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Abstract

In this paper, we investigate sparsity regularization for electrical impedance tomography (EIT). Here, we combine sparsity regularization with the energy functional approach. The main results of our paper is the well-posedness and convergence rates of the sparsity regularization method.

Keywords: *Sparsity regularization, electrical impedance tomography.*

1 Introduction

The problem of identifying the conductivity coefficient σ in the elliptic equation

$$-\operatorname{div}(\sigma\nabla\phi) = 0 \text{ in } \Omega, \tag{1}$$

from the Neumann-to-Dirichlet map, is of interest in electrical impedance tomography (EIT). For surveys on the problem, we refer the reader to [1, 9, 8, 5, 38, 22]. This problem is well-known to be severely ill-posed and has to be stabilized by some regularization methods. There have been a few regularization methods for the problem in the literatures [2, 10, 11, 29, 30, 32, 35, 36, 40, 42]. However, the quality of reconstructed conductivity parameters is not satisfactory in comparison with those in other fields.

Let $\tilde{H}^1(\Omega)$ be a subspace of $H^1(\Omega)$ with zero mean on the boundary Γ, i.e.

$$\tilde{H}^1(\Omega) = \{v \in H^1(\Omega) : \int_\Gamma v ds = 0\}.$$

The spaces $\tilde{H}^{1/2}(\Gamma)$ and $\tilde{H}^{-1/2}(\Gamma)$ are defined similarly. These spaces are equipped with the usual norms.

We denote by

$$\mathcal{A} = \{\sigma \in L^\infty(\Omega) : \lambda \le \sigma \le \lambda^{-1} \text{ a.e and } \operatorname{supp}\left(\sigma - \sigma^0\right) \subset \Omega'\},$$

for some fixed $\lambda \in (0,1)$, where Ω' is an open set with the smooth boundary that contained compactly in Ω. The set $\mathcal{A}$ is endowed with the $L^q(\Omega)$ −norm $(1 \le q \le \infty)$.

The basis mathematical model for the forward problem in electrical impedance tomography is the elliptic partial differential equation

$$-\operatorname{div}(\sigma\nabla\phi) = 0 \text{ in } \Omega;\ \sigma\frac{\partial\phi}{\partial n}|_\Gamma = j \in \tilde{H}^{-1/2}(\Gamma). \tag{2}$$

To obtain the unique weak solution of this problem, we normalize the solution by requiring $\int_\Gamma u ds = 0$, i.e. $u \in \tilde{H}^1(\Omega)$ and define the Neumann operator $F_N(\cdot)j$ by

$$F_N(\cdot)j : \mathcal{A} \to \tilde{H}^1(\Omega), \sigma \mapsto F_N(\sigma)j \text{ is the weak solution of (2).}$$

Similarly, the Dirichlet operator $F_D(\cdot)g : \mathcal{A} \to \tilde{H}^1(\Omega), \sigma \mapsto F_D(\sigma)g$, the weak solution of the equation

$$-\operatorname{div}(\sigma\nabla\phi) = 0 \text{ in } \Omega;\ \phi|_\Gamma = g \in \tilde{H}^{1/2}(\Gamma) \tag{3}$$

and the Neumann-to-Dirichlet operator $NtD\left(\sigma\right)$ is defined by

$$NtD\left(\sigma\right):\tilde{H}^{-1/2}\left(\Gamma\right)\to\tilde{H}^{1/2}\left(\Gamma\right),j\mapsto NtD\left(\sigma\right)j=F_{N}\left(\sigma\right)j\,|_{\Gamma}\,. \tag{4}$$

An EIT experiment consists of applying an electrical current to the surface of the object and then measuring the resulting electrical potential on the boundary. In practice, the procedure is repeated several times with different currents, which yields partial information about the Neumann-to-Dirichlet map NtD. Thus, our inverse problem is stated as follow: Given the Neumann-to-Dirichlet operator NtD, find σ^* such that $NtD(\sigma^*)=NtD$.

Note that for any $\sigma\in\mathcal{A}$, if $NtD\left(\sigma\right)j=g$, then

$$F_{N}\left(\sigma\right)j-F_{D}\left(\sigma\right)g=0.$$

Thus, given the Neumann-to-Dirichlet operator NtD, we might identify the conductivity σ^* from solving the system of equations

$$F_{N}\left(\sigma\right)j_k-F_{D}\left(\sigma\right)g_k=0$$

with $g_k=NtDf_k$. This motivates our approach. The choice of currents j_k is crucial and has been investigated by many authors. In [25, 24, 23, 7, 13] the authors have investigated the so-called optimal current in some sense. Using several currents have also been examined in [28, 27]. For simplicity, we here assume that only one current j that is optimal in some sense is used. However, the results in this paper are still valid for several currents as in [28].

We assume that there exists some $\sigma^*\in\mathcal{A}_{ad}$ such that $NtD\left(\sigma^*\right)=NtD$. Fix $j\in\tilde{H}^{-1/2}\left(\Gamma\right)$ and denote $g=NtD\left(\sigma^*\right)j$ and assume that only noisy data $\left(j^\delta,g^\delta\right)\in\tilde{H}^{-1/2}\left(\Gamma\right)\times\tilde{H}^{1/2}\left(\Gamma\right)$ of (j,g) such that

$$\left\|j-j^\delta\right\|^2_{\tilde{H}^{-1/2}(\Gamma)}+\left\|g-g^\delta\right\|^2_{\tilde{H}^{1/2}(\Gamma)}\le\delta^2 \tag{5}$$

with $\delta>0$, are available. Our problem now is to identify σ^* from $\left(j^\delta,g^\delta\right)$.

To solve this problem, we minimize the energy functional

$$F_\delta\left(\sigma\right)=\int_\Omega\sigma|\nabla\left(F_N\left(\sigma\right)j^\delta-F_D\left(\sigma\right)g^\delta\right)|^2dx \tag{6}$$

over an admissible set $\mathcal{A}_{ad}$. Since the problem is ill-posed, sparsity regularization is used to solve it in a stable way. This leads to considering the minimization problem

$$\min_{\sigma\in\mathcal{A}_{ad}}F_\delta\left(\sigma\right)+\alpha\Phi\left(\sigma-\sigma^0\right), \tag{7}$$

where $\alpha>0$ is the regularization parameter and

$$\Phi\left(\vartheta\right):=\sum\omega_k|\left\langle\vartheta,\varphi_k\right\rangle|^p\;\;(1\le p\le 2) \tag{8}$$

with $\{\varphi_k\}$ being an orthonormal basis (or frame) of the Hilbert space $H^1_0\left(\Omega'\right)$ and $\omega_k\ge\omega_{min}$ for all k. Here, the admissible set $\mathcal{A}_{ad}:=\mathcal{A}\cap Q$ with $Q=\{\sigma\in\mathcal{A}:\sigma-\sigma^0\in H^1_0\left(\Omega'\right)\}$.

The energy functional $F_\delta\left(\cdot\right)$ in (6) has been used in [28, 27]. However, they aimed at constructing numerical algorithms to reconstruct the conductivity σ. Here, we aim at studying the well-posedness and convergence rates of the method. In order to obtain the well-posedness of the method, problem (7) is examined on $\mathcal{A}_{ad}$ instead of $\mathcal{A}$. The idea of choosing $\mathcal{A}_{ad}$ follows the paper of Jin and Maass [26]. We need this constraint to obtain the compactness of E_t defined below, which is sufficient for obtaining the well-posedness of the method. In order to obtain convergence rates, we follow the ideas of Hao and Quyen [19, 20].

Note that in EIT problem, it is very often that the conductivity coefficient σ^* consists of the background σ^0 plus several interesting features that have relatively simple mathematical descriptions, i.e. the number of nonzero components of $\sigma-\sigma^0$ are finite in a basis (or frame) of a space. Based on this prior information, there are advantages to use sparsity regularization.

