

# The Inside-Outside Duality for Inverse Scattering Problems with Near Field Data

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## Abstract

We derive an inside-outside duality for near field scattering data generated by time-harmonic scattering of acoustic point sources from a sound-soft scatterer. This duality in particular rigorously characterizes interior Dirichlet eigenvalues of the scattering object by near field operators for an interval of wave numbers. As a crucial new concept to prove this duality we exploit the numerical ranges of certain modifications of these near field operators. We also show that our theoretical results can be numerically used to approximate interior Dirichlet eigenvalues from multi-frequency near field measurements.

## 1 Introduction

Recently, there has been some effort to identify interior eigenvalues of unknown scattering objects from far field data by a technique known as inside-outside duality, see [8, 17, 13]. For scattering from impenetrable scattering objects with, e.g., Dirichlet or Robin Neumann conditions, this technique yields a rigorous characterization of these eigenvalues, that can also be exploited numerically, see [20, 21]. In this article, we extend this approach to scattering problems where near field data measured on some closed measurement surface surrounding the obstacle is available; such a model is relevant for any practical application involving field measurements taken merely few wavelengths away from the scatterer.

Important ingredients in the derivation of the inside-outside duality in a far field setting are compactness and normality of the far field operator, the special structure generated by its eigenvalues, in particular their arrangement in the complex plane, as well as a suitable factorization of the far field operator. Arguably, a natural approach for a near field setting would hence be to consider the eigenvalues of the near field operator. While this operator retains some important properties like compactness and denseness of its range, it fails to be normal, and, in contrast to the far field operator, its eigenvalues do not lie on a circle in the complex plane – indeed, they do not show any particular structure at all. This issue is somewhat related to the problem of deriving a proper factorization for the near field operator, suitable to construct a factorization method, see [1, 15, 16] for details. Since such a factorization is as much an ingredient for the inside-outside duality as the structure of the eigenvalues of the near field operator, we need to deal with both problems.

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To this end, we use an ansatz from [12], where the near field operator defined on a spherical (or circular) measurement surface is modified by a unitary operator such that the modification possesses a factorization similar to the well-known factorization of the far field operator, see [14]. In this article we show that these similarities can be used to base the inside-outside duality for near field data on the inside-outside duality for far field data. A new concept we exploit in this context is the numerical range of an operator, which helps to overcome several difficulties linked to the spectral theory of non-selfadjoint operators. Our main result in Corollary 19 then shows that interior Dirichlet eigenvalues can be characterized by the behavior of any element with the smallest phase in the numerical range of the modified near field operator: If this smallest phase tends to zero as the wave number tends to  $k_0 > 0$ , then  $k_0^2$  is a Dirichlet eigenvalue of the scattering object.

The ansatz in [12] relies on the fact that in spherical coordinates the Helmholtz equation can be solved explicitly using spherical harmonics and spherical Hankel functions. Consequently, one can easily transfer outgoing into incoming waves by complex conjugation of the Hankel function; the corresponding operator is unitary and provides the tool to tackle the above-mentioned problems linked to the factorization of the near field operator. Naturally, this spherical setting is crucial for this operator to be simply computable in the basis of spherical harmonics due to diagonalization, and shows why in most of this paper we work with near field data measured on a sphere. Nonetheless, in Section 3 we show how to tackle near field data for general measurement geometries.

Before introducing the setting, let us finally mention that the analytical and numerical results in this paper transfer straightforwardly to near field obstacle scattering from Neumann or Robin-type obstacles and as well to transmission problems, see [20, 17] for the corresponding analysis for far field data. Indeed, we prove the duality statement for near field data relying on the corresponding duality for far field data, exploiting a trick that is completely independent of the scattering problem.

In the remainder of this introduction we define first the underlying Dirichlet scattering problem and second what we understand by near- and far field data. To this end, we suppose that  $D \subset \mathbb{R}^3$  is the obstacle with Lipschitz boundary and  $k > 0$  the wave number under consideration. Given boundary data  $f \in H^{1/2}(\partial D)$ , we seek a solution  $u \in H_{\text{loc}}^1(\mathbb{R}^3 \setminus \overline{D})$  to the exterior boundary value problem

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \quad u = -f \quad \text{on } \partial D, \quad (1)$$

that additionally satisfies Sommerfeld's radiation condition

$$\frac{\partial u}{\partial |x|}(r\hat{x}) - iku(r\hat{x}) = \mathcal{O}(1/r^2) \quad \text{as } r \rightarrow \infty, \quad \text{uniformly in } \hat{x} := \frac{x}{|x|} \in \mathbb{S}_1 = \{x \in \mathbb{R}^3 : |x| = 1\}.$$

Since the field  $u$  is radiating, it is well-known [4, 23] that there exists a unique solution to this scattering problem for any  $f \in H^{1/2}(\partial D)$  and that the asymptotic behavior of this solution can be expressed in terms of its far field  $u^\infty : \mathbb{S}_1 \rightarrow \mathbb{S}_1$ , defined by the first-order expansion

$$u(x) = \frac{e^{ik|x|}}{4\pi|x|} u^\infty(\hat{x}) + \mathcal{O}(|x|^{-2}), \quad \hat{x} \in \mathbb{S}_1.$$

When choosing the boundary data  $f$  in (1) as the restriction of an incident field  $u^i$  to  $\partial D$ , the resulting field  $u$  is the corresponding scattered field. We consider in the following either

incident plane waves  $u^i(x, \theta) = \exp(ik \theta \cdot x)$  with direction  $\theta \in \mathbb{S}_1$  or radiating point-sources

$$u^i(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x \neq y \in \mathbb{R}^3, \quad \text{for source points } y \notin \overline{D}. \quad (2)$$

If the incident field  $u^i$  is a plane wave with direction  $\theta$ , we indicate the dependence of the far field pattern  $u^\infty$  on the incident direction by writing  $u^\infty(\hat{x}, \theta)$  for  $\hat{x}, \theta \in \mathbb{S}_1$  and define the far field operator  $F : L^2(\mathbb{S}_1) \rightarrow L^2(\mathbb{S}_1)$  by

$$Fg(\hat{x}) = \int_{\mathbb{S}_1} u^\infty(\hat{x}, \theta)g(\theta) dS(\theta), \quad \hat{x} \in \mathbb{S}_1. \quad (3)$$

It is well-known that the far field operator is compact and normal and that its eigenvalues  $\lambda_j$  lie on a circle of radius  $8\pi^2/k$  with center  $8\pi^2i/k$  in the complex plane  $\mathbb{C}$  and converge to zero from the left, i.e.,  $\text{Re}(\lambda_j) < 0$  for  $j$  large enough. This allows to write the eigenvalue  $\lambda_j$  in polar coordinates,

$$\lambda_j = r_j e^{i\vartheta_j} \quad \text{for all } j \in \mathbb{N}, \quad (4)$$

with radius  $r_j \geq 0$  and phase  $\vartheta_j \in (0, \pi]$  and to re-sort these eigenvalues in increasing order according to their phases. (If  $\lambda_j = 0$  we artificially set  $\vartheta_j = \pi$  to ensure monotonicity.) The special structure of the eigenvalues implies that there is one distinct eigenvalue with a smallest phase  $\vartheta_*$ , see [20]. Moreover, the inside-outside duality for the Dirichlet scattering problem, see [20, 8] states that the phase of this eigenvalue converges to zero if and only if the squared wave number  $k^2$  approaches a Dirichlet eigenvalue  $k_0^2$  of  $-\Delta$  from below. The proof is based on the well-known factorization  $F = -G_\infty S^* G_\infty^*$  of  $F$  involving the data-to-pattern operator  $G_\infty : H^{1/2}(\partial D) \rightarrow L^2(\mathbb{S}_1)$  mapping boundary data  $f$  to the far field  $u^\infty$  of the radiating solution to (1), its adjoint  $G_\infty^*$ , and the single-layer operator  $S = S_{\partial D} : H^{-1/2}(\partial D) \rightarrow H^{1/2}(\partial D)$ ; crucial ingredients for the inside-outside duality are that the two outer operators  $G_\infty$  and  $G_\infty^*$  are adjoint to each other, that  $G_\infty^*$  has dense range, and that  $S^*$  is coercive. Let us recall here that the single-layer operator is defined as restriction of the single-layer potential; for an arbitrary Lipschitz domain  $\Omega \subset \mathbb{R}^3$ , the latter operator is defined as

$$\text{SL}_{\partial\Omega} f(x) = \int_{\partial\Omega} \frac{e^{ik|x-y|}}{4\pi|x-y|} f(y) dS(y), \quad x \in \mathbb{R}^3 \setminus \partial\Omega,$$

and maps  $H^{-1/2}(\partial\Omega)$  continuously into  $H^1(B_R)$  for arbitrary balls  $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$  of radius  $R > 0$ , see [23].

If the boundary data  $f$  in (1) is the restriction of a point source at  $y \in \mathbb{R}^3 \setminus \overline{D}$ , see (2), we denote the solution to (1) by  $u(\cdot, y) \in H_{\text{loc}}^1(\mathbb{R}^3 \setminus \overline{D})$ . If  $\Gamma \subset \mathbb{R}^3 \setminus \overline{D}$  denotes the boundary of an arbitrary Lipschitz domain  $\Omega_\Gamma \ni D$  with connected complement, we define the near field operator  $N_\Gamma : L^2(\Gamma) \rightarrow L^2(\Gamma)$  corresponding to incident point sources on  $\Gamma$  and near field wave measurements on the same surface  $\Gamma$  by

$$N_\Gamma g(x) = \int_\Gamma u(x, y)g(y) dS(y), \quad x \in \Gamma.$$

As  $u(\cdot, y)$  solves (1) for boundary data  $f$  defined by  $f(x) = \exp(ik|x-y|)/(4\pi|x-y|)$  for  $x \in \Gamma$ , the linear combination  $u = \int_\Gamma u(\cdot, y)g(y) dS(y)$  solves (1) with  $f = \text{SL}_\Gamma g|_{\partial D}$ . Thus,  $N_\Gamma g$  equals the trace of the scattered field corresponding to the incident field  $\text{SL}_\Gamma g$ .

Moreover,  $u(\cdot, y) \in H_{\text{loc}}^1(\mathbb{R}^3 \setminus \overline{D})$  satisfies the homogeneous Helmholtz equation in  $\mathbb{R}^3 \setminus \overline{D}$ , such that well-known interior elliptic regularity results imply that  $u(\cdot, y) \in C^\infty(\mathbb{R}^3 \setminus \overline{D})$ . The near field reciprocity relation  $u(x, y) = u(y, x)$  for  $x \neq y \in \mathbb{R}^3 \setminus \overline{D}$ , that can be shown using Green's second identity, additionally implies that the kernel  $(x, y) \mapsto u(x, y)$  of  $N_\Gamma$  belongs to  $C^\infty(\Gamma \times \Gamma)$ . Thus, the near field operator is a compact linear operator on  $L^2(\Gamma)$ ; in contrast to the far field operator it generally fails to be normal (non-normality is further indicated by our numerical experiments). For this reason, it is not immediately clear whether this operator possesses eigenvalues at all. Analytically computing the eigenvalues of  $N_\Gamma$  when  $D$  is the unit ball shows that the eigenvalues, if they exist, do not need to show any distinct structure at all, which makes it impossible to sort them in any reasonable way other than according to their magnitude. Furthermore, as far as we know, there is no obvious factorization of  $N_\Gamma$  possessing the necessary properties for the inside-outside duality; arguably, the most obvious factorization of  $N_\Gamma$  into a triple of three products linking incident with scattered fields is

$$N_\Gamma = (\text{SL}_{\partial D}|_\Gamma) S^{-1} (\text{SL}_\Gamma|_{\partial D}).$$

However, as the outer operators of this factorization are *not* adjoint, it cannot be exploited to prove an inside-outside duality. To circumvent this problem, we generalize a trick from [12], where, roughly speaking, a unitary operator is composed with  $N_\Gamma$  such that the resulting product possesses a proper factorization. The underlying idea, roughly speaking, is to exploit that a spherical setting always allows to diagonalize a scattering problem in the basis of spherical harmonics. In [12], this trick has been exploited to rigorously construct a factorization method for the near field scattering problem; in the present paper, we exploit it to prove an inside-outside duality principle for a near field setting.

