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Modelling with Orthonormal Basis Functions

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Chapter 1

Discrete-Time System Modelling in L_p with Orthonormal Basis Functions

In this chapter, model sets for linear time–invariant systems spanned by fixed pole orthonormal bases are investigated. The obtained model sets are shown to be complete in $L_p(\mathbf{T})$ $(1 < p < \infty)$, the Lebesque spaces of functions on the unit circle $\mathbf{T}$, and in $C(\mathbf{T})$, the space of periodic continuous functions on $\mathbf{T}$. The L_p norm error bounds for estimating systems in $L_p(\mathbf{T})$ by the partial sums of the Fourier series formed by the orthonormal functions are computed for the case $1 < p < \infty$. Some inequalities on the mean growth of the Fourier series are also derived. These results have application in estimation and model reduction.

1.1 Introduction

The decomposing description of linear time–invariant infinite–dimensional dynamics in terms of an orthonormal basis is an important part of modern Systems Theory and has a long history in modelling and identification of dynamical systems dating back to the classical work of Lee [19] and Wiener [36]. This approach is greatest utility when accurate system descriptions are achieved with a small number of basis functions. The development of suitable basis functions that reflect the dominant charecteristics of the system has attracted considerable interest [26, 28, 30, 31, 32, 33, 34, 35, 22, 23, 17, 3, 5, 4, 7].

In particular, in the areas of control theory, signal processing and system identification, there has long been interest in the use of the finite–impulse response, the *Laguerre*, and the *two–parameter Kautz* functions to model stable linear dynamical systems [19, 18, 16]. The Laguerre and the Kautz models are special cases of the general orthonormal basis functions in [17], where the poles of the system transfer function are restricted to a finite set. The general orthonormal basis functions are generalized by the rational orthonormal basis functions with fixed poles considered in detail in [23, 3, 5, 4].

In [3] the rational orthonormal basis functions were shown to be complete in the disk algebra provided that the chosen basis poles satisy a mild condition and more recently in [4], it was established that the Fourier series formed by the rational orthonormal basis functions converges in the Hardy spaces.

In this chapter, a similar completeness result is obtained for the spaces $L_p(\mathbf{T})$ $(1 \leq p < \infty)$ and $C(\mathbf{T})$. As the orthonormal system, we consider a set of complex-valued rational functions $\{B_n\}$ defined by a choice of numbers $z_n \in \mathbf{D}$ and $x_n \in \mathbf{D}$, as $B_0 =$

$\sqrt{1-|z_0|^2}/\,(1-\overline{z_0}\,z)$ and for $n = 1, 2, \cdots$

$$B_n = \frac{\sqrt{1-|z_n|^2}}{1-\overline{z_n}\,z}\,\phi_n, \qquad \phi_n = \prod_{j=0}^{n-1} \frac{z-z_j}{1-\overline{z_j}\,z}, \tag{1.1}$$

$$B_{-n} = \frac{\sqrt{1-|x_n|^2}}{z-x_n}\,\phi'_{n-1}, \qquad \phi'_n = \prod_{j=1}^{n} \frac{1-\overline{x_j}\,z}{z-x_j}, \tag{1.2}$$

where $\phi'_0 = 1$. The orthonormality is with respect to the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\overline{g(e^{i\theta})}\,\mathrm{d}\theta.$$

We will establish the following completeness result.

Theorem 1.1.1 *The linear span of the functions $\{B_n\}$ defined by (1.1)–(1.2) are everywhere dense in $L_p(\mathbf{T})$ ($1 < p < \infty$) as well as in $C(\mathbf{T})$ if and only if*

$$\sum_{n=0}^{\infty}(1-|z_n|) = \infty, \qquad \sum_{n=1}^{\infty}(1-|x_n|) = \infty. \tag{1.3}$$

This result has interesting applications on the robust recovery of functions in $L_p(\mathbf{T})$ from noise–corrupted evaluations on the unit circle. An abstract framework that solves this type of problems is outlined in [24]. In the modelling of physical systems, it is necessary to ensure that the modelled impulse response is real valued. This issue is addressed in § 1.5.

The next result concerns the Fourier series of integrable functions on $\mathbf{T}$ with respect to the orthonormal system (1.1)–(1.2) whose partial sums are defined by

$$\mathcal{S}_n f(e^{i\theta}) = \sum_{k=-n}^{n} \langle f, B_k\rangle\, B_k(e^{i\theta}). \tag{1.4}$$

The L_p norm errors of the estimate (1.4) are computed quite accurately for the case $1 < p < \infty$. In establishing this, an essential role is played by the *Blaschke* products in (1.1)–(1.2). Relations between projection operators, conjugate functions, and the Fourier series are also displayed. Having computed the error bounds for the partial sums of the Fourier series (1.4), we provide bounds on the mean growth of the Fourier coefficients $\{\langle f, B_k\rangle\}$ and derive the so-called Hausdorff-Young inequalities.

Finally, a simulation example is given to illustrate the use of the basis functions defined by (1.1)–(1.2) for modelling.

1.2 Completeness of the orthonormal system

We will represent $\mathcal{S}_n f$ in terms of two Cauchy integrals of f when $f(e^{i\theta})$ is the restriction to $\mathbf{T}$ of a complex function which is analytic on a region that contains $\mathbf{T}$. This representation facilitates a simple proof of Theorem 1.1.1. The analysis of the estimate (1.4) will be based on these formulae. To this end, first we have the following lemma.

Lemma 1.2.1 *(Christoffel-Darboux formulae)*

$$\sum_{k=0}^{n} \overline{B_k(\zeta)}\, B_k(z) = \frac{1-\overline{\phi_{n+1}(\zeta)}\,\phi_{n+1}(z)}{1-\overline{\zeta}\,z}, \quad z\overline{\zeta} \neq 1 \tag{1.5}$$

$$\sum_{k=-n}^{-1} \overline{B_k(\zeta)}\, B_k(z) = \frac{1-\overline{\phi_n'(\zeta)}\,\phi_n'(z)}{\overline{\zeta}\,z-1}, \quad z\overline{\zeta} \neq 1. \tag{1.6}$$

Proof. The proof of (1.5) is by induction. For $n = 0$,

$$\frac{1-\overline{\phi_1(\zeta)}\phi_1(z)}{1-\overline{\zeta}\,z} = \frac{1-|z_0|^2}{(1-z_0\,\overline{\zeta})(1-\overline{z_0}\,z)} = \overline{B_0(\zeta)}\,B_0(z)$$

while for $n > 0$

$$\frac{1-\overline{\phi_{n+1}(\zeta)}\phi_{n+1}(z)}{1-\overline{\zeta}\,z} = \sum_{k=0}^{n-1} \overline{B_k(\zeta)}\,B_k(z) + (1-|z_n|^2)\,\frac{\overline{\phi_n(\zeta)}\phi_n(z)}{(1-z_n\,\overline{\zeta})(1-\overline{z_n}\,z)}.$$

The proof of (1.6) follows from (1.5) by the transformations and back transformations $z \mapsto 1/z$, $\zeta \mapsto 1/\zeta$, $x_j \mapsto \overline{z_{j-1}}$, $j = 1, 2, \cdots, n$. ∎

Hence from (1.5)–(1.6), we get for the two components of the sum in (1.4)

$$\sum_{k=0}^{n} \langle f, B_k\rangle\, B_k(z) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \frac{\phi_{n+1}(z)}{2\pi i}\int_{\gamma_0} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)\phi_{n+1}(\zeta)} \tag{1.7}$$

$$\sum_{k=-n}^{-1} \langle f, B_k\rangle\, B_k(z) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(\zeta)}{z-\zeta}\,\mathrm{d}\zeta - \frac{\phi_n'(z)}{2\pi i}\int_{\gamma_0} \frac{f(\zeta)\,\mathrm{d}\zeta}{(z-\zeta)\phi_n'(\zeta)} \tag{1.8}$$

where $\gamma_0(s) = e^{is}$ $(0 \le s \le 2\pi)$.

Let $\mathbf{A}(r_1, r_2)$ be the annulus $\{z : r_1 < |z| < r_2\}$, where $r_1 < 1$ and $r_2 > 1$ are two given positive numbers. Suppose that $f(z)$ is analytic in a region that contains $\mathbf{A}(r_1, r_2)$. Then the following Cauchy formula is valid on $\mathbf{A}(r_1, r_2)$

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \tag{1.9}$$

where

$$\gamma_1(s) = r_1\,e^{-is}, \qquad \gamma_2(s) = r_2\,e^{is} \qquad (0 \le s \le 2\pi).$$

The integrands in (1.7) are meromorphic functions on $\mathbf{A}(r_1, r_2)$ whose singularities are inside γ_0 and are encircled once by the contours γ_0 and γ_2. Hence by the residue theorem [27, Th. 10.42]

$$\sum_{k=0}^{n} \langle f, B_k\rangle\, B_k(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \frac{\phi_{n+1}(z)}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{(\zeta - z)\phi_{n+1}(\zeta)}\,\mathrm{d}\zeta$$

and letting $z \to e^{i\theta}$, we obtain

$$\sum_{k=0}^{n} \langle f, B_k\rangle\, B_k(e^{i\theta}) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - e^{i\theta}}\,\mathrm{d}\zeta - \frac{\phi_{n+1}(e^{i\theta})}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{(\zeta - e^{i\theta})\;\phi_{n+1}(\zeta)}\,\mathrm{d}\zeta. \tag{1.10}$$

Since the integrands in (1.8) are analytic on $\mathbf{A}(r_1, r_2)$, their integrals on the cycle $\gamma_0 \cup \gamma_1$ must vanish by the Cauchy theorem. Hence

$$\sum_{k=-n}^{-1} \langle f, B_k \rangle \, B_k(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \frac{\phi_n'(z)}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{(\zeta - z)\phi_n'(\zeta)} \, \mathrm{d}\zeta$$

and letting $z \to e^{i\theta}$, we get

$$\begin{aligned} \sum_{k=-n}^{-1} \langle f, B_k \rangle \, B_k(e^{i\theta}) &= \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - e^{i\theta}} \, \mathrm{d}\zeta \\ &\quad - \frac{\phi_n'(e^{i\theta})}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{(\zeta - e^{i\theta}) \; \phi_n'(\zeta)} \, \mathrm{d}\zeta. \end{aligned} \tag{1.11}$$

Thus from (1.9) and (1.11)

$$\begin{aligned} f(e^{i\theta}) - \mathcal{S}_n f(e^{i\theta}) &= \frac{\phi_{n+1}(e^{i\theta})}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{(\zeta - e^{i\theta}) \; \phi_{n+1}(\zeta)} \, \mathrm{d}\zeta \\ &\quad + \frac{\phi_n'(e^{i\theta})}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{(\zeta - e^{i\theta}) \; \phi_n'(\zeta)} \, \mathrm{d}\zeta. \end{aligned} \tag{1.12}$$

The third step is to bound $f - \mathcal{S}_n f$. First we have the following lemma.

Lemma 1.2.2

$$\sup_{z \in \gamma_2} \frac{1}{|\phi_{n+1}(z)|} \leq \exp\left(-\frac{r_2 - 1}{2r_2} \sum_{j=0}^{n} (1 - |z_j|) \right) \tag{1.13}$$

$$\sup_{z \in \gamma_1} \frac{1}{|\phi_n'(z)|} \leq \exp\left(-\frac{1 - r_1}{2} \sum_{j=1}^{n} (1 - |x_j|) \right). \tag{1.14}$$

Proof. Let $\overline{w} = z^{-1}$. Then

$$\frac{1}{|\phi_{n+1}(z)|} = |\phi_{n+1}(w)| \leq \prod_{j=0}^{n} \left| \frac{w - z_j}{1 - \overline{z_j}\, w} \right| \tag{1.15}$$

Let $w = re^{i\theta}$ and $z_j = R_j e^{i\theta_j}$ denote the polar decompositions of w and z_j. Then a simple algebraic manipulation yields

$$\left| \frac{w - z_j}{1 - \overline{z_j}\, w} \right|^2 \leq 1 - (1 - r)(1 - R_j) \leq \exp\left(-(1 - r)(1 - R_j)\right) \tag{1.16}$$

where the last inequality follows from the fact that $e^{-x} \geq 1 - x$ for all x. Consideration of (1.15) and (1.16) with $r = 1/r_2$ completes the proof of (1.13). The proof of (1.14) is similar. ∎

Hence from Lemma 1.2.2 and the integral formulation of the approximation error (1.12)

$$\begin{aligned} \|f - \mathcal{S}_n f\|_\infty \leq & \sup_{z \in \mathbf{A}(r_1, r_2)} |f(z)| \; \frac{r_2}{r_2 - 1} \; \exp\left(-\frac{r_2 - 1}{2r_2} \sum_{j=0}^{n} (1 - |z_j|) \right) \\ & + \sup_{z \in \mathbf{A}(r_1, r_2)} |f(z)| \; \frac{r_1}{1 - r_1} \; \exp\left(-\frac{1 - r_1}{2} \sum_{j=1}^{n} (1 - |x_j|) \right). \end{aligned} \tag{1.17}$$

Now we complete the proof of the sufficiency. Let $f \in L_p(\mathbf{T})$. Recall that the trigonometric system $\{e^{\pm ik\theta}\}$ is closed in $C(\mathbf{T})$ (Weierstrass' second theorem) and hence in $L_p(\mathbf{T})$ since $C(\mathbf{T})$ is a dense subset of $L_p(\mathbf{T})$. Thus we may assume without restriction that $f(e^{i\theta})$ is a trigononometric polynomial. Since f extends to an analytic function on the punctured plane $\mathbf{A}(0,\infty)$, it follows from the above inequality with $r_1 = 1/2$ and $r_2 = 2$ that

$$\lim_{n\to\infty} \sup_{0\le\theta\le 2\pi} \left| f(e^{i\theta}) - \mathcal{S}_n f(e^{i\theta}) \right| = 0$$

provided that the conditions in (1.3) are satisfied. This proves the sufficiency.

For the necessity, assume that

$$\sum_{n=0}^{\infty} (1 - |z_n|) < \infty.$$

Then the unimodulated finite Blaschke products $\phi_n(z)$ in (1.1) converge uniformly on $\mathbf{D}$ to a Blaschke product

$$\phi(z) = \prod_{n=0}^{\infty} \frac{z_n - z}{1 - \overline{z_n}\, z} \frac{|z_n|}{z_n}$$

(with the convention $|z_n|/z_n = 1$ when $z_n = 0$) which has zeros precisely at the points z_n. In this case, the linear functional Φ defined on $L_p(\mathbf{T})$ $(1 \le p < \infty)$ and $C(\mathbf{T})$ by $\Phi(f) = \langle f, \phi \rangle$ is clearly nontrivial and also bounded. However by Cauchy's theorem it also vanishes at every B_n as

$$\overline{\Phi(B_n)} = (-1)^{n+1} \prod_{k=0}^{n} \frac{|z_k|}{z_k} \frac{1}{2\pi i} \int_{\gamma_0} \frac{\sqrt{1-|z_n|^2}}{1-\overline{z_n}\,\zeta} \prod_{k=n+1}^{\infty} \frac{z_k - \zeta}{1 - \overline{z_k}\,\zeta} \frac{|z_k|}{z_k}\, \mathrm{d}\zeta = 0.$$

With the same reasoning we have $\Phi(B_n) = 0$ for all $n < 0$. Hence the linear span of the sets $\{B_n(e^{i\theta})\}$ is not dense in the spaces $C(\mathbf{T})$ and $L_p(\mathbf{T})$ $(p \ge 1)$. The other case

$$\sum_{n=1}^{\infty} (1 - |x_n|) < \infty$$

is similar and it suffices to consider the Blaschke product

$$\phi'(z) = \prod_{n=1}^{\infty} \frac{1 - \overline{x_n}\, z}{x_n - z} \frac{x_n}{|x_n|}$$

which is analytic on $\mathbf{A}(1,\infty)$ and has common zeros with the functions $B_n(z)$, $n < 0$ and the linear functional Φ' defined on $L_p(\mathbf{T})$ $(1 \le p < \infty)$ and $C(\mathbf{T})$ by $\Phi'(f) = \langle f, \phi'/z \rangle$.