Sparsity regularization has been of interest by many researchers for the last years. The well-posedness and some convergence rates of the method have been analyzed for linear inverse problems [12] as well as for nonlinear inverse problems [18]. It is shown that sparsity regularization is simple for use and very efficient for inverse problems with sparse solutions. This method has been investigated and applied very successfully to some fields such as for compressive imaging [16, 37, 39, 41]. Recently, sparsity regularization has been applied to EIT problem [27, 17, 26]. Numerical experiments in [27, 17] have demonstrated its great potentials. Following the least squares approach in [18], the well-posedness and some convergence rates of the method have been also obtained in [26]. Numerical algorithms have also been proposed [31, 12, 6, 4, 34, 3].

2 Auxiliary Results

Before proving the main results of sparsity regularization for EIT, we consider some properties of $F_N(\cdot)\,j$, $F_D(\cdot)\,g$ and $F_\delta(\cdot)$ on $\mathcal{A}$ with respect to the $L^q(\Omega)$-norm, which are needed for studying the well-posedness and convergence rates of the method as well as for numerical algorithms. Some of them have been proven in [26], which are derived by exploiting Meyers' gradient estimate [33] as follow.

Theorem 1 (Meyers' theorem) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ $(d \geq 2)$. Assume that $\sigma \in L^\infty(\Omega)$ satisfies $\lambda < \sigma < \lambda^{-1}$ for some fixed $\lambda \in (0,1)$. For $z \in (L^r(\Omega))^d$ and $y \in L^r(\Omega)$, let $\phi \in H^1(\Omega)$ be a weak solution of the equation*

$$-\operatorname{div}(\sigma \nabla \phi) = -\operatorname{div}(z) + y \text{ in } \Omega.$$

Then, there exists a constant $Q \in (2, +\infty)$ depending on λ and d only, $Q \to 2$ as $\lambda \to 0$ and $Q \to \infty$ as $\lambda \to 1$, such that for any $2 < r < Q$, $\phi \in W^{1,r}_{loc}(\Omega)$ and for any $\Omega' \subset\subset \Omega$

$$\|\nabla \phi\|_{L^r(\Omega')} \leq C' \left(\|\phi\|_{H^1(\Omega)} + \|z\|_{L^r(\Omega)} + \|y\|_{L^r(\Omega)} \right),$$

where the constant C' depends on λ, d, r, Ω' and Ω.

Remark 2

1. *By using Lax-Milgram's lemma, one can show that for any $\sigma \in \mathcal{A}$, there exist constants C_N and C_D (only depend on λ and Ω) such that*

$$\|F_N(\sigma)\,j\|_{H^1(\Omega)} \leq C_N \|j\|_{H^{-1/2}(\Gamma)}, \ \|F_D(\sigma)\,g\|_{H^1(\Omega)} \leq C_D \|g\|_{H^{1/2}(\Gamma)}.$$

2. *On the space $\tilde{H}^1(\Omega)$, the standard $H^1(\Omega)-$norm and the $H^1(\Omega)-$semi-norm are equivalent (see e.g. [26, Lemma 2.2]), which implies that for any $u \in \tilde{H}^1(\Omega)$, there exists a constant $\tilde{C}$ such that*

$$\|\nabla u\|_{L^2(\Omega)} \geq \tilde{C} \|u\|_{H^1(\Omega)}.$$

Lemma 3 *Let $q \in \left(\frac{2Q}{Q-2}, \infty\right]$, $j \in \tilde{H}^{-1/2}(\Gamma)$ and $g \in \tilde{H}^{1/2}(\Gamma)$. Then, for any $\sigma, \sigma + \vartheta \in \mathcal{A}$, we have*

$$\|F_N(\sigma + \vartheta)\,j - F_N(\sigma)\,j\|_{H^1(\Omega)} \leq C_1 \|\vartheta\|_{L^q(\Omega')} \|j\|_{\tilde{H}^{-1/2}(\Omega)}$$

and

$$\|F_D(\sigma + \vartheta)\,g - F_D(\sigma)\,g\|_{H^1(\Omega)} \leq C_2 \|\vartheta\|_{L^q(\Omega')} \|g\|_{\tilde{H}^{1/2}(\Omega)},$$

where the positive constants C_1 and C_2 depend on λ, d, q, Ω' and Ω.

Proof. For $F_N(\cdot)\,j$, the proof is in [26, Lemma 2.3]. For $F_D(\cdot)\,g$, the proof is similar. ∎

Remark 4 *By the above lemma, $F_D(\cdot)\,y$ is Lipschitz continuous on $\mathcal{A}$ with respect to the $L^q(\Omega)-$norm for $q \in \left(\frac{2Q}{Q-2}, \infty\right]$. Note that for $\sigma, \sigma + \vartheta \in \mathcal{A}$ and $1 \leq q_1 \leq q_2$, we have*

$$|\Omega|^{-1/q_1} \|\vartheta\|_{L^{q_1}(\Omega)} \leq |\Omega|^{-1/q_2} \|\vartheta\|_{L^{q_2}(\Omega)},$$

and

$$\|\vartheta\|^{q_2}_{L^{q_2}(\Omega)} \leq \left(2\lambda^{-1}\right)^{q_2 - q_1} \|\vartheta\|^{q_1}_{L^{q_1}(\Omega)}.$$

This means that the convergence of ϑ to zero with respect to the $L^{q_1}(\Omega)-$norm and the $L^{q_2}(\Omega)-$norm are equivalent. Therefore, the operators $F_N(\cdot)\,j$ and $F_D(\cdot)\,g$ are also continuous on $\mathcal{A}$ with respect to the $L^q(\Omega)$-norm for $q \geq 1$.

We now consider the differentiability of the operators $F_N(\cdot)\,j$ and $F_D(\cdot)\,g$. For $\sigma, \sigma + t\vartheta \in \mathcal{A}$ with $t > 0$, from the definition of $F_N(\sigma)\,j$ and $F_N(\sigma + t\vartheta)\,j$, we have

$$-\operatorname{div}(\sigma \nabla F_N(\sigma)\,j) = 0 \text{ and } -\operatorname{div}((\sigma + t\vartheta)\nabla F_N(\sigma + t\vartheta)\,j) = 0.$$

It implies that

$$-\operatorname{div}\left(\sigma \frac{\nabla (F_N(\sigma + t\vartheta)\,j - F_N(\sigma)\,j)}{t}\right) = \operatorname{div}(\vartheta \nabla F_N(\sigma + t\vartheta)\,j)$$

with $\sigma\frac{\partial}{\partial n}\left(F_N\left(\sigma+t\vartheta\right)j-F_N\left(\sigma\right)j\right)/t|_\Gamma=0$. Taking $t\to 0$, by the continuity of F_N we have $\phi'=F_N'\left(\sigma\right)j\left(\vartheta\right)$, the solution of the equation

$$-\operatorname{div}\left(\sigma\nabla\phi'\right)=\operatorname{div}\left(\vartheta F_N\left(\sigma\right)j\right)$$

with the Neumann boundary condition $\sigma\frac{\partial\phi'}{\partial n}=0$ on Γ.

Similarly, we also have $\phi=F_D'\left(\sigma\right)g\left(\vartheta\right)$ to be the solution of the equation

$$-\operatorname{div}\left(\sigma\nabla\phi\right)=\operatorname{div}\left(\vartheta F_D\left(\sigma\right)g\right)$$

with the Dirichlet boundary condition $\phi|_\Gamma=0$.

We have $F_N'\left(\sigma\right)j:L^q\left(\Omega'\right)\to\tilde{H}^1\left(\Omega\right),\vartheta\mapsto\phi'$ and $F_D'\left(\sigma\right)g:L^q\left(\Omega'\right)\to\tilde{H}^1\left(\Omega\right),\vartheta\mapsto\phi$. The following lemma shows that the operators $F_N\left(\cdot\right)j$ and $F_D\left(\cdot\right)g$ are not only directional differentiable but also the Fréchet differentiable.

