Inside-Outside Duality and the Determination of Electromagnetic Interior Transmission Eigenvalues

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Abstract

We introduce an inside-outside duality approach for the determination of interior transmission eigenvalues of a possibly anisotropic dielectric electromagnetic scattering object using time-harmonic electromagnetic far field data. To this end, we exploit a self-adjoint factorization of the far field operator to link the electromagnetic interior transmission eigenvalues to the maximal or minimal phase of the eigenvalues of the corresponding far field operator, depending whether the sign of the contrast function is positive or negative.

1 Introduction

The propagation of time-harmonic electromagnetic waves in $\mathbb{R}^3$ is governed by Maxwell's equations for the electric and magnetic field E and H. Given a circular frequency $\omega > 0$ and a dielectric medium with electric permittivity $\varepsilon > 0$, constant magnetic permittivity $\mu > 0$, and vanishing conductivity $\sigma > 0$, linear and time-harmonic electromagnetic waves are governed by the differential equations

$$\begin{aligned} \operatorname{curl} E - \mathrm{i}\omega\mu_0 H = 0\,, \\ \operatorname{curl} H + \mathrm{i}\omega\varepsilon E = 0 \end{aligned} \qquad \text{in } \mathbb{R}^3. \tag{1}$$

Denoting the constant background permittivity by ε_0 we introduce the wave number $k := \omega\sqrt{\varepsilon_0\mu_0}$, the relative permittivity $\varepsilon_\mathrm{r} = \varepsilon/\varepsilon_0$, which allows to reduce the system (1) to

$$\operatorname{curl}\left(\varepsilon_\mathrm{r}^{-1} \operatorname{curl} H\right) - k^2 H = 0 \qquad \text{in } \mathbb{R}^3. \tag{2}$$

We assume in the following that ε_r equals ε_0 outside some bounded scatterer $D \subset \mathbb{R}^3$. Considering the electromagnetic scattering problem governed by (2) and the Silver-Müller radiation condition (detailed in the subsequent section) we note that this scattering problem is as usual linked to an interior eigenvalue problem in D: In our context, this so-called interior transmission eigenvalue problem consists in finding an eigenvalue $k^2 \in \mathbb{C}$ and an eigenpair (u, w) such that

$$\operatorname{curl}\left(\varepsilon_\mathrm{r}^{-1} \operatorname{curl} u\right) - k^2 u = 0 \quad \text{in } D \qquad \text{and} \qquad \operatorname{curl}^2 w - k^2 w = 0 \quad \text{in } D, \tag{3}$$

subject to the constraint that the Cauchy data of u and v equal each other,

$$\nu \times (u - w)|_{\partial D} = 0 \qquad \text{and} \qquad \nu \times (\varepsilon_\mathrm{r}^{-1} \operatorname{curl} u - \operatorname{curl} w)\big|_{\partial D} = 0, \tag{4}$$

where ν is the exterior unit normal to ∂D. In this paper we show a tight link between interior transmission eigenvalues and the spectrum of the far field operator to the above-mentioned scattering problem via a conditional inside-outside duality.

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To detail this duality statement, recall that whenever the contrast $Q = I_3 - \varepsilon_{\mathrm{r}}^{-1}$ is real-valued then the far field operator F_k at wave number $k > 0$ to the above-mentioned scattering problem is compact and normal. Thus, F_k possesses eigenvalues $\{\lambda_j(k)\}_{j\in\mathbb{N}}$ that can be shown to lie on the circle $\{|z - 8\pi^2\mathrm{i}/k| = 8\pi^2/k\}$ in the complex plane and tend to zero as j tends to ∞. Whenever the contrast Q has a fixed sign in D then $\lambda_j(k)$ tend to zero as j tends to ∞ either from the left or from the right depending on the sign of the contrast. Given this setting, inside-outside duality roughly speaking states that whenever some eigenvalue $\lambda_j(k)$ tends to zero from the "wrong" side as $k \to k_0$, then $k_0 > 0$ is an interior transmission eigenvalue. Finally, under an implicit condition on a given transmission eigenvalue $k_0 > 0$ we also show that there exists an eigenvalue $\lambda_j(k)$ of F_k tending to zero from the "wrong" side as $k \to k_0$. We additionally transform this implicit condition into an explicit one for the contrast and the wave number that holds true at least for the smallest positive electromagnetic interior transmission eigenvalues if the contrast is large enough.

The latter result offers the possibility to determine at least some transmission eigenvalues from multi-spectral far field data by inspecting, e.g., the behavior of the smallest or largest phase of the eigenvalue of the far field operator (depending on the sign of the contrast Q). The knowledge of interior transmission eigenvalues is in particular of importance in the context of parameter identification for anisotropic materials from far field scattering data. Indeed, anisotropic material parameters are not uniquely identified by such data without a-priori knowledge, even if one possesses multi-frequency data [13, 1]. Such a-priori information can for instance be computed from transmission eigenvalues since [2, 3] show that these eigenvalues provide upper and lower bounds on the norm of the anisotropic material parameter.

The interior transmission eigenvalue problem is a non-selfadjoint and non-linear eigenvalue problem and it took about 20 years in between the first appearance of the problem in the literature [9, 7] and the first existence results of finitely or infinitely many eigenvalues for general (non-spherical) geometries [23, 4]. The growing interest in this eigenvalue problem, parameter identification methods exploiting transmission eigenvalues, and methods for their numerical computation is in particular indicated by the numerous recent references in the review articles [5] and the special issue introduced in [6].

Inside-outside duality is a well-known concept in billiard theory, see, e.g., [10]. A mathematically sound proof of the above-sketched duality for an exterior Dirichlet scattering problem has been given in [11]. The technique from the latter paper has been transferred to a scalar transmission problem for the Helmholtz equation in [19] and to a scalar transmission problem with anisotropic material coefficients in [21]. In this paper we actually follow the simplified approach from [20]. Let us further point out that we use, at least in the first parts of the paper the notation from [17, Chapter 5] since this allows to simplify the presentation by referring the certain parts of proofs in that reference. We further note that our results are restricted to *positive* interior transmission eigenvalues and that identification of complex transmission eigenvalues by an extension of the inside-outside duality is an open problem. Finally, we emphasize that the interior transmission eigenvalue to the eigenvalue problem (3–4) is by definition k, while some authors prefer to define k^2 to be the eigenvalue.

To give a brief outline of the rest of the paper, we first detail the electromagnetic scattering problem in the next Section 2. After rigorously defining transmission eigenvalues in Section 3 we link them to the far field operator in Section 4. Section 5 contains the first part of the inside-outside duality statement. After preparing some technical tools in Section 6 we prove the second part in Section 7 under a condition that is verified for small transmission eigenvalues in Section 8.

Notation: By $\mathbb{S}^2 = \{x \in \mathbb{R}^3,\ |x| = 1\}$ we denote the unit sphere in $\mathbb{R}^3$ and $B_R(x)$ is the ball of radius R about $x \in \mathbb{R}^3$. For any bounded Lipschitz domain $B \subset \mathbb{R}^3$ the Hilbert space $H(\mathrm{curl}, B)$ is defined by $H(\mathrm{curl}, B) := \big\{v \in L^2(B, \mathbb{C}^3),\ \mathrm{curl}\, v \in L^2(B, \mathbb{C}^3)\big\}$; its inner product is $(v, w)_{H(\mathrm{curl},B)} := (v, w)_{L^2(B)} + (\mathrm{curl}\, v, \mathrm{curl}\, w)_{L^2(B)}$. The closure of $C_0^\infty(B, \mathbb{C}^3)$ in the norm of $H(\mathrm{curl}, B)$ is $H_0(\mathrm{curl}, B) = \{v \in H(\mathrm{curl}, B),\ \nu \times v = 0 \text{ on } \partial B\}$. By abuse of notation, a duality pairing between the trace space of $H(\mathrm{curl}, B)$ and its dual (see, e.g., [22, Section 3.5.3]) will for simplicity always be written as a boundary integral over ∂B. Next, $H(\mathrm{div}, B) = \{v \in L^2(B, \mathbb{C}^3),\ \mathrm{div}\, v \in L^2(B, \mathbb{C}^3)\}$ is a Hilbert space for the inner product $(v, w)_{H(\mathrm{curl},B)} := (v, w)_{L^2(B)} + (\mathrm{div}\, v, \mathrm{div}\, w)_{L^2(B)}$ and $H(\mathrm{div}0, B)$ is the set of functions v in $H(\mathrm{div}, B)$ such that $\mathrm{div}\, v = 0$ in B. The closure of $C_0^\infty(B, \mathbb{C}^3)$ in the norm of $H(\mathrm{div}, B)$ is $H_0(\mathrm{div}, B) = \{v \in H(\mathrm{div}, B),\ v \cdot \nu = 0 \text{ on } \partial B\}$; we also define $H_0(\mathrm{div}0, B) = \{v \in H(\mathrm{div}0, B),\ v \cdot \nu = 0 \text{ on } \partial B\}$. Further,

$$H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3) := \left\{v \colon \mathbb{R}^3 \to \mathbb{C}^3,\ v|_B \in H(\mathrm{curl}, B) \text{ for all balls } B \subset \mathbb{R}^3\right\}.$$

Recall moreover that the space of functions in $H(\mathrm{curl}, B) \cap H(\mathrm{div}, B)$ with vanishing tangential trace,

$$X_N = \{\psi \in H(\mathrm{curl}, B) \cap H(\mathrm{div}, B),\ \nu \times \psi = 0 \text{ on } \partial B\} \subset H_0(\mathrm{curl}, B), \tag{5}$$

and norm $\|\psi\|_{X_N} = \|\psi\|_{L^2(B,\mathbb{C}^3)} + \|\mathrm{curl}\, \psi\|_{L^2(B,\mathbb{C}^3)} + \|\mathrm{div}\, \psi\|_{L^2(B)}$ embeds compactly into $L^2(B, \mathbb{C}^3)$, see, e.g., [22, Corollary 3.49].

2 Scattering from a Dielectric Medium

We consider the time-harmonic Maxwell's equations to model scattering of an incident electromagnetic wave from a non-magnetic dielectric medium modeled by space-dependent relative electric permittivity ε_{r}. Moreover, we suppose that the support of $I_3 - \varepsilon_{\mathrm{r}}$ is the closure of a bounded Lipschitz domain $D \subset \mathbb{R}^3$ with connected complement $\mathbb{R}^3 \setminus \overline{D}$. The material parameter $\varepsilon_{\mathrm{r}}^{-1} \in L^\infty(D, \mathrm{Sym}(3))$ takes values in the real-valued symmetric 3×3 matrices $\mathrm{Sym}(3)$ and is bounded from above and below on $\mathbb{R}^3$, i.e., $0 < c \leqslant \overline{\xi}^\top \varepsilon_{\mathrm{r}}^{-1}(x)\xi \in L^\infty(\mathbb{R}^3)$ for almost all $x \in \mathbb{R}^3$ and $\xi \in \mathbb{C}^3$. We denote the corresponding contrast function by $Q := I_3 - \varepsilon_{\mathrm{r}}^{-1}$; obviously, the support of Q equals $\overline{D}$. Note that we write $A < B$ whenever $A, B \in \mathrm{Sym}(3)$ satisfy $\overline{\xi}^\top (A - B)\xi < 0$ for all $\xi \in \mathbb{C}^3$.

