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Abstract: We derive a method for the computation of robust domains of attraction based on a recent generalization of Zubov's theorem on representing robust domains of attraction for perturbed systems via the viscosity solution of a suitable partial differential equation. While a direct discretization of the equation leads to numerical difficulties due to a singularity at the stable equilibrium, a suitable regularization enables us to apply a standard discretization technique for Hamilton-Jacobi-Bellman equations. We present the resulting fully discrete scheme and show a numerical example.

1 Introduction

The domain of attraction of an asymptotically stable fixed point has been one of the central objects in the study of continuous dynamical systems. The knowledge of this object is important in many applications modeled by those systems like e.g. the analysis of power systems [1] and turbulence phenomena in fluid dynamics [2, 8, 17]. Several papers and books discuss theoretical [19, 20, 5, 12] as well as computational aspects [18, 13, 1, 9] of this problem.

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Taking into account that usually mathematical models of complex systems contain model errors and that exogenous perturbations are ubiquitous it is natural to consider systems with deterministic time varying perturbations and look for domains of attraction that are robust under all these perturbations. Here we consider systems of the form

$$\dot{x}(t) = f(x(t), a(t)), \quad x \in \mathbb{R}^n$$

where $a(\cdot)$ is an arbitrary measurable function with values in some compact set $A \subset \mathbb{R}^m$. Under the assumption that $x^* \in \mathbb{R}^n$ is a locally exponentially stable fixed point for all admissible perturbation functions $a(\cdot)$ we try to find the set of points which are attracted to x^* for all admissible $a(\cdot)$.

This set has been considered e.g. in [14, 15, 4, 7]. In particular, in [14] and [7] numerical procedures based on optimal control techniques for the computation of robust domains of attraction are presented. The techniques in these papers have in common that a numerical approximation of the optimal value function of a suitable optimal control problem is computed such that the robust domain of attraction is characterized by a suitable sublevel set of this function. Whereas the method in [14] requires the numerical solution of several Hamilton-Jacobi-Bellman equations (and is thus very expensive) the method in [7] needs just one such solution, but requires some knowledge about the local behavior around x^* in order to avoid discontinuities in the optimal value functions causing numerical problems.

In this paper we use a similar optimal control technique, but start from recent results in [4] where the classical equation of Zubov [20] is generalized to perturbed systems. Under very mild conditions on the problem data this equation admits a continuous or even Lipschitz viscosity solution. The main problem in a numerical approximation is the inherent singularity of the equation at the fixed point which prevents the direct application of usual numerical schemes. Here we propose a regularization of this equation such that the classical schemes [6] and adaptive gridding techniques [11] are applicable without losing the main feature of the solution, i.e. the sublevel set characterization of the robust domain of attraction. It might be worth noting that in particular our approach is applicable to the classical Zubov equation (i.e. for unperturbed systems) and hence provides a way to compute domains of attraction also for unperturbed systems.

This paper is organized as follows: In Section 2 we give the setup and collect some facts about robust domains of attraction. In Section 3 we summarize the needed results from [4] on the generalization of Zubov's equation for perturbed system. In Section 4 we introduce the regularization technique and formulate the numerical scheme, and finally, in Section 5 we show a numerical example.

2 Robust domains of attraction

We consider systems of the following form

$$\begin{cases} \dot{x}(t) = f(x(t), a(t)), & t \in [0, \infty), \\ x(0) = x_0, \end{cases} \tag{2.1}$$

with solutions denoted by $x(t, x_0, a)$. Here $a(\cdot) \in \mathcal{A} = L^\infty([0, +\infty), A)$ and A is a compact subset of $\mathbb{R}^m$, f is continuous and bounded in $\mathbb{R}^n \times A$ and Lipschitz in x uniformly in $a \in A$. Furthermore the fixed point $x = 0$ is singular, that is $f(0, a) = 0$ for any $a \in A$.

We assume that the singular point 0 is uniformly locally exponentially stable for the system (2.1), i.e.

$$\text{(H1)} \quad \begin{array}{l} \text{there exist constants } C, \sigma, r > 0 \text{ such that } \|x(t, x_0, a)\| \le Ce^{-\sigma t}\|x_0\| \\ \text{for any } x_0 \in B(0, r) \text{ and any } a \in A. \end{array}$$

The following sets describe domains of attraction for the equilibrium $x = 0$ of the system (2.1).

Definition 2.1 For the system (2.1) satisfying (H1) we define the *robust domain of attraction* as

$$\mathcal{D} = \{x_0 \in \mathbb{R}^n \ : \ x(t, x_0, a) \to 0 \text{ as } t \to +\infty \text{ for any } a \in \mathcal{A}\} \ ,$$

and the *uniform robust domain of attraction* by

$$\mathcal{D}_0 = \left\{ x_0 \in \mathbb{R}^n \ : \ \begin{array}{l} \text{there exists a function } \beta(t) \to 0 \text{ as } t \to \infty \\ \text{s.th. } \|x(t, x_0, a)\| \le \beta(t) \text{ for all } t > 0,\ a \in \mathcal{A} \end{array} \right\}.$$

The following proposition summarizes several properties of (uniform) robust domains of attraction as proved in [4, Proposition 2.4]. Observe that several of these properties are very similar to those of the domain of attraction of an asymptotically stable fixed point of a time-invariant system, compare [12, Chap. IV].

Proposition 2.2 Consider system (2.1) and assume (H1), then

(i) $\mathrm{cl}B(0, r) \subset \mathcal{D}_0$.