Lemma 5 *For each $\sigma\in\mathcal{A}$, both $F_N\left(\cdot\right)j$ and $F_D\left(\cdot\right)g$ have the continuous Fréchet derivative at σ with respect to the $L^q\left(\Omega'\right)-$norms, $q\in\left(\frac{2Q}{Q-2},\infty\right]$. Moreover, let ϑ be a perturbation to σ belonging to $L^\infty\left(\Omega'\right)$ and extended by zero outside Ω', we have*

1) $F_N'\left(\sigma\right)j\left(\vartheta\right)=\phi'$ is the unique solution of the equation

$$-\operatorname{div}\left(\sigma\nabla\phi'\right)=\operatorname{div}\left(\vartheta\nabla F_N\left(\sigma\right)j\right)\tag{9}$$

with a homogeneous Neumann boundary condition.

2) $F_D'\left(\sigma\right)g\left(\vartheta\right)=\phi$ is the unique solution of the equation

$$-\operatorname{div}\left(\sigma\nabla\phi\right)=\operatorname{div}\left(\vartheta\nabla F_D\left(\sigma\right)g\right)\tag{10}$$

with a homogeneous Dirichlet boundary condition.

Moreover, the following estimations hold

$$\|F_N'\left(\sigma\right)j[\vartheta]\|_{L\left(L^q(\Omega'),\tilde{H}^1(\Omega)\right)}\leq C_3\|j\|_{\tilde{H}^{-1/2}(\Gamma)}\|\vartheta\|_{L^q(\Omega')},\tag{11}$$

$$\|F_D'\left(\sigma\right)g[\vartheta]\|_{L\left(L^q(\Omega'),\tilde{H}^1(\Omega)\right)}\leq C_4\|g\|_{\tilde{H}^{1/2}(\Gamma)}\|\vartheta\|_{L^q(\Omega')}.\tag{12}$$

Proof. The Fréchet differentiability of $F_N\left(\cdot\right)j$ is proven in [26, Lemma 2.4 and Theorem 2.2]. The Fréchet differentiability of $F_D\left(\cdot\right)g$ is proven similarly. We now prove two last inequalities. Since the proofs are similar to each other, we only prove for $F_N'\left(\cdot\right)j$. The weak solution formula of equation (9) is

$$\int_\Omega\sigma\nabla\phi'\cdot\nabla v\,dx=-\int_\Omega\vartheta\nabla F_N\left(\sigma\right)j\cdot\nabla v\text{ for all }v\in\tilde{H}^1\left(\Omega\right).\tag{13}$$

From (13), choosing $v=\phi'\in\tilde{H}^1\left(\Omega\right)$, using Holder's inequality, Theorem 1 and Remark 2, we obtain

$$\tilde{C}\lambda\|\phi'\|_{\tilde{H}^1(\Omega)}\leq\|\vartheta\|_{L^q(\Omega')}\;\|\nabla F_N\left(\sigma\right)j\|_{L^r(\Omega)}\text{ with }\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$$

$$\Rightarrow\|F_N'\left(\sigma\right)j\left(\vartheta\right)\|_{\tilde{H}^1(\Omega)}=\|\phi'\|_{\tilde{H}^1(\Omega)}\leq\frac{C_MC_N}{\tilde{C}\lambda}\|\vartheta\|_{L^q(\Omega')}\;\|j\|_{H^{-1/2}(\Gamma)}.$$

■

Next, we consider the continuity and differentiability of the energy functional $F_\delta\left(\sigma\right)$.

Lemma 6 *For any $(j,g)\in\tilde{H}^{-1/2}\left(\Gamma\right)\times\tilde{H}^{1/2}\left(\Gamma\right)$, the functional*

$$F\left(\sigma\right):=\int_\Omega\sigma|\nabla\left(F_N\left(\sigma\right)j-F_D\left(\sigma\right)g\right)|^2dx$$

has the following properties:

1. $F\left(\cdot\right)$ is Fréchet differentiable with respect to the $L^q\left(\Omega'\right)-$norm for $q\in\left(\frac{2Q}{Q-2},\infty\right]$ and

$$F'\left(\sigma\right)\vartheta=-\int_\Omega\vartheta\left(|\nabla F_N\left(\sigma\right)j|^2-|\nabla F_D\left(\sigma\right)g|^2\right)dx.$$

2. The second Fréchet derivative F'' of $F(\cdot)$ exists and is uniformly bounded with respect to the $L^q(\Omega')$-norm for $q \in \left(\frac{2Q}{Q-2}, \infty\right]$.

Proof.

1. $F(\cdot)$ is Fréchet differentiable since $F_N(\cdot)j$ and $F_D(\cdot)g$ are Fréchet differentiable. We have

$$F'(\sigma)\vartheta = \int_\Omega \vartheta |\nabla (F_N(\sigma)j - F_D(\sigma)g)|^2 dx$$
$$+ 2\int_\Omega \sigma (\nabla F'_N(\sigma)j(\vartheta) - \nabla F'_D(\sigma)g(\vartheta)) . (\nabla F_N(\sigma)j - \nabla F_D(\sigma)g)\, dx \quad (14)$$

Using the weak solution formulas of $F'_N(\sigma)j(\vartheta), F'_D(\sigma)g(\vartheta)$ and $F_N(\sigma)j$, we have

$$\int_\Omega \sigma \nabla F'_N(\sigma)j(\vartheta)\cdot \nabla F_N(\sigma)j dx = -\int_\Omega \vartheta |\nabla F_N(\sigma)j|^2 dx,$$

$$\int_\Omega \sigma \nabla F'_N(\sigma)j(\vartheta)\cdot \nabla F_D(\sigma)g dx = -\int_\Omega \vartheta \nabla F_N(\sigma)j \cdot \nabla F_D(\sigma)g dx,$$

$$\int_\Omega \sigma \nabla F'_D(\sigma)g(\vartheta)\cdot \nabla F_D(\sigma)g dx = -\int_\Omega \vartheta |\nabla F_D(\sigma)g|^2 dx,$$

$$\int_\Omega \sigma \nabla F_N(\sigma)\cdot \nabla F'_D(\sigma)g(\vartheta)\, dx = 0.$$

Inserting these equalities into (14) and simplifying, we get

$$F'(\sigma)\vartheta = -\int_\Omega \vartheta \left(|\nabla F_N(\sigma)j|^2 - |\nabla F_D(\sigma)g|^2\right) dx.$$

2. Clearly $F'(\cdot)$ has the Fréchet derivative and

$$F''(\sigma)(\vartheta,\vartheta) = -2\int_\Omega \vartheta (\nabla F_N(\sigma)j . \nabla F'_N(\sigma)j(\vartheta) + \nabla F_D(\sigma)g \cdot \nabla F'_D(\sigma)g(\vartheta))\, dx.$$

By the weak solution formulas of $F'_N(\sigma)g(\vartheta)$ and $F'_D(\sigma)j(\vartheta)$, it implies that

$$\int_\Omega \vartheta \nabla F_N(\sigma)j \cdot \nabla F'_N(\sigma)j(\vartheta)\, dx = -\int_\Omega \sigma |\nabla F'_N(\sigma)j(\vartheta)|^2 dx$$

and

$$\int_\Omega \vartheta \nabla F_D(\sigma)g \cdot \nabla F'_D(\sigma)g(\vartheta)\, dx = -\int_\Omega \sigma |\nabla F'_D(\sigma)g(\vartheta)|^2 dx.$$

Therefore,

$$F''(\sigma)(\vartheta,\vartheta) = 2\int_\Omega \sigma |\nabla F'_N(\sigma)j(\vartheta)|^2 dx - 2\int_\Omega \sigma |\nabla F'_D(\sigma)g(\vartheta)|^2 dx.$$

Finally, by (11) and (12), F'' is uniformly bounded. ∎

Remark 7 *From the uniform boundedness of F'', we deduce that F' is Lipschitz continuous on $\mathcal{A}$ with respect to the $L^q(\Omega')$ −norm for $q \in \left(\frac{2Q}{Q-2}, \infty\right]$. However, we can not show that F is a convex functional.*

3 The Well-posedness

We are now in a position to consider the well-posedness of sparsity regularization. To this end, the following property of Φ is necessary.