We have already derived in the introduction that the total magnetic field solves

$$\mathrm{curl}\left(\varepsilon_{\mathrm{r}}^{-1}\, \mathrm{curl}\, H\right) - k^2 H = 0 \qquad \text{in } \mathbb{R}^3. \tag{6}$$

On interfaces where ε_{r} is discontinuous, the tangential components of the magnetic field H and of $\varepsilon_{\mathrm{r}}^{-1}\, \mathrm{curl}\, H$ are continuous across the interface. In particular, if ε_{r} is discontinuous across ∂D, then

$$\nu \times [H]_{\partial D} = 0 \quad \text{ and } \quad \nu \times \left[\varepsilon_{\mathrm{r}}^{-1}\, \mathrm{curl}\, H\right]_{\partial D} = 0, \tag{7}$$

where $[\cdot]_{\partial D}$ denotes the jump of a function across ∂D. Assume that an incident time-harmonic electromagnetic plane wave

$$H^i(x, \theta; p) := p\, \mathrm{e}^{\mathrm{i}k\, x\cdot\theta}, \qquad x \in \mathbb{R}^3, \qquad \text{where } \theta \in \mathbb{S}^2,\ p \in \mathbb{C}^3, \text{ and } p \cdot \theta = 0,$$

with direction θ and polarization p propagates through the inhomogeneity D. Due to the different material parameters inside D there arises a scattered electromagnetic wave H^s such that the total field $H = H^i + H^s$ solves (6) and, moreover, H^s satisfies the Silver-Müller radiation condition

$$\mathrm{curl}\, H^s(x) \times \hat{x} - \mathrm{i}k H^s(x) = \mathcal{O}\left(|x|^{-2}\right), \quad \text{as } |x| \to \infty, \text{ uniformly with respect to } \hat{x} := \frac{x}{|x|} \in \mathbb{S}^2. \tag{8}$$

Any solution v to the Maxwell's equations $\mathrm{curl}\, \mathrm{curl}\, v - k^2 v = 0$ outside D that satisfies the latter condition is called radiating in the sequel. Since H^i solves $\mathrm{curl}^2 H^i - k^2 H^i = 0$ in $\mathbb{R}^3$, the radiating scattered field H^s is hence a solution to

$$\mathrm{curl}\left(\varepsilon_{\mathrm{r}}^{-1}\, \mathrm{curl}\, H^s\right) - k^2 H^s = \mathrm{curl}\left(Q\, \mathrm{curl}\, H^i\right) \qquad \text{in } \mathbb{R}^3. \tag{9}$$

For this and all subsequent scattering problems we consider weak solutions in $H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$. Before introducing the corresponding weak formulation, let us introduce a more general source term on the right of (9): For $f \in L^2(D, \mathbb{C}^3)$ we seek a weak radiating solution $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ to

$$\mathrm{curl}\left(\varepsilon_{\mathrm{r}}^{-1} \,\mathrm{curl}\, v\right) - k^2 v = \mathrm{curl}\,(Qf) \qquad \text{in } \mathbb{R}^3. \tag{10}$$

Note that setting $f = \mathrm{curl}\, H^i$ yields the original problem (9). The weak solution $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ thus needs to satisfy

$$\int_{\mathbb{R}^3} \left(\varepsilon_{\mathrm{r}}^{-1} \,\mathrm{curl}\, v \cdot \mathrm{curl}\,\overline{\psi} - k^2 v \cdot \overline{\psi}\right) \mathrm{d}x = \int_{\mathbb{R}^3} Q\, f \cdot \mathrm{curl}\,\overline{\psi}\, \mathrm{d}x \qquad \forall \psi \in H(\mathrm{curl}, \mathbb{R}^3) \tag{11}$$

with compact support and, additionally, the Silver-Müller radiation condition,

$$\mathrm{curl}\, v(x) \times \hat{x} - \mathrm{i} k v(x) = \mathcal{O}\left(|x|^{-2}\right), \quad \text{as } |x| \to \infty, \text{ uniformly with respect to } \hat{x} \in \mathbb{S}^2. \tag{12}$$

Remark. (a) Choosing $\psi = \nabla\varphi$ to be a gradient field, the equation $\mathrm{curl}\,\nabla\varphi = 0$ implies that $\int_{\mathbb{R}^3} v \cdot \nabla\overline{\varphi} = 0$ for all $\varphi \in H^1(\mathbb{R}^3)$ with compact support, i.e., $\mathrm{div}\, v = 0$ in $\mathbb{R}^3$.

(b) The Silver-Müller radiation condition is well-defined for any weak solution v to (11): Outside D the solution v solves $\mathrm{curl}^2 v - k^2 v = 0$ together with $\mathrm{div}\, v = 0$; thus, the identity $\Delta = \nabla\,\mathrm{div} - \mathrm{curl}^2$ implies that $\Delta v + k^2 v = 0$ and elliptic regularity results imply that v is a smooth function in $\mathbb{R}^3 \setminus \overline{D}$.

Using either a volume integral approach [15] or a variational formulation in involving the exterior Calderon operator [22] it is possible to show that (11) can be reduced to a Fredholm problem, i.e., uniqueness implies existence of solution.

Assumption 1. *We assume in the following that any solution to* (11) *for* $f \in L^2(D, \mathbb{C}^3)$ *is unique, such that existence and continuous dependence of this solution follow from uniqueness. This assumption is always satisfied if* ε_{r} *is globally Hölder continuous, since, under this smoothness assumption, unique continuation results for Maxwell's equations are applicable, see [25].*

Every radiating solution $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ to (11) has the asymptotic behavior

$$v(x) = \frac{\exp\left(\mathrm{i}k|x|\right)}{4\pi|x|} v^\infty(\hat{x}) + \mathcal{O}\left(|x|^{-2}\right), \qquad \text{as } |x| \to \infty,$$

uniformly in all directions $\hat{x} = x/|x| \in \mathbb{S}^2$, involving the far field pattern $v^\infty : \mathbb{S}^2 \to \mathbb{C}^3$ of v. It is well-known that v^∞ is an analytic and tangential vector field on the unit sphere, i.e.,

$$v^\infty(\hat{x}) \cdot \hat{x} = 0 \quad \text{for all } \hat{x} \in \mathbb{S}^2.$$

In particular, v^∞ belongs to the space of square-integrable tangential vector fields

$$L^2_t(\mathbb{S}^2) := \left\{v \in L^2(\mathbb{S}^2, \mathbb{C}^3),\, v(\hat{x}) \cdot \hat{x} = 0 \text{ for a.e. } \hat{x} \in \mathbb{S}^2\right\} \subset L^2(\mathbb{S}^2, \mathbb{C}^3).$$

For the above-introduced incident plane wave $H^i(\cdot, \theta; p)$ the far field pattern $H^\infty(\cdot, \theta; p)$ of $H^s(\cdot, \theta; p)$ depends both on the incident angle θ and the polarization $p \in \mathbb{C}$. The far field patterns $H^\infty(\cdot, \theta; p)$ define the far field operator $F : L^2_{\mathrm{t}}(\mathbb{S}^2) \to L^2_{\mathrm{t}}(\mathbb{S}^2)$, a linear integral operator defined by

$$(Fp)\,(\hat{x}) := \int_{\mathbb{S}^2} H^\infty(\hat{x}, \theta; p(\theta))\, \mathrm{d}S(\theta) \qquad \text{for } \hat{x} \in \mathbb{S}^2. \tag{13}$$

The far field operator is linear since H^∞ depends linearly on p, i.e. $H^\infty(\hat{x}, \theta; p) = \hat{H}^\infty(\hat{x}, \theta)p$ for all $p \in \mathbb{C}^3$ with $p \cdot \theta = 0$ and $\hat{H}^\infty(\hat{x}, \theta) \in \mathbb{C}^{3\times 3}$. Due to reciprocity relations, H^∞ is moreover a smooth function in both variables $\hat{x}$ and θ which implies that F is a compact operator on $L^2_{\mathrm{t}}(\mathbb{S}^2)$. Additionally, since ε_{r} is real-valued the scattering problem in non-absorbing hence F is a normal operator, see [8, Corollary 6.40]. Thus, F possesses a complete orthonormal eigensystem $(\lambda_j, g_j)_{j\in\mathbb{N}}$ of eigenvalues $\lambda_j \in \mathbb{C}$ and eigenfunctions $g_j \in L^2_{\mathrm{t}}(\mathbb{S}^2)$. From [17] we additionally know that all λ_j lie on the circle $\{\lambda \in \mathbb{C},\, |8\pi^2\mathrm{i}/k - \lambda| = 8\pi^2/k\}$ in the complex plane.

3 The Herglotz Operator, its Range, and Transmission Eigenvalues

To establish a link between electromagnetic transmission eigenvalues and certain eigenvalues of the far field operator F we will exploit a factorization of F based on the following linear, compact Herglotz operator $H : L^2_{\mathrm{t}}(\mathbb{S}^2) \to L^2(D, \mathbb{C}^3)$, defined by

$$Hg = \mathrm{curl}_x \, v_g|_D \,, \qquad v_g(x) := \int_{\mathbb{S}^2} \mathrm{e}^{\mathrm{i}k\, x\cdot\theta} g(\theta)\, \mathrm{d}S(\theta) \qquad \text{for } x \in D, \tag{14}$$

where v_g is a so-called Herglotz wave function. Since $g \in L^2_{\mathrm{t}}(\mathbb{S}^2)$ we note that v_g is smooth and divergence-free and thus solves both Maxwell's equations $\mathrm{curl}^2 v_g - k^2 v_g = 0$ and the vectorial Helmholtz equation $\Delta v_g + k^2 v_g = 0$ in $\mathbb{R}^3$ in the classical sense. If v_g vanishes in D, then analytic continuation and [8, Theorem 3.15] applied to each component of v_g implies that g vanishes, i.e., the Herglotz operator H is injective.

Proposition 2. *The adjoint $H^* : L^2(D, \mathbb{C}^3) \to L^2_{\mathrm{t}}(\mathbb{S}^2)$ of the Herglotz operator is given by*

$$(H^*\psi)(\theta) = \mathrm{i}k\, \theta \times \int_D \psi(x) \mathrm{e}^{-\mathrm{i}k\, x\cdot\theta}\, \mathrm{d}x \qquad \textit{for } \theta \in \mathbb{S}^2.$$

Proof. Recall first that $\mathrm{curl}(\varphi F) = \nabla\varphi \times F + \varphi \,\mathrm{curl}\, F$ for scalar functions φ and vector fields V. Thus, for $g \in L^2_{\mathrm{t}}(\mathbb{S}^2)$ it holds that $\mathrm{curl}_x(\exp(\mathrm{i}k\, x\cdot\theta) g(\theta)) = \mathrm{i}k\theta \exp(\mathrm{i}kx\cdot\theta) \times g(\theta)$. For arbitrary $\psi \in L^2(D, \mathbb{C}^3)$ we thus obtain

$$\begin{aligned}
(Hg,\, \psi)_{L^2(D,\mathbb{C}^3)} &= \int_D \left(\mathrm{curl}_x \int_{\mathbb{S}^2} g(\theta)\, \mathrm{e}^{\mathrm{i}k\, x\cdot\theta}\, \mathrm{d}S(\theta) \right) \overline{\psi(x)}\, \mathrm{d}x \\
&= \int_D \int_{\mathbb{S}^2} \mathrm{curl}_x \left(g(\theta)\, \mathrm{e}^{\mathrm{i}k\, x\cdot\theta} \right) \mathrm{d}S(\theta)\, \overline{\psi(x)}\, \mathrm{d}x \\
&= \int_{\mathbb{S}^2} g(\theta)\, (-\mathrm{i}k)\, \theta \times \int_D \overline{\psi(x)} \mathrm{e}^{\mathrm{i}k\, x\cdot\theta}\, \mathrm{d}x\, \mathrm{d}S(\theta) = (g,\, H^*\psi)_{L^2_{\mathrm{t}}(\mathbb{S}^2)} \,.
\end{aligned}$$

□

For the next result we recall the notation

$$\Phi(x, y) = \frac{\exp(\mathrm{i}k|x-y|)}{4\pi|x-y|}, \qquad \text{for } x, y \in \mathbb{R}^3,\ x \neq y,$$

for the radiating fundamental solution to the scalar Helmholtz equation in $\mathbb{R}^3$. The far field pattern of $x \mapsto \Phi(x, y)$ is well-known to be $\theta \mapsto \exp(-\mathrm{i}k\, \theta\cdot y)$ and the far field pattern of $x \mapsto \mathrm{curl}_x\, \Phi(x, y)$ equals $\theta \mapsto \mathrm{i}k\, \theta \times \exp(-\mathrm{i}k\, \theta \cdot y)$, see, e.g. [8]. By linearity, this implies the following proposition.

Proposition 3. *For $\psi \in L^2(D, \mathbb{C}^3)$ the function $H^*\psi \in L^2_{\mathrm{t}}(\mathbb{S}^2)$ is the far field pattern v^∞ to*

$$v(x) = \mathrm{curl}_x \int_D \Phi(x, y) \psi(y)\, \mathrm{d}y, \qquad x \in \mathbb{R}^3.$$

The closure of the range of H in $L^2(D, \mathbb{C}^3)$ plays an important role in the sequel.

Lemma 4. *For $k > 0$ we define the closed subspace*

$$X_k = \left\{ w \in L^2(D, \mathbb{C}^3),\ \int_D w\, (\mathrm{curl}^2\, \psi - k^2\psi)\, \mathrm{d}x = 0\ \forall \psi \in C_0^\infty(D, \mathbb{C}^3) \right\} \subset L^2(D, \mathbb{C}^3). \tag{15}$$

Then it holds that $X_k = \mathrm{closure}_{L^2(D,\mathbb{C}^3)} \mathcal{R}(H)$.

Proof. The definition of H in (14) implies that $Hg = \operatorname{curl} v_g|_D$ where both the Herglotz wave function v_g and its curl are smooth and entire solutions to Maxwell's equations in $\mathbb{R}^3$. In particular, two partial integrations imply that v_g satisfies $\int_D (Hg)\,(\operatorname{curl}^2\psi - k^2\psi)\,\mathrm{d}x = 0$ for all $\psi \in C_0^\infty(D,\mathbb{C}^3)$ and $g \in L^2_{\mathrm{t}}(\mathbb{S}^2)$. In consequence, $\mathcal{R}(H) \subset X_k$.