This rest of the paper is structured in the following way. In Section 2, we modify the near field operator, defined on a sphere, by multiplication with a unitary operator such that the resulting product possesses a factorization involving adjoint outer operators. Section 3 shows how to exploit this result for other measurement geometries. In Section 4, we derive a relation between the far field operator and the modified near field operator, which we will use in Section 5 to derive an inside-outside duality for near field data using the eigenvalues and the numerical range of the modified near field operator. In Section 6 we finally suggest a numerical algorithm to detect the interior eigenvalues from near field data and present several numerical experiments.

## 2 The Modified Near Field Operator

In this section, we modify the near field operator  $N_\Gamma : L^2(\Gamma) \rightarrow L^2(\Gamma)$  by, roughly speaking, composing it with a unitary operator  $T$ , such that the resulting product possesses a factorization with adjoint outer operators and a coercive middle operator. To this end, we assume first, for simplicity, that  $\Gamma = \mathbb{S}_R = \{x \in \mathbb{R}^3 : |x| = R\}$  is a sphere of radius  $R > 0$  and that the Dirichlet obstacle  $D$  is strictly included in the ball  $B_R$ . In the end of this section we generalize all results to arbitrary Lipschitz-smooth measurement surfaces.

Recall that the near field operator  $N_R := N_{\mathbb{S}_R}$  corresponding to the sphere  $\mathbb{S}_R$  as measurement surface is defined via the radiating solutions  $u(\cdot, y) \in H_{\text{loc}}^1(\mathbb{R}^3 \setminus \overline{D})$  to (1) for incident point sources on  $\mathbb{S}_R$ ,

$$N = N_R : L^2(\mathbb{S}_R) \rightarrow L^2(\mathbb{S}_R), \quad (N_R g)(x) = \int_{\mathbb{S}_R} u(x, y) g(y) \, dS(y), \quad x \in \mathbb{S}_R. \quad (5)$$

While in this section the wave number  $k$  is fixed, the dependence of functions and spaces both on  $k > 0$  and on the chosen radius  $R$  will become important later on; our notation indicates both quantities whenever necessary.

As a technical preparation, we recall that the spherical harmonics  $\{Y_n^m : n \in \mathbb{N}_0, -n \leq m \leq n\}$  form a complete orthogonal basis of the space  $L^2(\mathbb{S}_R)$  of square-integrable functions on the sphere  $\mathbb{S}_R$  for arbitrary  $R > 0$ , that is, every function  $g \in L^2(\mathbb{S}_R)$  expands as

$$g(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^n g_n^m Y_n^m(\hat{x}), \quad \text{where} \quad g_n^m = \frac{1}{R^2} \int_{\mathbb{S}_R} g(x) \overline{Y_n^m(\hat{x})} dS \quad \text{and} \quad \hat{x} = \frac{x}{|x|}. \quad (6)$$

Using this expansion we define  $\mathcal{P}_R : L^2(\mathbb{S}_R) \rightarrow \ell^2$  by

$$\mathcal{P}_R(g) = \mathbf{g}, \quad \mathbf{g} = \{g_n^m : n \in \mathbb{N}_0, |m| \leq n\} \in \ell^2. \quad (7)$$

(For simplicity, we do not explicitly introduce the corresponding index set of the sequence space  $\ell^2$ .) Its inverse  $\mathcal{P}_R^{-1} : \ell^2 \rightarrow L^2(\mathbb{S}_R)$  is then given by  $\mathcal{P}_R^{-1}(\mathbf{g}) = \sum_{n,m} g_n^m Y_n^m$ . Writing  $I_{\ell^2}$  and  $I_{L^2(\mathbb{S}_R)}$  for the identity operators on  $\ell^2$  and  $L^2(\mathbb{S}_R)$ , respectively, it is easy to compute that

$$\mathcal{P}_R \mathcal{P}_R^{-1} = I_{\ell^2}, \quad \mathcal{P}_R^{-1} \mathcal{P}_R = I_{L^2(\mathbb{S}_R)}, \quad \mathcal{P}_R^* = \frac{1}{R^2} \mathcal{P}_R^{-1}, \quad \text{and} \quad (\mathcal{P}_R^{-1})^* = R^2 \mathcal{P}_R.$$

We use  $\mathcal{P}_R$  to transform both the far field operator  $F$  and the near field operator  $N_R$  into operators acting on the sequence space  $\ell^2$  by defining

$$\mathcal{F} = \mathcal{P}_1 F \mathcal{P}_1^{-1} \quad \text{and} \quad \mathcal{N}_R = \mathcal{P}_R N_R \mathcal{P}_R^{-1}. \quad (8)$$

Thus, both  $\mathcal{F}$  and  $\mathcal{N}_R$  are compact operators on  $\ell^2$  representing  $F$  and  $N_R$  in the orthogonal basis of spherical harmonics. Since any solution  $u$  to the scattering problem (1) with boundary datum  $f$  can be expressed in terms of the spherical Hankel functions  $h_n^{(1)}$  on any sphere  $\mathbb{S}_\rho$  such that  $D \in B_\rho$ ,

$$u(x)|_{\mathbb{S}_\rho} = \sum_{n=0}^{\infty} \sum_{m=-n}^n b_n^m(f) h_n^{(1)}(k\rho) Y_n^m(\hat{x}) \quad \text{with coefficients } b_n^m(f) \in \mathbb{C}, \quad (9)$$

the asymptotic expansion of the Hankel function  $h_n^{(1)}$  for large arguments shows that the corresponding far field pattern is given by

$$u^\infty(\hat{x}) = \frac{1}{k} \sum_{n=0}^{\infty} \sum_{m=-n}^n \frac{1}{i^{n+1}} b_n^m(f) Y_n^m(\hat{x}).$$

The lifting  $\mathcal{N}_R$  of  $N_R$  into  $\ell^2$  now allows to modify the latter operator such that it possesses a factorization where the outer operators are adjoint to each other: Following the trick from [12], we define the unitary operator  $\mathcal{T}_R : \ell^2 \rightarrow \ell^2$  by

$$\mathcal{T}_R \mathbf{g} = \left\{ -\frac{\overline{h_n^{(1)}(kR)}}{h_n^{(1)}(kR)} g_n^m : n \in \mathbb{N}_0, |m| \leq n \right\}, \quad (10)$$

and the compact and linear operator  $\mathcal{G}_R : H^{1/2}(\partial D) \rightarrow \ell^2$  by

$$\mathcal{G}_R(f) = \left\{ b_n^m(f) h_n^{(1)}(kR) : n \in \mathbb{N}_0, |m| \leq n \right\}, \quad (11)$$

where the coefficients  $b_n^m(f)$  are defined in (9). The operator  $\mathcal{T}_R$  is well-defined since the spherical Hankel function cannot vanish for positive arguments and  $\mathcal{G}_R$  is compact, injective and has dense range in  $\ell^2$  (see [12, Lemma 3.5]). Note that  $\mathcal{P}_R^{-1}\mathcal{G}_R f$  is the evaluation of the solution to (1) on  $\mathbb{S}_R$ . Moreover, [12, Equation (3.7)] shows that the modified near field operator  $\mathcal{T}_R \mathcal{N}_R$  can be factorized as

$$\mathcal{T}_R \mathcal{N}_R = -R^2 (\mathcal{T}_R \mathcal{G}) S^* (\mathcal{T}_R \mathcal{G}_R)^*, \quad \text{i.e.,} \quad \mathcal{T}_R \mathcal{P}_R \mathcal{N}_R \mathcal{P}_R^{-1} = -R^2 (\mathcal{T}_R \mathcal{G}_R) S^* (\mathcal{T}_R \mathcal{G}_R)^*. \quad (12)$$

Lifting  $\mathcal{T}_R$  back into the space  $L^2(\mathbb{S}_R)$  yields  $T_R = \mathcal{P}_R^{-1} \mathcal{T}_R \mathcal{P}_R$ , a unitary operator on  $L^2(\mathbb{S}_R)$  and the factorization in (12) directly shows that  $T_R \mathcal{N}_R : L^2(\mathbb{S}_R) \rightarrow L^2(\mathbb{S}_R)$  factorizes into

$$T_R \mathcal{N}_R = -G_R S^* G_R^*, \quad \text{where } G_R = \mathcal{P}_R^{-1} \mathcal{T}_R \mathcal{G}_R. \quad (13)$$

This factorization hence features adjoint outer operators due to the replacement of  $\mathcal{P}_R^{-1} \mathcal{G}_R f$ , evaluating the solution to (1) on  $\mathbb{S}_R$ , by  $G_R = \mathcal{P}_R^{-1} \mathcal{T}_R \mathcal{G}_R$ , which conjugates the spherical Hankel functions in (11) before evaluation. The above-mentioned properties of  $\mathcal{G}_R$  clearly imply that  $G_R : H^{1/2}(\partial D) \rightarrow L^2(\mathbb{S}_R)$  is compact, injective and has dense range in  $L^2(\mathbb{S}_R)$ .

**Remark 1.** *If  $\Gamma = \{x_1^2/a^2 + x_2^2/b^2 + x_3^2/c^2 = 1\}$  is an ellipsoid with semi-principal axes of length  $a, b, c > 0$ , then separability of the Helmholtz equation in ellipsoidal coordinates [2] allows to transfer the above modification of near field operators on spheres  $\mathbb{S}_\rho$  to near field operators on ellipsoids, at the expense of drastically increased technicalities.*

We will later on use the factorization (13) to examine the structure of  $T_R \mathcal{N}_R$  more closely. Before, we show that the latter operator has infinitely many eigenvalues, following a technique from [3].

**Lemma 2.** *The operator  $T_R \mathcal{N}_R$  has an infinite number of eigenvalues tending to zero.*

*Proof.* We restrict  $T_R \mathcal{N}_R$  to an operator mapping the orthogonal complement  $\ker(N_R)^\perp \subset L^2(\mathbb{S}_R)$  of its kernel  $\ker(N_R)$  into the closure of its range  $\overline{\text{Rg}(T_R \mathcal{N}_R)} \subset L^2(\mathbb{S}_R)$  by defining  $A : \ker(N_R)^\perp \rightarrow \overline{\text{Rg}(T_R \mathcal{N}_R)} \subset L^2(\mathbb{S}_R)$  by  $Ag = T_R \mathcal{N}_R g$ . As  $T_R$  is unitary,  $A$  is hence injective and has dense range. Moreover, the factorization (13) implies that  $T_R \mathcal{N}_R$  is compact since  $G_R$  is compact, such that  $A$  is compact, too. We next show that  $\ker(N_R)$  is finite-dimensional, to conclude that the range of  $A$  is infinite-dimensional, too, due to injectivity of  $A$ :

If  $N_R g = 0$  for some  $g \neq 0$ , then the radiating solution  $u$  to (1) for  $f = \text{SL}_{\mathbb{S}_R} g|_{\partial D}$  vanishes on  $\mathbb{S}_R$ , and hence entirely in  $\mathbb{R}^3 \setminus \overline{D}$ , due to the radiation condition and Rellich's lemma. Thus, the single-layer potential  $\text{SL}_{\mathbb{S}_R} g$  vanishes on  $\partial D$ , such that  $v = \text{SL}_{\mathbb{S}_R} g|_D \in H_0^1(D)$  defines an Dirichlet eigenfunction of the (negative) Laplacian for the eigenvalue  $k^2$ . As the corresponding eigenspace is finite-dimensional due to Fredholm theory, there can at most exist a finite number of linearly independent  $g$  generating such eigenfunctions; consequently,  $\ker(N_R)$  is finite-dimensional.

We next define the subspace of principle functions of  $A$  by

$$P(A) = \text{span} \left\{ g \in \ker(N)^\perp : (\mu \text{Id} - A)^n g = 0 \text{ for some } n \in \mathbb{N} \text{ and } \mu \in \mathbb{C} \right\} \subset L^2(\mathbb{S}_R).$$

Assume for a moment that  $A$  is a trace class operator and that  $\text{Im } A \geq 0$ , i.e., that the non-selfadjoint part of  $A$  is non-negative. Due to [24, Theorem 3.5.1], these two properties imply that  $\overline{\text{Rg}(A)} = P(A)$ . We showed above that  $\text{Rg}(A)$  has infinite dimension and conclude

that there exist infinitely many linearly independent principle functions. As for each principle function there exists an associated eigenvalue and an eigenfunction due to Riesz theory, see [18], the infinitely many linearly independent principle functions guarantee the existence of infinitely many eigenfunctions of  $A$ . By definition of  $A$ , any eigenpair  $(\mu, g)$  satisfies  $\mu g = Ag = T_R N_R g$  and hence also  $T_R N_R$  possesses infinitely many eigenvalues. Since  $T_R N_R$  is compact, these eigenvalues tend to zero.

