with a sequence of discretizations F_N such that $\|F_N - F\|_{L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)}$ tends to zero in the operator norm as $n \to \infty$. (Such sequences could be constructed, e.g., using a sequence of regular surface meshes of $\mathbb{S}^2$ with mesh width tending to zero and a sequence of approximate far fields tending to the exact far-field patterns.) Under this assumption, standard perturbation results [10] imply that the spectra of F and F_N also converge to each other in the Hausdorff distance, that is,

$$\max \left[\sup_{j\in\mathbb{N}} \inf_{\ell\in\mathbb{N}} |\lambda_j - \lambda_\ell^N|,\ \sup_{\ell\in\mathbb{N}} \inf_{j\in\mathbb{N}} |\lambda_j - \lambda_\ell^N| \right] \le \|F_N - F\|_{L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)} \to 0 \qquad \text{as } N \to \infty. \tag{40}$$

Since both F and F_N together with their eigenvalues obviously depend on the wave number k, we write $F(k)$, $F_N(k)$, $\lambda_j(k)$ and $\lambda_j^N(k)$ from now on whenever this is appropriate.

In our experiments, we computed the numerical approximation to a scattered field using boundary integral equations and we briefly sketch here which equations we solved numerically. For the exterior Dirichlet problem, any radiating solution u^s to

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad u^s|^+_{\partial D} = \psi \in H^{1/2}(\partial D)$$

can be represented as a single layer potential $\mathrm{SL}\,\varphi$ if k^2 is not an interior Dirichlet eigenvalue. Indeed, under this assumption, the boundary integral equation of the first kind

$$S\varphi = \psi \qquad \text{in } H^{1/2}(\partial D) \tag{41}$$

is always uniquely solvable for $\psi \in H^{1/2}(\partial D)$. For all computations, we opted to use integral equations of the first kind since the resulting eigenvalue approximations showed in our experiments to be always more accurate than those computed via equations of the second kind. Except for values of k^2 closer than about $1e-4$ to an interior eigenvalue we did not observe stability problems of equations of the first kind at interior eigenvalues. (For the case of the cube, we used the normality error of $\|F_N^* F_N - F_N F_N^*\|/\|F_N^* F_N\|$ as error and stability indicator.) To illustrate that the accuracy of the eigenvalue computations does not depend on the choice of a direct or an indirect method, we use an integral equation of the first kind coming from a direct method to solve for radiating solutions to the exterior Neumann problem

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad \left.\frac{\partial u}{\partial \nu}\right|_{\partial D}^{+} = \phi \in H^{-1/2}(\partial D),$$

more precisely,

$$-N\psi = \frac{1}{2}\,\mathrm{Id}\,\phi + K'\phi \qquad \text{in } H^{-1/2}(\partial D), \tag{42}$$

which is uniquely solvable in $H^{1/2}(\partial D)$ if k^2 is not an interior Neumann eigenvalue.

We solved the boundary integral equations (41) and (42) using the software package BEM++ (see [16]). BEM++ discretizes (41) and (42) using a Galerkin discretization and solves the linear system using H-matrix compression and preconditioning techniques. The far-field pattern at points $\theta_j \subset \mathbb{S}^2$ of the numerical solution can directly be computed in BEM++ using its potential representation and yields the data $(u^\infty_{\mathrm{appr}}(\theta_j, \theta_\ell))_{j,\ell=1}^{120}$ we require to construct F_N as in (39). In the following examples, we always choose the same set of $N = 120$ uniformly distributed directions on the unit sphere. To indicate the good accuracy of the resulting eigenvalues of F_N, we plot in Figures 1(a) and (b) the analytically computed eigenvalues of $F(k)$ when the scatterer D is the open unit ball B, together with the N largest (that is, non-zero) eigenvalues of $F_N(k)$ for $k = 5$. Since later on we will investigate the stability of the eigenvalue computations with respect to synthetic noise, we also indicate in Figures 1(c) and (d) how the numerically computed eigenvalues behave under artificial noise. To this end, we perturb the numerically computed data $(u^\infty_{\mathrm{appr}}(\theta_j, \theta_\ell))_{j,\ell=1}^{120}$ by adding a random matrix of size 120×120 containing normally distributed entries with mean zero such that the relative noise level in the spectral matrix norm equals 10%. These figures indicate that it is difficult to obtain precise phase approximations for the eigenvalues close to zero. Below, we present a stabilization technique that is able to handle this problem.