To prove that $X_k \subset \operatorname{closure}_{L^2(D,\mathbb{C}^3)}\mathcal{R}(H)$ we assume that there exists $w_0 \in X_k$ such that w_0 is orthogonal to all elements in the range of H, i.e.,

$$0 = (w_0,\, Hg)_{L^2(D,\mathbb{C}^3)} = (H^* w_0,\, g)_{L^2_{\mathrm{t}}(\mathbb{S}^2)} \qquad \forall g \in L^2_{\mathrm{t}}(\mathbb{S}^2). \tag{16}$$

By Proposition 3 we know that $H^* w_0 = v^\infty$ is the far field pattern of the volume potential $v = \operatorname{curl}\int_D \Phi(\cdot, x) w_0(x)\,\mathrm{d}x$ in $\mathbb{R}^3$. Due to (16) the far field v^∞ vanishes and Rellich's lemma (see [8, Theorems 2.14 and 6.10] yields that $v = 0$ in $\mathbb{R}^3\backslash\overline{D}$. By [15] the volume potential v for $w_0 \in L^2(D,\mathbb{C}^3)$ solves

$$\int_{\mathbb{R}^3} \left[\operatorname{curl} v \cdot \operatorname{curl}\overline{\psi} - k^2 v\cdot\overline{\psi}\right]\,\mathrm{d}x = \int_D w_0 \cdot \operatorname{curl}\overline{\psi}\,\mathrm{d}x \tag{17}$$

for all $\psi \in H(\operatorname{curl},\mathbb{R}^3)$ with compact support, while $w_0 \in X_k$ solves

$$\int_D w_0 \cdot \left(\operatorname{curl}^2\overline{\psi} - k^2\overline{\psi}\right)\,\mathrm{d}x = 0 \qquad \forall \psi \in C_0^\infty(D,\mathbb{C}^3).$$

Thus, choosing the test function in (17) as $\left(\operatorname{curl}^2 - k^2\right)\psi$ for arbitrary $\psi \in C_0^\infty(D,\mathbb{R}^3)$ the right-hand side in (17) vanishes and $\int_{\mathbb{R}^3}\left(\operatorname{curl} v \cdot \operatorname{curl}\left(\operatorname{curl}^2 - k^2\right)\overline{\psi} - k^2 v\cdot\left(\operatorname{curl}^2 - k^2\right)\overline{\psi}\right)\mathrm{d}x = 0$. This shows that $(\operatorname{curl}^2 - k^2)(\operatorname{curl}^2 - k^2)v = 0$ holds in the distributional sense, i.e., in $\mathcal{D}'(\mathbb{R}^3,\mathbb{C}^3)$. Note that (17) moreover implies that $\operatorname{div} v = 0$. Exploiting the identity $\Delta = \nabla\operatorname{div} - \operatorname{curl}^2$ together with Schwartz's theorem, we find that

$$\begin{aligned}(\operatorname{curl}^2 - k^2)(\operatorname{curl}^2 - k^2)v =& (\nabla\operatorname{div} - \Delta - k^2)(\nabla\operatorname{div} - \Delta - k^2)v \\ =& (\nabla\operatorname{div} - \Delta - k^2)(-\Delta - k^2)v = (\Delta + k^2)(\Delta + k^2)v = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3,\mathbb{C}^3).\end{aligned}$$

The operator $(\Delta + k^2)(\Delta + k^2) = \Delta^2 + 2k^2\Delta + k^4$ is an elliptic differential operator of order four. Thus, Weyl's lemma for distributional solutions to elliptic partial differential equations with constant coefficients (see, e.g., [24, Corollary of Theorem 8.12]) applied to each of its components shows that $v \in C^\infty(\mathbb{R}^3,\mathbb{C}^3)$ is a smooth and compactly supported solution of $(\Delta + k^2)(\Delta + k^2)v = 0$ in $\mathbb{R}^3$. We multiply this equation by $\overline{v}$, integrate first over $\mathbb{R}^3$ and then twice by parts, and obtain that $(\Delta + k^2)v = 0$ in $\mathbb{R}^3$. Since v vanishes outside D, the analyticity of solutions the homogeneous Helmholtz equation shown in [8, Theorem 2.2] implies that $v = 0$ in all of $\mathbb{R}^3$.

By (17), the fact that v vanishes implies that w_0 satisfies $\int_D w_0 \cdot \operatorname{curl}\overline{\psi}\,\mathrm{d}x = 0$ for all $\psi \in H(\operatorname{curl}, D)$. Since $w_0 \in X_k$ is divergence-free, Theorem 3.5 in [12] shows the existence of a vector potential $V_0 \in H(\operatorname{curl}, D)$ such that $w_0 = \operatorname{curl} V_0$. Choosing $\psi = V_0$ hence yields $\int_D |w_0|^2\,\mathrm{d}x = \int_D w_0 \cdot \operatorname{curl}\overline{V_0}\,\mathrm{d}x = 0$. In consequence, every vector in X_k orthogonal to $\mathcal{R}(H)$ vanishes, which implies the claimed identity. □

Lemma 5. *For $k > 0$ it holds that*

$$X_k = \left\{ w \in L^2(D,\mathbb{C}^3),\ \int_D w\,(\operatorname{curl}^2\psi - k^2\psi)\,\mathrm{d}x = 0\ \forall\psi \in H_0(\operatorname{curl}, D) \text{ s.th. } \operatorname{curl}\psi \in H_0(\operatorname{curl}, D)\right\}.$$

Proof. Lemma A.1 in [14] shows that functions in $C_0^\infty(D,\mathbb{C}^3)$ are dense in $\{\psi \in H_0(\operatorname{curl}, D),\ \operatorname{curl}\psi \in H_0(\operatorname{curl}, D)\}$, equiped with the norm $\|\psi\| = \|\psi\|_{H(\operatorname{curl},D)} + \|\operatorname{curl}\psi\|_{H(\operatorname{curl},D)}$. Thus, the representation of X_k from Lemma 4 equals the claimed one by a density argument. □

Using the space X_k we rigorously define interior transmission eigenvalues.

Definition 6. The wave number $k > 0$ is an interior transmission eigenvalue if a non-trivial eigenpair $(v, w) \in H_0(\mathrm{curl}, D) \times X_k$ exists that satisfies

$$\begin{aligned} &\mathrm{curl}\left(\varepsilon_{\mathrm{r}}^{-1}\,\mathrm{curl}\, v\right) - k^2 v = \mathrm{curl}\,(Qw) \quad \text{in } D, \qquad \mathrm{curl}^2\, w - k^2 w = 0 \quad \text{in } D, \quad \text{and} \\ &\qquad \nu \times \varepsilon_{\mathrm{r}}^{-1}\,\mathrm{curl}\, v = \nu \times Qw \quad \text{on } \partial D. \end{aligned} \tag{18}$$

The differential equations and the boundary conditions are understood in a variational sense, i.e.,

$$\int_D \left[\varepsilon_{\mathrm{r}}^{-1}\,\mathrm{curl}\, v \cdot \mathrm{curl}\,\overline{\psi} - k^2 v \cdot \overline{\psi}\right]\,\mathrm{d}x = \int_D Qw \cdot \mathrm{curl}\,\overline{\psi}\,\mathrm{d}x \quad \forall \psi \in H(\mathrm{curl}, D). \tag{19}$$

Remark 7. In contrast to the formal introduction of transmission eigenvalues in (3), formulated using u and w, Definition 6 is formulated in terms of $v = u - w$ and $\mathrm{curl}\, w$. Of course, both formulations lead to precisely the same eigenvalues.

Lemma 8. *The eigenpair $(v, w) \in H_0(\mathrm{curl}, D) \times X_k$ belongs to $H_0^1(D, \mathbb{C}^3) \cap H_0(\mathrm{div}0, D) \times H(\mathrm{div}0, D)$.*

Proof. Since a function $w \in X_k$ belongs to $L^2(D, \mathbb{C}^3)$ and satisfies $[\mathrm{curl}^2 - k^2]w = 0$ in the distributional sense, the latter equation in particular holds in $L^2(D, \mathbb{C}^3)$. Since $\mathrm{div}\,\mathrm{curl} = 0$ we deduce that $\mathrm{div}\, w = 0$, i.e., $w \in H(\mathrm{div}0, D)$.

Choosing the test function $\psi \in H(\mathrm{curl}, D)$ in (19) to be a gradient field $\nabla\varphi$ for $\varphi \in H^1(D)$ we note that $v \in H_0(\mathrm{curl}, D)$ is divergence-free and that $v \cdot \nu = 0$ on ∂D: Indeed, a partial integration shows that

$$0 = \int_D v \cdot \nabla\overline{\varphi}\,\mathrm{d}x = \int_{\partial D} (v \cdot \nu)\overline{\varphi}\,\mathrm{d}S \qquad \forall \varphi \in H^1(D).$$

Thus, $v \in H_0(\mathrm{curl}, D) \cap H_0(\mathrm{div}, D) = H_0^1(D, \mathbb{C}^3)$ due to Lemma 2.5 in [12]. □

4 Linking Transmission Eigenvalues with the Far Field Operator

The characterization of transmission eigenvalues based on inside-outside duality relies on linking the interior eigenvalues with the far field operator F, more precisely, with a particular factorization of F. To state this factorization we introduce the operator

$$T = T_k : L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3), \qquad T_k f := Q\left(f + \mathrm{curl}\, v|_D\right), \tag{20}$$

where $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ is the unique radiating weak solution to $\mathrm{curl}\left(\varepsilon_{\mathrm{r}}^{-1}\,\mathrm{curl}\, v\right) - k^2 v = \mathrm{curl}\,(Qf)$ in $\mathbb{R}^3$, that is, for all $\psi \in H(\mathrm{curl}, \mathbb{R}^3)$ with compact support, v satisfies

$$\int_{\mathbb{R}^3} \left[\varepsilon_{\mathrm{r}}^{-1}\,\mathrm{curl}\, v \cdot \mathrm{curl}\,\overline{\psi} - k^2 v \cdot \overline{\psi}\right]\,\mathrm{d}x = \int_{\mathbb{R}^3} Qf \cdot \mathrm{curl}\,\overline{\psi}\,\mathrm{d}x \tag{21}$$

together with the Silver-Müller radiation condition (12).

Assumption 9. *From now on, we assume that $Q \in L^\infty(D, \mathrm{Sym}(3))$ either satisfies $Q(x) \geq c_0 I_3$ or $Q(x) \leq -c_0 I_3$ for some $c_0 > 0$ for almost all $x \in D$. We abbreviate these conditions as* $\mathrm{sign}(Q) = +1$ *or* $\mathrm{sign}(Q) = -1$. *In both cases, the inverse matrix $Q(x)^{-1}$ exists for almost every $x \in D$.*

Theorem 10. *(a) For $k > 0$ the factorization $F = H^* T H$ holds.*

(b) If $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ is the radiating weak solution to (21) *then the mapping $f \mapsto \mathrm{curl}\, v|_D$ is compact from $L^2(D, \mathbb{C}^3)$ into $L^2(D, \mathbb{C}^3)$.*

(c) For $k > 0$, $f \in L^2(D, \mathbb{C}^3)$, and v defined as the radiating weak solution to (21) *it holds that*

$$\mathrm{Im}\,(Tf,\, f)_{L^2(D,\mathbb{C}^3)} = \frac{k}{(4\pi)^2} \int_{\mathbb{S}^2} |v^\infty|^2\,\mathrm{d}S \geqslant 0. \tag{22}$$

Proof. In [17, Theorem 5.10], a slightly different factorization is shown for the isotropic case where $\varepsilon_\mathrm{r}^{-1} = 1 - q$ for a real-valued contrast q and that proof can be straightforwardly adapted to our setting. For the same isotropic setting, parts (b) and (c) are shown in [18, Theorem 5.12(d,e)] and [18, Theorem 5.12(a)], respectively, and those proofs can be adapted for our setting due to Assumption 9. □

The next theorem yields a first characterization of positive interior transmission eigenvalues using the above-introduced operator $T = T_k$. At this point, the k-dependence of the operators T_k, $H = H_k$ and $F = F_k$ as well as the dependence of X_k on the wave number becomes important. For this reason we denote this dependence explicitly from now on.

Theorem 11. *(a) If the wave number $k > 0$ is an interior transmission eigenvalue for the eigenpair $(v, w) \in H_0(\mathrm{curl}, D) \times X_k$ then $w \neq 0$ satisfies* $\mathrm{Im}\,(T_k w, w)_{L^2(D,\mathbb{C}^3)} = 0$.