(ii) $\mathcal{D}_0$ is an open, connected, invariant set. $\mathcal{D}$ is a pathwise connected, invariant set.

(iii) $\sup_{a \in \mathcal{A}}\{t(x, a)\} \to +\infty$ for $x \to x_0 \in \partial\mathcal{D}_0$ or $\|x\| \to \infty$,
where $t(x, a) := \inf\{t > 0 : \ x(t, x, a) \in B(0, r)\}$.

(iv) $\mathrm{cl}\mathcal{D}_0$, $\mathrm{cl}\mathcal{D}$ are invariant sets.

(v) $\mathcal{D}_0$, $\mathcal{D}$ are contractible to 0.

(vi) If for some $a_0 \in A$ $f(\cdot, a_0)$ is of class C^r, then $\mathcal{D}_0$ is C^r-diffeomorphic to $\mathbb{R}^n$.

(vii) If for every $x \in \partial\mathcal{D}_0$ there exists $a \in \mathcal{A}$ such that $x(t, x, a) \in \partial\mathcal{D}_0$ for all $t \ge 0$ then $\mathcal{D} = \mathcal{D}_0$.

(viii) If for all $x \in \mathcal{D}$ the set $\{f(x, a) \ : \ a \in \mathcal{A}\}$ is convex then $\mathcal{D}_0 = \mathcal{D}$.

3 Zubov's method for robust domains of attraction

In this section we discuss the following partial differential equation

$$\sup_{a\in A}\{Dv(x)f(x,a)+(1-v(x))g(x,a)\}=0 \qquad x\in\mathbb{R}^n \tag{3.1}$$

whose solution will turn out to characterize the uniform robust domain of attraction $\mathcal{D}_0$. This equation is a straightforward generalization of Zubov's equation [20]. In this generality, however, in order to obtain a meaningful result about solutions we have to work within the framework of viscosity solutions, which we recall for the convenience of the reader (for details about this theory we refer to [3]).

Definition 3.1 Given an open subset Ω of $\mathbb{R}^n$ and a continuous function $H:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, we say that a lower semicontinuous (l.s.c.) function $u:\Omega\to\mathbb{R}$ (resp. an upper semicontinuous (u.s.c.) function $v:\Omega\to\mathbb{R}$) is a viscosity supersolution (resp. subsolution) of the equation

$$H(x,u,Du)=0 \qquad x\in\Omega \tag{3.2}$$

if for all $\phi\in C^1(\Omega)$ and $x\in\operatorname{argmin}_\Omega(u-\phi)$ (resp., $x\in\operatorname{argmax}_\Omega(v-\phi)$) we have

$$H(x,u(x),D\phi(x))\geq 0 \qquad (\text{resp., } H(x,v(x),D\phi(x))\leq 0).$$

A continuous function $u:\Omega\to\mathbb{R}$ is said to be a viscosity solution of (3.2) if u is a viscosity supersolution and a viscosity subsolution of (3.2).

We now introduce the value function of a suitable optimal control problem related to (3.1). Consider the following nonnegative, extended value functional $G:\mathbb{R}^n\times\mathcal{A}\to\mathbb{R}\cup\{+\infty\}$

$$G^\infty(x,a):=\int_0^{+\infty}g(x(t),a(t))dt$$

and the optimal value function

$$v(x):=\sup_{a\in\mathcal{A}}1-e^{-G^\infty(x,a)}. \tag{3.3}$$

The function $g:\mathbb{R}^n\times A\to\mathbb{R}$ is supposed to be continuous and satisfies

(H2)
- (i) For any $a\in A$, $g(0,a)=0$ and $g(x,a)>0$ for $x\neq 0$.
- (ii) There exists a constant $g_0>0$ such that $\inf_{x\notin B(0,r),\,a\in A}g(x,a)\geq g_0$.
- (iii) For every $R>0$ there exists a constant L_R such that $\|g(x,a)-g(y,a)\|\leq L_R\|x-y\|$ for all $\|x\|$, $\|y\|\leq R$, and all $a\in A$.

Since g is nonnegative it is immediate that $v(x)\in[0,1]$ for all $x\in\mathbb{R}^n$. Furthermore, standard techniques from optimal control (see e.g. [3, Chapter III]) imply that v satisfy a dynamic programming principle, i.e. for each $t>0$ we have

$$v(x)=\sup_{a\in\mathcal{A}}\{(1-G(x,t,a))+G(x,t,a)v(x(t,x,a))\} \tag{3.4}$$

with

$$G(t,x,a) := \exp\left(-\int_0^t g(x(\tau,x,a),a(\tau))d\tau\right). \tag{3.5}$$

Furthermore, a simple application of the chain rule shows

$$(1 - G(x,t,a)) = \int_0^t G(x,\tau,a)g(x(\tau,x,a),a(\tau))d\tau$$

implying

$$v(x) = \sup_{a\in\mathcal{A}}\left\{\int_0^t G(x,\tau,a)g(x(\tau,x,a),a(\tau))d\tau + G(x,t,a)v(x(t,x,a))\right\} \tag{3.6}$$

The next proposition shows the relation between $\mathcal{D}_0$ and v, and the continuity of v. For the proof see [4, Proposition 3.1]

Proposition 3.2 Assume (H1), (H2). Then

(i) $v(x) < 1$ if and only if $x \in \mathcal{D}_0$.

(ii) $v(0) = 0$ if and only if $x = 0$.

(iii) v is continuous on $\mathbb{R}^n$.