Lemma 8 *The functional Φ defined by (8) has the following properties*

1) Φ is non-negative, convex and weakly lower semi-continuous.

2) There exists a positive constant C such that for any $u \in \mathcal{H}$,

$$\Phi(u) \geq \omega_{min} C^{p/2} \|u\|^p .$$

This implies that Φ is weakly coercive, i.e. $\Phi(u) \to \infty$ as $\|u\| \to \infty$.

3) If $\{u^n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ weakly converges to $u \in \mathcal{H}$ and $\Phi(u^n)$ converges to $\Phi(u)$, then $\Phi(u^n - u)$ converges to zero.

Proof. Φ is non-negative, convex and weakly lower semi-continuous because it is the sum of non-negative, convex and weakly continuous functionals. The proofs of 2) and 3) can be found in [18, Remark 3.] and [18, Lemma 2.], respectively. ∎

Lemma 9 *Let $\Phi : H_0^1(\Omega') \to \mathbb{R} \cup \{\infty\}$ be defined by (8). Then, the set $E_t := \{\vartheta := \sigma - \sigma^0 : \sigma \in \mathcal{A}_{ad}$ and $\Phi(\vartheta) \leq t\}$ is compact in $L^2(\Omega)$ for all $t \in \mathbb{R}$.*

Proof. Suppose that $\{\vartheta^n := \sigma^n - \sigma^0\} \subset E_t$ for some fixed $t \in \mathbb{R}_+$. From the coercivity of Φ, $\{\vartheta^n\}$ is bounded in $H_0^1(\Omega')$ and thus there exists a subsequence of $\{\vartheta^n\}$, denoted again by $\{\vartheta^n\}$, weakly converging to $\vartheta := \sigma - \sigma^0$ in $H_0^1(\Omega')$. By Kondrashov embedding theorem [15], it strongly converges in $L^q(\Omega)$ for any $q < 6$ in case of $d = 2, 3$. Thus, it strongly converges in $L^2(\Omega)$ and $\sigma \in \mathcal{A}_{ad}$ due to the closedness of $\mathcal{A}_{ad}$ in $L^2(\Omega)$. Since Φ is weakly lower semicontinuous in $H_0^1(\Omega)$, $\Phi(\vartheta) \leq \lim_n \inf \Phi(\vartheta^n) \leq t$. This implies that $\vartheta \in E_t$. Therefore, E_t is a compact set in $L^2(\Omega)$. ∎

Lemma 10 *For $j \in \tilde{H}^{-1/2}(\Gamma)$ and $g = NtD(\sigma^*) j$, the set*

$$\Pi_{\mathcal{A}_{ad}} := \{\sigma \in \mathcal{A}_{ad} : F_N(\sigma) j = F_D(\sigma) g\}$$

is nonempty, bounded and closed in the space $L^2(\Omega)$. Thus, the problem

$$\min_{\sigma \in \Pi_{\mathcal{A}_{ad}}} \Phi\left(\sigma - \sigma^0\right)$$

has at least one solution that is called Φ-minimizing solution of EIT. If $p > 1$ then Φ-minimizing solution is unique.

Proof. It is easy to show that $\Pi_{\mathcal{A}_{ad}}$ is nonempty and bounded. We now prove that it is a closed set. Suppose that the sequence $\{\sigma^n\} \subset \Pi_{\mathcal{A}_{ad}}$ converges to σ in $L^2(\Omega)$. From the weak solution formula of $F_N(\sigma^n) j$, we have

$$\int_\Gamma jv ds = \int_\Omega \sigma^n \nabla F_N(\sigma^n) j . \nabla v dx = \int_\Omega \sigma^n \nabla \phi^n . \nabla v dx,$$

for all $v \in \tilde{H}^1(\Omega)$. Here, $\phi^n = F_N(\sigma^n) j = F_D(\sigma^n) g$. From Remark 2, the sequence $\{\phi^n\}$ is bounded and thus there exists a subsequence, denoted again by $\{\phi^n\}$, which weakly converges to ϕ in $H^1(\Omega)$.

Since $\sigma^n \to \sigma$ in the $L^2(\Omega)$-norm and ϕ^n weakly converges to ϕ in $H^1(\Omega)$, we obtain

$$\int_\Omega \sigma^n \nabla \phi^n \cdot \nabla v dx - \int_\Omega \sigma \nabla \phi \cdot \nabla v dx$$
$$= \int_\Omega (\sigma^n - \sigma) \nabla \phi^n \cdot \nabla v dx + \int_\Omega \sigma \nabla (\phi^n - \phi) \cdot \nabla v dx \to 0, \text{ as } n \to \infty,$$

for all $v \in \tilde{H}^1(\Omega)$. Thus, we have

$$\int_\Omega \sigma \nabla \phi \cdot \nabla v dx = \int_\Gamma jv ds,$$

for all $v \in \tilde{H}^1(\Omega)$. It means that $\phi = F_N(\sigma) j$. Similarly, we also have $\phi = F_D(\sigma) g$. Thus, $\sigma \in \Pi_{\mathcal{A}_{ad}}$ or $\Pi_{\mathcal{A}_{ad}}$ is a closed set in $L^2(\Omega)$.

Finally, we prove that there exists at least one Φ-minimizing solution of EIT. Suppose that there does not exist a Φ-minimizing solution in $\Pi_{\mathcal{A}_{ad}}$. Then, there exists a sequence $\{\sigma^k\} \subset \Pi_{\mathcal{A}_{ad}}$ such that $\Phi\left(\sigma^k - \sigma^0\right) \to c$ and

$$c < \Phi\left(\sigma - \sigma^0\right) \text{ for all } \sigma \in \Pi_{\mathcal{A}_{ad}}. \tag{15}$$

Since $\Phi\left(\sigma^k-\sigma^0\right)\to c$ as $k\to\infty$, $\{\sigma^k-\sigma^0\}$ is bounded in $H_0^1(\Omega')$. Therefore, by Lemma 8, there exists a subsequence of $\{\sigma^k-\sigma^0\}$, denoted again by $\{\sigma^k-\sigma^0\}$, weakly converging to $\sigma-\sigma^0$ in $H_0^1(\Omega')$ and $\sigma\in\Pi_{\mathcal{A}_{ad}}$. From the weakly lower semi-continuity of Φ in $H_0^1(\Omega')$, it follows that $\Phi\left(\sigma-\sigma^0\right)\le\lim_{k\to\infty}\inf\Phi\left(\sigma^k-\sigma^0\right)=c$. This gives a contradiction to (15).

Note that if $p>1$, then Φ is strictly convex and thus the Φ-minimizing solution is unique. ■

Next, we consider the well-posedness of problem (7) that consists of existence, stability, convergence.

Theorem 11 (Existence) *For any $\left(j^\delta,g^\delta\right)\in\tilde{H}^{-1/2}(\Gamma)\times\tilde{H}^{1/2}(\Gamma)$, problem (7) has at least one solution.*

Proof. Suppose that $\{\sigma^n\}$ is a minimizing sequence. It implies that $\{\Phi\left(\sigma^n-\sigma^0\right)\}$ is uniformly bounded. By Lemma 8 there exists $t\in\mathbb{R}_+$ such that $\{\sigma^n-\sigma^0\}\subset E_t$ and $\left\|\sigma^n-\sigma^0\right\|^p_{H_0^1(\Omega')}\le Ct$. Since E_t is compact in $L^2(\Omega)$ and $\{\sigma^n-\sigma^0\}$ is bounded in $H_0^1(\Omega')$, there exist a subsequence of $\{\sigma^n\}$, denoted again by $\{\sigma^n\}$, and a $\sigma^*\in\mathcal{A}_{ad}$ such that $\sigma^n-\sigma^0$ weakly converges to $\sigma-\sigma^0$ in $H_0^1(\Omega')$ and $\sigma^n\to\sigma$ in $L^2(\Omega)$. Since F_δ is continuous with respect to the $L^2(\Omega)$ −norm and Φ is weakly lower semi-continuous in $H_0^1(\Omega')$, we have

$$F_\delta(\sigma)\le\liminf_n\left(F_\delta(\sigma^n)+\alpha\Phi\left(\sigma^n-\sigma^0\right)\right)=\inf_{\sigma\in\mathcal{A}_{ad}}F_\delta(\sigma)+\alpha\Phi\left(\sigma-\sigma^0\right).$$