(b) If $w \in X_k \setminus \{0\}$ satisfies $\mathrm{Im}\,(T_k w, w)_{L^2(D,\mathbb{C}^3)} = 0$ *the wave number $k > 0$ is an interior transmission eigenvalue and there is $v \in H_0(\mathrm{curl}, D)$ such that (v, w) is the corresponding eigenpair.*

Proof. (a) If $k > 0$ is a transmission eigenvalue with eigenpair (v, w) we extend v from D to all of $\mathbb{R}^3$ by zero. Due to (19) the extension satisfies

$$\int_{\mathbb{R}^3} \left[\varepsilon_\mathrm{r}^{-1} \,\mathrm{curl}\, v \cdot \mathrm{curl}\, \overline{\psi} - k^2 v \cdot \overline{\psi}\right] \,\mathrm{d}x = \int_D Q w \cdot \mathrm{curl}\, \overline{\psi} \,\mathrm{d}x \tag{23}$$

for all $\psi \in H(\mathrm{curl}, \mathbb{R}^3)$ with compact support. In particular, the definition of T_k in (20) shows that $T_k w = Q(w + \mathrm{curl}\, v)$ and since $v^\infty = 0$ we deduce from Theorem 10(c) that

$$\mathrm{Im}\,(T_k w, w) = \frac{k}{(4\pi)^2} \|v^\infty\|^2_{L^2_t(\mathbb{S}^2)} = 0.$$

(b) If $w \in X_k \setminus \{0\}$ satisfies $\mathrm{Im}\,(T_k w, w)_{L^2(D,\mathbb{C}^3)} = 0$ then define $v \in H_\mathrm{loc}(\mathrm{curl}, \mathbb{R}^3)$ to be the radiating weak solution to (21). Theorem 10(c) states that

$$\mathrm{Im}\,(T w, w)_{L^2(D,\mathbb{C}^3)} = \frac{k}{(4\pi)^2} \int_{\mathbb{S}^2} \|v^\infty\|^2_{L^2_t(\mathbb{S}^2)},$$

and hence the (tangential) far field pattern v^∞ vanishes on $\mathbb{S}^2$. Thus, Rellich's Lemma (see [8, Theorem 6.10]) implies that $v = 0$ in the exterior of D and (21) shows that

$$\int_D \left[\varepsilon_\mathrm{r}^{-1} \,\mathrm{curl}\, v \cdot \mathrm{curl}\, \overline{\psi} - k^2 v \cdot \overline{\psi}\right] \,\mathrm{d}x = \int_D Q w \cdot \mathrm{curl}\, \overline{\psi} \,\mathrm{d}x \qquad \forall \psi \in H(\mathrm{curl}, D).$$

Since v vanishes outside D we hence obtained a transmission eigenpair $(v, w) \in H_0(\mathrm{curl}, D) \times X_k$. □

Corollary 12. *The wave number $k > 0$ is an interior transmission eigenvalue if and only if there is $w \in X_k \setminus \{0\}$ such that* $(T_k w, w)_{L^2(D,\mathbb{C}^3)} = 0$.

Proof. If $(T_k w, w)_{L^2(D,\mathbb{C}^3)} = 0$ then $\mathrm{Im}\,(T_k w, w)_{L^2(D,\mathbb{C}^3)} = 0$ and Theorem 11 implies the claim. Moreover, if $k > 0$ is an interior transmission eigenvalue then $\mathrm{Im}\,(T_k w, w)_{L^2(D,\mathbb{C}^3)} = 0$ for $w \in X_k \setminus \{0\}$ due to Theorem 11. As in the proof of the latter theorem we exploit that $T_k w = Q\,(w + \mathrm{curl}\, v|_D)$ where $v \in H_\mathrm{loc}(\mathrm{curl}, \mathbb{R}^3) \cap H_0(\mathrm{curl}, D)$ solves (23), i.e.,

$$\int_{\mathbb{R}^3} \left[\mathrm{curl}\, v \cdot \mathrm{curl}\, \overline{\psi} - k^2 v \cdot \overline{\psi}\right] \,\mathrm{d}x = \int_D Q(w + \mathrm{curl}\, v) \cdot \mathrm{curl}\, \overline{\psi} \,\mathrm{d}x \quad \forall \psi \in H(\mathrm{curl}, D). \tag{24}$$

Since $w \in X_k$ belongs to the closure of $\mathcal{R}(H)$ in $L^2(D, \mathbb{C}^3)$ there exists a sequence $\{g_j\}_{j\in\mathbb{N}} \subset L^2_{\mathrm{t}}(\mathbb{S}^2)$ such that $w_j = H(g_j) \to w$ in $L^2(D, \mathbb{C}^3)$ as $j \to \infty$. We choose the test function ψ in (24) as $\operatorname{curl} w_j$,

$$\int_D Q(w + \operatorname{curl} v) \cdot \operatorname{curl}^2 \overline{w_j} \, \mathrm{d}x = \int_D \left[\operatorname{curl} v \cdot \operatorname{curl}^2 \overline{w_j} - k^2 v \cdot \operatorname{curl} \overline{w_j}\right] \, \mathrm{d}x.$$

and exploit the equation $\operatorname{curl}^2 w_j = k^2 w_j$ in D and integration by parts to find that

$$\int_D Q(w + \operatorname{curl} v) \cdot \overline{w_j} \, \mathrm{d}x = \int_D \left[\operatorname{curl} v \cdot \overline{w_j} - v \cdot \operatorname{curl} \overline{w_j}\right] \, \mathrm{d}x = \int_{\partial D} (\nu \times v) \cdot \overline{w_j} \, \mathrm{d}S = 0$$

because $v \in H_0(\operatorname{curl}, D)$. As $j \to \infty$ we obtain first that $\int_D Q(w + \operatorname{curl} v) \cdot \overline{w} \, \mathrm{d}x = 0$ and second that

$$(Tw, w)_{L^2(D,\mathbb{C}^3)} = \int_D Q(w + \operatorname{curl} v) \cdot \overline{w} \, \mathrm{d}x = 0.$$

□

In the end of Section 2 we already mentioned that the eigenvalues $\lambda_j = \lambda_j(k)$ of the far field operator $F = F_k$ lie on a circle with radius $8\pi^2/k$ centered at $8\pi^2\mathrm{i}/k$. Since F is compact on $L^2_{\mathrm{t}}(\mathbb{S}^2)$ these eigenvalues necessarily converge to zero as $j \to \infty$. We next show that if the contrast function $Q : D \to \mathrm{Sym}(3)$ has a fixed sign, the λ_j converge clockwise (i.e. from the right) or counter-clockwise (i.e. from the left) to zero as $j \to \infty$ (see Figure 1).

Theorem 13. *Assume that $k > 0$ is no interior transmission eigenvalue. If $\operatorname{sign}(Q) = \pm 1$, then $\operatorname{Re}(\lambda_j) \gtrless 0$ if $j \in \mathbb{N}$ is large enough.*

Proof. The claim follows from the factorization of the far field operator $F = H^*TH$, the orthonormality of its eigenfunctions $g_j \in L^2_{\mathrm{t}}(\mathbb{S}^2)$, and the fact that T is coercive up to a compact perturbation. These properties allow to prove the claim along the lines of, e.g., [17][Theorem 1.23], see also Section 5.4 in the same reference and Lemma 4.1 in [19]. □

If the far field operator F_k is not injective, then it is easy to show that the corresponding wave number is a transmission eigenvalue. Thus, if we assume that $k > 0$ is no interior transmission eigenvalue, then F is injective and all eigenvalues λ_j are non-zero and possess a unique polar representation,

$$\lambda_j = r_j \exp\left(\mathrm{i}\vartheta_j\right), \qquad \text{with } r_j \geqslant 0 \text{ and } \vartheta_j \in (0, \pi).$$

Theorem 13 directly determines the behavior of the phases ϑ_j,

$$\lim_{j\to\infty} \vartheta_j = \begin{cases} 0, & \text{if } \operatorname{sign}(Q) = +1, \\ \pi, & \text{if } \operatorname{sign}(Q) = -1. \end{cases}$$

Thus, if $\operatorname{sign}(Q) = +1$ we can define $\vartheta_+ = \max_{j\in\mathbb{N}} \vartheta_j$ and denote the corresponding eigenvalue of F with the largest phase by $\lambda_+ = r_+ \exp(\mathrm{i}\vartheta_+)$; if $\operatorname{sign}(Q) = -1$ we set $\vartheta_- = \min_{j\in\mathbb{N}} \vartheta_j$ and denote the corresponding eigenvalue by $\lambda_- = r_- \exp(\mathrm{i}\vartheta_-)$ (see Figure 1).

Theorem 14. *Assume that $k > 0$ is no interior transmission eigenvalue. If $\operatorname{sign}(Q) = +1$ or if $\operatorname{sign}(Q) = -1$ it holds that*

$$\cot\vartheta_+ = \min_{w\in X_k\setminus\{0\}} \frac{\operatorname{Re}(T_k w, w)_{L^2(D,\mathbb{C}^3)}}{\operatorname{Im}(T_k w, w)_{L^2(D,\mathbb{C}^3)}} \quad \text{or} \quad \cot\vartheta_- = \max_{w\in X_k\setminus\{0\}} \frac{\operatorname{Re}(T_k w, w)_{L^2(D,\mathbb{C}^3)}}{\operatorname{Im}(T_k w, w)_{L^2(D,\mathbb{C}^3)}}, \quad \textit{respectively.}$$

Remark. If $k > 0$ is no interior transmission eigenvalue, the denominator in the latter expressions is strictly positive due to Theorem 11 .

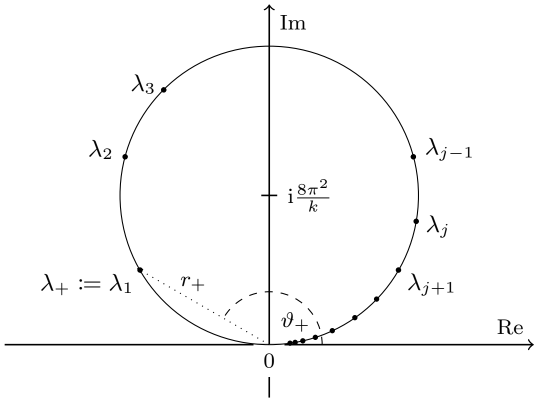

Figure 1: If $\operatorname{sign}(Q) = +1$ the eigenvalues λ_j of F converge clockwise to zero and the eigenvalue with the largest phase is $\lambda_+ = r_+ \exp(\mathrm{i}\vartheta_+)$.

Proof. Expressing F by his eigensystem, $Fg = \sum_{j\in\mathbb{N}} \lambda_j (g, g_j)_{L^2_t(\mathbb{S}^2)} g_j$, we note that

$$(Fg, g)_{L^2_t(\mathbb{S}^2)} = \sum_{j\in\mathbb{N}} \lambda_j \left|(g, g_j)_{L^2_t(\mathbb{S}^2)}\right|^2 \qquad \text{for } g \in L^2_t(\mathbb{S}^2).$$

From Euler's identity we know that $\operatorname{Re}(\lambda_j) = r_j \cos(\vartheta_j)$ and $\operatorname{Im}(\lambda_j) = r_j \sin(\vartheta_j)$. Furthermore, the function $h(\alpha) := \cos(\alpha)/\sin(\alpha) = \cot(\alpha)$ is a strictly monotonically decreasing function in the interval $(0, \pi)$. An application of [20, Lemma 4] thus shows that

$$\begin{aligned}\frac{\operatorname{Re}(Fg, g,)_{L^2_t(\mathbb{S}^2)}}{\operatorname{Im}(Fg, g)_{L^2_t(\mathbb{S}^2)}} &= \frac{\sum_{j\in\mathbb{N}} \operatorname{Re}(\lambda_j)|(g, g_j)|^2}{\sum_{j\in\mathbb{N}} \operatorname{Im}(\lambda_j)|(g, g_j)|^2} = \frac{\sum_{j\in\mathbb{N}} \cos(\vartheta_j) r_j |(g, g_j)|^2}{\sum_{j\in\mathbb{N}} \sin(\vartheta_j) r_j |(g, g_j)|^2} \\ &\leqslant \max_{g\in L^2_t(\mathbb{S}^2)} \frac{\sum_{j\in\mathbb{N}} \cos(\vartheta_j)|(g, g_j)|^2}{\sum_{j\in\mathbb{N}} \sin(\vartheta_j)|(g, g_j)|^2} = \frac{\cos(\vartheta_-)}{\sin(\vartheta_-)} = \cot(\vartheta_-) < \infty\end{aligned}$$

if $\operatorname{sign}(Q) = -1$. Lemma 4 in [20] moreover shows that the latter inequality becomes an equality if and only if $g = g_-$ is an eigenfunction of the eigenvalue λ_-. If $\operatorname{sign}(Q) = +1$ the claim follows analogously. Finally, because of the factorization of $F = H^*TH$ we obtain

$$(Fg, g)_{L^2_t(\mathbb{S}^2)} = (H^*THg, g)_{L^2_t(\mathbb{S}^2)} = (THg, Hg)_{L^2(D,\mathbb{C}^3)} = (Tw, w)_{L^2(D,\mathbb{C}^3)} \quad \text{with } w = Hg \in X_k.$$

Since the range of $H = H_k$ is by definition dense in X_k this implies the claim. □

5 Extremal Phases and Transmission Eigenvalues

Since the representation of the extremal phases $\vartheta_\pm$ in Theorem 14 relies on the k-depended spaces X_k we will from now on explicitly denote the dependence of the eigenvalues $\lambda_j = \lambda_j(k)$ and the extremal phases $\vartheta_\pm = \vartheta_\pm(k)$ of the far field operator F_k on the wave number $k > 0$ explicitly.

The next result shows the first part of the inside-outside duality holds without further assumptions: Whenever an eigenvalue corresponding to the smallest or largest phase tends to zero from the wrong side as $k \to k_0$ the limiting wave number k_0 is a transmission eigenvalue.

Theorem 15. *Choose $k_0 > 0$ such that $I := (k_0-\varepsilon, k_0+\varepsilon)\backslash\{k_0\}$ contains no transmission eigenvalue. If it holds that*

$$\lim_{I\ni k\to k_0} \vartheta_+(k) = \pi \text{ and } \operatorname{sign}(Q) = +1, \qquad \text{or if} \qquad \lim_{I\ni k\to k_0} \vartheta_-(k) = 0 \text{ and } \operatorname{sign}(Q) = -1, \tag{25}$$

then k_0 is an interior transmission eigenvalue.