In Achieser [1], Theorem 1.1.1 is proven for the rational functions in the form

$$\left\{ \frac{1}{e^{i\theta} - z_n} \right\}_{n=1}^{\infty} \qquad (0 \le \theta \le 2\pi)$$

where $\{z_n\}$ is a given sequnce of distinct complex numbers satisfying $|z_n| \ne 1$. These functions don't include the exponentials $\{e^{\pm in\theta}\}$ whereas the orthonormal functions defined by (1.1)–(1.2) include them in the special case $z_n = x_n = 0$ for all n.

The proof in Achieser builds on the solution of a certain extremal problem. When suited for the basis functions in (1.1)–(1.2), this extremal problem directly yields Theorem 1.1.1.

We omit the details. Our proof on the other hand is based on the integral formulation of the approximation error.

The completeness conditions (1.3) are very mild. For example removing a finite number of B_n's from the span does not destroy the completeness as the same conditions still apply. This stability property is not seen in the bases spanned by the complex exponentials $\{e^{i\lambda_n\theta}\}$ where $\{\lambda_n\}$ is a sequence of real or complex numbers. For example if $\{\lambda_n\}$ satisfies

$$|\lambda_n - n| \le \frac{1}{2p}, \qquad n = 0, \pm 1, \pm 2, \cdots$$

then $\{e^{i\lambda_n\theta}\}$ is complete in $L_p(\mathbf{T})$ $(1 < p < \infty)$ (Kadec's $\frac{1}{4}$–theorem). However, the constant $1/2p$ can not be replaced by any larger number.

1.3 Mean convergence of the Fourier series

In this section we show that the Fouries series formed by the orthonormal functions in (1.1)–(1.2) converges in the spaces $L_p(\mathbf{T})$ $(1 < p < \infty)$.

Let $S_n f$ denote the partial sums of the Fourier series of an integrable function f with respect to the exponential functions $\{e^{\pm ik\theta}\}$. It is well-known fact that every $f \in L_p(\mathbf{T})$ $(p \ge 1)$ has a Fourier series converging in $L_p(\mathbf{T})$ if and only if the operators S_n are uniformly bounded.

Now assume that $\sup_n \|S_n\| < \infty$ and consider the operators P_n which maps $\sum_{-\infty}^{\infty} c_k e^{ik\theta} \in X$ to $\sum_0^n c_k e^{ik\theta}$. The identity

$$P_{2n} f(e^{i\theta}) = e^{in\theta} S_n(e^{-in\theta} f) \tag{1.18}$$

shows that $\sup_n \|P_n\| < \infty$. Hence for each $f \in L_p(\mathbf{T})$, the sequence $P_n f$ converges in the norm and let $\mathcal{P}_+ f$ denote the limit, which is the projection of f as $\sum_{-\infty}^{\infty} c_k e^{ik\theta} \mapsto \sum_0^{\infty} c_k e^{ik\theta}$. In particular, $\|\mathcal{P}_+\| < \infty$. This implies that the complementary projection $\mathcal{P}_- : f(e^{i\theta}) \mapsto \sum_{k=-\infty}^{-1} c_k e^{ik\theta}$ is also bounded.

Let F denote the Cauchy integral of f defined as

$$F(z) = \frac{1}{2\pi i} \int_{\gamma_0} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta, \qquad \gamma_0 = e^{i\theta}\ (0 \le \theta \le 2\pi). \tag{1.19}$$

On the domains seperated by $\mathbf{T}$, $F(z)$ is analytic. Observe that the Cauchy integral of $\mathcal{P}_- f$ vanishes on $\mathbf{D}$. (This follows from the boundedness of $\|\mathcal{P}_-\|$ and the denseness of the trigonometric polynomials in $L_p(\mathbf{T})$ $(1 \le p < \infty)$). Thus F equals to the Cauchy integral of $\mathcal{P}_+ f$ on $\mathbf{D}$. This implies that $F(z)$ converges to $\mathcal{P}_+ f(e^{i\theta})$ for almost every $e^{i\theta} \in \mathbf{T}$ as $z \to e^{i\theta}$ nontangentially in $\mathbf{D}$. Hence in (1.7) letting $z \to e^{i\theta}$ nontangentially in $\mathbf{D}$, we get almost everywhere on $\mathbf{T}$

$$\sum_{k=0}^{n} \langle f, B_k \rangle\, B_k(e^{i\theta}) = \mathcal{P}_+ f(e^{i\theta}) - \phi_{n+1}(e^{i\theta})\, \mathcal{P}_+ \left[\frac{f}{\phi_{n+1}}\right](e^{i\theta}) \tag{1.20}$$

Next consider the Cauchy integral (1.19) on $\mathbf{A}(1,\infty)$, the complement of the closed unit disk. The conjugation and the change of variables $\zeta = e^{it}$ yield

$$\overline{F(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\overline{f(e^{it})}}{1 - \overline{z}\, e^{it}}\, \mathrm{d}t \tag{1.21}$$

In (1.21) substituting $\overline{w} = 1/z$ and changing the variables as $e^{it} = \zeta$ we obtain

$$-\frac{1}{\omega}\,\overline{F\left(\frac{1}{\overline{w}}\right)} = \frac{1}{2\pi i}\int_{\gamma_0} \frac{\overline{f(\zeta)}}{\zeta\,(\zeta - w)}\,\mathrm{d}\zeta, \qquad w \in \mathbf{D}.$$

This is recognised as the previously considered situation where $f(e^{i\theta})$ is replaced by $e^{-i\theta}\,\overline{f(e^{i\theta})}$. Consequently for almost every $e^{i\theta} \in \mathbf{T}$, as $w \to e^{i\theta}$ nontangentially in $\mathbf{D}$

$$-\frac{1}{\omega}\,\overline{F\left(\frac{1}{\overline{w}}\right)} \to \mathcal{P}_+\left[\frac{\overline{f(e^{i\theta})}}{e^{i\theta}}\right]$$

which implies $F(z) \to -\mathcal{P}_- f(e^{i\theta})$ as $z \to e^{i\theta}$ nontangentially in $\mathbf{A}(1,\infty)$. Thus in (1.8) letting $z \to e^{i\theta}$ nontangentially in $\mathbf{A}(1,\infty)$ we get almost everywhere on $\mathbf{T}$

$$\sum_{k=-n}^{-1} \langle f, B_k\rangle\, B_k(e^{i\theta}) = \mathcal{P}_- f(e^{i\theta}) - \phi_n'(e^{i\theta})\,\mathcal{P}_-\left[\frac{f}{\phi_n'}\right](e^{i\theta}). \tag{1.22}$$

Hence from (1.20) and (1.22)

$$f - \mathcal{S}_n f = \phi_{n+1}\,\mathcal{P}_+\,(f/\phi_{n+1}) + \phi_n'\,\mathcal{P}_-\,(f/\phi_n') \qquad \text{a.e. } \mathbf{T} \tag{1.23}$$

Let $\tilde{f}(e^{i\theta})$ denote the conjugate of $f(e^{i\theta})$. Recall that f and $\tilde{f}$ are recovered almost everywhere on $\mathbf{T}$ by taking nontangential limits of $u(z)$ and $\tilde{u}(z)$ defined by

$$(u + i\tilde{u})(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, f(e^{it})\,\mathrm{d}t \tag{1.24}$$

as $z \to e^{i\theta}$. Let $\mathcal{F}$ denote the map $f \mapsto f + i\tilde{f}$. Noting that $c_0 = \langle f, 1\rangle$, the operators $\mathcal{P}_+$ and $\mathcal{P}_-$can be written as

$$\begin{aligned} \mathcal{P}_+ f &= \frac{1}{2}\left(\mathcal{F}f + \langle f, 1\rangle\right) && (1.25)\\ \mathcal{P}_- f &= \overline{\mathcal{P}_+(\overline{f})} - c_0 = \frac{1}{2}\left(\overline{\mathcal{F}(\overline{f})} - \langle f, 1\rangle\right). \end{aligned}$$

Thus from (1.23) and the equalities $\langle f/\phi_k, 1\rangle = \langle f, \phi_k\rangle$ for all k

$$f - \mathcal{S}_n f = \frac{\phi_{n+1}}{2}\,\mathcal{F}\left(\frac{f}{\phi_{n+1}}\right) + \frac{\phi_n'}{2}\,\overline{\mathcal{F}\left(\frac{\overline{f}}{\overline{\phi_n'}}\right)} + \frac{\phi_{n+1}}{2}\,\langle f, \phi_{n+1}\rangle - \frac{\phi_n'}{2}\,\langle f, \phi_n'\rangle.$$

Hence

$$\|f - \mathcal{S}_n f\|_p \le (1 + \|\mathcal{F}\|)\ \|f\|_p, \qquad f \in L_p(\mathbf{T}). \tag{1.26}$$

We started with the assumption $\sup_n \|S_n\| < \infty$ and concluded via to the boundedness of $\mathcal{P}_+$ that $\|\mathcal{F}\| < \infty$. The converse is also true by the equalities (1.25) and (1.18).

Let X_n denote the linear space spanned by the sets $\{B_k(e^{it})\}_{k=-n}^{n}$ and define

$$e_n(f; L_p(\mathbf{T})) = \min_{g \in X_n} \|g - f\|_p, \qquad f \in L_p(\mathbf{T}). \tag{1.27}$$

Thus $e_n(f; L_p(\mathbf{T}))$ is the best approximation error of $f \in L_p(\mathbf{T})$ by functions in X_n. Since (1.1)–(1.2) is closed in $L_p(\mathbf{T})$ $(1 \le p < \infty)$ and $C(\mathbf{T})$, the quantity $e_n(f; L_p(\mathbf{T}))$ defined by (1.27) monotonically tends to zero as $n \to \infty$.

Let f be a given function in $L_p(\mathbf{T})$ and let g be the minimizing solution in (1.27). Let $\psi = f - g$ denote the approximation error. Observe that $\mathcal{S}_n g = g$ since $g \in X_n$. Due to the linearity of $\mathcal{S}_n$ notice also that $\mathcal{S}_n \psi = \mathcal{S}_n f - \mathcal{S}_n g$. Thus from (1.26)

$$\begin{aligned} \|f - \mathcal{S}_n f\|_p &= \|\psi - \mathcal{S}_n \psi\|_p \\ &\leq (1 + \|\mathcal{F}\|)\, e_n(f; L_p(\mathbf{T})). \end{aligned} \tag{1.28}$$

The error bound in (1.28) expressed in terms of $\|\mathcal{F}\|$ is rather tight and without further assumptions on f and the orthonormal system (1.1)–(1.2) it does not seem possible to improve upon. In the special case $f \in H_p(\mathbf{T})$, the Hardy space of functions g which are analytic on $\mathbf{D}$ and such that $g(e^{i\theta}) \in L_p(\mathbf{T})$, we have instead of (1.28)

$$\|f - \mathcal{S}_n f\|_p \leq \frac{1}{2}\,(1 + \|\mathcal{F}\|)\, e_n(f; H_p(\mathbf{T})) \tag{1.29}$$

where $\mathcal{S}_n f = \sum_{k=0}^{n} \langle f, B_k \rangle\, B_k$.

We need the following lemma to compute an upper bound for $\|\mathcal{F}\|$.

Lemma 1.3.1 *Let $f = f_R + i f_I$ where f_R and f_I are real-valued functions. Then*

$$\|f_R\|_p + \|f_I\|_p \leq B_p\, \|f\|_p. \tag{1.30}$$

where

$$B_p = \begin{cases} 2^{1/2}, & 1 \leq p \leq 2 \\ 2^{(p-1)/p}, & p \geq 2. \end{cases} \tag{1.31}$$

Proof. Note the following inequalities whose proofs can be found for example in Duren [11, § 4.2]

$$\begin{aligned} 2^{p-1}(a^p + b^p) &\leq (a+b)^p \leq a^p + b^p, & 0 < p \leq 1 \\ a^p + b^p &\leq (a+b)^p \leq 2^{p-1}(a^p + b^p), & p \geq 1 \end{aligned}$$

where a and b are two arbitrary nonnegative numbers. Put $a = \|f_R\|_p$ and $b = \|f_I\|_p$ in the above inequalities. Then for $1 \leq p \leq 2$

$$\begin{aligned} (\|f_R\|_p + \|f_I\|_p)^p &\leq 2^{p-1} \int \left([f_R^2]^{p/2} + [f_I^2]^{p/2} \right) \quad (p \geq 1) \\ &\leq 2^{p-1} 2^{1-\frac{p}{2}} \int \left(f_R^2 + f_I^2 \right)^{p/2} \quad (p/2 \leq 1) \\ &= 2^{p/2}\, \|f\|_p^p \end{aligned}$$

while for $p \geq 2$

$$\begin{aligned} (\|f_R\|_p + \|f_I\|_p)^p &\leq 2^{p-1} \int \left([f_R^2]^{p/2} + [f_I^2]^{p/2} \right) \quad (p \geq 1) \\ &\leq 2^{p-1} \int \left(f_R^2 + f_I^2 \right)^{p/2} \quad (p/2 \geq 1) \\ &= 2^{p-1}\, \|f\|_p^p \end{aligned}$$

■

When p equals to 1 or 2, the top equality in (1.31) is attained for complex-valued functions in the form $f = (1+i) f_R$. Observe that when $p = \infty$, the bottom equality is attained by complex-valued functions with real and imaginary parts disjointly supported on $\mathbf{T}$.

If $1 < p < \infty$ and f is real-valued function, it is known [12] that

$$\|\mathcal{F}f\|_p \le C_p \, \|f\|_p \tag{1.32}$$

where C_p is the best possible constant given by

$$C_p = \begin{cases} [\cos(\pi/2p)]^{-1}, & 1 < p \le 2 \\ [\sin(\pi/2p)]^{-1}, & 2 < p < \infty. \end{cases} \tag{1.33}$$

Write f as $f = f_R + i f_I$ where f_R and f_I are real valued. Then from (1.32) and (1.30) due to the linearity of $\mathcal{F}$

$$\begin{aligned} \|\mathcal{F}f\|_p &\le \|\mathcal{F}f_R\|_p + \|\mathcal{F}f_I\|_p \\ &\le C_p \left(\|f_R\|_p + \|f_I\|_p\right) \\ &\le C_p B_p \, \|f\|_p. \end{aligned} \tag{1.34}$$

Using (1.28) and (1.34), the following result can now be established.

Theorem 1.3.2 *Consider the partial sums of the Fourier series defined by (1.4). Let $e_n(f; L_p(\mathbf{T}))$, B_p, and C_p be as in (1.27), (1.31), and (1.33). Then for all $1 < p < \infty$ and $f \in L_p(\mathbf{T})$*

$$\|f - \mathcal{S}_n f\|_p \le (1 + B_p C_p) \; e_n(f; L_p(\mathbf{T})) \tag{1.35}$$

and if the conditions in (1.3) are satisfied

$$\lim_{n\to\infty} \|f - \mathcal{S}_n f\|_p = 0.$$

From (1.29), (1.34), (1.31), and (1.33), observe that $\|f - \mathcal{S}_n f\|_2 \le (3/2)\, e_n(f; H_2(\mathbf{T}))$ while the best value is $e_n(f; H_2(\mathbf{T}))$.