To verify the main assertions of this paper from Theorem 8 and Theorem 17 we compute the eigenvalues $\lambda_j^N(k)$, $j = 1, \dots, N$, of $F_N(k)$ for several k and examine how their phases depend on the wave number.

Theorem 8 states, roughly speaking, that k_0^2 is an interior Dirichlet eigenvalue if and only if the eigenvalue $\lambda_*(k)$ of $F(k)$ with smallest phase converges to zero as k tends to k_0 from below. To verify this statement, we convert the positions of the eigenvalues in polar coordinates and plot the resulting phases. For eigenvalues close to zero, small position errors produce large phase errors, as we already discussed above. Since we are interested only in the phase behavior, we hence must stabilize the phase computations and proceed as follows: Assuming that the noise level $\|F_N(k) - F(k)\| =: \varepsilon(k)$ is known, the perturbation bound (40) implies that eigenvalues can be perturbed at most by a distance of $\varepsilon(k)$. We hence omit all eigenvalues $\lambda_j^N(k)$ such that

$$\lambda_j^N(k) \in R_+(\varepsilon(k)) := \{z \in \mathbb{C},\, |z| \le \varepsilon(k),\, \mathrm{Re}\, z \ge 0\} \subset \mathbb{C}.$$

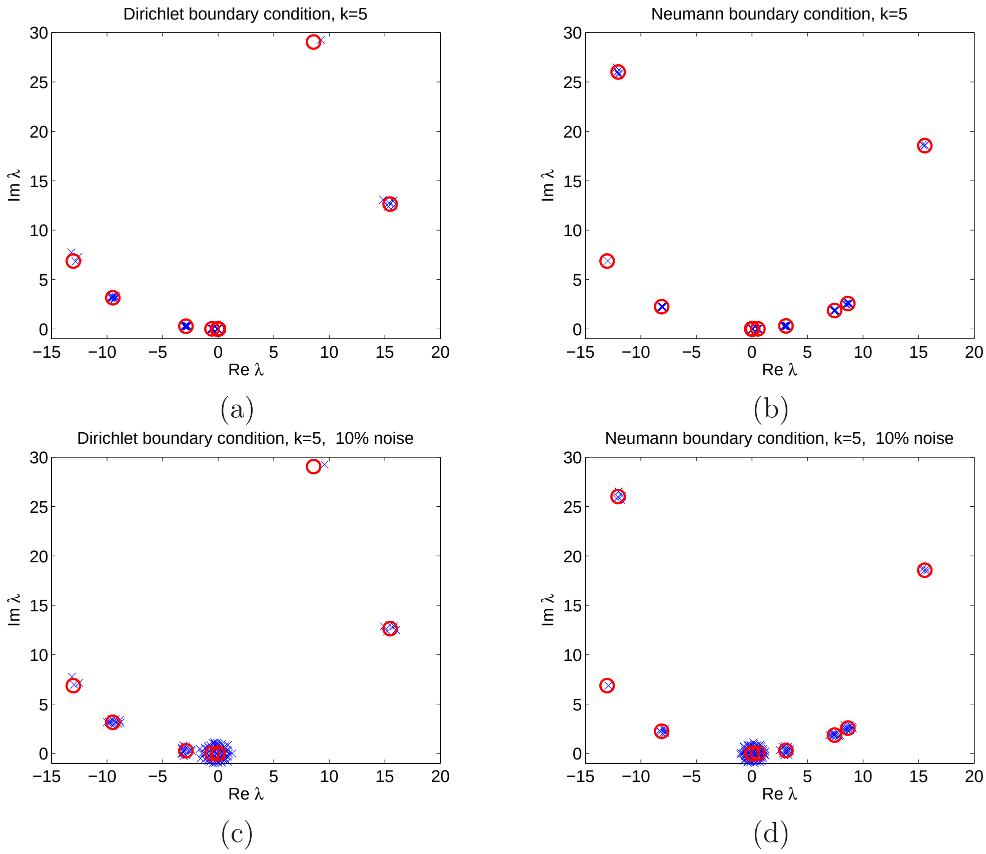


Figure 1: Eigenvalues of the far field operator $F(k)$ and of $F_N(k)$ for $k = 5$, $N = 120$, and $D = B$ (the unit ball). Red circles and blue crosses mark analytically computed eigenvalues of $F(k)$ and numerically computed eigenvalues of $F_N(k)$, respectively. For (c) and (d) we perturbed $F_N(k)$ by adding artificial noise with a relative noise level of 10%. (a) Dirichlet boundary conditions, no artificial noise. (b) Neumann boundary conditions, no artificial noise. (c) Dirichlet boundary conditions, relative noise level of 10%. (d) Neumann boundary conditions, relative noise level of 10%.