Proof. We merely treat the case that $\operatorname{sign}(Q) = -1$; the case of a positive contrast can be treated analogously. Assuming that k_0 is no transmission eigenvalue, Theorem 14 implies that

$$\cot\vartheta_-(k) = \max_{w\in X_k} \frac{\operatorname{Re}(T_k w, w)_{L^2(D,\mathbb{C}^3)}}{\operatorname{Im}(T_k w, w)_{L^2(D,\mathbb{C}^3)}} \longrightarrow \infty \qquad \text{as } k\to k_0,\ k\in I.$$

Thus, there is a sequence $\{k_j\}_{j\in\mathbb{N}} \subset I$ and functions $\{w_j\}_{j\in\mathbb{N}} \subset X_{k_j}$ with $\|w_j\|_{L^2(D,\mathbb{C}^3)} = 1$ such that $k_j \to k_0$,

$$0 < \operatorname{Im}(T_{k_j} w_j, w_j)_{L^2(D,\mathbb{C}^3)} \to 0 \quad \text{as } j\to\infty, \qquad \text{and} \quad \operatorname{Re}(T_{k_j} w_j, w_j)_{L^2(D,\mathbb{C}^3)} \geqslant 0 \tag{26}$$

for sufficiently large j. Let $v_j \in H_{\mathrm{loc}}(\operatorname{curl},\mathbb{R}^3)$ be the corresponding radiating weak solution to (11),

$$\int_{\mathbb{R}^3} \left[(I_3 - Q)\operatorname{curl} v_j \cdot \operatorname{curl}\overline{\psi} - k_j^2 v_j\cdot\overline{\psi}\right]\,\mathrm{d}x = \int_{\mathbb{R}^3} Q\, w_j\cdot\operatorname{curl}\overline{\psi}\,\mathrm{d}x, \tag{27}$$

for all $\psi \in H(\operatorname{curl},\mathbb{R}^3)$ with compact support. Since the sequence w_j is bounded, there exists a weakly convergent subsequence $w_j \rightharpoonup w_0$ in $L^2(D,\mathbb{C}^3)$; by abuse of notation, we denote this subsequence again by $\{w_j\}$. The weak limit w_0 belongs to X_{k_0} and the solutions v_j to (27) converge weakly as well: If $v_0 \in H_{\mathrm{loc}}(\operatorname{curl},\mathbb{R}^3)$ denotes the solution to (27) when w_j is replaced by w_0, then $v_j \rightharpoonup v_0$ in $H(\operatorname{curl}, B)$ for every ball $B \subset \mathbb{R}^3$. Plugging in $f = w_j$ into (22) we deduce that

$$(4\pi)^2 \operatorname{Im}(T_{k_j} w_j, w_j)_{L^2(D,\mathbb{C}^3)} = k_j\, \|v_j^\infty\|^2_{L^2(\mathbb{S}^2)}.$$

The left-hand side tends to zero by (26) and the right-hand side to $k_0\|v_0^\infty\|^2_{L^2(\mathbb{S}^2)}$. Thus, $v_0^\infty = 0$ and Rellich's Lemma implies that v_0 vanishes in the exterior of D, see [8, Theorem 6.10]. Thus, $(v_0, w_0) \in H_0(\operatorname{curl}, D) \times X_{k_0}$ satisfy the transmission eigenvalue problem (18–19).

Under the assumption that k_0 is no interior transmission eigenvalue we conclude that w_0 and v_0 vanish in D, i.e., $(v_j, w_j) \rightharpoonup 0$. Exploiting that $Q\,w_j = Tw_j - Q\operatorname{curl} v_j$ we infer that

$$\begin{aligned}
(T_{k_j} w_j, w_j)_{L^2(D,\mathbb{C}^3)} &= (T_{k_j} w_j, Q^{-1} T_{k_j} w_j)_{L^2(D,\mathbb{C}^3)} - (T_{k_j} w_j, \operatorname{curl} v_j)_{L^2(D,\mathbb{C}^3)}\\
&= (Q^{-1} T_{k_j} w_j, T_{k_j} w_j)_{L^2(D,\mathbb{C}^3)} - \int_D Q(w_j + \operatorname{curl} v_j)\cdot\operatorname{curl}\overline{v_j}\,\mathrm{d}x\\
&= (Q^{-1} T_{k_j} w_j, T_{k_j} w_j)_{L^2(D,\mathbb{C}^3)} - \int_{|x|<R} \left[|\operatorname{curl} v_j|^2 - k_j^2|v_j|^2\right]\,\mathrm{d}x\\
&\qquad\qquad - \int_{|x|=R} (\hat{x}\times\operatorname{curl} v_j)\cdot\overline{v_j}\,\mathrm{d}S,
\end{aligned}$$

where we choose $R > 0$ large enough such that $\overline{D} \subset B_R(0)$. Under the latter condition, v_j is a smooth function outside $\overline{D}$ and mappings $w_j \mapsto v_j|_{|x|=R}$ and $w_j \mapsto [\hat{x}\times\operatorname{curl} v_j]|_{|x|=R}$ are compact from $L^2(D,\mathbb{C}^3)$ into, e.g., $L^2(\partial B_R(0),\mathbb{C}^3)$. Consider now the real part of the latter equation for $(T_{k_j} w_j, w_j)$: Since $\operatorname{Re}(T_{k_j} w_j, w_j)_{L^2(D,\mathbb{C}^3)} \geqslant 0$ for j large enough and $\operatorname{sign}(Q) = -1$,

$$\begin{aligned}
\int_{|x|<R} |\operatorname{curl} v_j|^2\,\mathrm{d}x &= -\operatorname{Re}(T_{k_j} w_j, w_j)_{L^2(D,\mathbb{C}^3)} + \int_D Q^{-1} T_{k_j} w_j\cdot T_{k_j}\overline{w_j}\,\mathrm{d}x\\
&\qquad + \int_{|x|<R} k_j^2|v_j|^2\,\mathrm{d}x - \operatorname{Re}\int_{|x|=R} (\hat{x}\times\operatorname{curl} v_j)\cdot\overline{v_j}\,\mathrm{d}S\\
&\leqslant \int_{|x|<R} k_j^2|v_j|^2\,\mathrm{d}x - \operatorname{Re}\int_{|x|=R} (\hat{x}\times\operatorname{curl} v_j)\cdot\overline{v_j}\,\mathrm{d}S.
\end{aligned} \tag{28}$$

The second term in (28) tends to zero since $w_j \rightharpoonup 0$ and $w_j \mapsto (\hat{x} \times \operatorname{curl} v_j) \cdot \overline{v_j}$ is compact from $L^2(D, \mathbb{C}^3)$ into $L^1(\partial B_R(0), \mathbb{C}^3)$. Concerning the first term in (28), recall that $v_j \rightharpoonup 0$ in $H(\operatorname{curl}, B_R(0))$ and that $v_j \in H(\operatorname{div}0, B_R(0))$. Since the space $H(\operatorname{curl}, B_R(0)) \cap H(\operatorname{div}0, B_R(0))$, equipped with the norm of $H(\operatorname{curl}, D)$, is compactly embedded in $L^2(B_R(0), \mathbb{C}^3)$ we obtain that $v_j \to 0$ strongly in $L^2(B_R(0), \mathbb{C}^3)$, see [22, Theorem 4.7]. In consequence, the right-hand side in (28) tends to zero, which implies that v_j converges strongly to zero in $H(\operatorname{curl}, B_R(0))$ for arbitrarily large $R > 0$. From (27) we deduce that $w_j \to 0$ in $L^2(D, \mathbb{C}^3)$, which contradicts the assumption $\|w_j\|_{L^2(D,\mathbb{C}^3)} = 1$. Thus, the assumption that k_0 is no interior transmission eigenvalue was wrong. □

6 The Orthogonal Projection onto X_k

In Theorem 15 we showed that whenever the smallest or largest phase tends to zero or π, respectively, then the limiting wave number is a transmission eigenvalue. The reciprocal result is more difficult to prove. A crucial tool in our analysis is a projection P_k onto $X_k \subset L^2(D, \mathbb{C}^3)$ that we will construct in this section. To this end, we introduce

$$\begin{aligned} W &:= \{\psi \in H_0(\operatorname{curl}, D) \cap H_0(\operatorname{div}, D),\ \operatorname{curl}\psi \in H_0(\operatorname{curl}, D) \cap H_0(\operatorname{div}, D)\} \\ &\overset{(*)}{=} \left\{\psi \in H_0^1(D, \mathbb{C}^3),\ \operatorname{curl}\psi \in H_0^1(D, \mathbb{C}^3)\right\} \end{aligned}$$

with norm $\|\psi\|_W := \|\psi\|_{H(\operatorname{curl},D)} + \|\operatorname{div}\psi\|_{L^2(D,\mathbb{C}^3)} + \|\operatorname{curl}^2\psi\|_{L^2(D,\mathbb{C}^3)}$. The equality $(*)$ is due to Lemma 2.5 in [12] stating that $H_0(\operatorname{div}, D) \cap H_0(\operatorname{curl}, D) = H_0^1(D, \mathbb{C}^3)$.

Lemma 16. *For $k > 0$ there is $c > 0$ such that $\|(\operatorname{curl}^2 - k^2)\psi\|^2_{L^2(D,\mathbb{C}^3)} + \|\operatorname{div}\psi\|^2_{L^2(D,\mathbb{C}^3)} \geqslant c\|\psi\|^2_W$ for all $\psi \in W$.*

Proof. Assume, on the contrary, that there is no such constant $c > 0$. Then there exists a sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset W$ such that $\|\psi_j\|_W = 1$ and $\|(\operatorname{curl}^2 - k^2)\psi_j\|_{L^2(D,\mathbb{C}^3)} \to 0$ and $\|\operatorname{div}\psi_j\|_{L^2(D,\mathbb{C}^3)} \to 0$ as $j \to \infty$. We choose a weakly convergent subsequence, also denoted by $\{\psi_j\}$, such that $\psi_j \rightharpoonup \psi \in W$ weakly in W. Since $W \subset H_0^1(D, \mathbb{C}^3)$ the compact embedding of $H_0^1(D, \mathbb{C}^3)$ in $L^2(D, \mathbb{C}^3)$ implies that $\psi_j \to \psi$ in $L^2(D, \mathbb{C}^3)$. As $\|(\operatorname{curl}^2 - k^2)\psi_j\|_{L^2(D,\mathbb{C}^3)} \to 0$ we moreover obtain that $\operatorname{curl}^2\psi_j$ converges strongly in $L^2(D, \mathbb{C}^3)$. Since the only possible limit equals $\operatorname{curl}^2\psi$ the limit equation $(\operatorname{curl}^2 - k^2)\psi = 0$ holds in $L^2(D, \mathbb{C}^3)$. Since $\psi \in W$, the Stratton-Chu formula [22, Theorem 9.2] implies that

$$\psi = -\operatorname{curl}\int_{\partial D} (\nu \times \psi(y))\Phi(\cdot, y)\,\mathrm{d}S - \frac{1}{k^2}\int_{\partial D} (\nu \times \operatorname{curl}\psi(y))\Phi(\cdot, y)\,\mathrm{d}S = 0 \qquad \text{in } D,$$

because the tangential trace of $\psi \in H_0^1(D, \mathbb{C}^3)$ and $\operatorname{curl}\psi \in H_0^1(D, \mathbb{C}^3)$ vanishes. We already saw above that $\operatorname{div}\psi_j \to 0$ in $L^2(D, \mathbb{C}^3)$ and deduce that $\|\psi_j\|_W \to \|\psi\|_W = 0$ as $j \to \infty$, contradicting our assumption that $\|\psi_j\|_W = 1$ for all $j \in \mathbb{N}$. □

From now on we adopt the following assumption on D to avoid the appearance of cohomology spaces in the Helmholtz decomposition when defining the projection P_k, cf., e.g., [22, Section 3.7].

Assumption 17. *D is a Lipschitz domain with connected complement and each connected component of D is simply connected. In particular, the boundary of each connected component is connected.*

Due to the Helmholtz decomposition (see, e.g., [22, Theorem 3.45]) and the geometric Assumption 17, a function $g \in L^2(D, \mathbb{C}^3)$ can be decomposed as $g = \operatorname{curl} A_g + \nabla p_g$ with a uniquely determined vector potential $A_g \in H(\operatorname{curl}, D) \cap H(\operatorname{div}0, D)$ such that $A_g \cdot \nu = 0$ on ∂D and a unique scalar potential $p_g \in H_0^1(D)$. Moreover, both A_g and p_g depend continuously on $g \in L^2(D, \mathbb{C}^3)$ in their natural norms. This allows to define the operator P_k for $k > 0$ by

$$P_k : L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3), \qquad P_k g := g - \left(\operatorname{curl}^2 - k^2\right)\hat{A}_g - \nabla p_g, \tag{29}$$

where $\hat{A}_g \in W$ solves the following variational problem for all $\psi \in W$,

$$\int_D (\operatorname{curl}^2 - k^2)\, \hat{A}_g \cdot (\operatorname{curl}^2 - k^2)\, \overline{\psi}\, \mathrm{d}x + \int_D \operatorname{div} \hat{A}_g \cdot \operatorname{div} \overline{\psi}\, \mathrm{d}x = \int_D \operatorname{curl} A_g \cdot (\operatorname{curl}^2 - k^2)\, \overline{\psi}\, \mathrm{d}x. \qquad (30)$$

Lemma 18. *(a) The mapping $P_k : L^2(D, \mathbb{C}^3) \to X_k$ is well-defined and represents the orthogonal projection from $L^2(D, \mathbb{C}^3)$ onto X_k. The function $\hat{A}_g \in W$, defined in* (30)*, is divergence-free and*

$$\int_D (\operatorname{curl}^2 - k^2)\, \hat{A}_g \cdot (\operatorname{curl}^2 - k^2)\, \overline{\psi}\, \mathrm{d}x = \int_D \operatorname{curl} A_g \cdot (\operatorname{curl}^2 - k^2)\, \overline{\psi}\, \mathrm{d}x \qquad \forall \psi \in W. \qquad (31)$$

(b) For $g \in L^2(D, \mathbb{C}^3)$ the function $k \mapsto P_k g$ from $\mathbb{R}_{>0}$ into $L^2(D, \mathbb{C}^3)$ is continuously differentiable.