The inequality (1.35) shows that the approximation error of the Fourier series is in the order of the best achievable error for every choice of orthonormal system of functions when the approximated function lies in $L_p(\mathbf{T})$ $(1 < p < \infty)$. The choice of orthonormal functions on the other hand depends on the class of functions being approximated. This subject is not investigated here.

In Theorem 1.3.2, the spaces $L_1(\mathbf{T})$ and $C(\mathbf{T})$ can not be included since the projection operator $\mathcal{P}_+$ is not bounded on these spaces.

1.4 Mean growth of the Fourier coefficients

In this section we will derive two inequalities which are analogous to the Hausdorff-Young inequalities for the trigonometric basis $\{e^{\pm int}\}$.

Theorem 1.4.1 *Let $1 \le p \le 2$ and let q be the conjugate exponent,* that is, $q = p/(p-1)$. *Suppose that the basis defined by (1.1)–(1.2) is bounded,* i.e.

$$\sup_n \{|z_n|, |x_n|\} = r < 1. \tag{1.36}$$

If $f \in L_p(\mathbf{T})$ then

$$\left(\sum_{n=-\infty}^{\infty} |\langle f, B_n\rangle|^q\right)^{1/q} \le \left(\frac{1+r}{1-r}\right)^{(q-2)/2q} \|f\|_p. \tag{1.37}$$

If $\{a_n\} \in \ell_p$ *then there exists a function* $f \in L_q(\mathbf{T})$ *such that* $a_n = \langle f, B_n \rangle$. *Moreover,*

$$\|f\|_q \leq \left(\frac{1+r}{1-r}\right)^{(q-2)/2q} \left(\sum_{n=-\infty}^{\infty} |a_n|^p\right)^{1/p}. \tag{1.38}$$

Proof. The mapping $\mathcal{S} : f \mapsto \{\langle f, B_n \rangle\}$ is a linear transformation of functions on the measure space $(\mathbf{T}, dt)$ into functions on (Z, dn), Z being the group of integers and dn the so-called counting measure. The norm of the mapping as $L_1(\mathbf{T}) \mapsto \ell_\infty$ is

$$\|\mathcal{S}\|_{L_1,\ell_\infty} = \sup_{\|f\|_1 \leq 1} \|\langle f, B_n \rangle\|_\infty = \sqrt{\frac{1+r}{1-r}}.$$

The mapping $\mathcal{S}$ is an isometry of $L_2(\mathbf{T})$ onto ℓ_2. Hence $\|\mathcal{S}\|_{L_2,\ell_2} = 1$. Then by the Riesz-Thorin interpolation theorem [25, Th. IX.17] the mapping $\mathcal{S}$ from $L_p(\mathbf{T})$ into ℓ_q is bounded as

$$\|\mathcal{S}\|_{L_p,\ell_q} \leq \|\mathcal{S}\|_{L_1,\ell_\infty}^{(q-2)/q} \|\mathcal{S}\|_{L_2,\ell_2}^{2/q} = \left(\frac{1+r}{1-r}\right)^{(q-2)/2q}.$$

This proves (1.37). The proof of (1.38) is again by interpolation. For this consider the mapping $\mathcal{T} : \{a_n\} \mapsto f(t) = \sum a_n B_n(e^{it})$. If $\{a_n\} \in \ell_1$ then $f(t) = \sum a_n B_n(e^{it}) \in C(\mathbf{T})$ and $\langle f, B_n \rangle = a_n$. Moreover

$$\|\mathcal{T}\|_{\ell_1,L_\infty} = \sup_{\|a\|_1 \leq 1} \|f\|_\infty = \sqrt{\frac{1+r}{1-r}}.$$

The equality $\|T\|_{\ell_2,L_2} = 1$ is obvious. Thus (1.38) follows from

$$\|\mathcal{T}\|_{\ell_p,L_q} \leq \|\mathcal{T}\|_{\ell_1,L_\infty}^{(q-2)/q} \|\mathcal{T}\|_{\ell_2,L_2}^{2/q} = \left(\frac{1+r}{1-r}\right)^{(q-2)/2q}.$$

∎

Theorem 1.4.1 can not be extended to the case $p > 2$. For example with the trigonometric basis $z_n = x_n = 0$ for all n, there exist continuous functions f such that

$$\sum_{k=-\infty}^{\infty} |\langle f, e^{ik\theta} \rangle|^{2-\epsilon} = \infty, \qquad \text{for all } \epsilon > 0.$$

The uniformly bounded basis assumption can be relaxed if $f(e^{i\theta})$ extends to a function that is analytic on a region which contains $\mathbf{T}$.

In the next result, we restrict the attention to $H_p(\mathbf{D})$.

Corollary 1.4.2 *Let* $1 \leq p \leq 2$. *Suppose that* $\sup_n |z_n| = r < 1$. *Then*

$$\left(\sum_{n=0}^{\infty} |\langle f, B_n \rangle|^q\right)^{1/q} \leq \left(\frac{1+r}{1-r}\right)^{(q-2)/2q} \|f\|_p, \qquad f \in H_p(\mathbf{D}) \tag{1.39}$$

If $c = \{c_0, c_1, \cdots\} \in \ell_p$, *then there exists a function* $f \in H_q(\mathbf{D})$ *such that* $c_n = \langle f, B_n \rangle$. *Moreover,*

$$\|f\|_q \leq \left(\frac{1+r}{1-r}\right)^{(q-2)/2q} \|c\|_p. \tag{1.40}$$

Proof. Let $f \in H_p(\mathbf{D})$. Then $f(e^{i\theta}) \in L_p(\mathbf{T})$. Notice that $\langle f, B_n \rangle = 0$ for all $n < 0$ since $\{B_n\}_{n \geq 0}$ is a basis for $H_p(\mathbf{D})$. Thus (1.39) follows from (1.37) in Theorem 1.4.1. Conversely, if $c \in \ell_p$ $(1 \leq p \leq 2)$, then $c \in \ell_2$ and $\sum_{k=0}^{n} c_k B_k$ converges to some $f \in H_2(\mathbf{T})$. The numbers c_n are the Fourier coefficients of $f(e^{i\theta})$. The inequality (1.38) in Theorem 1.4.1 tells us that $f(e^{i\theta}) \in L_q(\mathbf{T})$, which implies $f \in H_q(\mathbf{D})$. ∎

1.5 Modelling of physical systems

Up to now, we have not imposed any restriction on pole location save for the conditions in (1.3). However, in any appplication involving the modelling of a physical system, it is necessary to ensure that the underlying modelled impulse response is real valued. A requirement is that the sets $\{z_0, z_1, \cdots, z_n\}$ and $\{x_1, x_2, \cdots, x_n\}$ used to define basis via (1.1)–(1.2) always contain complex conjugates. Then the constraint of realness of impulse response is easily accommodated by taking suitable linear combinations of the basis functions (1.1)–(1.2). The idea in the following basis construction is taken from [23].

Suppose that $z_0, \cdots, z_{n-1}$ are real so that the basis functions $B_0,\cdots,B_{n-1}$ have real-valued impulse responses. Now we wish to include a complex pole at $1/\overline{z_n}$. Then two new basis functions $\widetilde{B}_n, \widetilde{B}_{n+1}$ with real impulse responses should be formed as a linear combination of B_n and B_{n+1} generated by (1.1) with $z_{n+1} = \overline{z_n}$. These new functions then replace B_n and B_{n+1}. The suggested linear combination can be expressed as

$$\begin{bmatrix} \widetilde{B}_n \\ \widetilde{B}_{n+1} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \begin{bmatrix} B_n \\ B_{n+1} \end{bmatrix}. \tag{1.41}$$

Considering only $\widetilde{B}_n$ for the moment given by

$$\widetilde{B}_n(z) = \frac{\sqrt{1-|z_n|^2}\,(\beta z + \mu)}{1-(z_n+\overline{z_n})z + |z_n|^2 z^2}\ \phi_n(z)$$

where $\phi_n(z)$ has real-valued impulse response and the real coefficients β, μ are related to the choice of c_1, c_2 by

$$c_1 = \frac{\mu + \beta z_n}{1 - z_n^2}, \qquad c_2 = \frac{\mu z_n + \beta}{1 - z_n^2},$$

to ensure a unit norm for $\widetilde{B}_n$, β and μ must be chosen according to the constraint that $|c_1|^2 + |c_2|^2 = 1$ which becomes

$$x^T M x = |1 - z_n^2|^2 \tag{1.42}$$

where

$$x = (\beta, \mu)^T, \qquad M = \begin{bmatrix} 1+|z_n|^2 & z_n + \overline{z_n} \\ z_n + \overline{z_n} & 1+|z_n|^2 \end{bmatrix}.$$

Now, suppose we make two pairs of choices: $x = (\beta, \mu)^T$ giving a basis function $\widetilde{B}_n$ and $y = (\beta', \mu')^T$ giving another basis function $\widetilde{B}_{n+1}$. These two choices correspond to two pairs of complex numbers $\{c_1, c_2\}$ and $\{c_3, c_4\}$. The requirement $c_1\overline{c_3} + c_2\overline{c_4} = 0$ ensuring orthogonality of $\widetilde{B}_n$ and $\widetilde{B}_{n+1}$ can be expressed as

$$x^T M y = 0. \tag{1.43}$$

All solutions to (1.42) are given by

$$x = \frac{1}{\sqrt{2}} \begin{bmatrix} |1-z_n|\cos\theta + |1+z_n|\sin\theta \\ |1-z_n|\cos\theta - |1+z_n|\sin\theta \end{bmatrix}, \qquad 0 \le \theta < 2\pi.$$

Then for a fixed θ, a unique y that satisfies (1.42) and (1.43) is found by substituting $\theta + \pi/2$ above:

$$y = -\frac{1}{\sqrt{2}} \begin{bmatrix} |1-z_n|\sin\theta - |1+z_n|\cos\theta \\ |1-z_n|\sin\theta + |1+z_n|\cos\theta \end{bmatrix}.$$

Let $\theta = 0$. Then the basis functions $\widetilde{B}_n$ and $\widetilde{B}_{n+1}$ are found as

$$\begin{aligned}\widetilde{B}_n(z) &= \frac{2^{-1/2}(1-|z_n|^2)^{1/2}\,|1-z_n|\,(z+1)}{1-(z_n+\overline{z_n})z+|z_n|^2z^2}\,\phi_n(z),\\ \widetilde{B}_{n+1}(z) &= \frac{2^{-1/2}(1-|z_n|^2)^{1/2}\,|1+z_n|\,(z-1)}{1-(z_n+\overline{z_n})z+|z_n|^2z^2}\,\phi_n(z).\end{aligned}$$

These real-valued impulse response basis vectors $\widetilde{B}_n$ and $\widetilde{B}_{n+1}$ are then used for modelling instead of B_n and B_{n+1}. If we require further basis functions with complex modes then we repeat the process in (1.41) by forming $\widetilde{B}_{n+2}$ and $\widetilde{B}_{n+3}$ from linear combinations of B_{n+2} and B_{n+3} and so on, and in this way arbitrary complex pole configurations may be accommodated.

For example, when $z_n = \overline{z_{n+1}} = \cdots = z_{n+2m} = \overline{z_{n+2m+1}}$, the above basis construction process yields for $j = 0, \cdots, m$

$$\widetilde{B}_{n+2j}(z) = \frac{a\,(z+1)}{1-bz+cz^2}\left(\frac{z^2-bz+c}{1-bz+cz^2}\right)^j \phi_n(z), \tag{1.44}$$

$$\widetilde{B}_{n+2j+1}(z) = \frac{a_1\,(z-1)}{1-bz+cz^2}\left(\frac{z^2-bz+c}{1-bz+cz^2}\right)^j \phi_n(z) \tag{1.45}$$

where $b = z_n + \overline{z_n},\ c = |z_n|^2$, and

$$a = \sqrt{\frac{(1-c)(1-b+c)}{2}}, \qquad a_1 = \sqrt{\frac{(1-c)(1+b+c)}{2}}. \tag{1.46}$$

With $n = 0$, this is the defining formula for the two-parameter Kautz functions.

The old basis functions B_n and B_{n+1} can be written in terms of the new basis functions as

$$\begin{bmatrix} B_n \\ B_{n+1}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \dfrac{1-z_n}{|1-z_n|} & -\dfrac{1+z_n}{|1+z_n|} \\ \dfrac{1-z_n}{|1-z_n|} & \dfrac{1+z_n}{|1+z_n|}\end{bmatrix}\begin{bmatrix} \widetilde{B}_n \\ \widetilde{B}_{n+1}\end{bmatrix}.$$

From this, we derive the following identity

$$\langle f, B_n\rangle B_n + \langle f, B_{n+1}\rangle B_{n+1} = \langle f, \widetilde{B}_n\rangle \widetilde{B}_n + \langle f, \widetilde{B}_{n+1}\rangle \widetilde{B}_{n+1}. \tag{1.47}$$

Having shown how to construct new basis functions with real-valued impulse responses from the basis functions $B_n, n = 0, 1, \cdots$, we will next study the same problem for the basis functions in (1.2). For the new basis functions $\widetilde{B}_{-n}$ and $\widetilde{B}_{-n-1}$, we seek a linear transformation of the old basis functions B_{-n} and B_{-n-1} expressed as

$$\begin{bmatrix} \widetilde{B}_{-n}(z) \\ \widetilde{B}_{-n-1}(z)\end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4\end{bmatrix}\begin{bmatrix} B_{-n}(z) \\ B_{-n-1}(z)\end{bmatrix}. \tag{1.48}$$

The substitutions $\overline{x_j} \mapsto z_{j-1},\ \forall j$ and $z \mapsto z^{-1}$ transform this problem to the previously considered case. Thus when $x_n = \overline{x_{n+1}} = \cdots = x_{n+2m} = \overline{x_{n+2m+1}}$, we have for $j = 0, \cdots, m$

$$\begin{aligned}\widetilde{B}_{-n-2j}(z) &= \frac{a'\,(1+z)}{z^2-b'z+c'}\left(\frac{1-b'z+c'z^2}{z^2-b'z+c'}\right)^j \phi'_n(z),\\ \widetilde{B}_{-n-2j-1}(z) &= \frac{a'_1\,(1-z)}{z^2-b'z+c'}\left(\frac{1-b'z+c'z^2}{z^2-b'z+c'}\right)^j \phi'_n(z)\end{aligned}$$

where $b' = x_n + \overline{x_n}$, $c' = |x_n|^2$ and a', a_1' are computed from the formulae in (1.46) with b' and c'. Furthermore

$$\begin{bmatrix} B_{-n} \\ B_{-n-1} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1-\overline{x_n}}{|1-x_n|} & -\frac{1+\overline{x_n}}{|1+x_n|} \\ \frac{1-\overline{x_n}}{|1-x_n|} & \frac{1+\overline{x_n}}{|1+x_n|} \end{bmatrix} \begin{bmatrix} \tilde{B}_{-n} \\ \tilde{B}_{-n-1} \end{bmatrix}$$

which leads to

$$\langle f, B_{-n}\rangle B_{-n} + \langle f, B_{-n-1}\rangle B_{-n-1} = \langle f, \tilde{B}_{-n}\rangle \tilde{B}_{-n} + \langle f, \tilde{B}_{-n-1}\rangle \tilde{B}_{-n-1}. \qquad (1.49)$$

The unitary equivalance of the bases $\{B_k\}$ and $\{\tilde{B}_k\}$ shows that the latter is complete in $L_p(\mathbf{T})$ $(1 < p < \infty)$ and $C(\mathbf{T})$ if the conditions in (1.3) hold. Moreover from (1.47) and (1.49),

$$\mathcal{S}_n f = \sum_{k=-n}^{n} \langle f, \tilde{B}_k\rangle \tilde{B}_k = \tilde{\mathcal{S}}_n f$$

whenever the sequence $\{z_0, z_1, x_1, \cdots, z_n, x_n\}$ contains complex conjugates as well. In this case, if f has a real-valued impulse response, then both $\mathcal{S}_n f$ and $\tilde{\mathcal{S}}_n f$ will have real-valued impulse responses. This identity shows also that approximation properties of $\mathcal{S}_n f$ and $\tilde{\mathcal{S}}_n f$ are identical.