(iv) $v(x) \to 1$ for $x \to x_0 \in \partial\mathcal{D}_0$ and for $|x| \to \infty$.

We now turn to the relation between v and equation (3.1). Recalling that v is locally bounded on $\mathbb{R}^n$ an easy application of the dynamic programming principle (3.4) (cp. [3, Chapter III]) shows that and v is a viscosity solution of (3.1). The more difficult part is to obtain uniqueness of the solution, since equation (3.1) exhibits a singularity at the origin.

In order to get a uniqueness result we use the following super- and suboptimality principles, which essentially follow from Soravia [16, Theorem 3.2 (i)], see [4, Proposition 3.5] for details.

Proposition 3.3

(i) Let w be a l.s.c. supersolution of (3.1) in $\mathbb{R}^n$, then for any $x \in \mathbb{R}^n$

$$w(x) = \sup_{a\in\mathcal{A}}\sup_{t\geq 0}\{(1 - G(x,t,a)) + G(x,t,a)w(x(t))\}. \tag{3.7}$$

(ii) Let u be a u.s.c. subsolution of (3.1) in $\mathbb{R}^n$, and $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ be a continuous function with $u \leq \tilde{u}$. Then for any $x \in \mathbb{R}^n$ and any $T \geq 0$

$$u(x) \leq \sup_{a\in\mathcal{A}}\inf_{t\in[0,T]}\{(1 - G(x,t,a)) + G(x,t,a)\tilde{u}(x(t))\}. \tag{3.8}$$

Remark 3.4 If u is continuous or the set of the control functions $\mathcal{A}$ is replaced by the set of relaxed control laws $\mathcal{A}^r$, assertion (ii) can be strengthened to

$$u(x) = \sup_{\mu\in\mathcal{A}^r} \inf_{t\geq 0} \{(1 - G(x,t,\mu)) + G(x,t,\mu)u(x(t))\},$$

which follows from [16, Theorem 3.2(iii)].

We can now apply these principles to the generalized version of Zubov's equation (3.1) in order to obtain comparison principles for sub and supersolutions. Since these are the essential properties for our theory, we recall the proofs from [4].

Proposition 3.5 Let w be a bounded l.s.c. supersolution of (3.1) on $\mathbb{R}^n$ with $w(0) \geq 0$. Then $w \geq v$ for v as defined in (3.3).

Proof: First observe that the lower semicontinuity of w and the assumption $w(0) \geq 0$ imply that for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$w(x) \geq -\epsilon \text{ for all } x \in \mathbb{R}^n \text{ with } \|x\| \leq \delta. \tag{3.9}$$

Furthermore, the upper optimality principle (3.7) implies

$$w(x_0) \geq \sup_{a\in\mathcal{A}} \inf_{t\geq 0} \{1 + G(x_0,t,a)(w(x(t,x_0,a)) - 1)\}. \tag{3.10}$$

Now we distinguish two cases:
(i) $x_0 \in \mathcal{D}_0$: In this case we know that for each $a \in \mathcal{A}$ we have $x(t,x_0,a) \to 0$ as $t \to \infty$. Thus from (3.9) and (3.10), and using the definition of v we can conclude

$$w(x_0) \geq \sup_{a\in\mathcal{A}} \left\{ \lim_{t\to\infty} (1 - G(x_0,t,a)) \right\} = v(x_0).$$

which shows the claim.
(ii) $x_0 \notin \mathcal{D}_0$: Since $v(x) \in [0,1]$ for all $x \in \mathbb{R}^n$ it is sufficient to show that $w(x_0) \geq 1$. Now consider the time $t(x,a)$ as defined in Proposition 2.2(iii). By the definition of $\mathcal{D}_0$ we know that for each $T > 0$ there exists $a_T \in \mathcal{A}$ such that $t(x_0,a_T) > T$, which implies $G(x_0,T,a_T) \leq \exp(-Tg_0)$ which tends to 0 as $T \to \infty$. Thus denoting the bound on $|w|$ by $M > 0$ the inequality (3.10) implies

$$w(x_0) \geq (1 - \exp(-Tg_0)) - \exp(-Tg_0)M$$

for every $T > 0$ and hence $w(x_0) \geq 1$. □

Proposition 3.6 Let u be a bounded u.s.c. subsolution of (3.1) on $\mathbb{R}^n$ with $u(0) \leq 0$. Then $u \leq v$ for v defined in (3.3).

Proof: By the upper semicontinuity of u and $u(0) \le 0$ we obtain that for every $\epsilon > 0$ there exists a $\delta > 0$ with $u(x) \le \epsilon$ for all $x \in \mathbb{R}^n$ with $\|x\| \le \delta$. Thus for each $\epsilon > 0$ we find a bounded and continuous function $\tilde{u}_\epsilon : \mathbb{R}^n \to \mathbb{R}$ with

$$\tilde{u}_\epsilon(0) < \epsilon \text{ and } u \le \tilde{u}_\epsilon. \tag{3.11}$$

Now the lower optimality principle (3.8) implies for every $t \ge 0$ that

$$u(x_0) \le \sup_{a \in \mathcal{A}} \{1 + G(x_0, t, a)(\tilde{u}_\epsilon(x(t, x_0, a)) - 1)\} . \tag{3.12}$$