Therefore, σ is a solution of (7). ■

Theorem 12 (Stability) *For a fixed regularization $\alpha>0$, let the sequence (j^n,g^n) converge to $\left(j^\delta,g^\delta\right)$ in $\tilde{H}^{-1/2}(\Gamma)\times\tilde{H}^{1/2}(\Gamma)$ and let*

$$\sigma^n\in\operatorname*{argmin}_{\sigma\in\mathcal{A}_{ad}}\int_\Omega\sigma|\nabla\left(F_N(\sigma)j^n-F_D(\sigma)g^n\right)|^2dx+\alpha\Phi\left(\sigma-\sigma^0\right).$$

Then there exist a subsequence $\{\sigma^{n_k}\}$ of the sequence $\{\sigma^n\}$ and a minimizer $\sigma^p_{\alpha,\delta}$ of (7) such that

$$\left\|\sigma^{n_k}-\sigma^p_{\alpha,\delta}\right\|_{H_0^1(\Omega')}\to0\ \text{as}\ k\to\infty.$$

In addition, if the minimizer $\sigma^p_{\alpha,\delta}$ is unique, then $\{\sigma^n-\sigma^0\}$ converges to $a^p_{\alpha,\delta}-\sigma^0$ in the Hilbert space $H_0^1(\Omega')$.

Proof. Denote $F_n(\sigma)=\int_\Omega\sigma|\nabla\left(F_N(\sigma)j^n-F_D(\sigma)g^n\right)|^2dx$. By the definition of σ^n, we have

$$\begin{aligned}F_n(\sigma^n)+\alpha\Phi\left(\sigma^n-\sigma^0\right)&\le F_n(\sigma)+\alpha\Phi\left(\sigma-\sigma^0\right)\\&\le\lambda^{-1}\left(\|F_N(\sigma)j^n\|^2_{H^1(\Omega)}+\|F_D(\sigma)g^n\|^2_{H^1(\Omega)}\right)+\alpha\Phi\left(\sigma-\sigma^0\right)\\&\le\lambda^{-1}\left(C_N^2\|j^n\|^2_{\tilde{H}^{-1/2}(\Gamma)}+C_D^2\|g^n\|^2_{\tilde{H}^{1/2}(\Gamma)}\right)+\alpha\Phi\left(\sigma-\sigma^0\right)\\&\le\lambda^{-1}C_1\max\left(C_N^2,C_D^2\right)+\alpha\Phi\left(\sigma-\sigma^0\right)\end{aligned}\tag{16}$$

for any $\sigma\in\mathcal{A}_{ad}$, where the constants C_N,C_D are given in Remark 2 and C_1 is independent of n such that $\|(j^n,g^n)\|^2_{\tilde{H}^{-1/2}(\Gamma)\times\tilde{H}^{1/2}(\Gamma)}\le C_1$ for all n. This follows that $\{\Phi\left(\sigma^n-\sigma^0\right)\}$ is uniformly bounded and thus there exists $t\in\mathbb{R}_+$ such that $\{\vartheta^n:=\sigma^n-\sigma^0\}\subset E_t$ and $\|\vartheta^n\|^p_{H_0^1(\Omega')}\le Ct$ for all n. Since E_t is compact in $L^2(\Omega)$ and $\{\vartheta^n\}$ is bounded in $H_0^1(\Omega')$, there exist a subsequence of $\{\sigma^n\}$ denoted by $\{\sigma^{n_k}\}$ and an element $\sigma^p_{\alpha,\delta}\in L^2(\Omega)$ such that ϑ^{n_k} weakly converges to $\sigma^p_{\alpha,\delta}-\sigma^0$ in $H_0^1(\Omega')$ and $\{\sigma^{n_k}\}$ strongly converges to $\sigma^p_{\alpha,\delta}$ in $L^2(\Omega)$. Since $\mathcal{A}_{ad}$ is closed in $L^2(\Omega)$, $\sigma^p_{\alpha,\delta}\in\mathcal{A}_{ad}$. On the other hand, since F_δ is continuous in $L^2(\Omega)$ and Φ is weakly lower semi-continuous in $H_0^1(\Omega')$, we have

$$F_\delta\left(\sigma^p_{\alpha,\delta}\right)=\lim_k F_\delta(\sigma^{n_k})\tag{17}$$

and

$$\Phi\left(\sigma^p_{\alpha,\delta}-\sigma^0\right)\le\liminf_k\Phi\left(\sigma^{n_k}-\sigma^0\right).\tag{18}$$

Moreover,

$$\begin{aligned}&F_\delta(\sigma)-F_{n_k}(\sigma)\\&\quad=\int_\Omega\sigma\nabla[F_N(\sigma)\left(j^\delta-j^{n_k}\right)-F_D(\sigma)\left(g^\delta-g^{n_k}\right)]\cdot\nabla\theta dx,\end{aligned}\tag{19}$$

where $\theta = F_N(\sigma)\left(j^\delta + j^{n_k}\right) - F_D(\sigma)\left(g^\delta + g^{n_k}\right)$. Since $\left(j^{n_k}, g^{n_k}\right) \to \left(j^\delta, g^\delta\right)$ in $\tilde{H}^{-1/2}(\Gamma) \times \tilde{H}^{1/2}(\Gamma)$, the right-hand side of (19) uniformly converges in $\mathcal{A}$ to zero as $k \to \infty$. Therefore,

$$F_\delta(\sigma) = \lim_k F_{n_k}(\sigma), \ \liminf_k F_\delta\left(\sigma^{n_k}\right) = \liminf_k F_{n_k}\left(\sigma^{n_k}\right). \tag{20}$$

From (20), (16), (17) and (18), we obtain

$$\begin{aligned}
F_\delta\left(\sigma^p_{\alpha,\delta}\right) + \alpha\Phi\left(\sigma^p_{\alpha,\delta} - \sigma^0\right) &\overset{(17),(18)}{=} \liminf_k F_\delta\left(\sigma^{n_k}\right) + \alpha \liminf_k \Phi\left(\sigma^{n_k} - \sigma^0\right) \\
&\overset{(20)}{\leq} \liminf_k F_{n_k}\left(\sigma^{n_k}\right) + \alpha \liminf_k \Phi\left(\sigma^{n_k} - \sigma^0\right) \\
&\leq \liminf_k \left(F_{n_k}\left(\sigma^{n_k}\right) + \alpha\Phi\left(\sigma^{n_k} - \sigma^0\right)\right) \\
&\leq \limsup_k \left(F_{n_k}\left(\sigma^{n_k}\right) + \alpha\Phi\left(\sigma^{n_k} - \sigma^0\right)\right) \\
&\overset{(16)}{\leq} \limsup_k \left(F_{n_k}(\sigma) + \alpha\Phi\left(\sigma - \sigma^0\right)\right) \\
&\overset{(20)}{=} F_\delta(\sigma) + \alpha\Phi\left(\sigma - \sigma^0\right)
\end{aligned} \tag{21}$$

for all $\sigma \in A_{ad}$. It means that $\sigma^p_{\alpha,\delta}$ is a minimizer of (7).