1.6 Example

In this section, we use a simulation example to illustrate the use of the basis functions defined by (1.1) for modelling. We consider the identification of a fifth order system with poles (in the usual stability notion) $0.95 \pm 0.20i$, $0.85 \pm 0.10i$, 0.55 and zeros $0.96 \pm 0.28i$, $0.96 \pm 0.17i$. The transfer function of the system is normalized so that its H_∞ norm satisfies $\|G\|_\infty = 1$. This system was studied in [10] to illustrate the use of the generalized orthonormal basis functions for the time–domain identification.

We assume $N = 500$ frequency response measurements

$$E_k = G(e^{i\omega_k}) + \eta_k, \qquad k = 1, \ldots, N \qquad (1.50)$$

are available where ω_k are equally spaced on the interval $[0, 3]$ and the disturbances η_k are bounded random variables as

$$\eta_k = 0.1 e^{i\alpha_k}$$

where α_k are independent and uniformly distributed random variables in the interval $[0, 2\pi]$. Note that by this choice of frequencies, frequency response are not on a uniform grid of frequencies.

The basis functions in (1.1) were chosen with $z_0 = 0$ and

$$z_k = \begin{cases} 0.2, & k \text{ odd} \\ 0.9, & k \text{ even.} \end{cases}$$

This simple choice represents both slow and fast dynamics in the model structure via to the Laguerre functions. We will estimate G from the data (1.50) by two algorithms.

In the first algorithm, a high-order model is computed from the data (1.50) by the simple least-squares method as

$$\widetilde{G}_N(z) = \sum_{k=0}^{100} [\Phi^\dagger E]_k B_k(z) \tag{1.51}$$

where $\Phi^\dagger$ is the Moore-Penrose pseudoinverse of Φ defined by

$$\Phi^\dagger = \left(\Phi^* \Phi\right)^{-1} \Phi^*$$

and

$$\Phi(\omega) = \begin{bmatrix} 1 & \cdots & B_{100}(e^{i\omega_1}) \\ \vdots & \ddots & \vdots \\ 1 & \cdots & B_{100}(e^{i\omega_N}) \end{bmatrix}. \tag{1.52}$$

The estimated linear–in parameters model was reduced to a $5th$ order final model by using the subspace-based identification algorithm in [21] for model reduction purpose. The input to the algorithm in [21] were 2048 equally spaced frequency response data on $[0, 2\pi]$. Note that this amounts to evaluating Φ on a uniform grid of 2048 frequencies for which fast algorithms are known to exist. The size of the Hankel matrix in the subspace algorithm was chosen 128 by 128. The returned models by this algorithm are almost balanced and they converge to balanced truncations of the approximated system as the number of the supplied data tends to infinity. The step prior to forming a Hankel matrix was 2048-point inverse fast Fourier transform.

In Figure 1.1, the magnitudes of E, $\widetilde{G}_N(e^{i\omega})$, the final model transfer function denoted by $\widehat{G}_N(e^{i\omega})$, and the measured errors $\widetilde{G}_N(e^{i\omega}) - E$, $\widehat{G}_N(e^{i\omega}) - E$ are plotted. The poles of $\widehat{G}_N$ are $0.95 \pm 0.19i$, $0.85 \pm 0.11i$, 0.54 and the four significant zeros are $0.97 \pm 0.17i$, $0.96 \pm 0.28i$. They all agree well with the system poles and zeros.

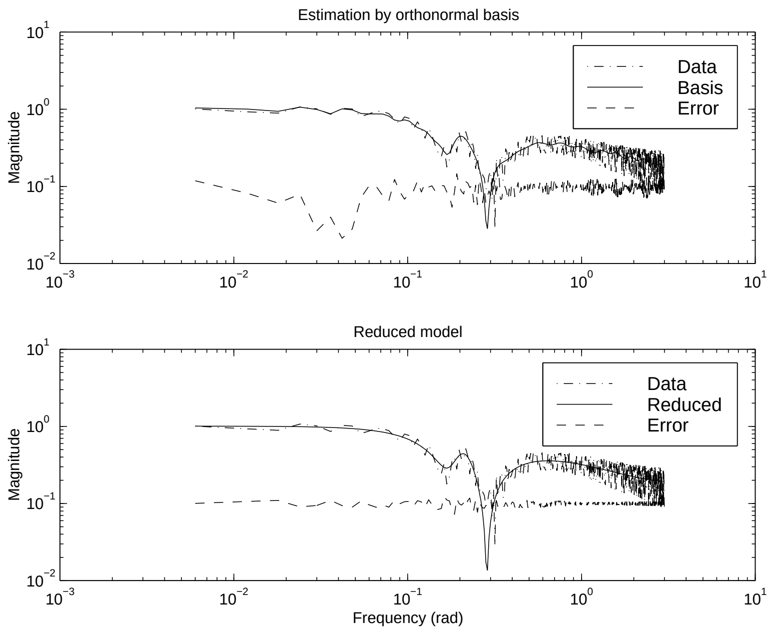

Figure 1.1: The magnitude plots of E, $\widetilde{G}_N(e^{i\omega})$, $\widetilde{G}_N(e^{i\omega}) - E$ (on the top) and E, $\widehat{G}_N(e^{i\omega})$, $\widehat{G}_N(e^{i\omega}) - E$ (on the bottom) using the linear estimate in (1.51).

Next we will compare this algorithm with the minimax algorithm in [3]. In the minimax algorithm, the coefficient vector $\hat{\Lambda} \in \mathbf{R}^{101}$ in the linearly parameterized model

$$\widehat{G}_N(z) = \sum_{k=0}^{101} \widehat{\Lambda}_k \, B_k(z) \tag{1.53}$$

is obtained by solving the following min–max problem

$$\widehat{\Lambda} = \arg \min_{\Lambda \in \mathbf{R}^{101}} \left\| \begin{bmatrix} \Phi_R \\ \Phi_I \end{bmatrix} \Lambda - \begin{bmatrix} E_R \\ E_I \end{bmatrix} \right\|_\infty \tag{1.54}$$

where E_R and E_I are respectively the real and imaginary parts of E in (1.50) and Φ_R and Φ_I are the real and imaginary parts of Φ.

The min–max solution in (1.54) is obtained from the following linear programming problem:

$$\min_{\mu} \, [\mathrm{O} \ 1] \begin{bmatrix} \Lambda \\ \mu \end{bmatrix}$$

subject to

$$\begin{bmatrix} \Phi_R & -\mathrm{J} \\ \Phi_I & -\mathrm{J} \\ -\Phi_R & -\mathrm{J} \\ -\Phi_I & -\mathrm{J} \end{bmatrix} \begin{bmatrix} \Lambda \\ \mu \end{bmatrix} \leq \begin{bmatrix} E_R \\ E_I \\ -E_R \\ -E_I \end{bmatrix}$$

where $\mathrm{O} \in \mathbf{R}^{1\times 101}$ and $\mathrm{J} \in \mathbf{R}^{N\times 1}$ are respectively row and column vectors of zeros and ones. This program is implemented by the **lp** command in the MATLAB's Optimization Toolbox.

In Figure 1.2, the simulation results are plotted for the minimax algorithm. We followed the same model reduction procedure as in the previous algorithm. The poles of the final model are $0.95 \pm 0.20i$, $0.87 \pm 0.10i$, 0.54 and the four significant zeros are $0.96 \pm 0.17i$, $0.96 \pm 0.28i$. They are in very good agreement with the system poles and zeros. This increase in accuracy was offset by the fact that computing (1.53) took about two orders of magnitude more time than needed to compute (1.51).

1.7 Summary

In this chapter completeness and approximation properties of a general class of fixed pole rational orthonormal basis functions in the $L_p(\mathbf{T})$ $(1 < p < \infty)$ and $C(\mathbf{T})$ spaces were studied and a fairly complete analysis of the convergence properties of the Fourier series formed by the orthonormal basis functions was carried out for the case $(1 < p < \infty)$.

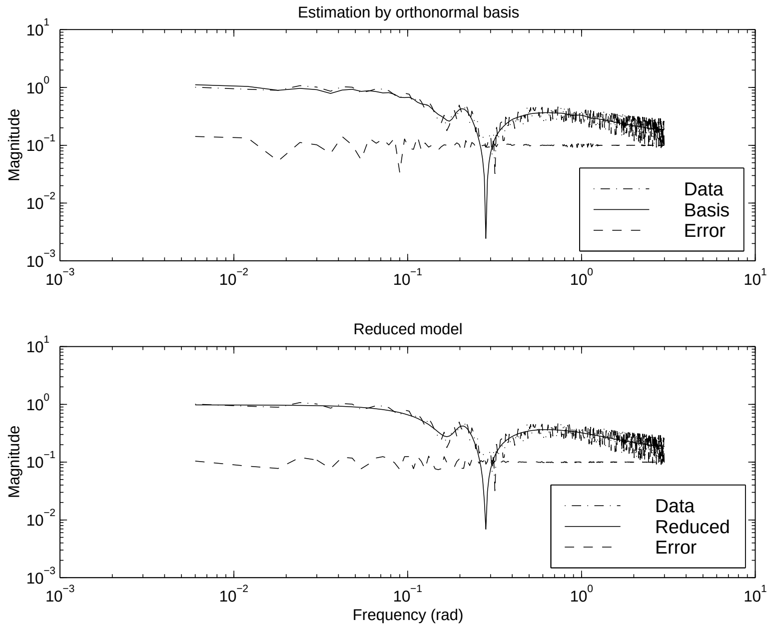

Figure 1.2: The magnitude plots of E, $\widetilde{G}_N(e^{i\omega})$, $\widetilde{G}_N(e^{i\omega}) - E$ (on the top) and E, $\widehat{G}_N(e^{i\omega})$, $\widehat{G}_N(e^{i\omega}) - E$ (on the bottom) using the min–max estimate in (1.53).

Chapter 2

On the uniform approximation of discrete–time systems by generalized Fourier Series

In this chapter, model sets for linear time-invariant discrete-time systems spanned by fixed orthonormal bases are studied. It is shown that the Fourier series formed by bounded orthonormal basis functions converges uniformly in the space of Dini-Lipschitz continuous functions.

2.1 Introduction

In this chapter, we will assume that the basis defined by (1.1) and (1.2) is uniformly bounded, i.e. it satisfies

$$\sup_n \{|z_n|, |x_n|\} = r < 1. \tag{2.1}$$

Since the completeness conditions

$$\sum_{k=0}^{\infty}(1-|z_k|) = \infty, \qquad \sum_{k=1}^{\infty}(1-|x_k|) = \infty$$

are obviously satisfied by the basis functions in (1.1) and (1.2) subject to (2.1), they are complete in $L_p(\mathbf{T})$ $(1 \leq p < \infty)$, the Lebesque spaces on $\mathbf{T}$, and $C(\mathbf{T})$ [2]. The partial sums of the generalized Fourier series of an integrable function f are defined by

$$\mathcal{S}_{n,m} f(z) = \sum_{k=-m}^{n} \langle f, B_k \rangle B_k(z). \tag{2.2}$$

We will study approximation of functions from a particular subset of $C(\mathbf{T})$ by the sums in (2.2) in the supremum norm

$$\|f\|_\infty = \sup_\theta \left| f(e^\theta) \right|.$$

When f is a continuous function on $\mathbf{T}$, we write

$$\omega_f(\delta) = \sup_{|x-y|\leq\delta} |f(x) - f(y)|$$

for the *modulus of continuity* of f. A function f is said *Dini-Lipschitz continuous* if

$$\omega_f(\delta)\,\ln(1/\delta) \to 0 \qquad (\delta \to 0).$$

For the trigonometric basis functions $B_k = z^k$, $k = 0, \pm 1, \pm 2, \cdots$, it is well known that $\mathcal{S}_{n,m} f \to f$ uniformly as $n, m \to \infty$ if f is Dini-Lipschitz continuous. The main result of this chapter is to establish an analogous result for the basis functions defined by (1.1) and (1.2) as follows.

Theorem 2.1.1 *Suppose $m = O(n^\gamma)$ for some $0 < \gamma$. Let $\mathcal{S}_{n,m} f$ be as in (2.2). Assume that the orthonormal functions in (1.1) and (1.2) are uniformly bounded. If f is Dini-Lipschitz continuous, then*

$$\|\mathcal{S}_{n,m} f - f\|_\infty \to 0 \qquad (n \to \infty).$$

In the course of proving this theorem, we show that $\|\mathcal{S}_{n,m}\| = O\left(\ln[n+m]\right)$. This implies that the orthonormal functions in (1.1) and (1.2) can not form a basis for the space $C(\mathbf{T})$ if they are uniformly bounded. The above results hold also for $A(\mathbf{D})$ with the obvious modification $m = 0$.

2.2 Proof of the Theorem

Note that the partial sums in (2.2) can be written as

$$\mathcal{S}_{n,m} f(e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{iy})\, L_{n,m}(y;\theta)\, \mathrm{d}y$$

where $L_{n,m}(y;\theta)$ is the so-called Dirichlet kernel defined by

$$L_{n,m}(y;\theta) = \sum_{k=-m}^{n} \overline{B_k(e^{iy})}\, B_k(e^{i\theta}).$$

Hence

$$\|\mathcal{S}_{n,m}\| = \sup_\theta \|L_{n,m}(\cdot\,;\theta)\|_1.$$

The following lemma will be instrumental in proving the theorem.

Lemma 2.2.1 *Let $e^{i\theta} - z_k = r_k(\theta)\, e^{i a_k(\theta)}$. Then*

$$a_k(\theta) = a_k(s) + \frac{\theta - s}{2} + \frac{1}{2} \int_s^\theta \left|B_k(e^{iy})\right|^2 \mathrm{d}y.$$

Proof. Choose $[0, 2\pi)$ branch for the variables θ and $a_k(\theta)$. For the notation, we refer to Fig. 1. Let β denote the unique zero of $a_k(\beta) = 0$. A variety of situations arises from different arrangements of the four points $s, \theta, 0, \beta \in [0, 2\pi)$. We will prove only one case here. The other cases follow from this. For the sake of completeness, they are given in § 2.4.

Case 1. $0 \notin [s, \theta]$ and $\beta \notin [s, \theta]$

Let us first show that if $t \neq 0$ and $a_k(t) \neq 0$,

$$2 \lim_{s \to t} \frac{a_k(t) - a_k(s)}{t - s} = 1 + |B_k(e^{it})|^2. \tag{2.3}$$

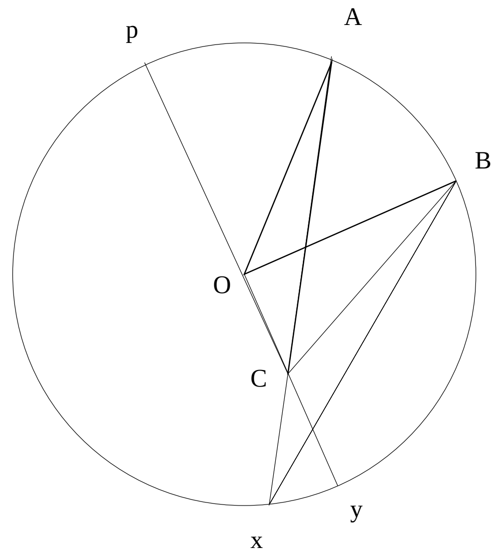

Figure 2.1: The points A, B, C, O denote respectively e^{it}, e^{is}, z_k, 0. The points p, y, and x lying on the unit circle are obtained by extending the chords OC and AC.