Again, we distinguish two cases:
(i) $x_0 \in \mathcal{D}_0$: In this case $\|x(t, x_0, a)\| \to 0$ as $t \to \infty$ uniformly in $a \in \mathcal{A}$. Hence for each $\epsilon > 0$ there exists $t_\epsilon > 0$ such that

$$\tilde{u}_\epsilon(x(t_\epsilon, x_0, a)) \le \epsilon \quad \text{and} \quad |G(x_0, t_\epsilon, a) - G(x_0, \infty, a)| \le \epsilon$$

for all $a \in \mathcal{A}$. Thus from (3.11) and (3.12), and using the definition of v we can conclude

$$u(x_0) \le \sup_{a \in \mathcal{A}} \{1 - (1 - \epsilon)G(x_0, t_\epsilon, a)\} \le v(x_0) + \epsilon(1 - v(x_0)) + \epsilon\,,$$

which shows the claim since v is bounded and $\epsilon > 0$ was arbitrary.
(ii) $x_0 \notin \mathcal{D}_0$: Since in this case $v(x_0) = 1$ (by Proposition 3.2(i)) it is sufficient to show that $u(x_0) \le 1$. By (i) we know that $u(y) \le v(y) < 1$ for each $y \in \mathcal{D}_0$, hence analogous to (3.11) for each $\epsilon > 0$ we can conclude the existence of a continuous $\tilde{u}_\epsilon$ with $u \le \tilde{u}_\epsilon$ and $\tilde{u}_\epsilon(y) \le 1 + \epsilon$ for each $y \in \mathcal{D}_0$; w.l.o.g. we may choose $\tilde{u}_\epsilon$ such that $M := \inf_{x \in \mathbb{R}^n \setminus \mathcal{D}_0} \tilde{u}_\epsilon(x) \ge 1$. Now fix $\epsilon > 0$ and consider a sequence $t_n \to \infty$. Then (3.12) implies that there exists a sequence $a_n \in \mathcal{A}$ with

$$u(x_0) - \epsilon \le 1 + G(x_0, t_n, a_n)(\tilde{u}_\epsilon(x(t_n, x_0, a_n)) - 1).$$

If $x(t_n, x_0, a_n) \in \mathcal{D}_0$ we know that $\tilde{u}_\epsilon(x(t_n, x_0, a_n)) \le 1 + \epsilon$, and since $G \le 1$ we obtain $u(x_0) - \epsilon \le 1 + \epsilon$. If $x(t_n, x_0, a_n) \notin \mathcal{D}_0$ then $G(x_0, t_n, a_n) \le \exp(-g_0 t_n)$, thus

$$1 + G(x_0, t_n, a_n)(\tilde{u}_\epsilon(x(t_n, x_0, a_n)) - 1) \le 1 + \exp(-g_0 t_n)(M - 1).$$

Thus for each $n \in \mathbb{N}$ we obtain

$$u(x_0) \le 2\epsilon + 1 + \exp(-g_0 t_n)(M - 1)\,,$$

which for $n \to \infty$ implies $u(x_0) \le 1 + 2\epsilon$. This proves the assertion since $\epsilon > 0$ was arbitrary.

□

Using these propositions we can now formulate an existence and uniqueness theorem for the generalized version of Zubov's equation (3.1).

Theorem 3.7 Consider the system (2.1) and a function $g : \mathbb{R}^n \times A \to \mathbb{R}$ such that (H1) and (H2) are satisfied. Then (3.1) has a unique bounded and continuous viscosity solution v on $\mathbb{R}^n$ satisfying $v(0) = 0$.

This function coincides with v from (3.3). In particular the characterization $\mathcal{D}_0 = \{x \in \mathbb{R}^n \mid v(x) < 1\}$ holds.

Proof: This is immediate from Propositions 3.5 and 3.6. □

The following theorem is an immediate consequence of Theorem 3.7. It shows that we can restrict ourselves to a proper open subset $\mathcal{O}$ of the state space and still obtain our solution v, provided $\mathcal{D}_0 \subseteq \mathcal{O}$. This is in particular important for our computational approach as we will not be able to approximate v on the whole $\mathbb{R}^n$.

Theorem 3.8 Consider the system (2.1) and a function $g : \mathbb{R}^n \times A \to \mathbb{R}$. Assume (H1) and (H2). Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set containing the origin, and let $v : \mathrm{cl}\mathcal{O} \to \mathbb{R}$ be a bounded and continuous function which is a viscosity solution of (3.1) on $\mathcal{O}$ and satisfies $v(0) = 0$ and $v(x) = 1$ for all $x \in \partial\mathcal{O}$.

Then v coincides with the restriction $v|_{\mathcal{O}}$ of the function v from (3.3). In particular the characterization $\mathcal{D}_0 = \{x \in \mathbb{R}^n \mid v(x) < 1\}$ holds.

Proof: Any solution $\tilde{v}$ meeting the assumption can be continuously extended to a viscosity solution of (3.1) on $\mathbb{R}^n$ by setting $\tilde{v}(x) = 1$ for $x \in \mathbb{R}^n \setminus \mathcal{O}$. Hence the assertion follows. □

We end this section by stating several additional properties of v as proved in [4, Sections 4 and 5].

Theorem 3.9 Assume (H1) and (H2) and consider the unique viscosity solution v of (3.1) with $v(0) = 0$. Then the following statements hold.

(i) The function v is a robust Lyapunov function for the system (2.1). More precisely we have

$$v(x(t, x_0, a(\cdot))) - v(x_0) \le \left[1 - \exp\left(-\int_0^T g(x(t), a(t))dt\right)\right] (v(x(t, x_0, a(\cdot))) - 1) < 0$$

for all $x_0 \in \mathcal{D}_0 \setminus \{0\}$ and all $a(\cdot) \in \mathcal{A}$.