Proof. (a) The variational problem (30) is well-posed as the sesquilinear form on the right of (30) is coercive on W due to Lemma 16. Since $\operatorname{curl} A_g$ is bounded in term of $g \in L^2(D, \mathbb{C}^3)$ the solution $\hat{A}_g \in W$ to (30) is hence uniquely defined and bounded in terms of g as well.

We further show that $\hat{A}_g$ is divergence-free: Plugging in $\nabla \varphi$ for $\varphi \in C_0^\infty(D)$ into (30) we exploit that $\int_D \operatorname{curl}^2 \hat{A}_g \cdot \nabla \overline{\varphi}\, \mathrm{d}x = 0$ by partial integration and obtain that $k^4 \int_D \hat{A}_g \cdot \nabla \overline{\varphi}\, \mathrm{d}x + \int_D \operatorname{div} \hat{A}_g \cdot \operatorname{div} \nabla \overline{\varphi}\, \mathrm{d}x = 0$ for all $\varphi \in C_0^\infty(D)$. The Helmholtz decomposition $\hat{A}_g = \operatorname{curl} A + \nabla p$ with $p \in H_0^1(D)$ implies that $\Delta p = \operatorname{div} \hat{A}_g \in L^2(D)$, i.e., $p \in H^1_{0,\Delta}(D) = \{q \in H_0^1(D),\, \Delta q \in L^2(D)\}$. Arguing as in [22, Chapter 7.4], we find that

$$k^4 \int_D (\operatorname{curl} A + \nabla p) \cdot \nabla \overline{\varphi}\, \mathrm{d}x + \int_D \Delta p \cdot \operatorname{div} \nabla \overline{\varphi}\, \mathrm{d}x = 0 \qquad \forall \varphi \in C_0^\infty(D).$$

Again, partial integration shows that $\int_D \operatorname{curl} A \cdot \nabla \overline{\varphi}\, \mathrm{d}x = 0$, that is, $p \in H^1_{0,\Delta}(D)$ solves

$$\int_D (\Delta p - k^4 p) \Delta \overline{\varphi}\, \mathrm{d}x = 0 \qquad \forall \varphi \in C_0^\infty(D).$$

Thus, $p \in H^1_{0,\Delta}(D)$ satisfies $-\Delta p = -k^4 p$ in D and p is an eigenfunction of $-\Delta$ for a negative eigenvalue. The negative Dirichlet Laplacian is however well-known to be a positive operator which implies first that p necessarily vanishes and second that $\hat{A}_g = \operatorname{curl} A$ is a divergence-free function that satisfies (31).

To check that P_k maps into X_k we choose $g \in L^2(D, \mathbb{C}^3)$ and consider $w = P_k g = g - (\operatorname{curl}^2 - k^2)\hat{A}_g - \nabla p_g = \operatorname{curl} A_g - (\operatorname{curl}^2 - k^2)\hat{A}_g$. Due to (31),

$$\int_D w \cdot (\operatorname{curl}^2 - k^2)\, \overline{\psi}\, \mathrm{d}x = \int_D \Big(\operatorname{curl} A_g - (\operatorname{curl}^2 - k^2)\hat{A}_g\Big) \cdot (\operatorname{curl}^2 - k^2)\, \overline{\psi}\, \mathrm{d}x = 0 \qquad \forall \psi \in W.$$

Since $C_0^\infty(D, \mathbb{C}^3) \subset W$, Lemma 4 implies that $w \in X_k$.

To check that P_k is a projection we choose $w \in X_k$ and recall from Lemma 8 that w is divergence-free. Hence, the scalar potential $p_w \in H_0^1(D)$ from the Helmholtz decomposition $w = \operatorname{curl} A_w + \nabla p_w$ of w vanishes since it is a weak solution to the Laplace equation in D with homogeneous Dirichlet boundary data. In consequence, the right-hand side of (30) vanishes,

$$\int_D \operatorname{curl} A_w \cdot (\operatorname{curl}^2 - k^2)\, \overline{\psi}\, \mathrm{d}x = \int_D w \cdot (\operatorname{curl}^2 - k^2)\, \overline{\psi}\, \mathrm{d}x = 0$$

for all $\psi \in W$ since, as above, by definition of W it holds that $\psi \in H_0(\operatorname{curl}, D)$ and $\operatorname{curl} \psi \in H_0(\operatorname{curl}, D)$ and Lemma 5 states that the latter integral vanishes for $w \in X_k$. Thus, the solution $\hat{A}_w \in W$ to (30) vanishes and $P_k w = w$, i.e., P_k is a projection onto X_k. This projection is even orthogonal: Consider $w \in X_k$ and $g \in L^2(D, \mathbb{C}^3)$ with Helmholtz decomposition $g = \operatorname{curl} A_g + \nabla p_g$, where again $p_g \in H_0^1(D)$. Since $w \in X_k$ is divergence-free to obtain and since $\hat{A}_g \in W$ it follows that

$$\big(P_k g - g, w\big)_{L^2(D,\mathbb{C}^3)} = -\big(\nabla p_g, w\big)_{L^2(D,\mathbb{C}^3)} - \big((\operatorname{curl}^2 - k^2)\hat{A}_g, w\big)_{L^2(D,\mathbb{C}^3)} = 0.$$

(b) We note that $k \mapsto P_k g$ is a differentiable function for every $g \in L^2(D, \mathbb{C}^3)$ since

$$P_k' g := \frac{\mathrm{d}}{\mathrm{d}k}(P_k g) = \frac{\mathrm{d}}{\mathrm{d}k}\Big[w - (\operatorname{curl}^2 - k^2)\hat{A}_g + \nabla p_g\Big] = -(\operatorname{curl}^2 - k^2)\hat{A}_g' + 2k\hat{A}_g,$$

where $\hat{A}_g \in W$ solves (30) and $\hat{A}_g' := \mathrm{d}\hat{A}_g/\mathrm{d}k \in W$ solves

$$\int_D (\operatorname{curl}^2 - k^2)\hat{A}_g' \cdot (\operatorname{curl}^2 - k^2)\overline{\psi}\,\mathrm{d}x + \int_D \operatorname{div}\hat{A}_g' \cdot \operatorname{div}\overline{\psi}\,\mathrm{d}x = 2k\int_D \hat{A}_g \cdot (\operatorname{curl}^2 - k^2)\overline{\psi}\,\mathrm{d}x$$
$$+ 2k\int_D (\operatorname{curl}^2 - k^2)\hat{A}_g \cdot \overline{\psi}\,\mathrm{d}x - 2k\int_D \operatorname{curl} A_g \cdot \overline{\psi}\,\mathrm{d}x \qquad \forall \psi \in W.$$

The latter formula for $\hat{A}_g'$ follows from the polynomial dependence on k of the left- and right-hand side of the coercive variational formulation (30). The above proof that $\hat{A}_g$ is divergence-free transfers to $\hat{A}_g'$ which shows that $\hat{A}_g'$ solves

$$\int_D (\operatorname{curl}^2 - k^2)\hat{A}_g' \cdot (\operatorname{curl}^2 - k^2)\overline{\psi}\,\mathrm{d}x = 2k\int_D \hat{A}_g \cdot (\operatorname{curl}^2 - k^2)\overline{\psi}\,\mathrm{d}x$$
$$+ 2k\int_D (\operatorname{curl}^2 - k^2)\hat{A}_g \cdot \overline{\psi}\,\mathrm{d}x - 2k\int_D \operatorname{curl} A_g \cdot \overline{\psi}\,\mathrm{d}x \qquad \forall \psi \in W. \tag{32}$$

□

7 Inside-Outside Duality

In this section, we apply the projection P_k from (29) to show that under a certain condition the reciprocal result to Theorem 15 holds: If $k_0 > 0$ is an interior transmission eigenvalue, then the smallest or largest phase tends to zero or π, respectively. Together, these two statements yield the so-called inside-outside duality. We emphasize that our results merely show that this duality holds under certain conditions. While these conditions are not explicitly related to the contrast Q in this section, in the subsequent section we will derive explicit conditions on the contrast such that the duality holds at least for the smallest positive interior transmission eigenvalues.

Theorem 19. *Let be $k_0 > 0$ be a transmission eigenvalue and $w_0 \in X_{k_0}$ such that $w_0 \neq 0$ and $(T_{k_0} w_0, w_0)_{L^2(D,\mathbb{C}^3)} = 0$. Choose $\varepsilon > 0$ such that $(k_0 - \varepsilon, k_0 + \varepsilon)$ contains no other transmission eigenvalue. If $k \mapsto (T_k P_k w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)}$ is differentiable in k at $k = k_0$ and if the derivative*

$$\alpha'(k_0) := \frac{\mathrm{d}}{\mathrm{d}k}\left(T_k P_k w_0, P_k w_0\right)_{L^2(D,\mathbb{C}^3)}\Big|_{k=k_0} \in \mathbb{R} \setminus \{0\}$$

is real and non-zero, then it holds for $\operatorname{sign}(Q) = +1$ *that*

$$\lim_{k_0 - \varepsilon > k \nearrow k_0} \vartheta_+(k) = \pi \text{ if } \alpha'(k_0) > 0 \qquad \text{and} \qquad \lim_{k_0 + \varepsilon < k \searrow k_0} \vartheta_+(k) = \pi \text{ if } \alpha'(k_0) < 0$$

and for $\operatorname{sign}(Q) = -1$ *that*

$$\lim_{k_0 + \varepsilon > k \searrow k_0} \vartheta_-(k) = 0 \text{ if } \alpha'(k_0) > 0 \qquad \text{and} \qquad \lim_{k_0 - \varepsilon > k \nearrow k_0} \vartheta_-(k) = 0 \text{ if } \alpha'(k_0) < 0.$$

Proof. We merely prove the claim in case that $\operatorname{sign}(Q) = -1$ since the case of a positive contrast can be treated analogously. Choose $\varepsilon > 0$ such that $I := (k_0 - \varepsilon, k_0 + \varepsilon)$ contains no interior transmission eigenvalue different from k_0. In Theorem 14 we saw that for $k \in I \setminus \{k_0\}$ it holds that

$$\cot \vartheta_-(k) = \max_{w \in X_k \setminus \{0\}} \frac{\operatorname{Re}(T_k w, w)_{L^2(D,\mathbb{C}^3)}}{\operatorname{Im}(T_k w, w)_{L^2(D,\mathbb{C}^3)}} = \max_{g \in L^2(D,\mathbb{C}^3) \setminus \{0\}} \frac{\operatorname{Re}(T_k P_k g, P_k g)_{L^2(D,\mathbb{C}^3)}}{\operatorname{Im}(T_k P_k g, P_k g)_{L^2(D,\mathbb{C}^3)}}.$$

Define $\alpha(k) := (T_k P_k w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)}$ for $k \in I$ and note that $\alpha(k_0) = (T_{k_0} w_0, w_0)_{L^2(D,\mathbb{C}^3)}$ vanishes by Theorem 11 since k_0 is a transmission eigenvalue. Thus, Taylor's theorem implies that

$$\alpha(k) = \alpha(k_0) + \alpha'(k_0)(k - k_0) + r(k) = \alpha'(k_0)(k - k_0) + r(k) \quad \text{where } r(k) = \mathrm{o}(|k - k_0|) \text{ as } k \to k_0.$$

Since $\alpha'(k_0)$ is real and $\operatorname{Im}(\alpha(k)) = \operatorname{Im}(T_k P_k w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)} > 0$ for $k \in I \setminus \{k_0\}$ by Theorem 10(c) we obtain that

$$\cot \vartheta_-(k) \geqslant \frac{\operatorname{Re}(T_k P_k w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)}}{\operatorname{Im}(T_k P_k w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)}} = \frac{\alpha'(k_0)\,(k - k_0) + \operatorname{Re}(r(k))}{\operatorname{Im}(r(k))}, \qquad k \in I \setminus \{k_0\}.$$

If $\alpha'(k_0) > 0$ and if $k_0 + \varepsilon > k \searrow k_0$ then $\alpha'(k_0)\,(k - k_0) > 0$ tends to zero linearly whereas $\operatorname{Re}(r(k))$ and $\operatorname{Im}(r(k))$ both tend to zero faster than linearly in $k - k_0$. In consequence, $\cot \vartheta_-(k) \to \infty$ as $k_0 + \varepsilon > k \searrow k_0$, i.e., $\vartheta_-(k) \to 0$. The same technique applies in case that $\alpha'(k_0) < 0$. ◻

Corollary 20 (Conditional Inside-Outside Duality)**.** *If there exist wave numbers $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_{>0}$ such that $k_j \to k_0 > 0$, $k_j \neq k_0$, and $\vartheta_+(k_j) \to \pi$ or $\vartheta_-(k_j) \to 0$ as $j \to \infty$ in case that* $\operatorname{sign}(Q) = +1$ *or* $\operatorname{sign}(Q) = -1$, *respectively, then k_0 is an interior transmission eigenvalue.*

If $k_0 > 0$ is an interior transmission eigenvalue such that the derivative $\alpha'(k_0)$ is non-zero, then there exists $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_{>0}$ such that $k_j \to k_0 > 0$, $k_j \neq k_0$, and $\vartheta_+(k_j) \to \pi$ or $\vartheta_-(k_j) \to 0$ as $j \to \infty$ in case that $\operatorname{sign}(Q) = +1$ *or* $\operatorname{sign}(Q) = -1$, *respectively.*

Proof. Due to Theorems 15 and 19 it merely remains to show that $k \mapsto (T_k P_k w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)}$ is differentiable at $k = k_0$, which will be shown independently in Lemma 22 below. ◻

The remaining crucial task is hence to compute the derivative $\alpha'(k_0)$ from the last theorem. Before doing so we show the following auxiliary result.