Fig. 1 depicts the case $s \nearrow t$. From elementary geometry notice the similarity of the two triangles $\overset{\triangle}{\mathrm{pCA}} \sim \overset{\triangle}{\mathrm{xCy}}$, which is due to the pairwise equality of the six angles

$$\widehat{\mathrm{pAx}} = \widehat{\mathrm{pyx}}, \qquad \widehat{\mathrm{ypA}} = \widehat{\mathrm{Axy}}, \qquad \widehat{\mathrm{pCA}} = \widehat{\mathrm{xCy}}\ .$$

Thus

$$\mathrm{Cx} \ = \ \frac{\mathrm{Cy}\cdot\mathrm{pC}}{\mathrm{AC}} \ = \ \frac{(1-|z_k|)\,(1+|z_k|)}{r_k(t)}.$$

A second fact from elementary geometry provides

$$\widehat{\mathrm{AxB}} = \frac{\widehat{\mathrm{AOB}}}{2} \ = \ \frac{t-s}{2}. \tag{2.4}$$

A third fact from trigonometry yields

$$\sin\,\widehat{\mathrm{CBx}} = \frac{\mathrm{Cx}}{\mathrm{CB}}\,\sin\,\widehat{\mathrm{CxB}} = \frac{(1-|z_k|^2)}{r_k(t)\,r_k(s)}\,\sin\left(\frac{t-s}{2}\right). \tag{2.5}$$

Finally

$$a_k(t) - a_k(s) \ = \widehat{\mathrm{ACB}} = \widehat{\mathrm{AxB}} \ + \ \widehat{\mathrm{CBx}}\ . \tag{2.6}$$

Hence from (2.4)–(2.6)

$$\begin{aligned} 2\lim_{s\nearrow t}\frac{a_k(t)-a_k(s)}{t-s} \ &= \ \lim_{s\nearrow t}\frac{\sin\left(a_k(t)-a_k(s)\right)}{\sin\left(\frac{t-s}{2}\right)} \\ &= \ \lim_{s\nearrow t}\frac{\sin\,\widehat{\mathrm{AxB}} + \sin\,\widehat{\mathrm{CBx}}}{\sin\left(\frac{t-s}{2}\right)} \end{aligned}$$

$$\begin{aligned} &= 1 + \frac{1-|z_k|^2}{r_k^2(t)} \\ &= 1 + |B_k(e^{it})|^2. \end{aligned}$$

When $s \searrow t$, reverse the roles of t and s and note the reflexity in (2.3). Hence a_k is differentiable on $[s,\theta]$ and integrating (2.3), we obtain the following formula for $a_k(\theta)$

$$a_k(\theta) = a_k(s) + \frac{\theta - s}{2} + \frac{1}{2}\int_s^\theta \left|B_k(e^{iy})\right|^2 \, \mathrm{d}y. \tag{2.7}$$

∎

Corollary 2.2.2

$$\begin{aligned} \phi_n(e^{i\theta})\,\overline{\phi_n(e^{is})} &= \exp\left(i\int_s^\theta \sum_{k=0}^{n-1} |B_k|^2 \,\mathrm{d}y.\right), \\ \phi'_m(e^{i\theta})\,\overline{\psi_m(e^{i\theta})} &= \exp\left(-i\int_s^\theta \sum_{k=-m}^{-1} |B_k|^2 \,\mathrm{d}y.\right). \end{aligned}$$

Proof. Write the numerator and denumerator factors of $\phi_{n+1}(e^{i\theta})$ and $\phi'_m(e^{i\theta})$ in polar forms as

$$e^{i\theta} - z_k = r_k(\theta)\, e^{ia_k(\theta)}, \qquad e^{i\theta} - x_k = r'_k(\theta)\, e^{ia_{-k}(\theta)}.$$

Since

$$1 - \overline{z_k}\, e^{i\theta} = r_k(\theta)\ \exp\left(i\left[\theta - a_k(\theta)\right]\right),$$

we have

$$\phi_n(e^{i\theta}) = \exp\left(i\left[-n\,\theta + 2\sum_{k=0}^{n-1} a_k(\theta)\right]\right)$$

and

$$\phi'_m(e^{i\theta}) = \exp\left(i\left[m\,\theta - 2\sum_{k=-m}^{-1} a_k(\theta)\right]\right).$$

Now the previous lemma completes the proof. ∎

A key consequence of this result is that it facilitates a simple formulation of the Dirichlet kernel.

Lemma 2.2.3

$$L_{n,m}(s;\theta) = e^{iA}\,\frac{\sin\left[\lambda_\theta(s)\right]}{\sin\left(\frac{\theta - s}{2}\right)}. \tag{2.8}$$

where

$$\begin{aligned} A &= \frac{1}{2}\int_s^\theta \left(\sum_{k=0}^{n} |B_k|^2 - 1 - \sum_{k=-m}^{-1} |B_k|^2\right)\,\mathrm{d}y, \\ \lambda_\theta(s) &= \frac{1}{2}\int_s^\theta \sum_{k=-m}^{n} |B_k|^2 \,\mathrm{d}y. \end{aligned} \tag{2.9}$$

Proof. This follows from the corollary, (1.5)–(1.6), and $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$. ∎

Now we are ready to prove the main result of this chapter. From Lemma 2.2.3 we derive bounds for $\|\mathcal{S}_{n,m}\|$. Let $N = n + m + 1$ and a given positive number x, let $\lfloor x \rfloor$ denote the nearest integer rounding x down.

Lemma 2.2.4 *Let $\mathcal{S}_{n,m}f$ be as in (2.2). Suppose that the basis defined by (1.1) and (1.2) satisfies (2.1). Then*

$$\begin{aligned} \frac{1}{4\pi}(1-r)^2 \ln\left(\lfloor (1-r)N \rfloor\right) \le \|L_{n,m}(\cdot\,;\theta)\|_1 \\ \|L_{n,m}(\cdot\,;\theta)\|_1 \le \frac{\pi}{(1-r)^2} + \ln\left[\frac{2N}{1-r}\right]. \end{aligned} \tag{2.10}$$

Proof. We start with the lower bound. For each fixed θ, $\lambda_\theta(\theta - x)$ defined by (2.9) is a strictly increasing function of x. Let $x_k \in [-\pi, \pi]$, $k = -M, \cdots, N$ denote the roots of the equations $\lambda_\theta(\theta - x_k) = \pi k/2$. Then after a change of variables $\theta - s = x$ and by the fact that $|\sin(x)| \le |x|$ for all x, we have

$$\begin{aligned} \|L_{n,m}(\cdot\,;\theta\|_1 &\ge \frac{1}{2\pi}\sum_{k=-M}^{N-1}\int_{x_k}^{x_{k+1}} \left|\frac{\sin\left[\lambda_\theta(\theta - x)\right]}{\sin\left(\frac{x}{2}\right)}\right| \mathrm{d}x \\ &\ge \frac{1}{\pi}\sum_{k=-M}^{N-1} \frac{\int_0^{x_{k+1}-x_k} \left|\sin\left[\lambda_\theta\left(\zeta_k(x)\right)\right]\right| \,\mathrm{d}x}{\max\left\{|x_k|\,, |x_{k+1}|\right\}}. \end{aligned} \tag{2.11}$$

where $\zeta_k(x) = \theta - x_k - x$. From the sin expansion $\sin(a+b) = \sin a \cos b + \cos a \sin b$, the last integrand above can be written as

$$\begin{aligned} \sin\left[\lambda_\theta\left(\zeta_k(x)\right)\right] &= \sin\left[\lambda_{\beta_k}\left(\zeta_k(x)\right)\right]\cos\left[\lambda_\theta\left(\beta_k\right)\right] + \cos\left[\lambda_{\beta_k}\left(\zeta_k(x)\right)\right]\sin\left[\lambda_\theta\left(\beta_k\right)\right] \\ &= \begin{cases} (-1)^{k/2}\sin\left[\lambda_{\beta_k}\left(\zeta_k(x)\right)\right], & k \text{ even} \\ (-1)^{(k-1)/2}\cos\left[\lambda_{\beta_k}\left(\zeta_k(x)\right)\right], & k \text{ odd} \end{cases} \end{aligned}$$

where $\beta_k = \theta - x_k$. Since $\lambda_\theta(\theta - x)$ is increasing on $[-\pi, \pi]$, we have $|\lambda_{\beta_k}(\zeta_k(x))| \le \pi/2$ for all $x \in [0, x_{k+1} - x_k]$. Thus from $\sin(x) \ge 2x/\pi$ valid for all $|x| \le \pi/2$,

$$\sin\left[\lambda_{\beta_k}\left(\zeta_k(x)\right)\right] \ge \frac{2}{\pi}\lambda_{\beta_k}\left(\zeta_k(x)\right) = \frac{1}{\pi}\int_0^x \sum_{k=-m}^{n}\left|B_k(e^{i(\zeta_k(x)-y)})\right|^2 \mathrm{d}y \ge \frac{R_2}{\pi}x$$

where

$$R_2 = \min_\theta \sum_{k=-m}^{n}\left|B_k(e^{i\theta})\right|^2.$$

Hence if k is even integer,

$$\int_0^{x_{k+1}-x_k} \left|\sin\left[\lambda_\theta\left(\zeta_k(x)\right)\right]\right| \,\mathrm{d}x \ge \frac{R_2}{2\pi}\left(x_{k+1} - x_k\right)^2. \tag{2.12}$$

Next from

$$\frac{\pi}{2} = \lambda_\theta(\beta_{k+1}) - \lambda_\theta(\beta_k) = \frac{1}{2}\int_{\beta_{k+1}}^{\beta_k}\sum_{k=-m}^{n}\left|B_k(e^{iy})\right|^2 \mathrm{d}y \ge \frac{R_2}{2}\left(x_{k+1} - x_k\right), \tag{2.13}$$

we have

$$x_{k+1} - x_k \le \frac{\pi}{R_2}. \tag{2.14}$$

Note also from (2.13) that

$$x_{k+1} - x_k \ge \frac{\pi}{R_1} \tag{2.15}$$

where

$$R_1 = \max_{\theta} \sum_{k=-m}^{n} \left| B_k(e^{i\theta}) \right|^2.$$

Summing (2.14) over $k = 0, 1, \cdots, p-1$ or $k = -p, \cdots, -1$ and noting that $x_0 = 0$, we derive

$$|x_p| \le \frac{\pi}{R_2} |p|, \qquad p = 0, \pm 1, \pm 2, \cdots. \tag{2.16}$$

The last inequality implies that $x_N \le (\pi/R_2)N$ and $x_{-M} \ge -(\pi/R_2)M$. Recall that x_N and x_M were chosen so that $x_{N+1} > \pi$ and $x_{-(M+1)} < -\pi$. Thus $\pi < (\pi/R_2)(N+1)$ and $\pi < (\pi/R_2)(M+1)$. Therefore $N + 1 > R_2$ and $M + 1 > R_2$. It follows that $N, M \ge \lfloor R_2 \rfloor$.

Considering terms in (2.11) that have even indices, we have from (2.15)–(2.16) and (2.12)

$$\frac{1}{\pi} \sum_{\substack{k=-M \\ k:\text{ even}}}^{N-1} \frac{1}{\max\{|x_k|, |x_{k+1}|\}} \int_0^{x_{k+1}-x_k} |\sin[\lambda_\theta(\zeta_k(x))]| \, \mathrm{d}x$$

$$\ge \frac{R_2^2}{2\pi R_1^2} \left(\sum_{\substack{k=-M \\ k:\text{ even}}}^{-1} \frac{1}{|k|} + \sum_{\substack{k=0 \\ k:\text{ even}}}^{N-1} \frac{1}{k+1} \right) \ge \frac{R_2^2}{2\pi R_1^2} \sum_{k=1}^{\lfloor R_2 \rfloor - 1} \frac{1}{k}. \tag{2.17}$$

Now we consider terms in (2.11) that have odd indices. The graph of $\cos(x)$ has the property $\cos(x) \ge 1 - (2x/\pi)$, $x \in [0, \pi/2]$. Thus for all $x \in [0, x_{k+1} - x_k]$,

$$\cos[\lambda_{\beta_k}(\zeta_k(x))] \ge 1 - \frac{2}{\pi} \lambda_{\beta_k}(\zeta_k(x))$$
$$= 1 - \frac{1}{\pi} \int_0^x \sum_{k=-m}^{n} \left| B_k(e^{i(\zeta_k(x) - y)}) \right|^2 \mathrm{d}y \ge 1 - \frac{R_1}{\pi} x.$$

Hence if k is odd integer,

$$\int_0^{x_{k+1}-x_k} |\sin[\lambda_\theta(\zeta_k(x))]| \, \mathrm{d}x \ge \int_0^{\pi/R_1} \left(1 - \frac{R_1}{\pi} x \right) \mathrm{d}x = \frac{\pi}{2R_1}. \tag{2.18}$$

Thus from (2.16) and (2.18)

$$\frac{1}{\pi} \sum_{\substack{k=-M \\ k:\text{ odd}}}^{N-1} \frac{1}{\max\{|x_k|, |x_{k+1}|\}} \int_0^{x_{k+1}-x_k} |\sin[\lambda_\theta(\zeta_k(x))]| \, \mathrm{d}x$$

$$\ge \frac{R_2}{2\pi R_1} \left(\sum_{\substack{k=-M \\ k:\text{ odd}}}^{-1} \frac{1}{|k|} + \sum_{\substack{k=0 \\ k:\text{ odd}}}^{N-1} \frac{1}{k+1} \right) \ge \frac{R_2}{2\pi R_1} \sum_{k=1}^{\lfloor R_2 \rfloor - 1} \frac{1}{k}. \tag{2.19}$$

It follows from (2.11), (2.17), and (2.19)

$$\|L_{n,m}(\cdot\,;\theta)\|_1 \geq \frac{1}{2\pi}\,\frac{R_2}{R_1}\,\left(1+\frac{R_2}{R_1}\right)\sum_{k=1}^{\lfloor R_2\rfloor-1}\frac{1}{k}. \tag{2.20}$$

Lower and upper bounds on R_2 and R_1 are obtained from

$$\sum_{k=-m}^{n}\left|B_k(e^{ix})\right|^2 = \sum_{k=-m}^{n}\frac{1-|z_k|^2}{|1-\overline{z_k}\,e^{ix}|^2} \geq \sum_{k=-m}^{n}\frac{1-|z_k|^2}{(1+|z_k|)^2} \geq N(1-r) \tag{2.21}$$

and

$$\sum_{k=-m}^{n}\left|B_k(e^{ix})\right|^2 \leq \sum_{k=-m}^{n}\frac{1+|z_k|}{1-|z_k|} \leq N\frac{1+r}{1-r}. \tag{2.22}$$

Hence from (2.20)–(2.22), and the following inequality

$$\sum_{k=1}^{N}\frac{1}{k} \geq \int_1^{N+1}\frac{\mathrm{d}x}{x} = \ln(N+1)$$

we obtain the lower bound on $\|L_{n,m}(\cdot\,;\theta)\|_1$ as follows

$$\|L_{n,m}(\cdot\,;\theta)\|_1 \geq \frac{1}{4\pi}\,(1-r)^2\,\ln\left(\lfloor(1-r)N\rfloor\right).$$

The upper bound on $\|L_{n,m}(\cdot\,;\theta)\|_1$ is obtained as follows

$$\begin{aligned}
\|L_{n,m}(\cdot\,;\theta)|_1 = \frac{1}{2\pi}\int_{x_{-1}}^{x_1}\left|\frac{\sin\lambda_\theta(\theta-x)}{\sin\left(\frac{x}{2}\right)}\right|\mathrm{d}x + \frac{1}{2\pi}\int_{x\notin[x_{-1},x_1]}\left|\frac{\sin\left[\lambda_\theta(\theta-x)\right]}{\sin\left(\frac{x}{2}\right)}\right|\mathrm{d}x \\
\leq \frac{1}{2\pi}\int_{x_{-1}}^{x_1}\frac{\pi R_1}{2}\,\mathrm{d}x + \frac{1}{2}\int_{x\notin[x_{-1},x_1]}\frac{\pi}{|x|}\,\mathrm{d}x \\
= \frac{R_1}{4}\,(x_1-x_{-1}) + \frac{1}{2}\,(2\ln\pi - \ln(-x_{-1}) - \ln x_1) \\
\leq \frac{\pi R_1}{2R_2} + \frac{1}{2}\,[2\ln\pi - 2\ln(\pi/R_1)] \\
\leq \pi(1-r)^{-2} + \ln\left[2N/(1-r)\right]
\end{aligned}$$

where in deriving the second inequality from the bottom (2.14)–(2.15) have been used. ∎

From the lemma, it follows that the orthonormal functions in (1.1) and (1.2) can not also form a basis for the space $L_1(\mathbf{T})$ if they are uniformly bounded.