In principle, we could also omit all eigenvalues inside the circle $\{|z| \leq \varepsilon(k)\}$. However, for the characterization of interior eigenvalues we are merely interested in eigenvalues with small phases and hence do not care about wrong phase information for eigenvalues in the left half-plane. To further stabilize the phase computations, we afterwards exploit the a-priori knowledge that the exact eigenvalues $\lambda_j(k)$ lie on the circle $\{z \in \mathbb{C}, |z - 8\pi^2\mathrm{i}/k| = 8\pi^2/k\}$ in the complex plane and project the eigenvalues $\lambda_j^N(k)$ outside $R_+(\varepsilon(k))$ orthogonally onto this circle, using the mapping

$$Q : \lambda \mapsto \frac{8\pi^2\mathrm{i}}{k} + \frac{8\pi^2}{k} \frac{\lambda - 8\pi^2\mathrm{i}/k}{|\lambda - 8\pi^2\mathrm{i}/k|}. \tag{43}$$

Finally, we compute the phases of the projected eigenvalues $Q[\lambda_j^N(k)]$ such that $\lambda_j^N(k) \notin R_+(\varepsilon(k))$. Following Theorem 8, interior eigenvalues are characterized by the fact that the exact eigenvalue $\lambda_*(k)$ with smallest phase tends to zero from the right. To be able to compare the resulting values of k in our computations with the true interior eigenvalues, we choose the scatterer to be either the unit ball B or the cube $C = (0, 1)^3$, such that the interior Dirichlet eigenvalues are known exactly: For the unit ball B, the eigenvalues are given as positive roots of spherical Bessel functions and the