It remains to show that  $A$  is a trace class operator and that  $\text{Im}(A) \geq 0$ . The second property follows immediately from the factorization (13), since for any  $g \in L^2(\mathbb{S}_R)$  it holds that

$$\text{Im}(Ag, g)_{L^2(\mathbb{S}_R)} = \text{Im}(T_R N_R g, g)_{L^2(\mathbb{S}_R)} = -\text{Im}(S^* G_R^* g, G_R^* g)_{L^2(\partial D)} \geq 0,$$

where we exploited that the non-selfadjoint part  $\text{Im} S$  of the single layer operator  $S$  is non-negative, see [16, Lemma 1.14]. Since  $T_R$  is unitary, it is further sufficient to show that  $N_R$  is a trace class operator to prove this property for  $T_R N_R$ . For  $N_R$ , this is essentially due to the smoothness of its kernel  $(x, y) \mapsto u^s(x, y) \in C^\infty(\mathbb{S}_R \times \mathbb{S}_R)$ , since this smoothness implies that  $N_R$  is a bounded linear operator from  $L^2(\mathbb{S}_R)$  into any Sobolev space  $H^s(\mathbb{S}_R)$  for arbitrary  $s \in \mathbb{R}$ . Choosing  $s > 2$  implies that the embedding of  $H^s(\mathbb{S}_R)$  in  $L^2(\mathbb{S}_R)$  is a trace class operator, see [9], and finally proves that  $N_R$  itself is a trace class operator on  $L^2(\mathbb{S}_R)$ .  $\square$

The following corollary shows that any eigenvalue of  $T_R N_R$  is contained in the upper half of the complex plane.

**Corollary 3.** *If  $k^2$  is no Dirichlet eigenvalue of  $-\Delta$  in  $D$ , then all eigenvalues of  $T_R N_R$  are contained in the upper half  $\{z \in \mathbb{C} : \text{Im}(z) > 0\}$  of the complex plane; if  $k^2$  is a Dirichlet eigenvalue, they are contained in  $\{z \in \mathbb{C} : \text{Im}(z) > 0\} \cup \{0\}$ .*

*Proof.* If  $\mu$  is an eigenvalue to a normalized eigenfunction  $g$ , then we compute as in the proof of Lemma 2 that

$$\text{Im}(\mu) = \text{Im}(T_R N_R g, g)_{L^2(\mathbb{S}_R)} = -\text{Im}(S G_R^* g, G_R^* g)_{L^2(\partial D)} \geq 0$$

due to the factorization (13) of  $T_R N_R$  and the properties of the single-layer operator  $S$ , see [16, Lemma 1.14]:  $\text{Im}(Sf, f)_{L^2(\partial S)} \leq 0$  for any  $f \in H^{-1/2}(\partial D)$  and the latter expression can only vanish if either  $f = 0$  or else if  $k^2$  is a Dirichlet eigenvalue of  $D$  and  $f$  is the normal derivative of a Dirichlet eigenfunction.  $\square$

### 3 Transfer to Non-Spherical Measurement Geometries

As a spherical measurement geometry is certainly a serious restriction for the theory introduced so far, we consider in this section the question how to treat near field operators defined on other measurement surfaces. Precisely, we consider the near field operator  $N_\Gamma$  defined on the boundary  $\Gamma \subset \mathbb{R}^3 \setminus \overline{D}$  of some connected  $C^{2,1}$  domain  $\Omega_\Gamma \subset \mathbb{R}^3$  such that the complement of  $\Omega_\Gamma$  is connected and such that  $D \Subset \Omega_\Gamma$ . We further fix a radius  $\rho$  such that the ball  $B_\rho$  strictly contains the Lipschitz domain  $\Omega_\Gamma$  (and hence also the obstacle  $D$ ). As usual, the near field operator defined on the sphere  $B_\rho \ni \Omega_\Gamma$  is denoted by  $N_\rho$  (defined as in (5), replacing  $R$  by  $\rho$ ).

As mentioned in the introduction, the crucial property of the operator  $T$  from (10), transferring outgoing to incoming solutions to the Helmholtz equation and vice versa, is linked to the diagonalization of the Helmholtz equation in spherical coordinates. This motivates to connect  $N_\Gamma$ , defined on a non-spherical surface  $\Gamma$ , to a near field operator defined on  $\mathbb{S}_\rho$ . We do

so by constructing auxiliary operators  $Q^\pm$  such that  $N_\rho = Q^+ N_\Gamma Q^-$ : The operator  $Q^+$  maps the boundary datum of the radiating solution to (1) on  $\Gamma$  to the trace of that solution on  $\mathbb{S}_\rho$ , whereas  $Q^-$  maps  $f \in L^2(\mathbb{S}_\rho)$  to  $\phi \in L^2(\Gamma)$  such that  $\text{SL}_{\mathbb{S}_\rho} f = \text{SL}_\Gamma \phi$  holds in  $H^1(\Omega_\Gamma)$ . Thus, transforming  $N_\Gamma$  into  $Q^+ N_\Gamma Q^- = N_\rho$  allows to transfer all results shown above to  $N_\rho$ , as none of the arguments in these proofs explicitly or implicitly required the scatterer to be spherical. Moreover, also all later results on the determination of interior Dirichlet eigenvalues of  $D$  via near field data can hence be applied to  $N_\Gamma$ , up to transforming  $N_\Gamma$  to  $N_\rho$ .

Both operators  $Q^\pm$  are explicitly constructed in the proof of the subsequent lemma, which crucially relies on the fact that  $\Omega_\Gamma \subset B(0, \rho)$ . After setting up these operators, we will discuss how to discretize them numerically.

**Lemma 4.** *Assume that  $\Omega_\Gamma$  is a  $C^{2,1}$  domain.*

(a) *There exists a compact operator  $Q^- : L^2(\mathbb{S}_\rho) \rightarrow L^2(\Gamma)$  mapping  $f \in L^2(\mathbb{S}_\rho)$  to  $\phi \in L^2(\Gamma)$  that  $\text{SL}_\Gamma \circ Q^- f = \text{SL}_{\mathbb{S}_\rho} f$  holds in  $H^1(\Omega_\Gamma)$ .*

(b) *The exterior boundary value problem with boundary datum  $g \in L^2(\Gamma)$ ,*

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega_\Gamma}, \quad v|_\Gamma = g, \quad (14)$$

*possesses a unique radiating solution  $v \in L^2_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\Omega_\Gamma})$  and the operator  $Q^+ : L^2(\Gamma) \rightarrow L^2(\mathbb{S}_\rho)$ , defined by  $Q^+ g = v|_{\mathbb{S}_\rho}$ , is compact.*

(c) *The operators  $Q^+$  and  $Q^-$  are transposed to each other, i.e.,  $\int_{\mathbb{S}_\rho} f Q^+ g \, dS = \int_\Gamma Q^- f g \, dS$  holds for all  $f \in L^2(\mathbb{S}_\rho)$  and  $g \in L^2(\Gamma)$ .*

If  $k^2$  is no interior Dirichlet eigenvalue of  $\Omega_\Gamma$ , and if the boundary datum  $g$  in (b) belongs to  $H^{1/2}(\Gamma)$ , then Lemma 4 is classical. Since the wave number is allowed to vary in Section 5 on the inside-outside duality, we avoid the assumption of  $k^2$  being no interior Dirichlet eigenvalue. This, however, increases the technical difficulties in the proof.

*Proof.* We rely on boundary integral equations and recall that the single-layer operator  $S_\Gamma$ , defined via the single-layer potential  $\text{SL}_\Gamma$  by  $S_\Gamma = \text{SL}_\Gamma(\cdot)|_{\partial\Omega}$ , is continuous from  $H^{-1/2+s}(\Gamma)$  into  $H^{1/2+s}(\Gamma)$  for  $-2 \leq s \leq 2$ , as  $\Omega_\Gamma$  is  $C^{2,1}$  smooth, see [23, Theorem 7.2]. Further,  $\langle \cdot, \cdot \rangle_\Gamma$  is the duality product between  $H^{\pm s}(\Gamma)$ ,  $-3 \leq s \leq 3$ , that extends the inner product of  $L^2(\Gamma)$ .

(a) For  $f \in L^2(\mathbb{S}_\rho)$ , we seek a solution to the interior boundary value problem

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega_\Gamma, \quad u|_\Gamma = g := \text{SL}_{\mathbb{S}_\rho} f|_\Gamma, \quad (15)$$

in the form  $u = \text{SL}_\Gamma \psi$  in  $\mathbb{R}^3 \setminus \Omega_\Gamma$  for some  $\psi \in L^2(\Gamma)$  required to satisfy  $S_\Gamma \psi = g$  in  $H^1(\Gamma)$ . Considering  $S_\Gamma$  as an operator from  $H^{-1/2}(\Gamma)$  into  $H^{1/2}(\Gamma)$ , we know from [20] that non-zero elements  $\psi_0$  in the kernel  $\Lambda := \ker(S_\Gamma)$  equal the normal derivative of some Dirichlet eigenfunction  $u_0 \in H_0^1(\Omega_\Gamma)$  of  $-\Delta$  in  $\Omega_\Gamma$ ; even more,  $\text{SL}_\Gamma$  is a bijection between  $\Lambda$  and the corresponding eigenspace, as every eigenfunction can be represented as  $\text{SL}_\Gamma \psi_0$  for a unique element in  $\psi_0 \in \Lambda$ . Computing the adjoint of  $S_\Gamma$  with respect to  $\langle \cdot, \cdot \rangle_\Gamma$  shows that the kernel of  $S_\Gamma^* : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$  equals  $\Lambda$ , too.

Considering  $S_\Gamma$  as an operator from  $H^{s-1/2}(\Gamma)$  into  $H^{s+1/2}(\Gamma)$  for  $-1 \leq s \leq 1$ , Theorem 7.17 in [23] states that  $S_\Gamma$  is a Fredholm operator of index zero and that the kernel  $\Lambda$  of these operators is independent of  $s$ . Consequently, the corresponding adjoints  $S_\Gamma^* : H^{-s-1/2}(\Gamma) \rightarrow H^{-s+1/2}(\Gamma)$  are Fredholm of index zero with kernels all equal to  $\Lambda$ , too.

The Fredholm alternative hence states that  $S_\Gamma \psi = g$  in  $H^1(\Gamma)$  is solvable if and only if  $\langle \psi_0, g \rangle_\Gamma = 0$  for all  $0 \neq \psi_0 \in \ker S_\Gamma^* = \Lambda$ . As  $\psi_0$  equals the normal derivative of some Dirichlet eigenfunction  $u_0 \in H_0^1(\Omega_\Gamma)$ , Green's second identity implies that

$$\langle \psi_0, g \rangle_\Gamma = \left\langle \frac{\partial u_0}{\partial \nu}, \text{SL}_{\mathbb{S}_\rho} f \right\rangle_\Gamma = \left\langle \frac{\partial}{\partial \nu} \text{SL}_{\mathbb{S}_\rho} f, u_0 \right\rangle_\Gamma = 0 \quad \text{for all } f \in L^2(\mathbb{S}_\rho). \quad (16)$$

We conclude that there exists an equivalence class of solutions  $\{\psi + \psi_0 : \psi_0 \in \Lambda\} \subset L^2(\Gamma)$  to  $S_\Gamma \psi = g$  in  $H^1(\Gamma)$ . For arbitrary  $\psi_0 \in \Lambda$ , the traces of the potentials  $\text{SL}_\Gamma(\psi + \psi_0)$  and  $\text{SL}_{\mathbb{S}_\rho} f$  on  $\Gamma$  equal each other. Consequently, the difference  $\text{SL}_\Gamma(\psi + \psi_0) - \text{SL}_{\mathbb{S}_\rho} f$  either vanishes in  $\Omega_\Gamma$  or equals some eigenfunction  $u_1 \in H_0^1(\Omega_\Gamma)$  of  $-\Delta$  with eigenvalue  $k^2$ . Since  $u_1 = \text{SL}_\Gamma \psi_1$  for some  $\psi_1 \in \Lambda$ , we hence choose  $\psi_0 = \psi_1$  such that  $\text{SL}_\Gamma \psi = \text{SL}_{\mathbb{S}_\rho} f$  in  $\Omega_\Gamma$ . This choice determines a linear and bounded solution operator  $S_\Gamma^- : g \mapsto \psi + \psi_0$  from the range of  $S_\Gamma : L^2(\Gamma) \rightarrow H^1(\Gamma)$  into  $L^2(\Gamma)$ , such that  $f \mapsto Q^- f = S_\Gamma^- \circ (\text{SL}_{\mathbb{S}_\rho} f|_\Gamma)$  satisfies  $\text{SL}_\Gamma Q^- f = \text{SL}_{\mathbb{S}_\rho} f$  in  $H^1(\Omega_\Gamma)$ . Moreover,  $Q^-$  is compact, because  $\text{SL}_{\mathbb{S}_\rho}$  is bounded from  $L^2(\mathbb{S}_\rho)$  into  $H^{3/2}(\Omega_\Gamma)$ .