Now we complete the proof of the main result.

Proof. Let $X_{n,m}$ denote the linear space spanned by the basis functions B_k, $k=-m,\cdots,n$ and define

$$e_{n,m}(f) = \min_{g\in X_{n,m}}\|f-g\|_\infty. \tag{2.23}$$

Thus $e_{n,m}(f)$ is the best $L_\infty(\mathbf{T})$-norm approximation error of f by functions in $X_{n,m}$. Let $\tau_{n,m}$ be the minimizing solution in (2.23). Let $E_q(f)$ denote the best $L_\infty(\mathbf{T})$-norm approximation error of f by trigonometric polynomials $h(z)=\sum_{k=-q}^{q}c_k z^k$ and let h_q^* be the unique minimizer. Note that

$$\|h_q^*\|_\infty \leq E_q(f) + \|f\|_\infty \leq 2\,\|f\|_\infty.$$

Hence $\|c\|_\infty \le 2\,\|f\|_\infty$. Let $\mathbf{A}(r_1,r_2)$ denote the annulus $\{z : r_1 < |z| < r_2\}$. Since $h_q^* \in \mathbf{A}(0,\infty) \subset \mathbf{A}(1/2,2)$,

$$\sup_{z\in\mathbf{A}(1/2,2)} |h_q^*(z)| \le 2^{q+1}\,\|c\|_\infty. \tag{2.24}$$

Let $\widetilde{n} = \min\{m,n\}$. Fix q as

$$q = \left\lfloor (1-r)\,\frac{\widetilde{n}}{4} \right\rfloor. \tag{2.25}$$

Then from (1.17) and (2.24)–(2.25), we have for some absolute constant $C_1 > 0$

$$\begin{aligned}\|h_q^* - S_{n,m}h_q^*\|_\infty &\le 2^{q+3}\,\|f\|_\infty \exp\left[-\frac{1}{4}\,(1-r)(n+1)\right] + 2^{q+2}\,\|f\|_\infty \exp\left[-\frac{1}{4}\,(1-r)m\right] \\ &\le 2\,\|f\|_\infty \exp\left[-C_1\,(1-r)\,\widetilde{n}\right]. \end{aligned}\tag{2.26}$$

Since $\mathcal{S}_{n,m}h_q^* \in X_{n,m}$, an application of the triangle inequality yields

$$e_{n,m}(f) \le E_q(f) + \|h_q^* - S_{n,m}h_q^*\|_\infty. \tag{2.27}$$

The first term on the right hand side of the above inequality is bounded from a theorem of Jackson [8, p. 144] as

$$E_q(f) \le \omega_f\left(\frac{\pi}{q+1}\right).$$

Hence if $m = O(n^\gamma)$, then for some absolute constant $C_2 > 0$

$$E_q(f)\,\ln N \le C_2\,\omega_f\left(\frac{\pi}{q+1}\right)\ln q$$

and thus

$$E_q(f)\,\ln N \to 0 \qquad (n\to\infty). \tag{2.28}$$

From (2.26) under the same condition $m = O(n^\gamma)$, we also have

$$\|h_q^* - S_{n,m}h_q^*\|_\infty\,\ln N \to 0 \qquad (n\to\infty). \tag{2.29}$$

It follows from (2.27)–(2.29)

$$e_{n,m}(f)\,\ln N \to 0 \qquad (n\to\infty).$$

Write f as $f = \tau_{n,m} + \chi$ with the approximation error $\chi = f - \tau_{n,m}$. Then $\mathcal{S}_{n,m}\tau_{n,m} = \tau_{n,m}$ since $\tau_{n,m} \in X_{n,m}$ and due to the linearity of the Fourier series $\mathcal{S}_{n,m}\chi = \mathcal{S}_{n,m}f - \tau_{n,m}$. Hence from Lemma 2.2.4 for some absolute constant $C_3 > 0$

$$\begin{aligned}\|f - \mathcal{S}_{n,m}f\|_\infty &= \|\chi - \mathcal{S}_{n,m}\chi\|_\infty \\ &\le e_{n,m}(f) + \|\mathcal{S}_{n,m}\|\,e_{n,m}(f) \\ &\le C_3\,e_{n,m}(f)\,\ln N \to 0 \qquad (n\to\infty).\end{aligned}$$

∎

2.3 Summary

This chapter has provided a preliminary study of the uniform convergence properties of certain general classes of rational orthonormal basis functions. The main result was to establish that the Fourier series formed by uniformly bounded orthonormal basis functions converged uniformly in the space of Dini-Lipscithz continuous functions.

2.4 Appendix

In this appendix, we study the other cases.

Case 2. $\beta \notin [s, 2\pi) \cup [0, \theta]$

Apply Case 1 to $[s, 2\pi - \epsilon)$ and $(\epsilon, \theta]$. Then from (2.7)

$$a_k(2\pi - \epsilon) = a_k(s) + \frac{2\pi - \epsilon - s}{2} + \frac{1}{2}\int_s^{2\pi-\epsilon} |B_k|^2 \,\mathrm{d}y,$$

$$a_k(\theta) = a_k(\epsilon) + \frac{\theta - \epsilon}{2} + \frac{1}{2}\int_\epsilon^\theta \left|B_k(e^{iy})\right|^2 \,\mathrm{d}y. \tag{2.30}$$

Since $a_k(0) \neq 0$, $a_k(y)$ is continuous at $y = 0$. Hence

$$\lim_{\epsilon\to 0} a_k(\epsilon) = a_k(0+) = a_k(0) = a_k(2\pi-) = \lim_{\epsilon\to 0} a_k(2\pi - \epsilon).$$

Thus

$$a_k(\theta) = \pi + a_k(s) + \frac{\theta - s}{2} + \frac{1}{2}\int_s^\theta \left|B_k(e^{iy})\right|^2 \,\mathrm{d}y - \frac{1}{2}\int_{2\pi}^0 \left|B_k(e^{iy})\right|^2 \,\mathrm{d}y$$

$$= 2\pi + a_k(s) + \frac{\theta - s}{2} + \frac{1}{2}\int_s^\theta \left|B_k(e^{iy})\right|^2 \,\mathrm{d}y \tag{2.31}$$

where the second equality follows from

$$\int_y^{y+2\pi} |B_k(e^{ix})|^2 \,\mathrm{d}x = 2\pi, \qquad \text{for all } y. \tag{2.32}$$

Case 3. $s \geq 0$ and $\beta \notin [s, \theta]$

If $s = 0$, consider only (2.30) in Case 2 which then yields (2.7) as $\epsilon \to 0$.

Case 4. $\beta \in [s, \theta]$ ($s \geq 0$)

1. $\beta \in (s, \theta)$: then $a_k(\beta-) = 2\pi$ and $a_k(\beta+) = 0$. Apply Case 3 and Case 1 to $[s, \beta)$ and $(\beta, \theta]$ to get from (2.7)

$$\begin{aligned} a_k(\beta-) &= a_k(s) + \frac{\beta - s}{2} + \frac{1}{2}\int_s^\beta \left|B_k(e^{iy})\right|^2 \,\mathrm{d}y, \\ a_k(\theta) &= a_k(\beta+) + \frac{\theta - \beta}{2} + \frac{1}{2}\int_\beta^\theta \left|B_k(e^{iy})\right|^2 \,\mathrm{d}y. \end{aligned}$$

Thus

$$a_k(\theta) = -2\pi + a_k(s) + \frac{\theta - s}{2} + \frac{1}{2}\int_s^\beta \left|B_k(e^{iy})\right|^2 \,\mathrm{d}y. \tag{2.33}$$

2. $\beta = s$: then $a_k(s) = a_k(\beta+) = 0$. Apply Case 1 to $(\beta, \theta]$ to get from (2.7)

$$\begin{aligned} a_k(\theta) &= a_k(\beta+) + \frac{\theta-\beta}{2} + \frac{1}{2}\int_\beta^\theta \left|B_k(e^{iy})\right|^2 \mathrm{d}y \\ &= a_k(s) + \frac{\theta-s}{2} + \frac{1}{2}\int_s^\theta \left|B_k(e^{iy})\right|^2 \mathrm{d}y. \end{aligned}$$

3. $\beta = \theta$: then $a_k(\theta) = 0$ and $a_k(\beta-) = 2\pi$. Apply Case 3 to $[s, \beta)$. Hence from (2.7)

$$a_k(\beta-) = a_k(s) + \frac{\beta-s}{2} + \frac{1}{2}\int_s^\beta \left|B_k(e^{iy})\right|^2 \mathrm{d}y$$

which is (2.33).

Case 5. $\beta \in [s, 2\pi) \cup [0, \theta]$

1. $\beta \in (s, 2\pi)$: then $a_k(\beta-) = 2\pi$ and $a_k(\beta+) = 0$. Apply Case 2 and Case 1 to $(\beta, 2\pi) \cup [0, \theta]$ and $[s, \beta)$ to get from (2.31) and (2.7)

$$\begin{aligned} a_k(\theta) &= 2\pi + a_k(\beta+) + \frac{\theta-\beta}{2} + \frac{1}{2}\int_\beta^\theta \left|B_k(e^{iy})\right|^2 \mathrm{d}y, \\ a_k(\beta-) &= a_k(s) + \frac{\beta-s}{2} + \frac{1}{2}\int_s^\beta \left|B_k(e^{iy})\right|^2 \mathrm{d}y. \end{aligned}$$

Hence summing the above equalities we get (2.7).

2. $\beta \in (0, \theta)$: then $a_k(\beta-) = 2\pi$ and $a_k(\beta+) = 0$. Consider $[s, 2\pi) \cup [0, \beta)$ and $(\beta, \theta]$. Thus the following equalities

$$\begin{aligned} a_k(\theta) &= a_k(\beta+) + \frac{\theta-\beta}{2} + \frac{1}{2}\int_\beta^\theta \left|B_k(e^{iy})\right|^2 \mathrm{d}y, \\ a_k(\beta-) &= 2\pi + a_k(s) + \frac{\beta-s}{2} + \frac{1}{2}\int_s^\beta \left|B_k(e^{iy})\right|^2 \mathrm{d}y \end{aligned}$$

yield again (2.7).

3. $\beta = 0$: then $a_k(2\pi-) = 2\pi$ and $a_k(0+) = 0$. Considering $[s, 2\pi)$ and $(0, \theta]$, we have

$$\begin{aligned} a_k(\theta) &= a_k(0+) + \frac{\theta-0}{2} + \frac{1}{2}\int_0^\theta \left|B_k(e^{iy})\right|^2 \mathrm{d}y, \\ a_k(2\pi-) &= a_k(s) + \frac{2\pi-s}{2} + \frac{1}{2}\int_s^{2\pi} \left|B_k(e^{iy})\right|^2 \mathrm{d}y \end{aligned}$$

which yield (2.7).

4. $\beta = s$: consider $(s, 2\pi) \cup [0, \theta]$. Then from (2.31)

$$a_k(\theta) = 2\pi + a_k(\beta+) + \frac{\theta-\beta}{2} + \frac{1}{2}\int_\beta^\theta \left|B_k(e^{iy})\right|^2 \mathrm{d}y$$

which is (2.31) since $a_k(s) = a_k(\beta+) = 0$.

5. $\beta = \theta$: consider $[s, 2\pi) \cup [0, \theta)$. Then $a_k(\theta) = 0$ and $a_k(\beta-) = 2\pi$ and from (2.31), we have

$$a_k(\beta-) = 2\pi + a_k(s) + \frac{\beta - s}{2} + \frac{1}{2}\int_s^\beta \left|B_k(e^{iy})\right|^2 \,\mathrm{d}y$$

which is (2.7).

The following equality defined for $s, \theta \in [0, 2\pi)$ unifies (2.7), (2.31), and (2.33)

$$a_k(\theta) = a_k(s) + \frac{\theta - s}{2} + \frac{1}{2}\int_s^\theta \left|B_k(e^{iy})\right|^2 \,\mathrm{d}y, \qquad \text{mod } (2\pi).$$

Due to (2.32) this equalitity is invariant to translations of θ and s by multiples of 2π. This completes the proof.

Chapter 3

Synthesis of Complete Orthonormal Bases with Prescribed Asymptotic Order

In this chapter, a method to construct complete orthonormal model sets for continuous–time systems, which have a prescribed asymptotic order, is presented. Two examples that illustrate the application of the method are provided.

3.1 Introduction

In a sequel of papers [5, 4], the basis functions defined by a choice of numbers a_n in the open right half-plane as

$$\begin{aligned} B_n(s) &= \frac{\sqrt{2\mathrm{Re}\{a_n\}}}{s+a_n}\,\psi_{n-1}(s), \qquad n=1,2,\cdots; \\ \psi_n(s) &= \prod_{k=1}^{n}\frac{s-\overline{a_k}}{s+a_k}, \qquad n=1,2,\cdots; \quad \psi_0(s)=1 \end{aligned} \tag{3.1}$$

which are orthonormal with respect to the inner product $\langle f, g\rangle = (1/2\pi)\int_{\infty}^{\infty} f\,\overline{g}$ have been considered in detail. The well-known *Laguerre* and *Kautz* models, and the more recently introduced general orthonormal basis functions are the special cases of the basis functions (3.1) where all the basis poles are restricted to a finite set.

In [5, 4], completeness of the basis defined by (3.1) in H_p, the Hardy spaces of functions $g(s)$ which are analytic on the open right half-plane and such that $\|g\|_p \triangleq [(1/2\pi)\int_{-\infty}^{\infty}|g(i\omega)|^p\,\mathrm{d}\omega]^{1/p} < \infty$, was studied. Completeness means that linear combinations of the basis functions (3.1) are capable of arbitrarily good approximation in the spaces H_p.

A function f is said to have *asymptotic order* m if $f(s) = O(s^{-m})$ as $s \to \infty$. The bases defined by (3.1) have asymptotic order 1. As illustrated in [22, 32], there are significant advantages in being able to construct bases of asymptotic order greater than one. In [22], a unified method for the construction of orthonormal sets with arbitrary asymptotic order was proposed. As pointed out in [32], orthonormal sets constructed by this method are not necessarily complete in H_2. Most recently in [4], a method to construct an infinite set of orthonormal bases, each of which have arbitrary asymptotic order, and whose linear span is everywhere dense in H_p for all $1 \le p < \infty$ was presented.