(ii) If $f(\cdot, a)$ and $g(\cdot, a)$ are uniformly Lipschitz continuous in $\mathbb{R}^n$, with constants L_f, $L_g > 0$ uniformly in $a \in A$, and if there exists a neighborhood N of the origin such that for all x, $y \in N$ the inequality

$$|g(x, a) - g(y, a)| \le K \max\{\|x\|, \|y\|\}^s \|x - y\|$$

holds for some $K > 0$ and $s > L_f/\sigma$ with $\sigma > 0$ given by (H1), then the function v is Lipschitz continuous in $\mathbb{R}^n$ for all g with $g_0 > 0$ from (H2) sufficiently large.

(iii) If $f(\cdot, a)$ is C^r for some $a \in A$ and $B \subset \mathcal{D}_0$ is such that $\mathrm{dist}(B, \partial\mathcal{D}_0) > 0$, then there exists a function $g : \mathbb{R}^n \to \mathbb{R}$ such that v is C^r on a neighborhood of B.

4 Numerical solution

A first approach to solve equation (3.1) is by directly adapting the first order numerical scheme from [6] to this equation. Considering a bounded domain Ω and a simplicid grid Γ with edges x_i covering $\mathrm{cl}\Omega$ this results in solving

$$\tilde{v}(x_i) = \max_{a \in A}\{(1 - hg(x_i, a))\tilde{v}(x_i + hf(x_i, a)) + hg(x_i, a)\} \tag{4.1}$$

where $\tilde{v}$ is continuous and affinely linear on each simplex in the grid and satisfies $\tilde{v}(0) = 0$ (assuming, of course, that 0 is a node of the grid) and $\tilde{v}(x_i) = 1$ for all $x_i \in \partial\Omega$. Unfortunately, since also (4.1) has a singularity in 0 the fixed point argument used in [6] fails here and hence convergence is not guaranteed. In fact, it is easy to see that in the situation of Figure 4.1 (showing one trajectory and the simplices surrounding the fixed point 0 in a two-dimensional example) the piecewise linear function $\tilde{v}$ with

$$\tilde{v}(x_i) = \begin{cases} 1, & x_i \neq 0 \\ 0, & x_i = 0 \end{cases}$$

satisfies (4.1), since for all nodes $x_i \neq 0$ the value $x_i + hf(x_i, a)$ lies in a simplex with nodes $x_j \neq 0$, hence $\tilde{v}(x_i + hf(x_i, a)) = 1$ implying

$$(1 - hg(x_i, a))\tilde{v}(x_i + hf(x_i, a)) + hg(x_i, a) = 1 = \tilde{v}(x_i),$$

i.e. (4.1). As this situation may occur for arbitrarily fine grids indeed convergence is not guaranteed.

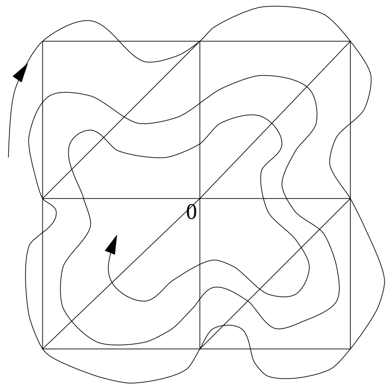

Figure 4.1: A situation of non-convergence

In order to ensure convergence we will therefore have to use a regularization of (3.1). The main idea in this is to change (3.1) in such a way that the "discount rate" (i.e. the factor $g(x)$ in front of the zero order term $v(x)$) becomes strictly positive, and thus the singularity disappears. To this end consider some parameter $\varepsilon > 0$ and consider the function

$$g_\varepsilon(x, a) = \max\{g(x, a), \varepsilon\}.$$

Using this g_ε we approximate (3.1) by

$$\sup_{a \in A} \{Dv(x)f(x, a) + g(x, a) - v(x)g_\varepsilon(x, a)\} = 0 \qquad x \in \mathbb{R}^n. \tag{4.2}$$

The following proposition summarizes some properties of (4.2). We state it in a global version on $\mathbb{R}^n$, the analogous statements hold in the situation of Theorem 3.8.

Proposition 4.1 Let the assumptions of Theorem 3.7 hold and let v be the unique solution of (3.1) with $v(0) = 0$. Then for each $\varepsilon > 0$ equation (4.2) has a unique continuous viscosity solution v_ε with the following properties.

(i) $v_\varepsilon(x) \le v(x)$ for all $x \in \mathbb{R}^n$

(ii) $v_\varepsilon \to v$ uniformly in $\mathbb{R}^n$ as $\varepsilon \to 0$

(iii) If $\varepsilon < g_0$ from (H2)(ii) then the characterization $\mathcal{D}_0 = \{x \in \mathbb{R}^n \mid v_\varepsilon(x) < 1\}$ holds

(iv) If $f(\cdot, a)$ and $g(\cdot, a)$ are uniformly Lipschitz on $\mathcal{D}_0$ (uniformly in A with Lipschitz constants L_f and L_g) and g is bounded on $\mathcal{D}_0$ and satisfies the inequalities

$$|g(x,a) - g(y,a)| \le K \max\{\|x\|,\ \|y\|\}^s \|x-y\| \tag{4.3}$$

for all $x,\ y \in B(0, Cr)$ and

$$|g(x,a)| \ge g_1 > L_f \tag{4.4}$$

for all $x \notin B(0, r/2)$ with C, σ and r from (H1), then the function v_ε is uniformly Lipschitz on $\mathbb{R}^n$.