From (21), setting $\sigma = \sigma^p_{\alpha,\delta}$ and by (20), we get

$$\lim_k \left(F_\delta\left(\sigma^{n_k}\right) + \alpha\Phi\left(\sigma^{n_k} - \sigma^0\right)\right) = F_\delta\left(\sigma^p_{\alpha,\delta}\right) + \alpha\Phi\left(\sigma^p_{\alpha,\delta} - \sigma^0\right).$$

Together with (17) and (18), we deduce that $\Phi\left(\sigma^{n_k} - \sigma^0\right) \to \Phi\left(\sigma^p_{\alpha,\delta} - \sigma^0\right)$. Finally, since $\left\{\sigma^{n_k} - \sigma^0\right\}$ weakly converges to $\sigma^p_{\alpha,\delta} - \sigma^0$ in $H^1_0(\Omega')$ and $\Phi\left(\sigma^{n_k} - \sigma^0\right) \to \Phi\left(\sigma^p_{\alpha,\delta} - \sigma^0\right)$ as $k \to \infty$, it implies that $\Phi\left(\sigma^{n_k} - \sigma^p_{\alpha,\delta}\right) \to 0$ as $k \to 0$ and thus $\left\|\sigma^{n_k} - \sigma^p_{\alpha,\delta}\right\|_{H^1_0(\Omega')} \to 0$ by Lemma 8.

In the case the minimizer $\sigma^p_{\alpha,\delta}$ is unique, the convergence of the original sequence $\{\sigma^n\}$ to $\sigma^p_{\alpha,\delta}$ follows by a subsequence argument. ∎

Theorem 13 (Convergence) *For any positive sequence $\{\delta_n\} \to 0$, let $\alpha_n := \alpha(\delta_n)$ be such that*

$$\alpha_n \to 0 \ \text{and} \ \frac{\delta_n^2}{\alpha_n} \to 0 \ \text{as} \ n \to \infty.$$

Furthermore, let $\{(j^n, g^n)\}$ be a sequence in $\tilde{H}^{-1/2}(\Gamma) \times \tilde{H}^{1/2}(\Gamma)$ satisfying

$$\|j^n - j\|^2_{\tilde{H}^{-1/2}(\Gamma)} + \|g^n - g\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq \delta_n^2$$

and

$$\sigma^n \in \operatorname*{argmin}_{\sigma \in \mathcal{A}_{ad}} \int_\Omega \sigma |\nabla\left(F_N(\sigma) j^n - F_D(\sigma) g^n\right)|^2 dx + \alpha_n \Phi\left(\sigma - \sigma^0\right).$$

Then, there exist a subsequence $\{\sigma^{n_k}\}$ of $\{\sigma^n\}$ and a Φ-minimizing solution σ^+ of EIT such that $\left\{\sigma^{n_k} - \sigma^0\right\}$ converges to $\sigma^+ - \sigma^0$ in $H^1_0(\Omega')$. Furthermore, if σ^+ is unique then the whole sequence converges.

Proof. Let $\overline{\sigma} \in \mathcal{A}_{ad}$ be a solution of $F_N(\overline{\sigma}) j = F_D(\overline{\sigma}) g$. The definition of σ^n implies that

$$\begin{aligned}
F_n(\sigma^n) + \alpha_n\Phi\left(\sigma^n - \sigma^0\right) &\leq F_n(\overline{\sigma}) + \alpha_n\Phi\left(\overline{\sigma} - \sigma^0\right) \\
&\leq \lambda^{-1} \int_\Omega |\nabla\left(F_N(\overline{\sigma}) j^n - F_D(\overline{\sigma}) g^n\right)|^2 + \alpha_n\Phi\left(\overline{\sigma} - \sigma^0\right) \\
&\leq \lambda^{-1}\left(\|F_N(\overline{\sigma})(j^n - j)\|^2_{H^1(\Omega)} + \|F_D(\overline{\sigma})(g^n - g)\|^2_{H^1(\Omega)}\right) + \alpha_n\Phi\left(\overline{\sigma} - \sigma^0\right) \\
&\leq \lambda^{-1} \max\left(C_N^2, C_D^2\right)\delta_n^2 + \alpha_n\Phi\left(\overline{\sigma} - \sigma^0\right).
\end{aligned} \tag{22}$$

In particular, when $\delta \to 0$ and $\delta^2/\alpha \to 0$,

$$F_n(\sigma^n) \to 0, \ \limsup_n \Phi\left(\sigma^n - \sigma^0\right) \leq \Phi\left(\overline{\sigma} - \sigma^0\right). \tag{23}$$

Since $F_n(\sigma^n) \to 0$, $F(\sigma^n) := \int_\Omega \sigma^n |\nabla (F_N(\sigma^n) j - F_D(\sigma^n) g)|^2 dx \to 0$, see (19).

By (23), $\{\Phi(\sigma^n - \sigma^0)\}$ is bounded and thus there exists $t \in \mathbb{R}_+$ such that $\{\vartheta^n = \sigma^n - \sigma^0\} \subset E_t$ and $\|\vartheta^n\|^p_{H^1_0(\Omega')} \leq Ct$ for all n. Since E_t is compact in $L^2(\Omega)$ and $\{\vartheta^n\}$ is bounded in $H^1_0(\Omega')$, there exist a subsequence $\{\sigma^{n_k}\}$ of $\{\sigma^n\}$ and $\sigma^+ \in \mathcal{A}_{ad}$ such that $\sigma^{n_k} - \sigma^0$ weakly converges to $\sigma^+ - \sigma^0$ in $H^1_0(\Omega')$ and $\sigma^{n_k} \to \sigma^+$ in $L^2(\Omega)$. Since F_N and F_D are continuous in $L^2(\Omega)$, we have

$$F_N(\sigma^{n_k}) j \to F_N(\sigma^+) j \text{ and } F_D(\sigma^{n_k}) g \to F_D(\sigma^+) g \tag{24}$$

On the other hand, by Remark 2

$$\begin{aligned} F(\sigma^{n_k}) &= \int_\Omega \sigma^{n_k} |\nabla (F_N(\sigma^{n_k}) j - F_D(\sigma^{n_k}) g)|^2 dx \\ &\geq \lambda \|\nabla (F_N(\sigma^{n_k}) j - F_D(\sigma^{n_k}) g)\|^2_{L^2(\Omega)} \\ &\geq \lambda \tilde{C} \|F_N(\sigma^{n_k}) j - F_D(\sigma^{n_k}) g\|^2_{\tilde{H}^1(\Omega)} \geq 0. \end{aligned} \tag{25}$$

From (24), (25) and $F(\sigma^{n_k}) \to 0$ as $k \to \infty$, we get $F_N(\sigma^+) j = F_D(\sigma^+) g$ or $\sigma^+ \in \Pi_{\mathcal{A}_{ad}}$. Moreover, since Φ is weakly lower semi-continuous in $H^1_0(\Omega')$ and (23), we get

$$\Phi(\sigma^+ - \sigma^0) \leq \liminf_k \Phi(\sigma^{n_k} - \sigma^0) \leq \limsup_k \Phi(\sigma^{n_k} - \sigma^0) \leq \Phi(\overline{\sigma} - \sigma^0). \tag{26}$$

Therefore, σ^+ is a Φ-minimizing solution of EIT.

Finally, choosing $\overline{\sigma} = \sigma^+$ in (26), we have $\Phi(\sigma^{n_k} - \sigma^0) \to \Phi(\sigma^+ - \sigma^0)$ as $k \to \infty$. Since $\{\sigma^{n_k} - \sigma^0\}$ weakly converges to $\sigma^+ - \sigma^0$ in $H^1_0(\Omega')$ and $\Phi(\sigma^{n_k} - \sigma^0) \to \Phi(\sigma^+ - \sigma^0)$ as $k \to \infty$, it implies that $\Phi(\sigma^{n_k} - \sigma^+) \to 0$ and $\|\sigma^{n_k} - \sigma^+\|_{H^1_0(\Omega')} \to 0$ as $k \to 0$ by Lemma 8.