(b) If  $\nu$  denotes the exterior unit normal to  $\Omega_\Gamma$ , we define the double layer potential by

$$\text{DL}_\Gamma \phi(x) = \int_\Gamma \frac{\partial}{\partial \nu(y)} \left[ \frac{e^{ik|x-y|}}{4\pi|x-y|} \right] \phi(y) \, dS(y), \quad x \in \mathbb{R}^3 \setminus \Gamma,$$

which yields a bounded operator from  $H^{1/2}(\Gamma)$  into  $H_{\text{loc}}^1(\mathbb{R}^3 \setminus \Gamma)$ . The traces  $\text{DL}_\Gamma \phi|_\Gamma^\pm$  on  $\Gamma$  from the inside (-) and outside (+) equal  $\pm \phi/2 + K_\Gamma \phi$ , where the double-layer operator  $K_\Gamma : H^{s+1/2}(\Gamma) \rightarrow H^{s+1/2}(\Gamma)$  is bounded for  $-2 \leq s \leq 2$ , as  $\Omega_\Gamma$  is  $C^{2,1}$ , see [23, Theorem 7.2]. As  $\Omega_\Gamma$  is  $C^{2,1}$ ,  $\text{DL}_\Gamma$  extends to a bounded operator from  $H^{-1/2}(\Gamma)$  into  $L^2(B_R)$  for all  $R > 0$  and defines distributional solutions to the homogeneous Helmholtz equation in  $\mathbb{R}^3 \setminus \Gamma$ ; the same statement holds for  $\text{SL}_\Gamma : H^{-3/2}(\Gamma) \rightarrow L^2(B_R)$ , see [5]. These distributional solutions are smooth away from  $\Gamma$  and radiating at infinity, because the kernel of both potentials is smooth when evaluated in domains with positive Hausdorff distance to  $\Gamma$ . The traces of these potential operators on  $\Gamma$  from the inside and outside are again equal to  $S_\Gamma$  and  $\pm \phi/2 + K_\Gamma \phi$ ; note that  $S_\Gamma : H^{-3/2}(\Gamma) \rightarrow L^2(\Gamma)$  and  $K_\Gamma : H^{-1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$  are bounded.

We seek a radiating solution to the exterior Dirichlet problem (14) with boundary data  $g \in L^2(\Gamma)$  in the form  $v = \text{DL}_\Gamma g - \text{SL}_\Gamma \psi$  for some  $\psi \in H^{-3/2}(\Gamma)$ , such that  $\psi$  needs to satisfy  $2S_\Gamma \psi = -g + 2K_\Gamma g$  in  $H^{-1/2}(\Gamma)$ . We recall from part (a) that  $S_\Gamma$  is a Fredholm operator of index zero from  $H^{-3/2}(\Gamma)$  into  $H^{-1/2}(\Gamma)$  and that the kernel of adjoint  $S_\Gamma^* : H^{1/2}(\Gamma) \rightarrow H^{3/2}(\Gamma)$  is a finite-dimensional space  $\Lambda$  given by the normal derivatives of eigenfunctions  $u_0$  of the (negative) Dirichlet Laplacian in  $\Omega_\Gamma$  with eigenvalue  $k^2$ . As all eigenfunctions can be represented as  $u_0 = \text{SL} \psi_0$  for a unique  $\psi_0 \in \Lambda$ , we hence check by Green's second identity that

$$\left\langle -g + 2K_\Gamma g, \frac{\partial u_0}{\partial \nu} \right\rangle_\Gamma = 2 \left\langle \text{DL}_\Gamma g|_\Gamma^-, \frac{\partial u_0}{\partial \nu} \right\rangle_\Gamma = 2 \left\langle \frac{\partial}{\partial \nu} \text{DL}_\Gamma g, u_0 \right\rangle_\Gamma = 0,$$

to conclude from Fredholm's alternative that there exists a solution  $\psi \in H^{-3/2}(\Gamma)$  to  $2S_\Gamma \psi = -g + 2K_\Gamma g$ . Choosing this solution to be linearly independent to non-zero elements in the kernel  $\Lambda$  of  $S_\Gamma$  allows to define a solution operator  $T : g \mapsto \psi$ , bounded from  $H^{-1/2}(\Gamma)$  into  $H^{-3/2}(\Gamma)$ , such that  $g \mapsto v(g) := \text{DL}_\Gamma g - \text{SL}_\Gamma \circ Tg$  is a linear solution operator to (14), bounded from  $H^{-1/2}(\Gamma)$  into  $L_{\text{loc}}^2(\mathbb{R}^3 \setminus \bar{\Omega}_\Gamma)$ . As the kernel functions of  $\text{SL}_\Gamma$  and  $\text{DL}_\Gamma$  are smooth away from their singularity at the origin, the restriction of  $v(g)$  to  $\mathbb{S}_\rho$  yields a bounded operator

$f \mapsto v|_{\mathbb{S}_\rho}$  from  $H^{-1/2}(\Gamma)$  into  $L^2(\mathbb{S}_\rho)$ , which becomes compact when changing its pre-image space to  $L^2(\Gamma)$ .

(c) For simplicity, we merely prove this fact if  $k^2$  is not a Dirichlet eigenvalue of  $-\Delta$  in  $\Omega_\Gamma$ , relying on the representations  $Q^- = S_\Gamma^{-1} \circ \text{SL}_{\mathbb{S}_\rho}$  and  $Q^+ = \text{SL}_\Gamma \circ S_\Gamma^{-1}$ . One computes that  $\int_{\mathbb{S}_\rho} g \text{SL}_\Gamma f \, dS = \int_\Gamma \text{SL}_{\mathbb{S}_\rho} g f \, dS$  holds for all  $g \in L^2(\mathbb{S}_\rho)$  and  $f \in L^2(\Gamma)$ , and, analogously, that  $\int_\Gamma f_1 S_\Gamma f_2 \, dS = \int_\Gamma S_\Gamma f_1 f_2 \, dS$  for  $f_{1,2} \in L^2(\Gamma)$ . As the single-layer operator is hence self-transposed, we deduce that its inverse  $S_\Gamma^{-1}$  is self-transposed, too. Hence,

$$\int_{\mathbb{S}_\rho} g Q^+ f \, dS = \int_{\mathbb{S}_\rho} g \text{SL}_\Gamma \circ S_\Gamma^{-1} f \, dS = \int_\Gamma \text{SL}_{\mathbb{S}_\rho} g S_\Gamma^{-1} f \, dS = \int_\Gamma S_\Gamma^{-1} \circ \text{SL}_{\mathbb{S}_\rho} g f \, dS = \int_\Gamma Q^- g f \, dS.$$

holds for  $g \in C^1(\mathbb{S}_\rho)$  and  $f \in C^1(\Gamma)$  (such that all integrals are well-defined). Density of  $C^1(\mathbb{S}_\rho)$  and  $C^1(\Gamma)$  in  $L^2(\mathbb{S}_\rho)$  and  $L^2(\Gamma)$ , respectively, then implies the claim.  $\square$

The following corollary provides the indicated factorization of  $N_\rho$ .

**Corollary 5.** *For any  $k > 0$ , the near field operator  $N_\rho$  can be factorized as  $N_\rho = Q^+ N_\Gamma Q^-$ .*

*Proof.* Due to Lemma 4(a), the solutions  $u_\rho(\cdot, y)$  and  $u_\Gamma(\cdot, y)$  to (1) that define the kernel of  $N_\rho$  and  $N_\Gamma$ , respectively, have the same trace on  $\partial D$ . Hence, both solutions equal each other. The construction of  $Q^+$  then shows that  $u_\rho(\cdot, y)|_{\mathbb{S}_\rho}$  equals  $Q^+(u_\Gamma(\cdot, y)|_\Gamma)$ .  $\square$

Let us now discuss how to numerically cope with  $Q^\pm$  in case one aims to compute  $N_\rho$  from  $N_\Gamma$ . Due to Lemma 4(c), it is obviously sufficient to discretize either  $Q^+$  or  $Q^-$ , as both are transposed operators; however, both require to discretize a solution operator for a boundary integral equation and the evaluation of a single layer potential, making this data post-processing rather costly. Since the application of the operator  $T_\rho$  to  $N_\rho$  anyway requires to compute the expansion (9) of the scattered field on  $\mathbb{S}_\rho$ , it seems, alternatively, attractive to compute  $Q^+$  directly in the basis of spherical harmonics to avoid the discretization of an operator inverse. Such a direct computation is in principle possible: If we suppose that  $0 \in \Omega_\Gamma$ , then restrictions of separation-of-variables solutions to the Helmholtz equation

$$V = \{\phi_n^m = v_n^m|_\Gamma : v_n^m(x) = h_n^{(1)}(k|x|)Y_n^m(\hat{x}), n \in \mathbb{N}_0, |m| \leq n\} \subset L^2(\Gamma)$$

form a complete set of linearly independent functions in  $L^2(\Gamma)$ , see [22]. For the boundary datum  $\phi_n^m \in L^2(\Gamma)$ , the solution to the exterior boundary value problem (14) equals  $v_n^m(x) = h_n^{(1)}(k|x|)Y_n^m(\hat{x})$ , such that  $Q^+(\phi_n^m) = v_n^m|_{\mathbb{S}_\rho}$ . Since  $\text{span}V$  is dense in  $L^2(\Gamma)$ , knowing  $Q^+$  on  $V$  determines  $Q^+$  uniquely and, by transposition,  $Q^-$ , too. However, to exploit this, roughly speaking, diagonalization of  $Q^+$  by this set of functions, one needs represent scattered fields on  $\Gamma$  via the set of functions in  $V$ ; despite the corresponding infinite or truncated linear system is well-known to be always uniquely solvable, see [22], the condition number of the truncated system increases rapidly in the truncation index, making systems of merely moderate size already rather ill-conditioned. This phenomenon is directly linked to instabilities of the so-called null-field method caused by, roughly speaking, the fact that the expansion (6) does in general not hold in all of  $\mathbb{R}^3 \setminus \Omega_\Gamma$ , and becomes increasingly important as  $\Gamma$  deviates from a sphere. Thus, this alternative option for the computation of  $Q^\pm$ , attractive at first glance, is in practice somewhat challenging to use, but in principle allows to bypass the discretization of the solution operators involved in  $Q^\pm$ .

## 4 Connecting Near- to Far Field Operators by Factorization

For the remainder of this paper we will merely work with near field operators defined on spheres, because the case where the measurement surface  $\Gamma$  is non-spherical can be reduced to a sphere by Corollary 5. Consequently, we fix a radius  $R$  and henceforth neglect the subscript  $R$  for better readability, such that, e.g.,  $N_R$  and  $T_R N_R$  become  $N$  and  $TN$ , respectively.