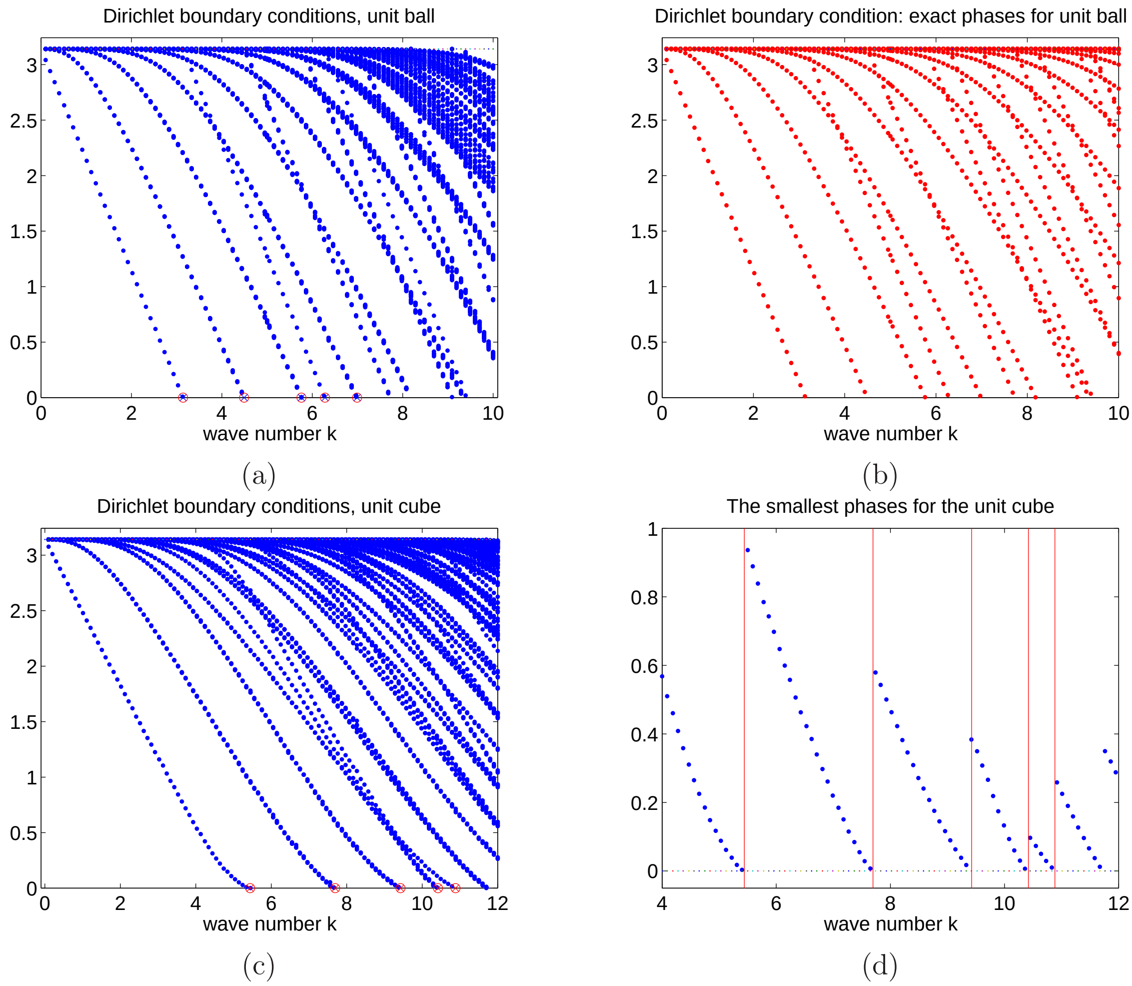


Figure 2: Blue dots mark the phases of the projected numerical eigenvalues $Q[\lambda_j^N(k)]$ with $\lambda_j^N(k) \notin R_+(\varepsilon(k))$ for Dirichlet boundary conditions, $N = 120$. Red dots make the exact phases ϑ_j. Red circles on the k-axis mark the exact positions of the smallest five interior Dirichlet eigenvalues. (a) Phases of the projected numerical eigenvalues for the unit ball B. (b) Phases of the analytically known eigenvalues of F for the unit ball B. (c) Phases of the projected numerical eigenvalues for the unit cube. (d) Only the smallest phase from (c) was plotted. Vertical red lines mark the smallest five interior Dirichlet eigenvalues.

first five eigenvalues appear at wave numbers

$$k_B^{(1)} = \pi, \quad k_B^{(2)} \approx 4.49, \quad k_B^{(3)} \approx 5.76, \quad k_B^{(4)} \approx 6.28, \quad k_B^{(5)} \approx 6.99.$$

For the cube $C = (0,1)^3$ the wave numbers k_C at which k_C^2 is an interior Dirichlet eigenvalue are given by $k_C = \sqrt{k_1 + k_2 + k_3}$ where $k_{1,2,3}$ is one of the numbers $\pi^2(n+1)^2$, $n \in \mathbb{N}_0$. Hence, the first five Dirichlet eigenvalues arise at the wave numbers

$$k_C^{(1)} = \sqrt{3}\pi, \quad k_C^{(2)} = \sqrt{6}\pi, \quad k_C^{(3)} = 3\pi, \quad k_C^{(4)} = \sqrt{11}\pi, \quad k_C^{(5)} = \sqrt{12}\pi.$$

Figure 2 shows plots of the phases of the projected eigenvalues $Q[\lambda_j^N(k)]$ such that $\lambda_j^N(k) \notin R_+(\varepsilon(k))$ against the wave number k. In these computations, the value of $\varepsilon(k)$ has been set to $10^{-4} \cdot 16\pi^2/k$. The phases of the projected eigenvalues plotted in Figure 2(a) for wave numbers in between 0 and, roughly speaking, 6 cannot be distinguished visually from the exact ones plotted in Figure 2(b).

Further, for wave numbers larger than 8 it is obvious that the numerical accuracy is not sufficient anymore to yield correct phases for eigenvalues lying in the left complex half-plane, that is, where the eigenvalues accumulate. However, Figures 2(a) and (c) show that the smallest phase tends to zero when k tends to an eigenvalue from below. Figure 2(d) shows that the location of the jumps in the curve of the smallest phase (that might, e.g., be found numerically using discrete derivatives) yield enclosures of the exact eigenvalues.