If the minimizer σ^+ is unique, the convergence of the original sequence $\{\sigma^n - \sigma^0\}$ to $\sigma^+ - \sigma^0$ follows from a subsequence argument. ■

4 Convergence Rates

For $\sigma \in \mathcal{A}_{ad}$ and $q \in \left(\frac{2Q}{Q-2}, \infty\right]$, the operators

$$F'_N(\sigma) j : L^q(\Omega') \to \tilde{H}^1(\Omega) \text{ and } F'_D(\sigma) g : L^q(\Omega') \to H^1_0(\Omega)$$

are linear and continuous. Denote by

$$(F'_N(\sigma) j)^* : \tilde{H}^{-1}(\Omega) \to L^{q_1}(\Omega') \text{ and } (F'_D(\sigma) g)^* : H^{-1}(\Omega) \to L^{q_1}(\Omega')$$

the dual operators of $F'_N(\sigma) j$ and $F'_D(\sigma) g$, respectively. Here, $\tilde{H}^{-1}(\Omega) := \left(\tilde{H}^1(\Omega)\right)^*$, $H^{-1}(\Omega) := \left(H^1_0(\Omega)\right)^*$ and q_1 is defined by $\frac{1}{q} + \frac{1}{q_1} = 1$. Note that since $H^1_0(\Omega) \subset \tilde{H}^1(\Omega)$, it implies $\tilde{H}^{-1}(\Omega) \subset H^{-1}(\Omega)$.

Then,

$$\begin{aligned} \left\langle (F'_N(\sigma) j)^* w^*_1, \vartheta \right\rangle_{(L^{q_1}(\Omega'), L^q(\Omega'))} &= \langle w^*_1, F'_N(\sigma) j(\vartheta) \rangle_{\left(\tilde{H}^{-1}(\Omega), \tilde{H}^1(\Omega)\right)} \\ \left\langle (F'_D(\sigma) g)^* w^*_2, \vartheta \right\rangle_{(L^{q_1}(\Omega'), L^q(\Omega'))} &= \langle w^*_2, F'_D(\sigma) g(\vartheta) \rangle_{\left(H^{-1}(\Omega), H^1_0(\Omega)\right)} \end{aligned} \tag{27}$$

with $w^*_1 \in \tilde{H}^{-1}(\Omega)$ and $w^*_2 \in H^{-1}(\Omega)$.

Some convergence rates of sparsity regularization for EIT are given in the following theorem. The ideas of the proof are similar to those in [19, 20]. However, we need more requirements on the source condition.

Theorem 14 *Let $q \in \left(\frac{2Q}{Q-2}, \infty\right]$, σ^+ be a Φ-minimizing solution of EIT and $a^p_{\alpha,\delta}$ be a solution of (7). Assume that there exists a function $w^* \in \tilde{H}^{-1}(\Omega)$ such that*

$$\xi := \left(F'_N(\sigma^+) j - F'_D(\sigma^+) g\right)^* w^* \in \partial\Phi(\sigma^+ - \sigma^0) \tag{28}$$

and

$$F'_N(\sigma^+) j(\vartheta) \in H^1_0(\Omega), \ \forall \vartheta \in L^\infty(\Omega'). \tag{29}$$

Then,

$$F_\delta\left(\sigma^p_{\alpha,\delta}\right) = O\left(\delta^2\right) \text{ and } D_\xi\left(\sigma^p_{\alpha,\delta}, \sigma^+\right) = O\left(\delta\right),$$

as $\delta \to 0$ *and* $\alpha \sim \delta$.

In particular, if $p \in (1, 2]$*, we have*

$$\left\|\sigma^p_{\alpha,\delta} - \sigma^+\right\|_{H^1_0(\Omega')} = O\left(\delta^{1/2}\right).$$

Proof. By the definition of $\sigma^p_{\alpha,\delta}$, we get

$$F_\delta\left(\sigma^p_{\alpha,\delta}\right) + \alpha\Phi\left(\sigma^p_{\alpha,\delta} - \sigma^0\right) \le F_\delta\left(\sigma^+\right) + \alpha\Phi\left(\sigma^+ - \sigma^0\right). \tag{30}$$

Then, we have

$$\begin{aligned} &F_\delta\left(\sigma^p_{\alpha,\delta}\right) + \alpha D_\xi\left(\sigma^p_{\alpha,\delta}, \sigma^+\right) \\ &\quad = F_\delta\left(\sigma^p_{\alpha,\delta}\right) + \alpha\left(\Phi\left(\sigma^p_{\alpha,\delta} - \sigma^0\right) - \Phi\left(\sigma^+ - \sigma^0\right) - \left\langle\xi, \sigma^p_{\alpha,\delta} - \sigma^+\right\rangle_{(L^{q_1}(\Omega'), L^q(\Omega'))}\right) \\ &\quad \le F_\delta\left(\sigma^+\right) - \alpha\left\langle\xi, \sigma^p_{\alpha,\delta} - \sigma^+\right\rangle_{(L^{q_1}(\Omega'), L^q(\Omega'))} \\ &\quad \le \lambda^{-1}\max\left(C_N^2, C_D^2\right)\delta^2 - \alpha\left\langle\xi, \sigma^p_{\alpha,\delta} - \sigma^+\right\rangle_{(L^{q_1}(\Omega'), L^q(\Omega'))}. \end{aligned} \tag{31}$$

On an other hand, denoting $\Psi := F'_N\left(\sigma^+\right)j - F'_D\left(\sigma^+\right)g$, from (27) and(28), we get

$$\begin{aligned} \left\langle\xi, \sigma^p_{\alpha,\delta} - \sigma^+\right\rangle_{(L^{q_1}(\Omega'), L^q(\Omega'))} &= \left\langle w^*, \Psi\left(\sigma^p_{\alpha,\delta} - \sigma^+\right)\right\rangle_{\left(\tilde{H}^{-1}(\Omega), \tilde{H}^1(\Omega)\right)} \\ &\overset{(29)}{=} \left\langle w^*, \Psi\left(\sigma^p_{\alpha,\delta} - \sigma^+\right)\right\rangle_{\left(H^{-1}(\Omega), H^1_0(\Omega)\right)}. \end{aligned} \tag{32}$$

By Riesz's representation theorem, there exists an element $w \in H^1_0\left(\Omega\right)$ such that

$$\left\langle w^*, \Psi\left(\sigma^p_{\alpha,\delta} - \sigma^+\right)\right\rangle_{\left(H^{-1}(\Omega), H^1_0(\Omega)\right)} = \left\langle w, \Psi\left(\sigma^p_{\alpha,\delta} - \sigma^+\right)\right\rangle_{H^1_0(\Omega)}. \tag{33}$$

Since $\sigma^+ \ge \lambda > 0$, the scalar product

$$[\phi, v]_{H^1_0(\Omega)} := \int_\Omega \sigma^+\nabla\phi\cdot\nabla v dx, \text{ for all } \phi, v \in H^1_0\left(\Omega\right)$$

is equivalent to $\langle\phi, v\rangle_{H^1_0(\Omega)}$ on $H^1_0\left(\Omega\right)$. Therefore, there exists an element $\hat{w} \in H^1_0\left(\Omega\right)$ independent of $\sigma^p_{\alpha,\delta}$ such that

$$\left\langle w, \Psi\left(\sigma^p_{\alpha,\delta} - \sigma^+\right)\right\rangle_{H^1_0(\Omega)} = \int_\Omega \sigma^+\nabla\hat{w}\cdot\nabla\Psi\left(\sigma^p_{\alpha,\delta} - \sigma^+\right)dx.$$

This implies that

$$\left\langle\xi, \sigma^p_{\alpha,\delta} - \sigma^+\right\rangle_{(L^{q_1}(\Omega'), L^q(\Omega'))} = \int_\Omega \sigma^+\nabla\hat{w}\cdot\nabla\Psi\left(\sigma^p_{\alpha,\delta} - \sigma^+\right)dx. \tag{34}$$