Since our goal is to prove an inside-outside duality for near field data relying on a corresponding duality for far field data, we derive a connection between the far field operator  $F$  and the modified near field operator  $TN$ . For this purpose we introduce a mapping  $\mathcal{Z}$ , which is later on used to relate far fields to near fields. For  $\mathbf{g} = \{g_n^m : n \in \mathbb{N}_0, |m| \leq n\} \in \ell^2$ , let

$$\mathcal{Z}\mathbf{g} = \left\{ -ki^{n+1} \overline{h_n^{(1)}(kR)} g_n^m : n \in \mathbb{N}_0, |m| \leq n \right\}.$$

This map is unbounded on  $\ell^2$  since  $n \mapsto |h_n^{(1)}(kR)|$  is an unbounded sequence, such that we restrict  $\mathcal{Z}$  to its domain

$$\text{dom}(\mathcal{Z}) = \{\mathbf{g} \in \ell^2 : \|\mathcal{Z}\mathbf{g}\|_{\ell^2} < \infty\}.$$

Then  $\mathcal{Z} : \ell^2 \supset \text{dom}(\mathcal{Z}) \rightarrow \ell^2$  is a well-defined unbounded linear operator.

**Remark 6.** *The domain  $\text{dom}(\mathcal{Z})$  contains precisely those sequences  $\mathbf{g} = (g_n^m)$  such that*

$$v(x) = k \sum_{n \in \mathbb{N}_0} i^{n+1} \sum_{m=-n}^n g_n^m h_n^{(1)}(k|x|) Y_n^m(\hat{x}), \quad |x| > R,$$

*is a radiating solution to the Helmholtz equation with trace in  $L^2(\mathbb{S}_R)$ , see [4, Theorem 2.17].*

**Lemma 7.** *The domain  $\text{dom}(\mathcal{Z})$  is dense in  $\ell^2$ , that is,  $\overline{\text{dom}(\mathcal{Z})} = \ell^2$ .*

*Proof.* To show that the space  $\text{dom}(\mathcal{Z})$  is dense in  $\ell^2$ , we choose an arbitrary  $\mathbf{g} \in \ell^2$  and define

$$\mathbf{g}_M = \begin{cases} g_n^m & \text{for } n \leq M, |m| \leq n, \\ 0 & \text{else.} \end{cases}$$

Clearly,  $\mathbf{g}_M \in \text{dom}(\mathcal{Z})$  for all  $M \in \mathbb{N}$ . Furthermore, for every  $\varepsilon > 0$  there exists  $M = M(\varepsilon) \in \mathbb{N}$  such that  $\|\mathbf{g} - \mathbf{g}_M\|_{\ell^2} < \varepsilon$ . This concludes the proof.  $\square$

The last lemma implies that the operator  $\mathcal{Z} : \ell^2 \supset \text{dom}(\mathcal{Z}) \rightarrow \ell^2$  is densely defined in  $\ell^2$ . We next prove further properties of  $\mathcal{Z}$  and its adjoint  $\mathcal{Z}^* : \ell^2 \supset \text{dom}(\mathcal{Z}^*) \rightarrow \ell^2$ , before we exploit these operators in Theorem 10 to establish a connection between the lifted far- and near field operators  $\mathcal{F}$  and  $\mathcal{TN}$ , defined in (8).

**Lemma 8.** *The operator  $\mathcal{Z} : \ell^2 \supset \text{dom}(\mathcal{Z}) \rightarrow \ell^2$  and its adjoint  $\mathcal{Z}^* : \ell^2 \supset \text{dom}(\mathcal{Z}^*) \rightarrow \ell^2$  are one-to-one and onto and  $\text{dom}(\mathcal{Z}) = \text{dom}(\mathcal{Z}^*)$ . Their inverse operators  $\mathcal{Z}^{-1} : \ell^2 \rightarrow \ell^2$  and  $(\mathcal{Z}^*)^{-1} : \ell^2 \rightarrow \ell^2$  are bounded and even compact on  $\ell^2$  with ranges  $\text{Rg}(\mathcal{Z}^{-1}) = \text{Rg}((\mathcal{Z}^*)^{-1}) = \text{dom}(\mathcal{Z})$ .*

*Proof.* The domain of  $\mathcal{Z}^*$  consists of those  $\mathbf{f} = (f_n^m) \in \ell^2$  for which there is a  $\mathbf{f}^* \in \ell^2$  such that

$$(\mathcal{Z}\mathbf{g}, \mathbf{f})_{\ell^2} = (\mathbf{g}, \mathbf{f}^*)_{\ell^2} \quad \text{for all } \mathbf{g} \in \text{dom}(\mathcal{Z}),$$

or, equivalently, such that

$$-k \sum_{n \in \mathbb{N}_0} \sum_{m=-n}^n i^{n+1} \overline{h_n^{(1)}(kR)} g_n^m \overline{f_n^m} = (\mathbf{g}, \mathbf{f}^*)_{\ell^2} \quad \text{for all } \mathbf{g} \in \text{dom}(\mathcal{Z}),$$

which implies that  $\mathbf{f}^* = \{k(-i)^{n+1} h_n^{(1)}(kR) f_n^m : n \in \mathbb{N}_0, |m| \leq n\}$ . In particular,  $\mathbf{f}^*$  exists in  $\ell^2$  if and only if  $\mathbf{f} \in \text{dom}(\mathcal{Z})$  and the adjoint  $\mathcal{Z}^* : \text{dom}(\mathcal{Z}^*) \rightarrow \ell^2$ , defined by  $\mathcal{Z}^* \mathbf{f} = \mathbf{f}^*$ , has the same domain as  $\mathcal{Z}$ . To show that  $\mathcal{Z}$  is onto, let  $\mathbf{f} \in \ell^2$  be arbitrary and set

$$\mathbf{g} = \left\{ -\frac{1}{ki^{n+1} h_n^{(1)}(kR)} f_n^m : n \in \mathbb{N}_0, |m| \leq n \right\}.$$

Clearly  $\mathbf{g} \in \ell^2$  and  $\mathcal{Z}\mathbf{g} = \mathbf{f}$ . For injectivity, we simply note that  $\mathcal{Z}$  is a diagonal operator with non-trivial entries. The inverse operator  $\mathcal{Z}^{-1} : \ell^2 \rightarrow \text{dom}(\mathcal{Z}) \subset \ell^2$  is given by

$$\mathcal{Z}^{-1}\mathbf{g} = \left\{ -\frac{1}{ki^{n+1} h_n^{(1)}(kR)} g_n^m : n \in \mathbb{N}_0, |m| \leq n \right\}.$$

This operator is bounded, since for any  $\mathbf{g} \in \ell^2$  it holds that

$$\|\mathcal{Z}^{-1}\mathbf{g}\|_{\ell^2}^2 = \frac{1}{k^2} \sum_{n \in \mathbb{N}} \sum_{m=-n}^n |h_n^{(1)}(kR)|^{-2} |g_n^m|^2 \leq c \sum_{n \in \mathbb{N}} \sum_{m=-n}^n |g_n^m|^2 = c \|\mathbf{g}\|_{\ell^2}^2,$$

because  $|h_n^{(1)}(kR)|^{-2} \rightarrow 0$  for  $n \rightarrow \infty$ . As  $\mathcal{Z}^{-1}$  is a diagonal operator with entries converging to zero, compactness of  $\mathcal{Z}^{-1}$  follows from Cantor's diagonal argument. Bijectivity and compactness of  $(\mathcal{Z}^*)^{-1}$  follow analogously.  $\square$

**Lemma 9.** *Assume that  $\mathcal{A} : H^{1/2}(\partial D) \rightarrow \ell^2$  is a bounded linear operator such that  $\text{Rg}(\mathcal{A}) \subset \text{dom}(\mathcal{Z})$  and such that  $\mathcal{Z}\mathcal{A} : H^{1/2}(\partial D) \rightarrow \ell^2$  is also a bounded operator. Then there holds that  $\text{dom}((\mathcal{Z}\mathcal{A})^*) \supset \text{dom}(\mathcal{A}^*\mathcal{Z}^*)$  and  $(\mathcal{Z}\mathcal{A})^*\mathbf{g} = \mathcal{A}^*\mathcal{Z}^*\mathbf{g}$  for all  $\mathbf{g} \in \text{dom}(\mathcal{Z}^*)$ .*

*Proof.* Since  $\text{dom}(\mathcal{A}^*\mathcal{Z}^*) = \text{dom}(\mathcal{Z}^*) = \text{dom}(\mathcal{Z})$  and  $\text{dom}((\mathcal{Z}\mathcal{A})^*) = \ell^2$ , it follows that  $\text{dom}(\mathcal{A}^*\mathcal{Z}^*) \subset \text{dom}((\mathcal{Z}\mathcal{A})^*)$ . If  $\mathbf{g} \in \text{dom}(\mathcal{Z}^*)$ , then for all  $f \in H^{1/2}(\partial D)$  we have that

$$((\mathcal{Z}\mathcal{A})^*\mathbf{g}, f)_{L^2(\partial D)} = (\mathbf{g}, \mathcal{Z}\mathcal{A}f)_{\ell^2} = (\mathcal{Z}^*\mathbf{g}, \mathcal{A}f)_{\ell^2} = (\mathcal{A}^*\mathcal{Z}^*\mathbf{g}, f)_{L^2(\partial D)},$$

which proves the assertion.  $\square$

Now we link the lifted far- and near field operators  $\mathcal{F}$  and  $\mathcal{N}$  with each other.

**Theorem 10.** *For all  $\mathbf{g} \in \text{dom}(\mathcal{Z}^*)$  it holds that  $\mathcal{T}\mathcal{N}\mathbf{g} = R^2 \mathcal{Z}\mathcal{F}\mathcal{Z}^*\mathbf{g}$  and for all  $\mathbf{g} \in \ell^2$  it holds that  $\mathcal{F}\mathbf{g} = R^{-2} \mathcal{Z}^{-1}\mathcal{T}\mathcal{N}(\mathcal{Z}^{-1})^*\mathbf{g}$ .*

*Proof.* In a first step, we lift the operators from the factorization  $F = -G_\infty S^* G_\infty^*$  to the sequence space. To this end, we define an operator  $\mathcal{G}_\infty : H^{1/2}(\partial D) \rightarrow \ell^2$  by

$$\mathcal{G}_\infty(f) = \left\{ \frac{1}{ki^{n+1}} b_n^m(f) : n \in \mathbb{N}_0, |m| \leq n \right\}, \quad (17)$$

where  $b_n^m(f)$  are the coefficients from the expansion (9). Then  $G_\infty f = \mathcal{P}_1^{-1} \mathcal{G}_\infty(f)$  holds for all  $f \in H^{1/2}(\partial D)$  and  $G_\infty^* = \mathcal{G}_\infty^* (\mathcal{P}_1^{-1})^* = \mathcal{G}_\infty^* \mathcal{P}_1$ . Thus, the far field operator can be written as

$$F = -G_\infty S^* G_\infty^* = -\mathcal{P}_1^{-1} \mathcal{G}_\infty S^* \mathcal{G}_\infty^* \mathcal{P}_1$$

and, in particular,

$$\mathcal{F} = -\mathcal{G}_\infty S^* \mathcal{G}_\infty^*. \quad (18)$$

Next recall the factorization from (12),

$$\mathcal{TN} = -R^2 (\mathcal{TG}) S^* (\mathcal{TG})^*, \quad (19)$$

where

$$\mathcal{TG}(f) = \left\{ -b_n^m(f) \overline{h_n^{(1)}(kR)} : n \in \mathbb{N}_0, |m| \leq n \right\}. \quad (20)$$

Comparing this to (17) yields  $\mathcal{ZG}_\infty = \mathcal{TG}$  and by Lemma 9 we get  $(\mathcal{TG})^* \mathbf{g} = \mathcal{G}_\infty^* \mathcal{Z}^* \mathbf{g}$  for all  $\mathbf{g} \in \text{dom}(\mathcal{Z}^*)$ . Inserting this equation into (19), we obtain

$$\mathcal{TN} \mathbf{g} = -R^2 \mathcal{ZG}_\infty S^* \mathcal{G}_\infty^* \mathcal{Z}^* \mathbf{g} = R^2 \mathcal{ZFZ}^* \mathbf{g}.$$

Finally setting  $\mathcal{G}_\infty = \mathcal{Z}^{-1} \mathcal{TG}$  and substituting  $\mathcal{G}_\infty$  into (18) yields the second factorization of the theorem.  $\square$

To establish a connection between  $TN$  and  $F$ , we first lift the operator  $\mathcal{Z}$  into  $L^2(\mathbb{S}_R)$ ,

$$Z : L^2(\mathbb{S}_1) \supset \text{dom}(Z) = \{\mathcal{P}_1^{-1} \mathbf{g} : \mathbf{g} \in \text{dom}(\mathcal{Z})\} \rightarrow L^2(\mathbb{S}_R), \quad Z = \mathcal{P}_R^{-1} \mathcal{Z} \mathcal{P}_1.$$

The adjoint  $Z^*$  of  $Z$  is characterized as follows,

$$Z^* : L^2(\mathbb{S}_R) \supset \text{dom}(Z^*) = \{\mathcal{P}_R^{-1} \mathbf{g} : \mathbf{g} \in \text{dom}(\mathcal{Z}^*)\} \rightarrow L^2(\mathbb{S}_1), \quad Z^* = \mathcal{P}_1^{-1} \mathcal{Z} \mathcal{P}_R.$$

Since  $\mathcal{P}_1$  and  $\mathcal{P}_R^{-1}$  are isomorphisms, we obtain the following corollaries from Lemmas 7 and 8 and Theorem 10.