Proof: Since the discount rate in (4.2) is strictly positive it follows by standard viscosity solution arguments [3, Chapter III] that there exists a unique solution v_ε which furthermore for all $t \ge 0$ satisfies the following dynamic programming principle

$$v_\varepsilon(x) = \sup_{a \in \mathcal{A}} \left\{ \int_0^t G_\varepsilon(x,\tau,a) g(x(\tau,x,a), a(\tau)) d\tau + G_\varepsilon(x,t,a) v_\varepsilon(x(t,x,a)) \right\} \tag{4.5}$$

with

$$G_\varepsilon(x,t,a) := \exp\left(- \int_0^t g_\varepsilon(x(\tau,x,a), a(\tau)) d\tau \right). \tag{4.6}$$

Since v satisfies the same principle (3.6) with $G(x,t,a) \ge G_\varepsilon(x,t,a)$ from (3.5) and $g > 0$ the stated inequality (i) follows.

In order to see (ii) observe that the continuity of g and v implies that for each $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\{x \in \mathbb{R}^n \mid g_\varepsilon(x,a) \ge g(x,a) \text{ for some } a \in A\} \subset \{x \in \mathbb{R}^n \mid v(x) \le \delta\}.$$

Now fix $\delta > 0$ and consider the corresponding $\varepsilon > 0$. Let $x \in \mathbb{R}^n$ and pick some $\gamma > 0$ and a control $a_\gamma \in \mathcal{A}$ such that

$$v(x) \le \int_0^\infty G(x,\tau,a_\gamma) g(x(\tau,x,a_\gamma), a_\gamma(\tau)) d\tau + \gamma.$$

Now let $T \ge 0$ be the (unique) time with $v(x(T,x,a_\gamma)) = \delta$. Then we can conclude that

$$\begin{aligned}
& v(x) - v_\varepsilon(x) - \gamma \\
\le & \int_0^\infty (G(x,\tau,a_\gamma) g(x(\tau,x,a_\gamma), a_\gamma(\tau)) - G_\varepsilon(x,\tau,a_\gamma) g(x(\tau,x,a_\gamma), a_\gamma(\tau))) d\tau \\
\le & \int_0^T \underbrace{(G(x,\tau,a_\gamma) g(x(\tau,x,a_\gamma), a_\gamma(\tau)) - G_\varepsilon(x,\tau,a_\gamma) g(x(\tau,x,a_\gamma), a_\gamma(\tau)))}_{=0} d\tau \\
& + G(x,T,a_\gamma) v(x(T,x,a_\gamma)) \quad \le \quad \delta.
\end{aligned}$$

Since $\gamma > 0$ and $x \in \mathbb{R}^n$ were arbitrary this shows (ii).

To prove (iii) let $\varepsilon < g_0$. Then for all $x \notin \mathcal{D}_0$ and all $T > 0$ there exists $a \in \mathcal{A}$ such that $G(x,t,a) = G_\varepsilon(x,t,a)$ for all $t \in [0,T]$ which immediately implies $\mathcal{D}_0 = \{x \in \mathbb{R}^n \mid v_\varepsilon(x) < 1\}$.

In order to see (iv) first note that (4.3) holds for g_ε for all $\varepsilon \geq 0$ (with the convention $g_0 = g$). Hence by straightforward integration using the exponential stability and (4.3) we can estimate

$$|\int_0^t g_\varepsilon(x(\tau,x,a),a(\tau))d\tau - \int_0^t g_\varepsilon(x(\tau,y,a),a(\tau))d\tau| \leq L_0\|x-y\|$$

for all $x, y \in B(0,r)$ and some $L_0 > 0$ independent of ε and a, which also implies

$$|G_\varepsilon(x,t,a) - G_\varepsilon(y,t,a)| \leq L_0\|x-y\|$$

for all $t \geq 0$ and consequently

$$\begin{aligned}
&\sup_{a\in\mathcal{A}}\left|\int_0^\infty G_\varepsilon(x,\tau,a)g(x(\tau,x,a),a(\tau))d\tau - \int_0^\infty G_\varepsilon(y,\tau,a)g(x(\tau,y,a),a(\tau))d\tau\right| \\
&\leq \sup_{a\in\mathcal{A}}\int_0^\infty |G_\varepsilon(x,\tau,a) - G_\varepsilon(y,\tau,a)|\underbrace{g(x(\tau,x,a),a(\tau))}_{\leq L_g C e^{-\sigma\tau}\|x\|}d\tau \\
&\quad + \int_0^\infty \underbrace{G_\varepsilon(y,\tau,a)}_{\leq 1}|g(x(\tau,x,a),a(\tau)) - g(x(\tau,y,a),a(\tau))|d\tau \\
&\leq L_1\|x-y\|
\end{aligned} \tag{4.7}$$

for some suitable $L_1 > 0$ and all $x, y \in B(0,r)$, implying in particular

$$|v_\varepsilon(x) - v_\varepsilon(y)| \leq L_1\|x-y\|.$$

For all $t > 0$ with $x(s,x,a) \notin B(0,r/2)$ and $x(s,y,a) \notin B(0,r/2)$ for all $s \in [0,t]$ we can estimate

$$|G_\varepsilon(x,t,a)| \leq e^{-tg_1}, \quad |G_\varepsilon(y,t,a)| \leq e^{-tg_1} \tag{4.8}$$

and using $|e^{-a} - e^{-b}| \leq \max\{e^{-a}, e^{-b}\}|a-b|$ it follows

$$\begin{aligned}
|G_\varepsilon(x,t,a) - G_\varepsilon(y,t,a)| &\leq e^{-tg_1}\int_0^t |g_\varepsilon(x(\tau,x,a),a(\tau)) - g_\varepsilon(x(\tau,y,a),a(\tau))|d\tau \\
&\leq e^{-tg_1}\int_0^t L_g e^{\tau L_f}\|x-y\|d\tau \\
&\leq e^{-tg_1}\frac{L_g}{L_f}e^{tL_f}\|x-y\| \quad = \quad e^{t(L_f-g_1)}\frac{L_g}{L_f}\|x-y\|.
\end{aligned} \tag{4.9}$$