By (13), we get

$$\begin{aligned} &\int_\Omega \sigma^+\nabla\hat{w}\cdot\nabla F'_N\left(\sigma^+\right)j\left(\sigma^p_{\alpha,\delta} - \sigma^+\right)dx = -\int_\Omega\left(\sigma^p_{\alpha,\delta} - \sigma^+\right)\nabla F_N\left(\sigma^+\right)j\cdot\nabla\hat{w}dx \\ &= -\int_\Omega \sigma^+\nabla F_N\left(\sigma^+\right)j\cdot\nabla\hat{w}dx + \int_\Omega \sigma^p_{\alpha,\delta}\nabla F_N\left(\sigma^+\right)j\cdot\nabla\hat{w}dx \\ &= -\int_\Omega \sigma^p_{\alpha,\delta}\nabla F_N\left(\sigma^p_{\alpha,\delta}\right)j\cdot\nabla\hat{w}dx + \int_\Omega \sigma^p_{\alpha,\delta}\nabla F_N\left(\sigma^+\right)j\cdot\nabla\hat{w}dx \\ &= \int_\Omega \sigma^p_{\alpha,\delta}\nabla\left(F_N\left(\sigma^+\right)j - F_N\left(\sigma^p_{\alpha,\delta}\right)j\right)\cdot\nabla\hat{w}dx. \end{aligned} \tag{35}$$

Similarly, since $\hat{w} \in H_0^1(\Omega)$, we have

$$\int_\Omega \sigma^+ \nabla \hat{w} \cdot \nabla F_D'(\sigma^+) g \left(\sigma_{\alpha,\delta}^p - \sigma^+\right) dx = \int_\Omega \sigma_{\alpha,\delta}^p \nabla \left(F_D(\sigma^+) g - F_D\left(\sigma_{\alpha,\delta}^p\right) g\right) \cdot \nabla \hat{w} dx. \tag{36}$$

Therefore, by (34), (35) and (36), we have

$$\begin{aligned}
\Sigma := \left\langle \xi, \sigma_{\alpha,\delta}^p - \sigma^+ \right\rangle_{(L^{q_1}(\Omega'), L^q(\Omega'))} &= \int_\Omega \sigma_{\alpha,\delta}^p \nabla \left(F_D\left(\sigma_{\alpha,\delta}^p\right) g - F_N\left(\sigma_{\alpha,\delta}^p\right) j\right) \cdot \nabla \hat{w} dx \\
&= \int_\Omega \sigma_{\alpha,\delta}^p \nabla \left(F_D\left(\sigma_{\alpha,\delta}^p\right) g - F_D\left(\sigma_{\alpha,\delta}^p\right) g^\delta\right) \cdot \nabla \hat{w} dx \\
&\quad - \int_\Omega \sigma_{\alpha,\delta}^p \nabla \left(F_N\left(\sigma_{\alpha,\delta}^p\right) j^\delta - F_D\left(\sigma_{\alpha,\delta}^p\right) g^\delta\right) \cdot \nabla \hat{w} dx \\
&\quad + \int_\Omega \sigma_{\alpha,\delta}^p \nabla \left(F_N\left(\sigma_{\alpha,\delta}^p\right) j^\delta - F_N\left(\sigma_{\alpha,\delta}^p\right) j\right) \cdot \nabla \hat{w} dx \\
&= \Sigma_1 + \Sigma_2 + \Sigma_3.
\end{aligned} \tag{37}$$

Using the Cauchy-Schwart inequality, Remark 2 and Lemma 3, with $q \in \left(\frac{2Q}{Q-2}, \infty\right]$ we get

$$\begin{aligned}
|\Sigma_1| &\leq \left\| \nabla \left(F_D\left(\sigma_{\alpha,\delta}^p\right) g - F_D\left(\sigma_{\alpha,\delta}^p\right) g^\delta\right) \right\|_{L^2(\Omega)} \left\| \sigma_{\alpha,\delta}^p \nabla \hat{w} \right\|_{L^2(\Omega)} \\
&\leq \frac{C_D}{\lambda} \|\nabla \hat{w}\|_{L^2(\Omega)} \left\| g - g^\delta \right\|_{\tilde{H}^{1/2}(\Gamma)}.
\end{aligned} \tag{38}$$

Similarly, we have the following estimates for Σ_2 and Σ_3

$$\begin{aligned}
|\Sigma_2| &\leq \left(\int_\Omega \sigma_{\alpha,\delta}^p |\nabla \left(F_N\left(\sigma_{\alpha,\delta}^p\right) j^\delta - F_D\left(\sigma_{\alpha,\delta}^p\right) g^\delta\right)|^2 dx \right)^{1/2} \left(\int_\Omega \sigma_{\alpha,\delta}^p |\nabla \hat{w}|^2 dx \right)^{1/2} \\
&\leq \left(F_\delta\left(\sigma_{\alpha,\delta}^p\right)\right)^{1/2} \lambda^{-1/2} \|\nabla \hat{w}\|_{L^2(\Omega)} \\
&\leq \frac{1}{2\alpha} F_\delta\left(\sigma_{\alpha,\delta}^p\right) + \frac{\alpha}{2} \lambda^{-1} \|\nabla \hat{w}\|_{L^2(\Omega)}^2
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
|\Sigma_3| &\leq \left\| \nabla \left(F_N\left(\sigma_{\alpha,\delta}^p\right) j^\delta - F_N\left(\sigma_{\alpha,\delta}^p\right) j\right) \right\|_{L^2(\Omega)} \left\| \sigma_{\alpha,\delta}^p \nabla \hat{w} \right\|_{L^2(\Omega)} \\
&\leq \lambda^{-1} \|\nabla \hat{w}\|_{L^2(\Omega)} C_N \left\| j^\delta - j \right\|_{\tilde{H}^{-1/2}(\Gamma)}.
\end{aligned} \tag{40}$$

By (37)-(40), we get

$$|\Sigma| \leq \lambda^{-1} \|\nabla \hat{w}\|_{L^2(\Omega)} \max(C_N, C_D) \delta + \frac{1}{2\alpha} F_\delta\left(\sigma_{\alpha,\delta}^p\right) + \frac{\alpha}{2} \lambda^{-1} \|\nabla \hat{w}\|_{L^2(\Omega)}^2. \tag{41}$$

From this inequality and (31), we have

$$\frac{1}{2} F_\delta\left(\sigma_{\alpha,\delta}^p\right) + \alpha D_\xi\left(\sigma_{\alpha,\delta}^p, \sigma^+\right) \leq \beta_1 \delta^2 + \beta_2 \delta \alpha + \beta_3 \alpha^2 := \Sigma_4, \tag{42}$$

where

$$\beta_1 = \lambda^{-1} \max\left(C_N^2, C_D^2\right),$$

$$\beta_2 = \lambda^{-1} \|\nabla \hat{w}\|_{L^2(\Omega)} \max(C_N, C_D), \quad \beta_3 = \frac{1}{2} \lambda^{-1} \|\nabla \hat{w}\|_{L^2(\Omega)}^2.$$

With $\alpha \sim \delta$, it follows that

$$F_\delta\left(\sigma_{\alpha,\delta}^p\right) = O\left(\delta^2\right) \text{ and } D_\xi\left(\sigma_{\alpha,\delta}^p, \sigma^+\right) = O(\delta).$$

In particular, for $p \in (1, 2]$ there exists a constant $C_p > 0$ such that

$$D_\xi\left(\sigma_{\alpha,\delta}^p, \sigma^+\right) \geq C_p \left\| \sigma_{\alpha,\delta}^p - \sigma^+ \right\|_{H_0^1(\Omega')}^2,$$

see [18, Lemma 10.]. Therefore, we have

$$\left\|\sigma^p_{\alpha,\delta} - \sigma^+\right\|_{H^1_0(\Omega')} = O\left(\sqrt{\delta}\right).$$

■

Remark 15 *1. To obtain the convergence rates, we do not require the smallness in the source condition [26, 14, 21, 18], which is often required in inverse problems when the least squares approach is used, but it requires (29). The reason is that it ensures the validity of the equality (32).*

2. In [26] the least squares approach incorporating with sparsity regularization is used for EIT. To obtain these convergence rates, the authors not only need the smallness in the source condition but also need the enough closeness of λ to 1, see [26, Theorem 4.7] and [26, Corollary 2.1]. Furthermore, their result does not include the case $p = 1$. Here, we only need the condition (29) and the convergence rates cover the case $p = 1$.

5 Conclusion

We have investigated sparsity regularization for electrical impedance tomography. The sparsity regularization method incorporated with the energy functional approach was analyzed and the well-posedness and convergence rates of the method was obtained under the source condition.

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