**Corollary 11.** *It holds that  $\overline{\text{dom}(Z)} = L^2(\mathbb{S}_1)$  and  $\overline{\text{dom}(Z^*)} = L^2(\mathbb{S}_R)$ .*

**Corollary 12.** *The operators  $Z : L^2(\mathbb{S}_1) \supset \text{dom}(Z) \rightarrow L^2(\mathbb{S}_R)$  and its adjoint  $Z^* : L^2(\mathbb{S}_R) \supset \text{dom}(Z^*) \rightarrow L^2(\mathbb{S}_1)$  are one-to-one and onto with bounded and compact inverse  $Z^{-1} : L^2(\mathbb{S}_R) \rightarrow L^2(\mathbb{S}_1)$  and  $(Z^*)^{-1} : L^2(\mathbb{S}_1) \rightarrow L^2(\mathbb{S}_R)$ , respectively. The ranges of their inverses are  $\text{Rg}(Z^{-1}) = \text{dom}(Z)$  and  $\text{Rg}((Z^*)^{-1}) = \text{dom}(Z^*)$ .*

**Theorem 13.** *For all  $\mathbf{g} \in \text{dom}(Z^*)$  it holds that  $TN \mathbf{g} = R^2 \mathcal{ZFZ}^* \mathbf{g}$ , whereas for all  $\mathbf{g} \in L^2(\mathbb{S}_1)$  it holds that  $F \mathbf{g} = R^{-2} \mathcal{Z}^{-1} TN (\mathcal{Z}^{-1})^* \mathbf{g}$ .*

*Proof.* One easily computes that

$$TN \mathbf{g} = \mathcal{P}_R^{-1} \mathcal{TN} \mathcal{P}_R \mathbf{g} = \mathcal{P}_R^{-1} \mathcal{TN} \mathbf{g} = R^2 \mathcal{P}_R^{-1} \mathcal{ZF}_1 \mathcal{Z}^* \mathbf{g} = R^2 \mathcal{P}_R^{-1} \mathcal{ZF}_1 \mathcal{Z}^* \mathbf{g} = R^2 \mathcal{ZFZ}^* \mathbf{g},$$

and a similar calculation yields the representation of  $F \mathbf{g}$ .  $\square$

## 5 Inside-Outside Duality for Near Field Data

In this section, we state and prove the main result of this paper on the characterization of interior Dirichlet eigenvalues of the scatterer  $D$  via the smallest phase in the numerical range of  $TN$ , thus proving an inside-outside duality for near field data. Let us recall from Corollary 3 that we have already shown that all eigenvalues  $(\mu_n)_{n \in \mathbb{N}}$  of  $TN$  lie in the upper half of the complex plain. Recall further from (4) that we have represented the eigenvalues  $\lambda_j = r_j \exp(i\vartheta_j)$  of the far field operator  $F$  in polar coordinates and that these eigenvalues are sorted in descending order according to their magnitude, i.e.,  $|\lambda_j| \geq |\lambda_{j+1}|$  for  $j \in \mathbb{N}$ . We further introduce the phases  $\delta_n \in (0, \pi]$  of the eigenvalues  $\mu_n$  of  $TN$  via polar coordinates, too, writing

$$\mu_n = |\mu_n| e^{i\delta_n},$$

where again we set  $\delta_n = \pi$  if  $\mu_n = 0$ . We also sort these eigenvalues by magnitude in descending order, i.e.  $|\mu_n| \geq |\mu_{n+1}|$  for all  $n \in \mathbb{N}$ . Although we have no further information about the structure of these eigenvalues, we can prove that all phases  $(\delta_n)_{n \in \mathbb{N}}$  are larger than or equal to the smallest phase  $\vartheta_* = \min_{j \in \mathbb{N}} \vartheta_j$ .

**Lemma 14.** *Let  $k^2$  be no Dirichlet eigenvalues of  $-\Delta$ . Let  $\vartheta_*$  be the smallest phase among all the phases of the eigenvalues of the far field operator  $F$  and let  $(\delta_n)_{n \in \mathbb{N}}$  be the phases of the eigenvalues  $(\mu_n)_{n \in \mathbb{N}}$  of  $TN$ . Then it holds that  $\delta_n \geq \vartheta_* > 0$  for all  $n \in \mathbb{N}$ .*

*Proof.* Let  $\mu_n$  be any eigenvalue of  $TN$  with eigenfunction  $f_n$  and phase  $\delta_n$ . Then we use the characterization of  $\vartheta_*$  from [20, Theorem 3] and the factorization of  $TN$  from Theorem 13 to get

$$\begin{aligned} \cot(\vartheta_*) &= \max_{g \in L^2(\mathbb{S}_1)} \frac{\operatorname{Re}(Fg, g)_{L^2(\mathbb{S}_1)}}{\operatorname{Im}(Fg, g)_{L^2(\mathbb{S}_1)}} = \max_{g \in L^2(\mathbb{S}_1)} \frac{\operatorname{Re}(TN(Z^{-1})^*g, (Z^{-1})^*g)_{L^2(\mathbb{S}_R)}}{\operatorname{Im}(TN(Z^{-1})^*g, (Z^{-1})^*g)_{L^2(\mathbb{S}_R)}} \\ &= \max_{f \in L^2(\mathbb{S}_R)} \frac{\operatorname{Re}(TNf, f)_{L^2(\mathbb{S}_R)}}{\operatorname{Im}(TNf, f)_{L^2(\mathbb{S}_R)}} \geq \frac{\operatorname{Re}(TNf_n, f_n)_{L^2(\mathbb{S}_R)}}{\operatorname{Im}(TNf_n, f_n)_{L^2(\mathbb{S}_R)}} = \cot(\delta_n) \end{aligned}$$

where we used the denseness of the image of  $(Z^{-1})^*$  in  $L^2(\mathbb{S}_R)$ . Note that all expressions in the last chain of equations are well-defined since  $\operatorname{Im}(Fg, g)$  and  $\operatorname{Im}(Tf, f)$  do not vanish. The assertion now follows from the strictly monotonic decrease of the cotangent.  $\square$

From now on the dependency of all quantities on the wave number  $k > 0$  becomes important, which we will indicate by writing, e.g.,  $\vartheta_* = \vartheta_*(k)$ ,  $\delta_n = \delta_n(k)$  and  $g_* = g_*(k)$  for numbers and vectors, respectively, and by  $TN = T_k N_k$  and  $F = F_k$  for operators. Let us first recall the inside-outside duality for the far field operator  $F = F_k$ .

**Theorem 15** (Theorem 8 in [20]). *Assume that  $k_0 > 0$  and that  $I = (k_0 - \varepsilon, k_0)$  contains no wave number  $k$  such that  $k^2$  is a Dirichlet eigenvalue of  $-\Delta$  in  $D$ . For  $k \in I$ , the smallest phase  $\vartheta_*(k) \in (0, \pi)$  of the eigenvalues of the far field operator satisfies  $\vartheta_*(k) \rightarrow 0$  as  $k$  tends to  $k_0$  from below if and only if  $k_0^2$  is a Dirichlet eigenvalue of  $-\Delta$  in  $D$ .*

We use the last result to formulate a first partial result for near field data.

**Corollary 16.** *Assume that  $k_0 > 0$  and that  $I = (k_0 - \varepsilon, k_0)$  contains no wave number  $k$  such that  $k^2$  is a Dirichlet eigenvalue of  $-\Delta$  in  $D$  and consider, for  $k \in I$ , the phase  $\delta_n(k) \in (0, \pi)$  of an arbitrary eigenvalue  $\mu_n(k)$  of  $T_k N_k$ . If  $\delta_n(k) \rightarrow 0$  as  $k$  tends to  $k_0$  from below, then  $k_0^2$  is a Dirichlet eigenvalue of  $-\Delta$  in  $D$ .*

*Proof.* As  $\delta_n(k) \rightarrow 0$  it follows that  $\vartheta_*(k) \rightarrow 0$  for  $k \rightarrow k_0$  by Lemma 14, which proves the claim due to Theorem 15.  $\square$

The latter corollary merely states a sufficient condition for  $k_0^2$  being a Dirichlet eigenvalue of  $-\Delta$  in  $D$ . To prove a necessary condition, and thus to arrive at a complete duality statement, we rely on the numerical range of an operator as further technical tool. If  $\mathcal{H}$  is a Hilbert space, then the numerical range  $W(B)$  of a bounded linear operator  $B : \mathcal{H} \rightarrow \mathcal{H}$  is a subset of the complex plane given by

$$W(B) = \{(Bg, g)_{\mathcal{H}} : g \in \mathcal{H}, \|g\|_{\mathcal{H}} = 1\}.$$

In Lemma 17 we gather some important, well-known results about the numerical range of the operator  $B$ , which can be found in [10, 7, 19, 11]. Let us recall before that the boundary of  $W(B)$  has infinite curvature at one of its points  $\beta \in \partial W(B)$  if there is no closed disc contained in  $W(B)$  that contains  $\beta$ . (As an illustrative example, any corner of a polygon hence has infinite curvature.)

**Lemma 17.** (a) *The numerical range of  $B$  is convex.*

(b) *If  $\beta \in W(B)$  is a boundary point at which  $\partial W(B)$  has infinite curvature, then  $\beta$  is an eigenvalue of  $B$ .*

(c) *The spectrum of  $B$  is contained in the closure of the numerical range of  $B$ .*

(d) *If  $B$  is compact and normal, then the numerical range is the convex hull of its eigenvalues.*

Due to the factorization of  $T_k N_k$  in (13), it is clear that its numerical range  $W(T_k N_k)$  is contained in the upper half of the complex plane. The factorizations shown in Theorem 13 will even allow to characterize the smallest phase of all elements of  $W(T_k N_k)$  in Theorem 18 below. To this end, we will compare the numerical ranges of  $T_k N_k$  and  $F_k$ , given by

$$W(T_k N_k) = \{(T_k N_k f, f)_{L^2(\mathbb{S}_R)} : f \in L^2(\mathbb{S}_R), \|f\|_{L^2(\mathbb{S}_R)} = 1\} \quad (21)$$

$$W(F_k) = \{(F_k g, g)_{L^2(\mathbb{S}_1)} : g \in L^2(\mathbb{S}_1), \|g\|_{L^2(\mathbb{S}_1)} = 1\}. \quad (22)$$

subsequent theorem, recall that  $\lambda_*(k)$  is the eigenvalue of  $F_k$  possessing the smallest phase  $\vartheta_*(k)$  among the phases of all eigenvalues of  $F_k$  (the phase of the origin equals  $\pi$ , by definition).