Now define $T(x,a) := \inf\{t > 0 : \ x(t,x,a) \in B(0,r/2)\}$. Then by continuous dependence on the initial value (recall that f is Lipschitz in x uniformly in $a \in A$) for each $x \in \mathcal{D}_0 \setminus B(0,r)$ there exists a neighborhood $\mathcal{N}(x)$ such that $x(t(x,a),y,a) \in B(0,r)$ and

$x(t(y,a),x,a) \in B(0,r)$ for all $y \in \mathcal{N}(x)$ and all $a \in \mathcal{A}$. Now pick some $x \in \mathcal{D}_0 \setminus B(0,r)$ and some $y \in \mathcal{N}(x)$. Then for each $\gamma > 0$ we find $a_\gamma \in \mathcal{A}$ such that

$$\begin{aligned} &|v_\varepsilon(x) - v_\varepsilon(y)| - \gamma \\ &\leq \left| \int_0^\infty G_\varepsilon(x,\tau,a_\gamma) g(x(\tau,x,a_\gamma),a_\gamma(\tau))d\tau - \int_0^\infty G_\varepsilon(y,\tau,a_\gamma) g(x(\tau,y,a_\gamma),a_\gamma(\tau))d\tau \right|. \end{aligned}$$

Now fix some $\gamma > 0$ and let $T := \min\{T(x,a_\gamma), T(y,a_\gamma)\}$. Abbreviating $x(t) = x(t,x,a_\gamma)$ and $y(t) = x(t,y,a_\gamma)$ we can conclude that $x(T) \in B(0,r)$ and $y(T) \in B(0,r)$. Hence we can continue

$$\begin{aligned} &|v_\varepsilon(x) - v_\varepsilon(y)| - \gamma \\ &\leq \left| \int_0^T G_\varepsilon(x,\tau,a_\gamma) g(x(\tau),a_\gamma(\tau))d\tau - \int_0^T G_\varepsilon(y,\tau,a_\gamma) g(y(\tau),a_\gamma(\tau))d\tau \right| \\ &\quad + G_\varepsilon(x,T,a_\gamma) \left| \int_0^\infty G_\varepsilon(x(T),\tau,a_\gamma(T+\cdot)) g(x(T+\tau),a_\gamma(T+\tau))d\tau \right. \\ &\qquad\qquad \left. - \int_0^\infty G_\varepsilon(y(T),\tau,a_\gamma(T+\cdot)) g(y(T+\tau),a_\gamma(T+\tau))d\tau \right| \\ &\leq \int_0^T |G_\varepsilon(x,\tau,a_\gamma) - G_\varepsilon(y,\tau,a_\gamma)| g(x(\tau),a_\gamma(\tau))d\tau \\ &\quad + \int_0^T G_\varepsilon(y,\tau,a_\gamma) |g(x(\tau,x,a_\gamma),a_\gamma(\tau)) - g(y(\tau),a_\gamma(\tau))| d\tau \\ &\quad + e^{-g_1 T} e^{L_f T} L_1 \|x-y\| \\ &\leq \int_0^T |G_\varepsilon(x,\tau,a_\gamma) - G_\varepsilon(y,\tau,a_\gamma)| \underbrace{g(x(\tau),a_\gamma(\tau))}_{\leq \sup_{x\in\mathcal{D}_0, a\in A} g(x,a) =: g^*} d\tau \\ &\quad + \int_0^T e^{-\tau g_1} \underbrace{|g(x(\tau),a_\gamma(\tau)) - g(y(\tau),a_\gamma(\tau))|}_{\leq L_g e^{tL_f} \|x-y\|} d\tau \\ &\quad + L_1 \|x-y\| \\ &\leq \left(g^* \frac{L_g}{L_f(g_1 - L_f)} + \frac{L_g}{g_1 - L_f} + L_1 \right) \|x-y\| \end{aligned}$$

since $g_1 > L_f$. Here the first inequality follows by splitting up the integrals using the triangle inequality, the second follows by the triangle inequality for the first term and using $x(T)$, $y(T) \in B(0,r)$, $\|x(T) - y(T)\| \leq e^{L_f T}$, and (4.7) for the second term, and the third and fourth inequality follow from (4.8) and (4.9).

Since $\gamma > 0$ was arbitrary the Lipschitz property follows on $\mathcal{D}_0$, thus also on $\mathrm{cl}\mathcal{D}_0$ and consequently on the whole $\mathbb{R}^n$ since $v_\varepsilon \equiv 1$ on $\mathbb{R}^n \setminus \mathcal{D}_0$. □

Remark 4.2 Note that in general the solution v_ε is not a robust Lyapunov function for the origin of (2.1) anymore. More precisely, we can only ensure decrease of v_ε along trajectories $x(t,x_0,a)$ as long as $g(x(t,x_0,a),a(t)) > \varepsilon$, i.e. outside the region where the regularization is effective. Hence although many properties of v are preserved in this regularization, some are nevertheless lost.