This chapter is continuation of the work initiated in [4]. First, a different version of the result in [4] that allows easier implementation of the method is presented. Next, it is shown in a special case that this method is equivalent to calculating certain Toeplitz determinants. The latter problem is reduced to finding solution of a linear finite–difference equation. Two simple examples illustrating application of the method are given. In the first example, this method is shown to produce the *generalized Laguerre functions* [29, 9, 20], which are known to form a complete set in H_2 and have asymptotic order two. In the second example, a restriction in the first example is removed. Extensions to more complicated situations are straightforward and they are briefly described.

3.2 Bases with Prescribed Asymptotic Order

In this section, we derive model sets that are complete in H_p for all $1 \leq p < \infty$ and for which the basis functions $f_k(s)$ defining the sets each have a prescribed asymptotic order. The following result provides a recipe to construct bases of arbitrary asymptotic order and with arbitrary pole locations.

Theorem 3.2.1 *Let $P(s)$ be an mth order real coefficient polynomial with roots in the open left half-plane. Consider the basis functions (3.1). Let $f_k(s) \triangleq B_{k+1}(s)/P(s)$, $k \geq 0$. Then $\{f_k\}_{k\geq 0}$ is complete in H_p for all $1 \leq p < \infty$ if $\sum_{n=1}^{\infty}(1 - |a_n|) = \infty$ and $f_k(s) = O(s^{-(m+1)})$ as $s \to \infty$.*

Proof. Let $f \in H_p$ be a given function. Let $\epsilon > 0$ be also a given number. Approximate f by a function g analytic on the open right half-plane that has the properties $\|f - g\|_p < \epsilon$ and

$$\lim_{|s|\to\infty} |s|^{m+2}\, |g(s)| = 0, \qquad \text{Re}\{s\} > 0.$$

This is possible since such functions form a dense subset of H_p for all $0 < p < \infty$ (for a proof, see Corollary 3.3 in Chapter II of Garnett [13]). Take $h(s) = P(s)g(s)$. Then for all $1 \leq p \leq \infty$,

$$h(s) = (s+1)^{m+2}\, g(s)\, \frac{P(s)}{(s+1)^{m+2}} \in H_p.$$

Since sp $\{B_1, B_2, \cdots\}$ is complete in H_p for all $1 \leq p < \infty$ (see Lemma 4 in [5] and Theorem 3.1 in [4]), there exists a function $F \in$ sp$\{B_1, B_2, \cdots\}$ such that

$$\|h - F\|_p < \epsilon$$

which implies that

$$\left\|g - \frac{F}{P}\right\|_p \leq \frac{\|h - F\|_p}{\min_\omega |P(i\omega)|} < \frac{\epsilon}{\min_\omega |P(i\omega)|}.$$

Therefore

$$\left\|f - \frac{F}{P}\right\|_p < \epsilon + \frac{\epsilon}{\min_\omega |P(i\omega)|}.$$

Since f and ϵ are arbitrary, this establishes the claim. ∎

The proof of Theorem 3.2.1 is adapted from [4]. This new result is stronger than Theorem 4.1 in [4] since the required order for $P(s)$ is one less. The next job is to orthonormalize the set $\{f_k\}$. In [4], the Gram–Schmidt orthonormalization procedure was applied to the set $\{f_k\}$ without obtaining explicit results. As will be shown shortly, in a special case, the orthonormalization problem is equivalent to computing certain Toeplitz determinants. This in turn is equivalent to finding solution of a linear finite–difference equation. An increase in the order of $P(s)$ by one means an increase in the order of this equation by two.

Examination of the proof of Theorem 3.2.1 raveals that the conclusion of Theorem 3.2.1 is valid for every complete base in H_p $(1 \leq p < \infty)$. A set of basis functions with real–valued impulse responses that is complete in the real Hardy spaces H_p $(1 \leq p < \infty)$ can be extracted from the basis functions in (3.1). This process is described in [5]. Then Theorem 3.2.1 tells us that the basis functions $\{f_k\}$ will have all real–valued impulse responses. The real–valued impulse response basis functions are used in applications involving the modelling of physical systems.

Since $\{B_k\}$ is linearly independent, $\{f_k\}$ is also linearly independent. Thus the following determinants

$$D_n = \begin{vmatrix} \langle f_0, f_0\rangle & \langle f_1, f_0\rangle & \cdots & \langle f_n, f_0\rangle \\ \langle f_0, f_1\rangle & \langle f_1, f_1\rangle & \cdots & \langle f_n, f_1\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_0, f_n\rangle & \langle f_1, f_n\rangle & \cdots & \langle f_n, f_n\rangle \end{vmatrix}, \quad n \geq 0 \tag{3.2}$$

are all positive. It is fairly easy to show that the functions defined by $\phi_0(s) = D_0^{-1/2} f_0(s)$ and for $n \geq 1$,

$$\phi_n(s) = (D_{n-1}D_n)^{-1/2} \begin{vmatrix} & \langle f_n, f_0\rangle \\ D_{n-1} & \vdots \\ & \langle f_n, f_{n-1}\rangle \\ f_0(s) & f_n(s) \end{vmatrix} \tag{3.3}$$

are orthonormal. The coefficient of $f_n(s)$ in $\phi_n(s)$ is $(D_{n-1}/D_n)^{1/2}$. If this positivity is enforced in the orthonormalization, then $\phi_k,\ k \geq 0$ are the unique functions which orthonormalize the given functions $f_k,\ k \geq 0$. The inner products above are calculated from (3.1) as follows

$$\langle f_k, f_l\rangle = \frac{1}{\pi i} \oint_\Gamma \frac{[\mathrm{Re}\{a_{k+1}\}\mathrm{Re}\{a_{l+1}\}]^{1/2}\ [\psi_k(s)/\psi_l(s)]}{(s + a_{k+1})(s - \overline{a_{l+1}})P(s)P(-s)}\,\mathrm{d}s \tag{3.4}$$

where the closed path Γ consists of the imaginary axis and the infinity radius semicircle in the right half-plane centered at the origin and is traversed counter clockwise.

The equations (3.2)–(3.4) are greatly simplified by enforcing a periodicity condition on the basis poles

$$a_{k+jM} = a_k, \qquad k = 1, 2, \cdots, M; \ \ j = 1, 2, \cdots, \tag{3.5}$$

which corresponds to the choice of general orthonormal basis functions, and choosing the roots of $P(s)$ from a subset of basis poles. The condition (3.5) implies that

$$\langle f_{k+jM}, f_{l+jM}\rangle = \langle f_k, f_l\rangle, \qquad j = 0, 1, 2, \cdots$$

and thus D_n defined by (3.2) are determinants of Hermitian (block) Toeplitz matrices. Recall that elements of a Toeplitz matrix are constant along any stripe parallel to the main diagonal. The second assumption that the roots of $P(s)$ lie in the set $\{-a_k\}$ forces these matrices to be band matrices. Indeed, if k is sufficiently larger than l, the term $(s - \overline{a_{l+1}})P(-s)$ is cancelled by the term $\psi_k(s)/\psi_l(s)$ and the integrand in (3.4) becomes analytic on the open right half-plane; thus by the Cauchy's formula the right hand side of (3.4) vanishes.

In § 3.3, in order to simplify the computations and also to gain insight in the orthonormalization procedure we will consider the *Laguerre* functions defined by a fixed real parameter $a > 0$ as

$$B_k(s) = \frac{\sqrt{2a}}{s+a}\left(\frac{s-a}{s+a}\right)^{k-1}, \qquad k = 1, 2, \cdots \tag{3.6}$$

and take $P(s) = (2a^2)^{-1/2}(s+a)$. The factor $(2a^2)^{-1/2}$ is for normalization. This choice of the basis functions and $P(s)$ corresponds to $m = M = 1$. In § 3.4, we will remove the restriction on the pole location of $P(s)$. The case $M > 1$ with no restriction on the poles of $P(s)$ is not difficult but tedious; it requires some lengthy notation and substantial algebra.

3.3 Construction of Single Pole Orthonormal Bases with Asymptotic Order 2

When applied to the Laguerre functions (3.6), equation (3.4) with $P(s) = (2a^2)^{-1/2}(s+a)$ plugged in admits the following simple form

$$\langle f_k, f_l \rangle = -\frac{2a^3}{\pi i}\oint_\Gamma \frac{(s-a)^{k-l-2}}{(s+a)^{k-l+2}}\,\mathrm{d}s, \qquad k \geq l.$$

Hence $\langle f_k, f_l \rangle = 0$ if $|k - l| \geq 2$. Moreover from the following identity [14, Formula 2.148.3]

$$\int \frac{\mathrm{d}x}{(1+x^2)^n} = \frac{x}{(2n-2)(1+x^2)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{\mathrm{d}x}{(1+x^2)^{n-1}}, \tag{3.7}$$

we have

$$\langle f_0, f_0 \rangle = \frac{2a^3}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{(\omega^2+a^2)^2} = 1.$$

For the term $\langle f_1, f_0 \rangle$, using (3.7) twice for $n = 3$ and the following identity [14, Formula 2.147.2]

$$\int \frac{x^m\,\mathrm{d}x}{(1+x^2)^n} = -\frac{x^{m-1}}{(2n-m-1)(1+x^2)^{n-1}} + \frac{m-1}{2n-m-1}\int \frac{x^{m-2}\,\mathrm{d}x}{(1+x^2)^n},$$

we get

$$\langle f_1, f_0 \rangle = -\frac{2a^3}{\pi}\int_{-\infty}^{\infty} \frac{a^2-\omega^2}{(\omega^2+a^2)^3}\,\mathrm{d}\omega = -\frac{1}{2}.$$

Let T_n denote the matrix in (3.2) whose determinant is D_n. Then it has the following form

$$T_n = \begin{bmatrix} c_0 & c_1 & 0 & \cdots & 0 \\ c_1 & c_0 & c_1 & \cdots & 0 \\ 0 & c_1 & c_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_0 \end{bmatrix}, \quad n \geq 1 \tag{3.8}$$

where $c_0 = 1$ and $c_1 = -1/2$. The determinants D_n can be computed recursively from

$$D_n = D_{n-1} - \frac{1}{4}D_{n-2}, \qquad n = 2, 3, \cdots \tag{3.9}$$

with the initial conditions $D_0 = 1$ and $D_1 = 3/4$. If we set $D_{-1} = 1$, then this recursion is also valid for $n = 1$. The equation (3.9) has a multiple root at $z = 1/2$. Thus its solution is found as

$$D_n = \left(1 + \frac{n}{2}\right) 2^{-n}, \qquad n \geq 0. \tag{3.10}$$

Expanding the determinant in (3.3) with respect to the last row and taking into account the structure in (3.8), we derive the following expression for $\phi_n(s)$

$$\begin{aligned} \phi_n(s) &= \sum_{k=0}^{n} (-1)^{n+k} \frac{c_1^{n-k} D_{k-1}}{(D_{n-1} D_n)^{1/2}} f_k(s) \\ &= \left[\frac{8a^3}{(n+1)(n+2)}\right]^{1/2} \frac{1}{(s+a)^2} \sum_{k=0}^{n} (k+1) \left(\frac{s-a}{s+a}\right)^k, \quad n \geq 0 \end{aligned} \tag{3.11}$$

which simplifies to

$$\phi_{n-1}(s) = -\frac{1}{\sqrt{n^2+n}} \left\{ \frac{\sqrt{2a}\, n}{s+a} \left(\frac{s-a}{s+a}\right)^n - \frac{1}{\sqrt{2a}} \left[1 - \left(\frac{s-a}{s+a}\right)^n\right] \right\}, \quad n \geq 1 \tag{3.12}$$

where we used the identity [14, Formula 0.113]

$$\sum_{k=0}^{n} (a + bk)\, x^k = \frac{a - (a+nb)\, x^{n+1}}{1-x} + \frac{bx\,(1-x^n)}{(1-x)^2} \tag{3.13}$$

which is valid for all $x \neq 1$ and $n > 0$.

The functions $-\phi_{n-1}$, $n \geq 1$ are the *generalized Laguerre functions* considered in [29, 9, 20], which are known to form a complete set in H_2. Then Theorem 3.2.1 tells us that the set of generalized Laguerre functions is complete in the spaces H_p for all $1 \leq p < \infty$ and each function in this set has an asymptotic order 2.

It should be noted that generating orthonormal sets with prescribed asymptotic order is not difficult. As a matter of fact, consider the orthonormal basis functions (3.1) and let $\phi_n(s) = B_n^m / \|B_n^m\|_2$, $n \geq 1$ where m is an integer satisfying $m > 1$. Then by an application of the residue theorem, observe that $\{\phi_n\}_{n\geq 1}$ is orthonormal. However this set will not be complete. This implies that any finite collection of such sets generated by different values of m will not be complete. Then it follows that the method proposed by Mendel [22] does not necessarily produce complete orthonormal sets. On the other hand, our recipe guarantees completeness. To show incompleteness of the set $\{\phi_n\}_{n\geq 1}$ for each fixed $m > 1$, first by an application of the residue theorem note that B_1 is orthogonal to the functions B_n^m, $n \geq 2$ and $\langle B_1, B_1^m \rangle \neq 0$. Set

$$\zeta = \|B_1^m\|_2\, B_1 - \langle B_1, B_1^m \rangle\, \phi_1.$$

Then $\zeta \neq 0$ and it is orthogonal to the set $\{\phi_n\}_{n\geq 1}$.