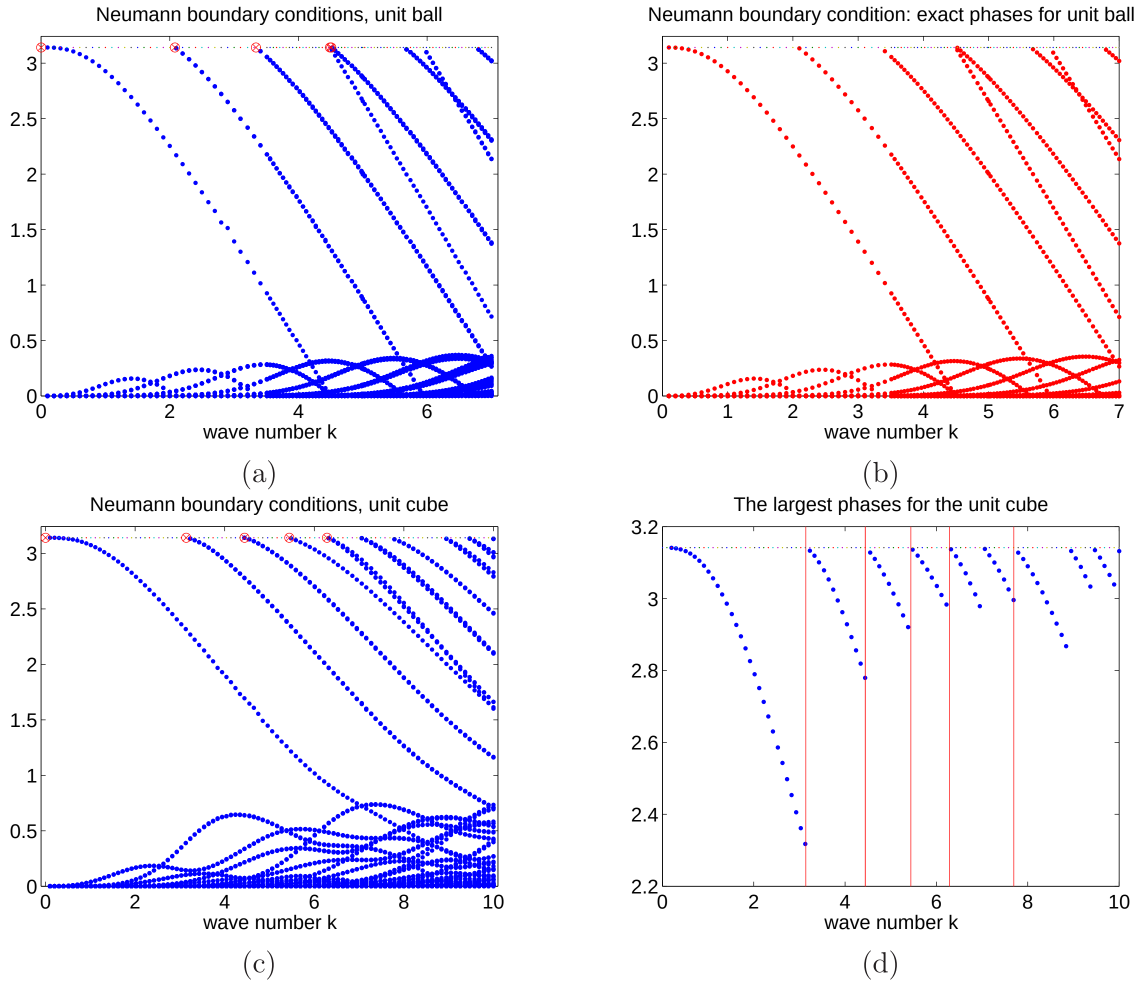


Figure 3: Blue dots mark the phases of the projected numerical eigenvalues $Q[\lambda_j^N(k)]$ with $\lambda_j^N(k) \notin R_-(\varepsilon(k))$ for Neumann boundary conditions, $N = 120$. Red dots make the exact phases ϑ_j. Red circles on the k-axis mark the exact positions of the smallest five interior Neumann eigenvalues. (a) Phases of the projected numerical eigenvalues for the unit ball B. (b) Phases of the analytically known eigenvalues of F for the unit ball B. (c) Phases of the projected numerical eigenvalues for the unit cube. (d) Only the smallest phase from (c) was plotted. Vertical red lines mark the exact positions of the smallest five non-zero interior Neumann eigenvalues.