We now apply the numerical scheme from [6] to (4.2). Thus we end up with

$$\tilde{v}_\varepsilon(x_i) = \max_{a\in A}\{(1 - hg_\varepsilon(x_i, a))\tilde{v}_\varepsilon(x + hf(x_i, a)) + hg(x_i, a)\} \tag{4.10}$$

where again $\tilde{v}_\varepsilon$ is continuous and affinely linear on each simplex in the grid and satisfies $\tilde{v}_\varepsilon(0) = 0$ and $\tilde{v}_\varepsilon(x_i) = 1$ for all $x_i \in \partial\Omega$.

A straightforward modification of the arguments in [3, 6] yields that there exists a unique solution $\tilde{v}_\varepsilon$ converging to v_ε as h and the size of the simplices tends to 0. Note that the adaptive gridding techniques from [11] also apply to this scheme, and that a number of different iterative solvers for (4.10) are available, see e.g. [6, 10, 11].

Remark 4.3 The numerical examples show good results also in the case where we cannot expect a globally Lipschitz continuous solution v_ε of (4.2). The main reason for this seems to be that in any case v_ε is locally Lipschitz on $\mathcal{D}_0$. In order to explain this observation in a rigorous way a thorough analysis of the numerical error is currently under investigation.

5 A numerical example

We illustrate our algorithm with a model adapted from [17]

$$\dot{x} = \begin{pmatrix} -1/25 & 1 \\ 0 & -2/25 \end{pmatrix} x + \|x\| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ ax_1x_2 \end{pmatrix}$$

where $x = (x_1, x_2)^T \in \mathbb{R}^2$. The unperturbed equation (i.e. with $a = 0$) is introduced in order to explain the existence of turbulence in a fluid flow with Reynolds number $R = 25$ despite the stability of the linearization at the laminar solution. In [17] simulations are made in order to estimate the domain of attraction of the locally stable equilibrium at the origin. Here we compute it entirely in a neighborhood of 0, and in addition determine the effect of the perturbation term ax_1x_2 for time varying perturbation with different ranges A. Figure 5.1 shows the corresponding results obtained with the fully discrete scheme (4.10), setting $g(x, a) = \|x\|^2$, $\varepsilon = 10^{-10}$, $h = 1/20$. The grid was constructed adaptively using the techniques from [11] with a final number of about 20000 nodes. Note that due to numerical errors in the approximate solution it is not reasonable to take the "exact" sublevel sets $\tilde{v}_\varepsilon(x) < 1$, instead some "security factor" has to be added. The domains shown in the figures are the sublevel sets $\tilde{v}_\varepsilon(x) \leq 0.95$.

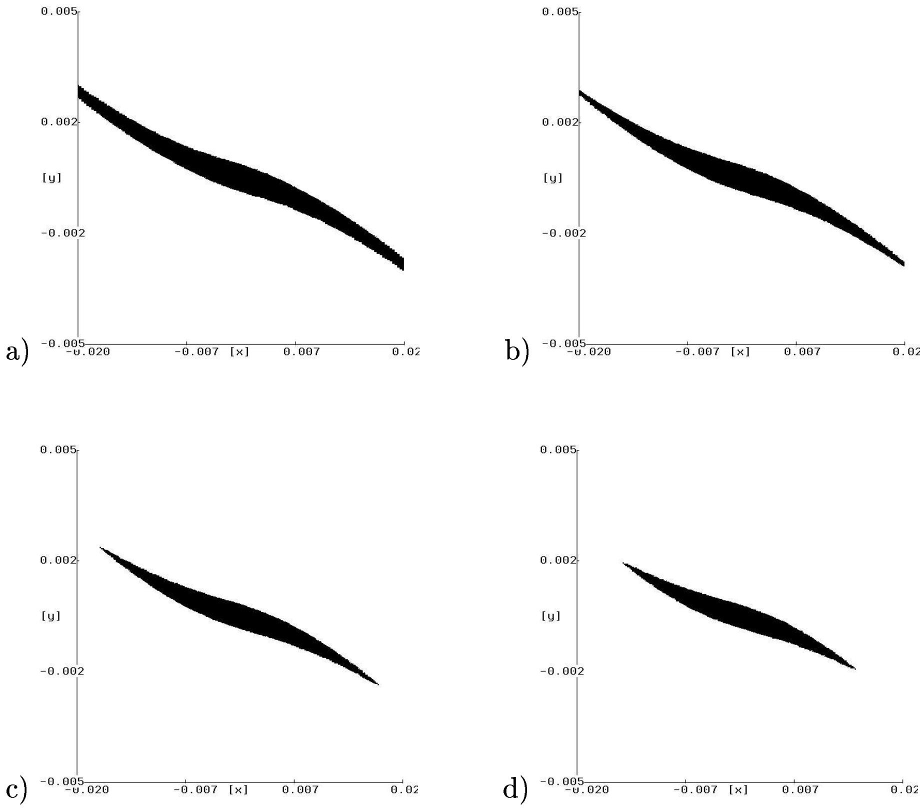

Figure 5.1: Approximation of $\mathcal{D}_0$ for a) $A = \{0\}$, b) $A = [-1, 1]$, c) $A = [-2, 2]$, and d) $A = [-3, 3]$

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