3.4 Construction of Double Pole Orthonormal Bases with Asymptotic Order 2

In this section, we remove the restriction on the pole of $P(s)$ and choose

$$P(s) = \left[\frac{a}{b(a+b)^2}\right]^{1/2} (s+b)$$

where the constant factor has been introduced for normalization. Then the basis functions to be orthonormalized are

$$f_k(s) = \frac{(a+b)\sqrt{2b}}{(s+a)(s+b)} \left(\frac{s-a}{s+a}\right)^k, \qquad k \geq 0.$$

We start the orthonormalization process by computing inner products in (3.2). We have for $k > l$,

$$\begin{aligned} \langle f_k, f_l \rangle &= -\frac{1}{2\pi i} \oint_\Gamma \frac{2b(a+b)^2 (s-a)^{k-l-1}}{(s+a)^{k-l+1}(s+b)(s-b)} \mathrm{d}s \\ &= -2b(a+b)^2 \frac{(b-a)^{k-l-1}}{2b(b+a)^{k-l+1}} \\ &= -\left(\frac{b-a}{b+a}\right)^{k-l-1}, \end{aligned} \tag{3.14}$$

where the closed path Γ is as defined in (3.4) and the second equality follows from the residue theorem, and for $k = l$,

$$\begin{aligned} \langle f_k, f_k \rangle &= \frac{b(a+b)^2}{\pi} \int_\infty^\infty \frac{\mathrm{d}\omega}{(\omega^2+a^2)(\omega^2+b^2)} \\ &= \frac{a+b}{a}. \end{aligned} \tag{3.15}$$

Letting

$$\lambda \triangleq \frac{a+b}{a}, \qquad \mu \triangleq \frac{b-a}{b+a},$$

we obtain from (3.14) and (3.15), the following expression for T_n $(n \geq 1)$

$$T_n = \begin{bmatrix} \lambda & -1 & -\mu & \cdots & -\mu^{n-1} \\ -1 & \lambda & -1 & \cdots & -\mu^{n-2} \\ -\mu & -1 & \lambda & \cdots & -\mu^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mu^{n-1} & -\mu^{n-2} & -\mu^{n-3} & \cdots & \lambda \end{bmatrix}. \tag{3.16}$$

As opposed to (3.8), T_n is not a band matrix. However it does not pose much difficulty for the orthonormalization process. This is due to the special structure in (3.16). We will show that as in § 3.3, the determinants of T_n can be calculated by solving a second order finite–difference equation. Indeed, first substract μ times the second row of T_n from the first row of T_n. Then substract μ times the second column of the resulting matrix from the first column of it. These operations leave D_n invariant and the result is

$$\begin{aligned} D_n &= \begin{vmatrix} 2(\lambda-1) & 1-\lambda & 0 & \cdots & 0 \\ 1-\lambda & \lambda & -1 & \cdots & -\mu^{n-2} \\ 0 & -1 & \lambda & \cdots & -\mu^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\mu^{n-2} & -\mu^{n-3} & \cdots & \lambda \end{vmatrix} \\ &= 2(\lambda-1)D_{n-1} - (\lambda-1)^2 D_{n-2}, \qquad n \geq 2 \end{aligned} \tag{3.17}$$

with the initial conditions $D_0 = \lambda$ and $D_1 = \lambda^2 - 1$. The solution of this equation is found as

$$D_n = (\lambda + n)(\lambda - 1)^n, \qquad n \geq 0. \tag{3.18}$$

To construct orthonormal base, we proceed as in § 3.3. Let

$$Q_n \triangleq \begin{bmatrix} \lambda & -1 & -\mu & \cdots & -\mu^{n-1} \\ -1 & \lambda & -1 & \cdots & -\mu^{n-2} \\ -\mu & -1 & \lambda & \cdots & -\mu^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mu^{n-2} & -\mu^{n-3} & -\mu^{n-4} & \cdots & -1 \\ f_0 & f_1 & f_2 & \cdots & f_n \end{bmatrix}.$$

Let $Q_{n-1}^{(k)}$ denote the matrix obtained by striking out the row and column of Q_n containing $f_k(s)$. Let $Q_{n-2}^{(k)}$ denote the matrix obtained by striking out the first row and the first column of $Q_{n-1}^{(k)}$. Then $Q_{n-3}^{(k)}$ is obtained from $Q_{n-2}^{(k)}$ by striking out its first row and first column and so on. We define another sequence of matrices starting with $S_{n-1}^{(k)}$, which denotes the matrix obtained from Q_n by striking out the row and column of Q_n containing $f_0(s)$. The second matrix $S_{n-2}^{(k)}$ is obtained from $S_{n-1}^{(k)}$ by striking out its first row and first column. The matrices $S_{n-3}^{(k)}, S_{n-4}^{(k)}, \cdots$ are defined similarly.

First, substract μ times the second row of $Q_{n-1}^{(k)}$ from the first row of $Q_{n-1}^{(k)}$. Then substract μ times the second column of this matrix from the first row of it. The result for $3 \leq k < n$ is

$$\begin{aligned} |Q_{n-1}^{(k)}| &= \begin{vmatrix} 2(\lambda-1) & 1-\lambda & 0 & \cdots & 0 \\ 1-\lambda & \lambda & -1 & \cdots & -\mu^{n-2} \\ 0 & -1 & & & \\ \vdots & \vdots & & Q_{n-3}^{(k)} & \\ 0 & -\mu^{n-3} & & & \end{vmatrix} \\ &= 2(\lambda-1)|Q_{n-2}^{(k)}| - (\lambda-1)^2|Q_{n-3}^{(k)}|. \end{aligned}$$

Hence for $n - k + 2 \leq j < n$ and $3 \leq k < n$, we obtain the following finite–difference equation

$$|Q_j^{(k)}| = 2(\lambda-1)|Q_{j-1}^{(k)}| - (\lambda-1)^2|Q_{j-2}^{(k)}|.$$

The solution of this equation is found as

$$|Q_j^{(k)}| = (C_1 + jC_2)(\lambda-1)^j \tag{3.19}$$

where the constants C_1, C_2 are to be determined from

$$\begin{aligned} |Q_{n-k}^{(k)}| &= [C_1 + C_2(n-k)]\,(\lambda-1)^{n-k}, \\ |Q_{n-k+1}^{(k)}| &= [C_1 + C_2(n-k+1)]\,(\lambda-1)^{n-k+1}. \end{aligned} \tag{3.20}$$

Thus from (3.19) and (3.20), we get for $3 \leq k < n$

$$|Q_{n-1}^{(k)}| = -\frac{(k-2)|Q_{n-k}^{(1-k)}|}{(\lambda-1)^{1-k}} + \frac{(k-1)|Q_{n-k+1}^{(k)}|}{(\lambda-1)^{2-k}}. \tag{3.21}$$

Next, assuming $3 \le k < n$, we find a recursion formula for $|Q^{(k)}_{n-k+1}|$. Letting $t = n-k+1$, we have

$$Q^{(k)}_t = \begin{bmatrix} \lambda & -1 & -\mu^2 & \cdots & -\mu^t \\ -1 & \lambda & -\mu & \cdots & -\mu^{t-1} \\ -\mu & -1 & & & \\ \vdots & \vdots & & S_{t-2} & \\ -\mu^{t-1} & -\mu^{t-2} & & & \end{bmatrix}.$$

On this matrix, repeating the same row and column operations as before, we arrive

$$\begin{aligned} |Q^{(k)}_t| &= \begin{vmatrix} 2(\lambda-1) & 1-\lambda & 0 & \cdots & 0 \\ 1-\lambda & \lambda & -\mu & \cdots & -\mu^{t-1} \\ 0 & -1 & & & \\ \vdots & \vdots & & S_{t-2} & \\ 0 & -\mu^{t-2} & & & \end{vmatrix} \\ &= 2(\lambda-1)|Q^{(k)}_{t-1}| - (\lambda-1)^2|S_{t-2}|. \end{aligned}$$

Note that this equation holds also for $k = 2$. Thus we have shown that for all $2 \le k < n$,

$$|Q^{(k)}_{n-k+1}| = 2(\lambda-1)|Q^{(k)}_{n-k}| - (\lambda-1)^2|S_{n-k-1}|. \tag{3.22}$$

Now we find a recursion formula for $|Q^{(k)}_{n-k}|$. Substract μ times the second row of $Q^{(k)}_{n-k}$ from the first row of $Q^{(k)}_{n-k}$ to obtain for $2 \le k < n$,

$$\begin{aligned} |Q^{(k)}_u| &= \begin{vmatrix} \lambda+\mu & 0 & 0 & \cdots & 0 \\ -1 & -1 & -\mu & \cdots & -\mu^{u-1} \\ -\mu & \lambda & -1 & \cdots & -\mu^{u-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mu^{u-1} & -\mu^{u-2} & \mu^{u-3} & \cdots & -1 \end{vmatrix} \\ &= (\lambda+\mu)|S_{u-1}| \end{aligned}$$

where $u = n-k$. This formula holds also for $k = 1$. Thus for all $1 \le k < n$,

$$|Q^{(k)}_{n-k}| = (\lambda+\mu)|S_{n-k-1}|. \tag{3.23}$$

Finally, we obtain a recursion formula for $|S_j|$. Substract μ times the second row of S_j from the first row of S_j. Then $S_0 = -1$ and for $1 \le j < n$,

$$\begin{aligned} |S_j| &= \begin{vmatrix} 1-\lambda & 0 & \cdots & 0 \\ \lambda & -1 & \cdots & -\mu^{j-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\mu^{j-2} & \mu^{j-3} & \cdots & -1 \end{vmatrix} \\ &= (1-\lambda)|S_{j-1}| \\ &= -(1-\lambda)^j. \end{aligned} \tag{3.24}$$

Hence from (3.23) and (3.24), we have for $1 \le k < n$,

$$|Q^{(k)}_{n-k}| = \frac{\lambda+2}{\lambda}(1-\lambda)^{n-k}. \tag{3.25}$$

Using (3.25) and (3.24) in (3.22), we get for $2 \leq k < n$

$$|Q^{(k)}_{n-k+1}| = -\frac{\lambda+4}{\lambda}\,(1-\lambda)^{n-k+1}. \tag{3.26}$$

Plugging (3.25) and (3.26) in (3.21), we obtain for $3 \leq k < n$

$$|Q^{(k)}_{n-1}| = \frac{\lambda+2k}{\lambda}\,(-1)^{n-k}\,(\lambda-1)^{n-1}. \tag{3.27}$$

This formula contains the cases $k = 1$ in (3.25) and $k = 2$ in (3.26). For $k = 0$, we have

$$|Q^{(0)}_{n-1}| = |S_{n-1}| = -(1-\lambda)^{n-1},$$

which is recovered by the formula (3.27) with $k = 0$ plugged in. Thus (3.27) is valid for all $k < n$. Note that

$$|Q^{(n)}_{n-1}| = D_{n-1}. \tag{3.28}$$

Expanding the determinant in (3.3) with respect to the last row, we have $\phi_0(s) = f_0(s)/\sqrt{\lambda}$ since $D_0 = \lambda$ and for $n \geq 1$

$$\phi_n(s) = \sum_{k=0}^{n}(-1)^{n+k}\,\left(D_{n-1}D_n\right)^{-1/2}\,Q^{(k)}_{n-1}\,f_k(s). \tag{3.29}$$

Thus from (3.18), (3.27), and (3.28), we get

$$\phi_n(s) = \frac{\beta_n}{(s+a)(s+b)}\left[\sum_{k=0}^{n-1}(\lambda+2k)\left(\frac{s-a}{s+a}\right)^k + \lambda(\lambda+n-1)\left(\frac{s-a}{s+a}\right)^n\right]$$

where

$$\beta_n \triangleq \left[\frac{2a^3}{(\lambda+n)(\lambda+n-1)}\right]^{1/2}. \tag{3.30}$$

From (3.13), we can write $\phi_n(s)$ as

$$\begin{aligned}\phi_n(s) \;=\; & \frac{\beta_n}{(s+a)(s+b)}\left\{\lambda(\lambda+n-1)\left(\frac{s-a}{s+a}\right)^n + \frac{(s+a)^2}{4a^2}\left[\lambda+(2-\lambda)\,\frac{s-a}{s+a}\right.\right.\\ & \left.\left.-(\lambda+2n)\left(\frac{s-a}{s+a}\right)^n + (\lambda+2n-2)\left(\frac{s-a}{s+a}\right)^{n+1}\right]\right\}.\end{aligned} \tag{3.31}$$

Although tedious, it is clear how to proceed for higher asymptotic orders. For example, consider the single–pole case in § 3.3 for deg$P(s) = m > 1$. Then T_n in (3.8) will have $2m$ nonzero stripes parallel to the main diagonal and the order of the finite–difference equation in (3.9) will be $2m$. Once this equation is solved, $\phi_n(s)$ can be calculated in terms of its solution as in (3.11). When $m > 1$, the multiple–pole case considered in this section requires taking more than one row or column at a time for pivoting. In the most general case, *i.e.*, the basis–pole set is not singleton, pivoting operations are to be performed on matrices. The details are omitted.

3.4.1 Asymptotic Toeplitz Results

The examples we have studied in this chapter are essentialy of determining Toeplitz determinants generated by nonnegative rational functions. A lot of results are available in the literature for Toeplitz determinants. In the rest of the chapter, we will review some of these results.

The nonnegative function on the unit circle that generates the Toeplitz matrices in (3.16) is

$$\begin{aligned} F(e^{i\theta}) &= \lambda - \sum_{k\neq 0} \mu^{|k-1|}\, e^{ik\theta} \\ &= \frac{b(z-1)^2}{a(z\mu-1)(z-\mu)}, \qquad z = e^{i\theta}. \end{aligned} \tag{3.32}$$

This rational function has double zeros at $z_{1,2} = 1$ and two poles at $p_1 = \mu$ and $p_2 = 1/\overline{\mu}$. The symmetry of the poles and zeros with respect to the unit circle is in agreement with the positivity of $F(e^{i\theta})$. When $\mu = 0$, we have $F(e^{i\theta}) = -(1-z)^2/z$, which is the case considered in § 3.3 after a rescaling of f_k's.

The *first Szegö limit theorem* [15] tells us that

$$\lim_{n\to\infty} \frac{D_n}{D_{n-1}} = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln F(e^{i\theta})\,\mathrm{d}\theta\right) \triangleq \mathcal{M}_F \tag{3.33}$$

where the symbol $\mathcal{M}_F$ stands for the geometric mean of F. Now consider the case in § 3.3. Then $F(e^{i\theta}) = 1 - \cos\theta$ and

$$\mathcal{M}_F = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(1-\cos\theta)\,\mathrm{d}\theta\right) = \frac{1}{2}$$

where we used the identity [14, Formula 4.224.9]

$$\int_0^{\pi} \ln(a + b\cos x)\,\mathrm{d}x = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \tag{3.34}$$

valid for all $a \geq b > 0$. Hence

$$\lim_{n\to\infty} \frac{D_n}{D_{n-1}} = \frac{1}{2}.$$

From (3.10), we have the same result as follows

$$\lim_{n\to\infty} \frac{D_n}{D_{n-1}} = \lim_{n\to\infty} \frac{n+2}{2(n+1)} = \frac{1}{2}.$$

When F is positive and satisfies certain smoothness conditions, the following refined version of (3.33), known as the *strong Szegö limit theorem* [15], holds

$$\lim_{n\to\infty} \frac{D_n}{\mathcal{M}_F^{n+1}} = \exp\left(2\int_0^1 \int_0^{2\pi} \left|\frac{\mathcal{F}'(re^{i\theta})}{\mathcal{F}(re^{i\theta})}\right|^2 r\,\mathrm{d}\theta\mathrm{d}r\right) \tag{3.35}$$

where $\mathcal{F}(z)$ is the unique analytic function that satisies $\mathrm{Re}\mathcal{F}(e^{i\theta}) = F(e^{i\theta})$ and $\mathrm{Im}\mathcal{F}(0) = 0$. Recall that if the Fourier series of F is

$$F(e^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k\, e^{ik\theta},$$

then its conjugate Fourier series is

$$\widetilde{F}(e^{i\theta}) = -i \sum_{k=-\infty}^{\infty} \operatorname{sgn}(k)\, c_k\, e^{ik\theta}$$

where sgn is the signum function defined by

$$\operatorname{sgn}(n) = \begin{cases} -n/|n|, & n \neq 0, \\ 0, & n = 0. \end{cases}$$

Thus for the case in § 3.3, we have $\mathcal{F}(z) = 1 - z$. Hence $\mathcal{F}'(z) = -1$ for all $|z| \leq 1$ and the right hand side of (3.35) is

$$\int_0^1 \int_0^{2\pi} \frac{(1-r^2)\,\mathrm{d}\theta}{1 - 2r\cos\theta + r^2} \frac{2r\,\mathrm{d}r}{1-r^2} = \int_0^1 \frac{4\pi r\,\mathrm{d}r}{1-r^2} = \infty$$

which equals to the left hand side of (3.35), where we used the fact that the integral of the Poisson kernel which is the first integrand on the left hand side of the above equation equals to 2π.

More elaborate asymptotic results on Toeplitz determinants can be found in the comprehensive book [6]. Notice that the coefficients of $f_k(s)$ in (3.10) increase slowly as $O(k/n)$. This implies that asymptotic analysis does not have much to offer in the problems considered here. However it is an indispensible technique when exact solutions as in (3.10) or (3.18) are not possible.

3.5 Summary

This chapter has illustrated application of a method to construct complete orthonormal model sets which have a prescribed asymptotic order.

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