**Theorem 18.** *If  $0 \notin W(T_k N_k)$  then the union of the phases of all elements of  $W(T_k N_k)$  is the interval  $[\vartheta_*(k), \pi)$ . If  $0 \in W(T_k N_k)$  then the union of the phases of all elements of  $W(T_k N_k)$  is the interval  $[\vartheta_*(k), \pi]$ .*

*Proof.* Assume first that  $0 \notin W(T_k N_k)$ . Let us introduce the set

$$W_{Z,k} = \{(T_k N_k f, f)_{L^2(\mathbb{S}_R)} : f \in \text{dom}(Z_k^*), \|f\|_{L^2(\mathbb{S}_R)} = 1\} \subset \mathbb{C}$$

and note that  $W_{Z,k}$  is dense in  $W(T_k N_k)$  due to the denseness of  $\text{dom}(Z_k^*)$  in  $L^2(\mathbb{S}_R)$  and the continuity of both  $T_k N_k$  and the inner product of  $L^2(\mathbb{S}_1)$ . Now we use the factorization of  $T_k N_k$  from Theorem 13,

$$\begin{aligned} W_{Z,k} &= \{(T_k N_k f, f)_{L^2(\mathbb{S}_R)} : f \in \text{dom}(Z_k^*), \|f\|_{L^2(\mathbb{S}_R)} = 1\} \\ &= \{R^2 (F_k Z_k^* f, Z_k^* f)_{L^2(\mathbb{S}_1)} : f \in \text{dom}(Z_k^*), \|f\|_{L^2(\mathbb{S}_R)} = 1\} \\ &= \left\{ R^2 \frac{(F_k g, g)_{L^2(\mathbb{S}_1)}}{\|f\|_{L^2(\mathbb{S}_R)}^2} : g = Z_k^* f, f \in \text{dom}(Z_k^*), \|f\|_{L^2(\mathbb{S}_R)} = 1 \right\} \\ &= \left\{ R^2 \frac{(F_k g, g)_{L^2(\mathbb{S}_1)}}{\|(Z_k^*)^{-1} g\|_{L^2(\mathbb{S}_R)}^2} : g \in L^2(\mathbb{S}_1), \|g\|_{L^2(\mathbb{S}_1)} = 1 \right\}, \end{aligned} \quad (23)$$

where we exploited that  $Z_k^*$  is one-to-one and onto from  $\text{dom}(Z_k^*)$  into  $L^2(\mathbb{S}_R)$  to obtain the last equality.

Note that since  $0 \notin W(T_k N_k)$ , it follows that  $0 \notin W_{Z,k}$  and therefore no eigenvalue of  $F$  vanishes, due to equation (23). By Lemma 17(d), the numerical range  $W(F_k)$  is the convex hull of the eigenvalues  $(\lambda_n(k))_{n \in \mathbb{N}}$  of  $F_k$ . Since the eigenvalues of  $F_k$  have phases in the interval  $[\vartheta_*(k), \pi)$  and tend to the origin from the left, we conclude that for any phase in  $[\vartheta_*(k), \pi)$  there is an element of  $W(F_k)$  possessing that phase. Now we compare (23) and (22) and note that to each element  $\gamma = (T_k N_k f, f)_{L^2(\mathbb{S}_R)}$  in  $W_{Z,k}$  there corresponds an element  $(F_k g, g)_{L^2(\mathbb{S}_1)}$  in  $W(F_k)$  that possesses the same phase, and vice versa. In particular, the union  $[\vartheta_*(k), \pi)$  of the phases of all elements in  $W(F_k)$  equals the union of the phases of all the elements in  $W_{Z,k}$ .

Denote now by  $g_*(k) \in L^2(\mathbb{S}_1)$  an eigenfunction for the eigenvalue  $\lambda_*(k)$  of  $F_k$  with the smallest phase  $\vartheta_*(k)$ . Since  $\vartheta_*(k)$ , which is also the phase of, e.g., the element

$$\gamma_*(k) = \frac{(F_k g_*(k), g_*(k))_{L^2(\mathbb{S}_1)}}{\|(Z_k^*)^{-1} g_*(k)\|_{L^2(\mathbb{S}_R)}^2} \in W_{Z,k},$$

is a distinct lower bound of the phases of the elements of  $W_{Z,k}$ , it follows from the density of  $W_{Z,k}$  in  $W(T_k N_k)$  that  $\vartheta_*(k)$  is also a lower bound of the phases of the elements of  $W(T_k N_k)$ . Since  $0 \notin W(T_k N_k)$  the union of all phases of this set is indeed  $[\vartheta_*(k), \pi)$ .

If  $0 \in W(T_k N_k)$ , the phase  $\pi$  is included in the set of phases, so that by the same arguments, the set of phases is  $[\vartheta_*(k), \pi]$ .  $\square$

Finally, we formulate an inside-outside duality between the interior eigenvalues of the Laplacian and the smallest phase of the numerical range of the near field operator.

**Corollary 19** (Inside-Outside Duality). *Assume that  $k_0 > 0$  and that  $I = (k_0 - \varepsilon, k_0)$  contains no  $k$  such that  $k^2$  is a Dirichlet eigenvalue of  $-\Delta$  in  $D$  and denote by  $[\delta_*(k), \pi)$  the union of phases of elements from  $W(T_k N_k)$ . Then it holds that  $k_0^2$  is a Dirichlet eigenvalue of  $-\Delta$  if and only if  $\delta_*(k)$  converges to zero as  $k$  approaches  $k_0$  from below.*

*Proof.* We have shown in Theorem 18 that the union of phases of elements from  $W(T_k N_k)$  is the half-open interval  $[\vartheta_*(k), \pi)$ , such that  $\delta_*(k)$  equals the smallest phase of the eigenvalues  $(\lambda_n(k))_{n \in \mathbb{N}}$  of  $F_k$ . The assertion now follows directly from the inside-outside duality for far field data, see [20, Theorem 8].  $\square$

## 6 Numerical Examples for Spherical Measurement Geometries

In this section we provide numerical examples to verify the theoretical results from the previous section. In particular, we show that it is possible to numerically compute the Dirichlet eigenvalues of the negative Laplacian in a domain  $D$  from the modified near field operators  $T_k N_k$  in a spherical setting given, for a sufficiently dense grid of wave numbers  $k$ . For simplicity, we assume that sources and measurements are done on the sphere  $\mathbb{S}_R$  and drop the index  $R$  from now on; the index  $k$  will be dropped whenever this causes no confusion.

From the point of view of applications, we require measured data consisting of an  $M_1 \times M_1$  matrix containing measurements  $(u(y_i, y_j))_{i,j=1}^{M_1}$  of scattered fields  $u(\cdot, y_j)$  that are radiating solutions of the exterior Dirichlet scattering problem

$$\Delta u(\cdot, y_j) + k^2 u(\cdot, y_j) = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D}, \quad u(\cdot, y_j) = -u^i(\cdot, y_j) \quad \text{on } \partial D,$$

see (2). In our experiments, we simulate this data set by computing numerical approximations  $u_{\text{appr}}(\cdot, y_j)$  to the solution  $u(\cdot, y_j)$  of this problem for the  $M_1$  source points  $y_j \in \mathbb{S}_R$ , relying the boundary element software package BEM++, see [25]. (We use a single-layer potential ansatz that is discretized by a Galerkin discretization.)

To indicate how the data matrix  $(u(y_i, y_j))_{i,j=1}^{M_1}$  can be understood as an approximation to the near field operator  $N = N_R$ , we assume that each of the points  $\{y_i : i = 1, \dots, M_1\} \subset \mathbb{S}_R$  belongs to a patch  $\Gamma_j$  of a regular surface mesh  $\Gamma = \{\Gamma_j : j = 1, \dots, M_1\}$  of  $\mathbb{S}_R$  consisting of  $M_1$  patches. Then we denote by  $P_{M_1}$  the interpolation projection on the space of bounded functions on  $\mathbb{S}_R$  that are constant on each patch  $\Gamma_j$ , by  $P_{M_1}[g](j)$ , the value of  $P_{M_1}[g]$  on  $\Gamma_j$ , and by  $\mathbf{1}_{\Gamma_j} : \mathbb{S}_R \rightarrow \mathbb{C}$  the indicator function of  $\Gamma_j$ . For this setting, one can show that

$$N_{M_1}g = \sum_{i=1}^{M_1} \mathbf{1}_{\Gamma_j} \sum_{j=1}^{M_1} u_{\text{appr}}^s(y_i, y_j) P_{M_1}[g](\ell) \quad (24)$$

is a converging finite-dimensional approximation to the near field operator. (An approximation to  $F_{M_1}$  to  $F$  is defined analogously, see [20, Section 4].)

To further discretize the modification  $TN$  to an element  $g \in L^2(\mathbb{S}_R)$  we develop  $N_{M_1}g$  into its coefficients  $((N_{M_1}g)_n^m)$  for the orthogonal basis of spherical harmonics by numerical integration on  $\mathbb{S}_R$  and truncate the series expression defining  $T$ , see [12, Equation (3.12)], at  $M_2 \in \mathbb{N}$ , such that

$$(T_{M_2}N_{M_1})g(x) = \sum_{n=0}^{M_2} \sum_{m=-n}^n \left( \frac{\overline{h_n^{(1)}(kR)}}{h_n^{(1)}(kR)} (N_{M_1}g)_n^m \right) Y_n^m(\hat{x}), \quad x \in \mathbb{S}_R. \quad (25)$$

yields an approximation of  $TN$ . This approximation is then discretized by evaluating it for all indicator functions  $\mathbf{1}_{\Gamma_j}$ ,  $j = 1, \dots, M_1$ , at the source points  $\{y_i, i = 1, \dots, M_1\}$  to obtain an  $M_1 \times M_1$  matrix. For our experiments later on we have chosen  $M_1 = 120$  and  $M_2 = 12$ , which corresponds to values where the estimates for interior Dirichlet eigenvalues presented below are insensitive to further increasing these parameters. (Choosing  $M_2 = 12$  corresponds to  $M_2^2 = 144$  series terms in (25).)

If the scattering object  $D$  is the unit ball  $B_1(0)$ , then the operators  $N$  and  $TN$  are diagonalizable in the basis of the spherical harmonics and their eigenvalues can be explicitly calculated. In Figure 1(a) we computed these eigenvalues for measurements on  $\mathbb{S}_2$ , i.e., for  $R = 2$ , and compared them to the numerically computed eigenvalues of the approximated near field operator  $N_{120}$ . We note that the numerically computed eigenvalues to  $N$  are sufficiently accurate in the visible norm; however, they do not share any visibly apparent structure. In Figure 1(b) we computed the analytic eigenvalues of  $TN_k$  for the same setting and compared them to the numerically computed eigenvalues of the approximated modified near field operator  $T_{12}N_{120}$ . Although the discretization of  $T_{12}$  visibly increases the inaccuracy of the approximated eigenvalues, one can see that they accumulate at zero from the left, corresponding to the eigenvalues of the far field operator, and that they lie approximately on a contour in the upper half of the complex plane. Expanding on that point, we note that the eigenvalues  $\mu_n$  of  $TN$  and  $\lambda_n$  of  $F$  are given by

$$\mu_n = ikR^2 |h_n^{(1)}(kR)|^2 \frac{j_n(k)}{h_n^{(1)}(k)} \quad \text{and} \quad \lambda_n = \frac{(4\pi)^2 i}{k} \frac{j_n(k)}{h_n^{(1)}(k)}, \quad n \in \mathbb{N},$$

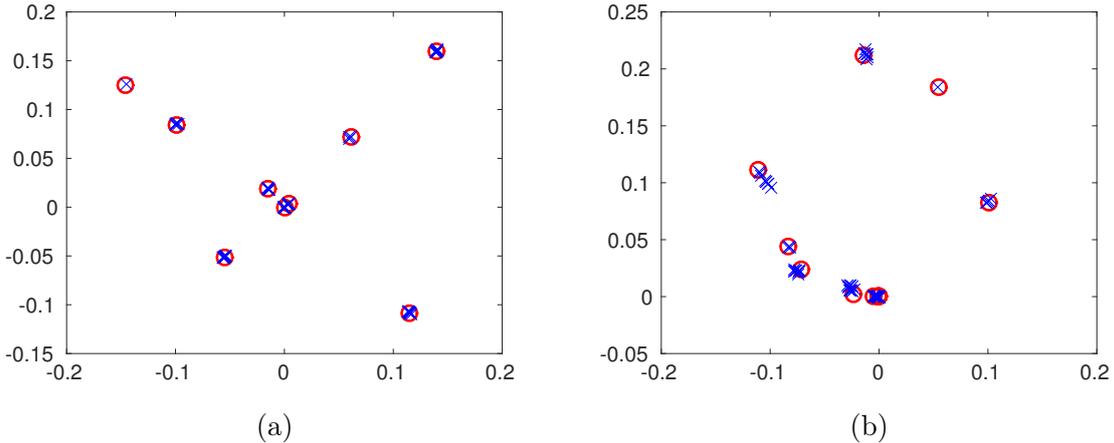


Figure 1: Eigenvalues in the complex plane for wave number  $k = 5$  and radius  $R = 2$ . Red circles mark analytically calculated eigenvalues and blue crosses numerically computed eigenvalues of discretizations. (a) Eigenvalues of the near field operator  $N$  and its discretization  $N_{120}$ . (b) Eigenvalues of the modified near field operator  $TN$  and its discretization  $T_{12}N_{120}$ .

respectively, see [12, Section 3.3] and [16, Section 1.5]. Comparing both expressions, we find that scaling the radii of the eigenvalues  $\lambda_n$  by  $k^2 |h_n^{(1)}(kR)|^2 / (4\pi)^2$  precisely yields the eigenvalues  $\mu_n$ . Note that this factor could also be derived from Theorem 10, since  $\mathbf{g} \in \text{dom}(\mathcal{Z})$  we have that  $\mathcal{Z}\mathcal{Z}^*\mathbf{g} = \{k^2 |h_n^{(1)}(kR)|^2 g_n^m, n \in \mathbb{N}_0, |m| \leq n\}$ . Obviously, the scaling factor does not change the phases of the eigenvalues. We would further like to point out that even for the other scatterers considered below the eigenvalues of the discretization of  $TN$  retained the same phases as the eigenvalues of the discretization of  $F$ . In particular, the smallest phase among all eigenvalues of  $T_{M_2}N_{M_1}$  and  $F_{M_1}$  larger than the noise level of these discretization always agreed roughly up to discretization error.