Lemma 21. *Assume that $k_0 > 0$ is an interior transmission eigenvalue with eigenfunction $(v_0, w_0) \in H_0(\operatorname{curl}, D) \times X_{k_0}$. Then the mapping $k \mapsto (T_k w_0, w_0)_{L^2(D,\mathbb{C}^3)}$ is differentiable in k at $k = k_0$ and*

$$\frac{\mathrm{d}}{\mathrm{d}k} (T_k w_0, w_0)_{L^2(D,\mathbb{C}^3)} \Big|_{k=k_0} = 2k_0 \int_D |v_0|^2 \,\mathrm{d}x. \tag{33}$$

Proof. Define v_k for $k > 0$ as the unique radiating solution to the variational formulation

$$\int_{\mathbb{R}^3} \left[(I_3 - Q) \operatorname{curl} v_k \cdot \operatorname{curl} \overline{\psi} - k^2 v_k \cdot \overline{\psi} \right] \mathrm{d}x = \int_D Q\, w_0 \cdot \operatorname{curl} \overline{\psi} \,\mathrm{d}x \qquad \forall \psi \in H(\operatorname{curl}, \mathbb{R}^3) \tag{34}$$

with compact support and note that $v_0 = v_{k_0} \in H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3) \cap H_0(\operatorname{curl}, D)$. Since this variational problem depends polynomially on k and since $v_{k_0} \in H_0(\operatorname{curl}, D)$ we note that the derivative $v_0' := \mathrm{d}v_k / \mathrm{d}k|_{k=k_0}$ of v_k with respect to $k > 0$ at $k = k_0$ satisfies

$$\int_D \left[(I_3 - Q) \operatorname{curl} v_0' \cdot \operatorname{curl} \overline{\psi} - k_0^2 v_0' \cdot \overline{\psi} \right] \mathrm{d}x = 2k_0 \int_D v_0 \cdot \overline{\psi} \,\mathrm{d}x \qquad \forall \psi \in H(\operatorname{curl}, D). \tag{35}$$

Now we compute the derivative of $k \mapsto (T_k w_0, w_0)_{L^2(D,\mathbb{C}^3)}$ with respect to k at $k = k_0$:

$$\frac{\mathrm{d}}{\mathrm{d}k}(T_k w_0, w_0)_{L^2(D,\mathbb{C}^3)}\bigg|_{k=k_0} = \frac{\mathrm{d}}{\mathrm{d}k}(Q(w_0 + v_k|_D), w_0)_{L^2(D,\mathbb{C}^3)}\bigg|_{k=k_0} = \int_D Q \operatorname{curl}(v_0')\overline{w_0}\,\mathrm{d}x.$$

Choosing $\psi = v_0'$ in (34) and taking the complex conjugate of this equation shows that

$$\frac{\mathrm{d}}{\mathrm{d}k}(T_k w_0, w_0)_{L^2(D,\mathbb{C}^3)}\bigg|_{k=k_0} = \int_D \left[(I_3 - Q)\operatorname{curl}(v_0') \cdot \operatorname{curl}(\overline{v_0}) - k_0^2 v_0' \cdot \overline{v_0}\right] \mathrm{d}x = 2k_0 \int_D |v_0|^2 \,\mathrm{d}x.$$

□

Lemma 22. *Assume that $k_0 > 0$ is an interior transmission eigenvalue with eigenpair $(v_0, w_0) \in H_0(\operatorname{curl}, D) \times X_{k_0}$. Then the mapping $k \mapsto (T_k P_k w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)}$ is differentiable in k at k_0 and*

$$\frac{\mathrm{d}}{\mathrm{d}k}\left(T_k P_k w_0, P_k w_0\right)_{L^2(D,\mathbb{C}^3)}\bigg|_{k=k_0} = 2k_0 \int_D |v_0|^2 \,\mathrm{d}x + \frac{4}{k_0}\mathrm{Re}\int_D \operatorname{curl} v_0 \cdot \overline{w_0}\,\mathrm{d}x. \tag{36}$$

Proof. Recall from Lemma 18 that $k \mapsto P_k w_0$ is continuously differentiable with derivative

$$P_k' w_0 = \frac{\mathrm{d}}{\mathrm{d}k}(P_k w_0) = -(\operatorname{curl}^2 - k^2)\hat{A}_{w_0}' + 2k\hat{A}_{w_0}, \tag{37}$$

where $\hat{A}_{w_0} \in W$ solves (30) for A_{w_0} instead of A_g and $\hat{A}_{w_0}' \in W$ solves (32) with g replaced by w_0. As in the proof of Lemma 18 we exploit that $w_0 = \operatorname{curl} A_{w_0}$ for $A_{w_0} \in H(\operatorname{curl}, D) \cap H(\operatorname{div}0, D)$ because $w_0 \in X_{k_0}$ is divergence-free. Now, integrate by parts twice to rewrite (32) for $\hat{A}_{w_0}'$ as

$$\int_D (\operatorname{curl}^2 - k^2)\hat{A}_{w_0}' \cdot (\operatorname{curl}^2 - k^2)\overline{\psi}\,\mathrm{d}x = 4k \int_D \hat{A}_{w_0} \cdot (\operatorname{curl}^2 - k^2)\overline{\psi}\,\mathrm{d}x - 2k \int_D w_0 \cdot \overline{\psi}\,\mathrm{d}x \qquad \forall \psi \in W \tag{38}$$

and note that no boundary terms occur since $\hat{A}_{w_0} \in W \subset H_0(\operatorname{curl}, D)$ and thus $\operatorname{curl}\hat{A}_{w_0} \in X_N \subset H_0(\operatorname{curl}, D)$. We compute the derivative $\alpha'(k_0)$ by the chain rule,

$$\begin{aligned}
\alpha'(k_0) &= \left[\frac{\mathrm{d}}{\mathrm{d}k}(T_k P_k w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)}\right]\bigg|_{k=k_0} \\
&= \left[(T_k' P_k w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)} + (T_k P_k' w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)} + (T_k P_k w_0, P_k' w_0)_{L^2(D,\mathbb{C}^3)}\right]\big|_{k=k_0} \\
&= 2k_0 \int_D |v_0|^2 \,\mathrm{d}x + \overline{(T_{k_0}^* w_0, P_{k_0}' w_0)}_{L^2(D,\mathbb{C}^3)} + (T_{k_0} w_0, P_{k_0}' w_0)_{L^2(D,\mathbb{C}^3)},
\end{aligned}$$

and show next that $T_{k_0}^* w_0 = T_{k_0} w_0$. To this end, recall that $T_{k_0} w_0 = Q(w_0 + v_0)$ where the first component $v_0 \in H_0(\operatorname{curl}, D)$ of the eigenpair (v_0, w_0) to the transmission eigenvalue k_0 solves

$$\int_D \left[(I_3 - Q)\operatorname{curl} v_0 \cdot \operatorname{curl}\overline{\psi} - k_0^2 v_0 \cdot \overline{\psi}\right] \mathrm{d}x = \int_D Q\, w_0 \cdot \operatorname{curl}\overline{\psi}\,\mathrm{d}x \qquad \forall \psi \in H(\operatorname{curl}, D).$$

Obviously, extending v_0 by zero outside D yields a radiating solution to (11). Moreover,

$$\begin{aligned}
(T_{k_0} w_0, w_0)_{L^2(D,\mathbb{C}^3)} &= (Q w_0, w_0)_{L^2(D,\mathbb{C}^3)} + \int_D \operatorname{curl} v_0 \cdot (Q\overline{w_0})\,\mathrm{d}x \\
&= \int_D \overline{w_0}^\top Q w_0 \,\mathrm{d}x + \int_D \left[(\operatorname{curl}\overline{v_0})^\top (I_3 - Q)\operatorname{curl} v_0 - k_0^2 |v_0|^2\right] \mathrm{d}x.
\end{aligned}$$

Since the latter expression is real-valued, T_{k_0} is self-adjoint on the kernel of $w_0 \mapsto (T_{k_0} w_0, w_0)$, i.e., $T_{k_0} w_0 = T_{k_0}^* w_0$, and

$$\frac{\mathrm{d}}{\mathrm{d}k} \left(T_k P_k w_0, P_k w_0\right)_{L^2(D,\mathbb{C}^3)} \bigg|_{k=k_0} = 2k_0 \int_D |v_0|^2 \,\mathrm{d}x + 2\mathrm{Re}\, \left(T_{k_0} w_0, P'_{k_0} w_0\right)_{L^2(D,\mathbb{C}^3)}.$$

To compute the last term on the right we recall that $w_0 \in X_{k_0}$ implies $P_{k_0} w_0 = w_0$, that is, the term $\hat{A}_{w_0}$ from (37) vanishes and $P'_{k_0} w_0 = -(\mathrm{curl}^2 - k_0^2)\hat{A}'_{w_0}$ where $\hat{A}'_{w_0} \in W$ solves

$$\int_D (\mathrm{curl}^2 - k_0^2)\hat{A}'_{w_0} \cdot (\mathrm{curl}^2 - k_0^2)\overline{\psi} \,\mathrm{d}x = -2k_0 \int_D w_0 \cdot \overline{\psi} \,\mathrm{d}x \qquad \forall \psi \in W. \tag{39}$$

Since $w_0 \in X_k$ is divergence-free one shows as in the proof of Lemma 18 that $\hat{A}'_{w_0}$ is divergence-free. In consequence, $(\mathrm{curl}^2 - k_0^2)\hat{A}'_{w_0} \in L^2(D,\mathbb{C}^3)$ is also divergence-free and Theorem 3.6 in [12] implies that there exists a unique vector potential $A_0 \in H_0(\mathrm{curl}, D) \cap H(\mathrm{div}0, D)$ such that $\mathrm{curl}\, A_0 = (\mathrm{curl}^2 - k_0^2)\hat{A}'_{w_0} = -P'_{k_0} w_0$. This allows to show that

$$\begin{aligned}
(T_{k_0} w_0, P'_{k_0} w_0) &= -\int_D Q(w_0 + \mathrm{curl}\, v_0) \cdot \overline{(\mathrm{curl}^2 - k_0^2)\hat{A}'_{w_0}} \,\mathrm{d}x \\
&= -\int_D Q(w_0 + \mathrm{curl}\, v_0) \cdot \mathrm{curl}\, \overline{A_0} \,\mathrm{d}x \overset{(34)}{=} -\int_D \left[\mathrm{curl}\, v_0 \cdot \mathrm{curl}\, \overline{A_0} - k_0^2 v_0 \cdot \overline{A_0}\right] \mathrm{d}x.
\end{aligned}$$

Since $v_0 \in H_0^1(D,\mathbb{C}^3) \cap H_0(\mathrm{div}0, D)$ by Lemma 8, we can exploit Theorem 3.6 in [12] another time to obtain the existence of a unique vector potential $V_0 \in H_0(\mathrm{curl}, D) \cap H(\mathrm{div}0, D)$ such that $\mathrm{curl}\, V_0 = v_0$. Obviously, $\mathrm{curl}\, V_0 \in H_0^1(D,\mathbb{C}^3)$, which allows to continue the last computation by a partial integration,

$$\begin{aligned}
&(T_{k_0} w_0, P'_{k_0} w_0) = -\int_D \left[\mathrm{curl}\, v_0 \cdot \mathrm{curl}\, \overline{A_0} - k_0^2 \,\mathrm{curl}\, V_0 \cdot \overline{A_0}\right] \mathrm{d}x \\
&= -\int_D \left[\mathrm{curl}^2\, V_0 \cdot \overline{(\mathrm{curl}^2 - k_0^2)\hat{A}'_{w_0}} - k_0^2 V_0 \cdot \overline{(\mathrm{curl}^2 - k_0^2)\hat{A}'_{w_0}}\right] \mathrm{d}x = \int_D \left[\mathrm{curl}^2 - k_0^2\right] V_0 \cdot P'_{k_0} \overline{w_0} \,\mathrm{d}x.
\end{aligned}$$