In the case of Neumann boundary conditions on ∂D, Theorem 17 states that the phase $\vartheta^*(k)$ of the eigenvalue $\lambda^*(k)$ of the far field operator with largest phase converges to π if and only if k tends to an interior Neumann eigenvalue from above. In Figure 3 we show plots of the phases of the projected eigenvalues $Q[\lambda_j^N(k)]$ for

$$\lambda_j^N(k) \notin R_-(\varepsilon(k)) := \{z \in \mathbb{C},\ |z| \leq \varepsilon(k),\ \mathrm{Re}\, z \leq 0\} \subset \mathbb{C}$$

for Neumann boundary conditions against the wave number k, again for the unit ball B and the cube

C. As in the Dirichlet case, the simplicity of the domain allows to compute the interior Neumann eigenvalues explicitly. For the unit ball, the wave numbers k at which interior eigenvalues arise are given by the roots of the derivative of the spherical Hankel function. The first few of those wave numbers are

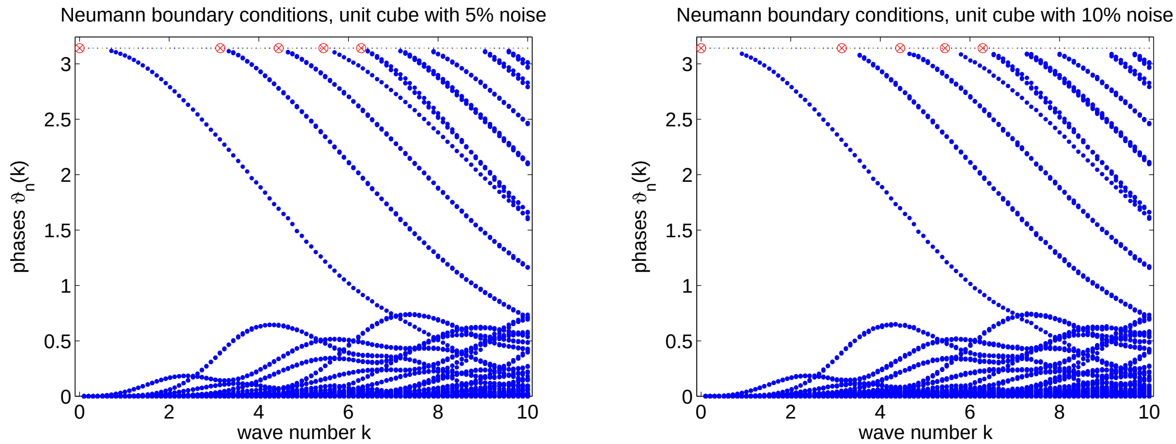
$$k_B^{(1)} = 0, \quad k_B^{(2)} \approx 2.08, \quad k_B^{(3)} \approx 3.34, \quad k_B^{(4)} \approx 4.49, \quad k_B^{(5)} \approx 4.51.$$

For the cube C, the wave numbers k_C at which k_C^2 is an interior Neumann eigenvalue are given by $k_C = \sqrt{k_1 + k_2 + k_3}$ where $k_{1,2,3}$ is one of the numbers $\pi^2 n^2$ for $n \in \mathbb{N}_0$. Therefore the first few Neumann eigenvalues arise at the wave numbers

$$k_C^{(1)} = 0, \quad k_C^{(2)} = \pi, \quad k_C^{(3)} = \sqrt{2}\pi, \quad k_C^{(4)} = \sqrt{3}\pi, \quad k_C^{(5)} = 2\pi.$$

Figure 3 shows that both for the unit ball B and the cube C these values correspond to the wave numbers for which the largest phase tends to π. Again, the jumps in the curve of the largest phase shown in Figure 3(d) can be used to derive enclosures of the exact interior eigenvalues.