Finally, we numerically verify the inside-outside duality for near field data. For that purpose we need to calculate the smallest phase of all the elements of the numerical range of the matrix representation  $A$  of  $T_{M_2}N_{M_1}$ , given by  $W(A) = \{(Av, v) : v \in \mathbb{C}^n, \|v\| = 1\}$ . The algorithm we use to compute this numerical range follows [6]. The essential idea is to first rotate  $A$  by multiplying a factor  $\exp(-it)$  to  $A$  and second to decompose the rotation  $\exp(-it)A$  into its real and imaginary part, i.e.  $\exp(-it)A = H_t + iK_t$ , with self-adjoint operators

$$H_t = \frac{\exp(-it)A + (\exp(-it)A)^*}{2}, \quad K_t = \frac{\exp(-it)A - (\exp(-it)A)^*}{2i}.$$

We denote by  $\mu_{\max}(t)$  the largest eigenvalue of  $H_t$  and by  $P_t$  the orthogonal projection from  $\mathbb{C}^{M_1}$  onto the eigenspace  $\{v \in \mathbb{C}^{M_1} : H_t v = \mu_{\max}(t)v\}$  and calculate (not necessarily different) eigenvectors  $v_t^+$  and  $v_t^-$  corresponding to  $\mu_{\max}(t)$ , which are also eigenvectors of the (not necessarily different) smallest and largest eigenvalue of  $P_t K_t P_t$ . For  $t \in [0, 2\pi]$ , the numbers  $(Av_t^+, v_t^+)$  and  $(Av_t^-, v_t^-)$  then belong to the boundary of the numerical range of  $A$ , and  $W(A)$  is the convex hull of all these numbers, see [6, Theorem 3].

Due to numerical inaccuracies, finding the smallest phase in this set is not an obvious task, as becomes apparent when comparing the numerical ranges of (the matrix representations of)  $T_{12}N_{120}$  and  $F_{120}$ . As a scattering object, we choose the unit cube  $[0, 1]^3$  and plot the the

boundaries of these two numerical ranges in Figure 2(a) and (b). While the boundary of the

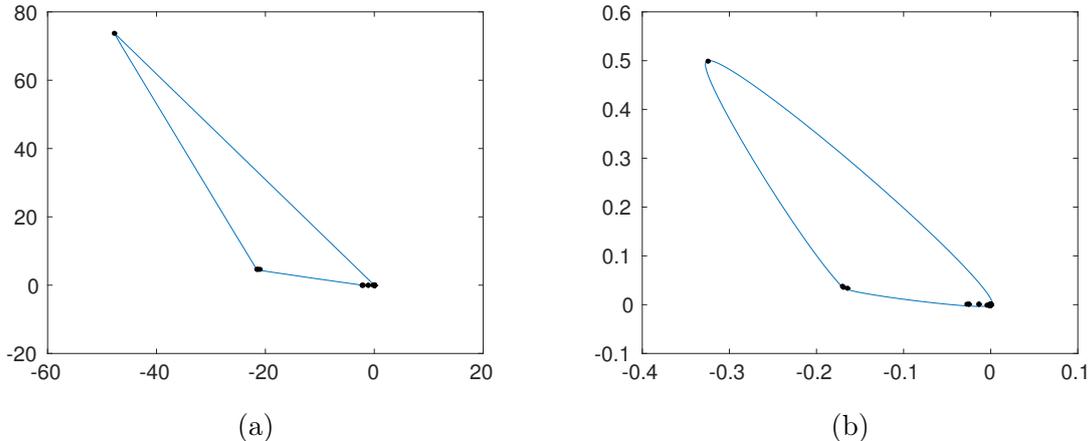


Figure 2: The numerical ranges of  $F_{120}$  and  $T_{12}N_{120}$  for the unit cube as obstacle, wave number  $k = 1.5$  and measurements taken on the sphere with radius  $R = 2$ . (a) Boundary of the numerical range of  $F_{120}$ . Black dots mark the numerically computed eigenvalues of  $F_{120}$ . (b) Boundary of the numerical range of  $T_{12}N_{120}$ . Black dots mark the numerically computed eigenvalues of  $T_{12}N_{120}$ .

numerical range of  $F_{120}$  in Figure 2(a) shows that the numerical range is indeed the convex hull of the eigenvalues of  $F_{120}$ , see Lemma 17(d), the inaccuracies in the computation of the numerical approximation of the operator  $T_{12}$  show up in the plot of the numerical range of  $T_{12}N_{120}$  in Figure 2(b). In particular, the boundary of the numerical range of  $T_{12}N_{120}$  between 0 and the corner with smallest phase fails to be straight, such that it is not obvious how to stably determine the element in  $W(T_{12}N_{120})$  possessing the smallest phase.

For this reason, we opted for the simple idea to use that eigenvalue of  $T_{12}N_{120}$  as an indicator for interior eigenvalues that possesses the smallest phase among all eigenvalues of  $T_{12}N_{120}$  larger than the discretization error. (The discretization error is estimated via the absolute value of the smallest negative imaginary part of these eigenvalues.) In all our computations, this eigenvalue coincided with that boundary point of the numerical range of  $T_{12}N_{120}$  possessing the smallest phase among all corner-like boundary points where boundary curvature peaks. This not surprising, since points in the boundary of the numerical range with infinite curvature are eigenvalues of the corresponding operator by Lemma 17(b).

The subsequent Figure 3 indicates that replacing the smallest phase of the numerical range by smallest phase of the eigenvalues of  $T_{12}N_{120}$  yields simple-to-compute and accurate indicator for Dirichlet eigenvalues of the unit sphere and the unit ball. For these particular scattering objects, we can analytically calculate the square root of the Dirichlet eigenvalues  $k_B^{(j)}$  of the unit ball. For the unit ball  $B$ , the eigenvalues are given as positive roots of spherical Bessel functions and the first three eigenvalues are the squares of

$$k_B^{(1)} = \pi, \quad k_B^{(2)} \approx 4.49, \quad k_B^{(3)} \approx 5.76.$$

For the cube  $C = [0, 1]^3$  the wave numbers  $k_C$  at which  $k_C^2$  is an interior Dirichlet eigenvalue are given by  $k_C = \sqrt{k_1 + k_2 + k_3}$  where  $k_{1,2,3}$  is one of the numbers  $\pi^2(n+1)^2$ ,  $n \in \mathbb{N}_0$ . Hence,

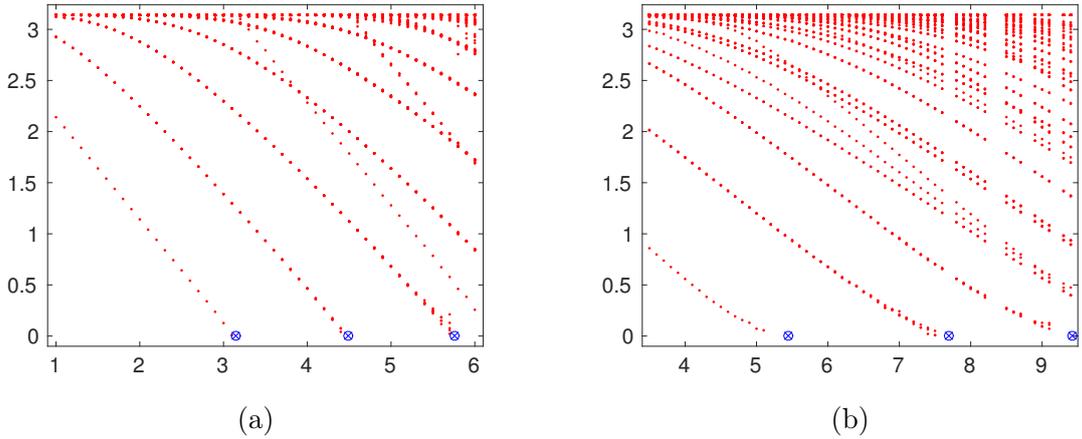


Figure 3: Phases of the eigenvalues of  $T_{12,k}N_{120,k}$  for varying wave number  $k$ . Near field data is measured on the sphere  $\mathbb{S}_2$  of radius two. (a) Scattering from the unit ball  $B_1(0)$ . Blue circles mark position of the exact square roots  $k_B$  to Dirichlet eigenvalues. (b) Scattering from the unit cube  $[0, 1]^3$ . Blue circles mark position of the exact square roots  $k_C$  to Dirichlet eigenvalues.

the first three Dirichlet eigenvalues are the square of the numbers

$$k_C^{(1)} = \sqrt{3}\pi \approx 5.44, \quad k_C^{(2)} = \sqrt{6}\pi \approx 7.70, \quad k_C^{(3)} = 3\pi \approx 9.42.$$

Indeed, one can see in Figure 3 that the smallest phase converges to zero if and only if  $k_0^2$  is a Dirichlet eigenvalue.

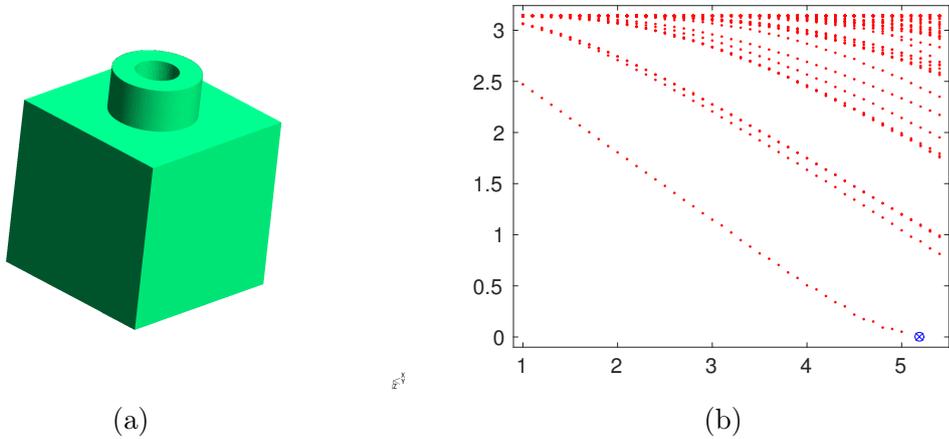


Figure 4: (a) The non-convex scattering object. (b) Phases of the eigenvalues of  $TN_M(k)$  for varying wave number  $k$ . The blue circle on the 0-axis marks the extrapolated position of the square root of the Dirichlet eigenvalue.

Finally, we provide an example for a non-convex scatterer  $D$  for which the Dirichlet eigenvalues of  $-\Delta$  are not known analytically; the object is plotted in Figure 4(a) and, roughly speaking, consists of the unit square with a smaller cylinder on top. Due to numerical inaccuracies at larger wave numbers, we only aim to approximate the smallest wave number  $k_0$

such that  $k_0^2$  is a Dirichlet eigenvalue. For this purpose, we take the last two smallest phases before the first phase jump at about 5.25, see Figure 4(b), and linearly extrapolate the line through these points with the 0-axis. This technique, which showed to yield stable results in [21], then provides the approximation  $k_0 \approx 5.19$  for the smallest Dirichlet eigenvalues of the plotted domain.

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