Recall that the projection $P_k w_0$ onto X_k satisfies by Lemma 5 that

$$\int_D P_k w_0 \cdot \left[\mathrm{curl}^2\, \psi - k^2 \psi\right] \mathrm{d}x = 0 \qquad \forall \psi \in H_0(\mathrm{curl}, D) \text{ such that } \mathrm{curl}\, \psi \in H_0(\mathrm{curl}, D)$$

and hence in particular for all $\psi \in W$. Differentiating the latter variational equation with respect to $k > 0$ we obtain that $P'_{k_0} w_0$ satisfies

$$\int_D P'_{k_0} w_0 \left[\mathrm{curl}^2\, \psi - k_0^2 \psi\right] \mathrm{d}x = 2k_0 \int_D P_{k_0} w_0 \cdot \psi \,\mathrm{d}x \qquad \text{for all } \psi \in W.$$

Since $P_{k_0} w_0 = w_0 \in X_{k_0}$ and since $V_0 \in W$ satisfies $V_0 \in H_0(\mathrm{curl}, D)$ and $\mathrm{curl}\, V_0 = v_0 \in H_0(\mathrm{curl}, D)$ it holds by Lemma 5 that $\int_D \overline{w_0} \cdot [\mathrm{curl}^2\, V_0 - k_0^2 V_0] \,\mathrm{d}x = 0$. In particular,

$$\begin{aligned}
(T_{k_0} w_0, P'_{k_0} w_0) &= \int_D \left[\mathrm{curl}^2 - k_0^2\right] V_0 \cdot P'_{k_0} \overline{w_0} \,\mathrm{d}x = 2k_0 \int_D P_{k_0} \overline{w_0} \cdot V_0 \,\mathrm{d}x \\
&= \frac{2}{k_0} \int_D \mathrm{curl}^2\, V_0 \cdot \overline{w_0} \,\mathrm{d}x = \frac{2}{k_0} \int_D \mathrm{curl}\, v_0 \cdot \overline{w_0} \,\mathrm{d}x.
\end{aligned}$$

□

8 Explicit Conditions for the Contrast

Now we derive a condition on the contrast function Q guaranteeing that the derivative $\alpha'(k_0)$ is non-zero at least for a couple of the smallest positive interior transmission eigenvalues $k_0 > 0$ that are below a certain bound. This critical bound is large enough to guarantee existence of transmission eigenvalues smaller than this bound. For all such transmission eigenvalues the conditional duality statement of Corollary 20 thus applies.

The following tools and results from [16] will be important in this section: As we saw in Lemma 8, the function v_0 from a transmission eigenpair (v_0, w_0) belongs in particular to

$$V = \left\{ v \in H_0(\mathrm{curl}, D), \int_D v \cdot \nabla\varphi \,\mathrm{d}x = 0 \text{ for all } \varphi \in H^1(D) \right\}. \tag{40}$$

Further, by $\mu_0 > 0$ we denote the smallest eigenvalue of the eigenvalue problem to find $(\mu, v) \in \mathbb{R} \times V$ such that $\int_D \mathrm{curl}\, v \cdot \mathrm{curl}\, \overline{\psi} \,\mathrm{d}x = \mu \int_D v\overline{\psi} \,\mathrm{d}x$ for all $\psi \in V$. By the min-max-principle, $\mu_0 = \min_{\psi \in V} \| \mathrm{curl}\, \psi\|^2_{L^2(D,\mathbb{C}^3)} / \|\psi\|^2_{L^2(D,\mathbb{C}^3)}$ and

$$\mu_0 \|\psi\|^2_{L^2(D,\mathbb{C}^3)} \leqslant \| \mathrm{curl}\, \psi\|^2_{L^2(D,\mathbb{C}^3)} \qquad \forall \psi \in V. \tag{41}$$

From [22, Corollary 3.51] we deduce the existence of $\rho_0 > 0$ such that

$$\|v\|_{L^2(D,\mathbb{C}^3)} \leqslant \rho_0 \| \mathrm{curl}\, v\|_{L^2(D,\mathbb{C}^3)} \qquad \forall v \in H(\mathrm{curl}, D) \cap H_0(\mathrm{div}0, D). \tag{42}$$

Since $V \subset H(\mathrm{curl}, D) \cap H_0(\mathrm{div}0, D)$ the Poincaré-type estimate (41) implies $\rho_0^2 \mu_0 \geqslant 1$. Finally, from [16, Eq. (4.34)] we know that if $Q \leqslant q_0 I_3 < 0$ in D and if $M > 0$ satisfies

$$\left(1 + \frac{2}{|q_0|}\right) \mu_0 \leqslant M^2 \left(1 - \frac{2\rho_0^2}{|q_0|} M^2\right), \tag{43}$$

then there exists at least one interior transmission eigenvalue $k_0 > 0$ less than or equal to M.

We will now first consider the case of a constant contrast $Q = q_0 I_3$ for $q_0 \in (-\infty, 0)$ and derive a condition guaranteeing that the set of transmission eigenvalues for which the implicit condition of Corollary 20 applied is non-empty.

Theorem 23. *If $Q = q_0 I_3$ and if $q_0 < -(1+\sqrt{5})$ satisfies that $8\rho_0^2\mu_0 \leqslant (2-q_0)((1+q_0)^2-5)/(1-q_0)^2$, then there exists at least one transmission eigenvalue $k_0 > 0$ such that*

$$k_0^2 < \frac{2\mu_0(1-q_0)}{2-q_0}.$$

For any transmission eigenvalue below the latter bound it holds that $\alpha'(k_0) < 0$ such that the duality statement from Corollary 20 holds true.

Proof. Assume that $k_0 > 0$ is a transmission eigenvalue with eigenpair $(v_0, w_0) \in H_0(\mathrm{curl}, D) \times X_{k_0}$. Due to Lemma 22, the derivative $\alpha'(k_0)$ for an interior transmission eigenvalue $k_0 > 0$ is given by

$$\alpha'(k_0) = 2k_0 \|v_0\|^2_{L^2(D,\mathbb{C}^3)} + \frac{4}{k_0} \mathrm{Re}\, (\mathrm{curl}\, v_0, w_0)_{L^2(D,\mathbb{C}^3)}, \tag{44}$$

where $0 \neq v_0 \in H_0(\mathrm{curl}, D)$ solves (34). Choosing $\psi = v_0$ in (34) shows that

$$\int_D \left[(1-q_0)|\mathrm{curl}\, v_0|^2 - k_0^2 |v_0|^2\right] \mathrm{d}x = \int_D Q\, w_0 \cdot \mathrm{curl}\, \overline{v_0} \,\mathrm{d}x = q_0 \int_D w_0 \cdot \mathrm{curl}\, \overline{v_0} \,\mathrm{d}x \tag{45}$$

and that

$$\alpha'(k_0) = \left(2k_0 - \frac{4k_0}{q_0}\right) \|v_0\|^2_{L^2(D,\mathbb{C}^3)} + \frac{4(1-q_0)}{k_0 q_0}\|\operatorname{curl} v_0\|^2_{L^2(D,\mathbb{C}^3)}$$

$$\overset{(41)}{\leqslant} \left[\left(2k_0 - \frac{4k_0}{q_0}\right) + \frac{4(1-q_0)}{k_0 q_0}\mu_0\right] \|v_0\|^2_{L^2(D,\mathbb{C}^3)} < 0 \quad \text{if and only if} \quad k_0^2 < \frac{2\mu_0(1-q_0)}{2-q_0} = C(q_0)^2.$$

The latter condition is equivalent to $(2\mu_0 - k_0^2)q_0 < 2\mu_0 - 2k_0^2$, which can only hold for $q_0 < 0$ if $k_0^2 < 2\mu_0$. Under this assumption we conclude that $q_0 < \min((\mu_0 - k_0^2)/(\mu_0 - k_0^2/2), 0)$. To ensure the existence of transmission eigenvalues satisfying $k_0 < C(q_0)$ we need to ensure by (43) that

$$\left(1 + \frac{2}{|q_0|}\right)\mu_0 \leqslant C(q_0)^2 \left(1 - \frac{2\rho_0^2}{|q_0|}C(q_0)^2\right) \quad \Leftrightarrow \quad 8\rho_0^2\mu_0 \leqslant \frac{2-q_0}{(1-q_0)^2}((1+q_0)^2 - 5). \tag{46}$$

The latter condition in particular implies that $q_0 < -(1+\sqrt{5})$. □

We finally derive an analogous condition for a transmission eigenvalue $k_0 > 0$ with eigenpair (v_0, w_0) for a variable contrast of the form

$$Q := q_0 I_3 + Q' \qquad \text{with } q_0 < -(1+\sqrt{5}) \text{ and } 0 \geqslant Q' \in L^\infty(D, \operatorname{Sym}(3)).$$

Plugging this representation of q into (45) shows that

$$\begin{aligned} \operatorname{Re}(\operatorname{curl} v_0, w_0)_{L^2(D,\mathbb{C}^3)} =& \frac{1-q_0}{q_0}\|\operatorname{curl} v_0\|^2_{L^2(D,\mathbb{C}^3)} - \frac{k_0^2}{q_0}\|v_0\|^2_{L^2(D,\mathbb{C}^3)} \\ &- \frac{1}{q_0}\int_D Q' \operatorname{curl} v_0 \cdot \operatorname{curl}\overline{v_0}\,\mathrm{d}x - \frac{1}{q_0}\operatorname{Re}\int_D Q' w_0 \cdot \operatorname{curl}\overline{v_0}\,\mathrm{d}x. \end{aligned}$$

In the next estimate we denote the essential supremum of the spectral matrix norm of Q' over D by $\||Q'|_2\|_{L^\infty(D)}$. Plugging in the last estimate into (44) we deduce that

$$\begin{aligned} \alpha'(k_0) =& \left(2k_0 - \frac{4k_0}{q_0}\right)\|v_0\|^2_{L^2(D,\mathbb{C}^3)} + \frac{4(1-q_0)}{q_0 k_0}\|\operatorname{curl} v_0\|^2_{L^2(D,\mathbb{C}^3)} \\ &- \frac{4}{q_0 k_0}\int_D Q' \operatorname{curl} v_0 \cdot \operatorname{curl}\overline{v_0}\,\mathrm{d}x - \frac{4}{q_0 k_0}\operatorname{Re}\int_D Q' w_0 \cdot \operatorname{curl}\overline{v_0}\,\mathrm{d}x \\ \leqslant& \left[\frac{1}{\mu_0}\left(2k_0 - \frac{4k_0}{q_0}\right) + \frac{4}{q_0 k_0}(1-q_0) + \frac{4}{|q_0| k_0 \sqrt{\mu_0}}\||Q'|_2\|_{L^\infty(D)}\right]\|\operatorname{curl} v_0\|^2_{L^2(D,\mathbb{C}^3)}. \end{aligned} \tag{47}$$

The latter expression is negative if and only if

$$k_0^2 < \frac{2\mu_0}{2-q_0}\left[1 - q_0 - \||Q'|_2\|_{L^\infty(D)}/\sqrt{\mu_0}\right] := C(Q)^2.$$

To guarantee the existence of such a transmission eigenvalue we need to check condition (43), i.e., whether $C(Q)^2$ satisfies $\mu_0(2-q_0)/|q_0| \leqslant C(Q)^2(1 - 2\rho_0^2 C(Q)^2/|q_0|)$, or equivalently whether

$$\frac{(2-q_0)^2}{2|q_0|} \leqslant \left[1 - q_0 - \frac{\||Q'|_2\|_{L^\infty(D)}}{\sqrt{\mu_0}}\right]\left[1 - \frac{4\rho_0^2\mu_0}{(2-q_0)|q_0|}\left[1 - q_0 - \frac{\||Q'|_2\|_{L^\infty(D)}}{\sqrt{\mu_0}}\right]\right]. \tag{48}$$

If $Q' = 0$ then the latter condition reduces to condition (46) from the last lemma and is hence satisfied whenever $q_0 < -(1+\sqrt{5})$ satisfies (46) and $\||Q'|_2\|_{L^\infty(D)}$ is small enough.

Lemma 24. *If $q_0 < -(1+\sqrt{5})$ and if $Q = q_0 I_3 + Q'$ satisfy (48) then there exists an interior transmission eigenvalue $k_0 > 0$ such that*

$$k_0^2 < \frac{2\mu_0}{2-q_0}\left[1 - q_0 - \||Q'|_2\|_{L^\infty(D)}/\sqrt{\mu_0}\right].$$

For any transmission eigenvalue satisfying this condition the derivative $\alpha'(k_0)$ is strictly negative such that the duality statement from Corollary 20 applies.

9 Acknowledgments

The research of AL was supported through an exploratory project granted by the University of Bremen in the framework of its institutional strategy, funded by the excellence initiative of the federal and state governments of Germany.

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