Finally we want to test the stability of the computation of interior eigenvalues via the behavior of the smallest or largest phase when adding artificial noise to the data $(u^\infty_{\mathrm{appr}}(\theta_j, \theta_\ell))_{j,\ell=1}^{120}$. As a test case we choose the unit cube with Neumann boundary conditions as a test object. To obtain two instances of noisy data from the numerically computed data $(u^\infty_{\mathrm{appr}}(\theta_j, \theta_\ell))_{j,\ell=1}^{120}$, we added a matrix with random numbers following a normal distribution with mean zero and variance such that the relative error in the spectral matrix norm equals once 5% and once 10%. For the phase computations, we applied the same stabilization technique used above: We first omitted the eigenvalues $\lambda_j^N(k)$ in $R_-(\varepsilon(k)) := \{|z| \leq \varepsilon(k), \operatorname{Re} z \leq 0\}$ and then projected the remaining eigenvalues onto the circle $\{|z - 8\pi^2 \mathrm{i}/k| = 8\pi^2/k\}$ using the projection Q from (43). The number $\varepsilon(k)$ was set to $0.025 \cdot 16\pi^2/k$ and $0.05 \cdot 16\pi^2/k$. Of course, the interior Neumann eigenvalues are not as precisely identifiable as in Figure 3(c). However, by, e.g., choosing the jump of the largest phase as an approximation to the exact interior eigenvalues yields an acceptable absolute error of less than 0.1 and 0.2 for $\lambda_C^{(j)}$, $j = 2, \ldots, 5$, for the two noise levels of 5% and 10%, respectively.



(a) (b)

Figure 4: Computed phase curves after adding synthetic noise to the numerically computed far field data for the cube C with Neumann boundary conditions, $N = 120$. Blue dots mark the phases of the projected numerical eigenvalues $Q[\lambda_j^N(k)]$ with $\lambda_j^N(k) \notin R_-(\varepsilon(k))$. Red circles on the k-axis mark the exact positions of the smallest five interior Neumann eigenvalues. (a) Relative noise level 5%. (b) Relative noise level 10%.

Acknowledgements

The research of AL and SP was supported through an exploratory project granted by the University of Bremen in the framework of its institutional strategy, funded by the excellence initiative of the federal and state governments of Germany.

References

[1] F. Cakoni, D. Colton, and H. Haddar. On the determination of Dirichlet and transmission eigenvalues from far field data. *C. R. Math. Acad. Sci. Paris, Ser. I*, 348:379–383, 2010.

[2] F. Cakoni, D. Colton, and P. Monk. On the use of transmission eigenvalues to estimate the index of refraction from far field data. *Inverse Problems*, 23:507–522, 2007.

[3] F. Cakoni, D. Colton, and P. Monk. *The linear sampling method in inverse electromagnetic scattering*. SIAM, Philadelphia, 2011.

[4] D. Colton and R. Kress. Eigenvalues of the far field operator and inverse scattering theory. *SIAM J. Math. Anal.*, 26:601–615, 1995.

[5] D. Colton and P. Monk. A new method for solving the inverse scattering problem for acoustic waves in an inhomogeneous medium. *Inverse Problems*, 5:1013–1026, 1989.

[6] David L. Colton and Rainer Kress. *Inverse acoustic and electromagnetic scattering theory.* Springer, 3rd edition, 2013.

[7] J.-P. Eckmann and C.-A. Pillet. Spectral duality for planar billiards. *Commun. Math. Phys.*, pages 283–313, 1995.

[8] J.-P. Eckmann and C.-A. Pillet. Zeta functions with dirichlet and neumann boundary conditions for exterior problems. *Helv. Phys. Acta*, pages 44–65, 1997.

[9] Masaru Ikehata. Inverse obstacle scattering problems with a single incident wave and the logarithmic differential of the indicator function in the enclosure method. *Inverse Problems*, 27(8):085006, 2011.

[10] T. Kato. *Perturbation theory for linear operators.* Springer, repr. of the 1980 edition, 1995.

[11] A. Kirsch and N. I. Grinberg. *The Factorization Method for Inverse Problems.* Oxford Lecture Series in Mathematics and its Applications 36. Oxford University Press, 2008.

[12] A. Kirsch and A. Lechleiter. The inside-outside duality for scattering problems by inhomogeneous media. *Inverse Problems*, 29(10):104011, 2013.

[13] A. Lechleiter. The Factorization method is independent of transmission eigenvalues. *Inverse Problems and Imaging*, 3:123–138, 2009.

[14] W. McLean. *Strongly Elliptic Systems and Boundary Integral Operators.* Cambridge University Press, Cambridge, UK, 2000.

[15] S. A. Sauter and C. Schwab. *Boundary Element Methods.* Springer, New York, 2013.

[16] Betcke T. Arridge S. Phillips J. Smigaj, W. and Schweiger. Solving boundary integral problems with BEM++. *Submitted to ACM Trans. Math. Softw*, 2013.