UNIVERSITÄT BREMEN

Zentrum für Technomathematik

Fachbereich 3 – Mathematik und Informatik

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Report 16–02

Berichte aus der Technomathematik

Report 16–02 Februar 2016

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Abstract

This work deals with mathematical modeling of processes involved in the quenching process of steel. Some comprehensive models which integrate the complex behavior of steel materials in general models with phase transitions in steel for small deformations and deal with the modeling of the mathematical problem of linear thermo-elasto-plasticity, taking into account phase transitions and transformation-induced plasticity are described. The main objective is to give a short review on existence and uniqueness results for the corresponding mathematical problems.

Kurzfassung

Diese Arbeit befasst sich mit der mathematischen Modellierung von Abschreckprozessen bei Stahlbauteilen. Es werden einige Gesamtmodelle für kleine Deformationen diskutiert, welche das komplexe physikalische Materialverhalten von Stahl in allgemeinere Modelle der Thermo-Elasto-Plastizität integrieren und sich mit der Modellierung des mathematischen Problems der linearen Thermo-Elasto-Plastizität unter Berücksichtigung von Phasentransformationen und Umwandlungsplastizität befassen. Das Hauptziel ist es, einen kurzen Überblick über Existenz- und Eindeutigkeitsresultate für die entsprechenden mathematische Probleme zu geben.

Keywords

Material behavior of steel, distortion engineering, heat treatment, phase transitions, TRIP, coupling of TRIP and plasticity, thermo-elasticity, thermo-mechanics, existence and uniqueness, system of coupled partial differential equations

AMS subject classifications

34B60, 35Q74, 74A05, 74A15, 74C05, 74C10, 74F05, 74H20, 74H25, 74H30, 74N99

Acknowledgments

This work has partially been supported by the University of Bremen via the post-graduate program ‘Scientific Computing in Engineering’ (SCiE) and the German Research Foundation (DFG) via the Collaborative Research Center (SFB) 570 ‘Distortion Engineering’ at the University of Bremen, Germany.

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1 Introduction

Heat treatment of steel (and some other metals) is a complex process in which heat conduction and thermo-elastic deformations are accompanied by classical plastic deformations and a change of the crystalline structure. The latter one induces a further kind of inelastic deformation – the transformation-induced plasticity (TRIP).

Such a process can be modeled by a system of hyperbolic, parabolic, ordinary and integral equations and a variational inequality, describing the displacement, heat conduction, phase transformations and TRIP and the classical-plasticity strains, resp. (cf. [Hau02, Šil97, Wil98], e.g. for a more general background. Moreover, there are numerous publications which refer to modeling of phase transformations and TRIP. Without claim to completeness we mention [DAA+99, Fis97, FSS03, HHR07, HHR10, LMD86a, LMD86b, Höm04, MSA09, MWSB12, TP06, WBH08, WBMS11], e.g., which relate to the situation we are dealing with here.).

In this note we give an overview of some results for the well-posedness of the corresponding initial- and boundary value problems. Models, like the one considered here, are the base for corresponding simulations aimed at forecasting the material behavior and small deformations under loading and cooling.

The *point* of this paper is the *simultaneous* treatment of *all* the effects mentioned above.

1.1 Outline

We begin with a summary of the model components in the whole section 1. Section 2 provides an overview of some results of well-posedness for models of thermo-elasto-plasticity with phase transitions in steel. Section 3 concludes with some discussion and an outlook.

1.2 Mathematical notation

The notations are standard, but for convenience of the reader we summarize them here.

Let $k, m, n \in \mathbb{N}$, $p \in [1, \infty]$, $\lambda \in [0, 1]$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ – set of all positive and non-negative reals, resp. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with (at least) $C^{0,1}$-boundary, $\boldsymbol{\nu} : \Gamma \to S^{n-1}(0, 1)$ – outward unit normal field to the boundary $\Gamma := \partial\Omega$, $\mathcal{C}^{k,p}$ – class of domains whose boundary is locally representable as a graph of a $C^{k,p}$-function, $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ (matrices, tensors), Id – identity tensor, $\mathbf{C} = (C_{ij}) : \Omega \to \mathbb{R}^{n\times n}$, $\mathbf{u} = (u_1, \ldots, u_n) : \Omega \to \mathbb{R}^n$. $\mathbf{u} \cdot \mathbf{v} := \sum_{i=1}^n u_i v_i$ and $\mathbf{A} : \mathbf{B} := \sum_{i,j=1}^n A_{ij}B_{ij}$ denote the usual scalar products, $\mathbf{u}^T$ and $\mathbf{A}^T$ are the transposed vectors and tensors, resp., $\mathrm{tr}(\mathbf{A})$ – trace of $\mathbf{A}$, $\mathbf{A}^* := \mathbf{A} - \frac{1}{n}\,\mathrm{tr}(\mathbf{A})\,\mathrm{Id}$ – deviator of $\mathbf{A}$, $\mathbb{R}^{n\times n}$ ($\mathbb{R}^{n\times n}_{\mathrm{sym}}$) – set of all real (symmetric) $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathrm{meas}(K)$ – Lebesgue measure of a set K, $\mathrm{h}(A, B)$ – Hausdorff distance between the two sets A and B, χ_K – indicator function of a set K, $\frac{\partial}{\partial t}$ $\left(\text{resp. } \frac{\mathrm{d}}{\mathrm{d}t}\right)$ – partial (resp. total) derivative w.r.t. t, $\nabla\mathbf{u}$ – gradient (Jacobian) of the function $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathrm{div}(\mathbf{q})$ – divergence of the vector field $\mathbf{q}$, $\mathrm{div}(\mathbf{A}) := \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} A_{ij}\right)_{i=1,\ldots,n}$ – divergence of the matrix field $\mathbf{A}$, $\partial\psi$ – sub-differential of the convex function $\psi : X \to \mathbb{R} \cup \{+\infty\}$.

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ – normed spaces. Then: $(\cdot, \cdot)$ or $(\cdot, \cdot)_X$ – scalar product on X (if there is one); $(X^*, \|\cdot\|_*)$ – dual space, $X \times Y$, $X \cap Y$ and X^m – normed by the corresponding sum norms unless otherwise required, $\|\cdot\|_\infty$ – maximum norm on $\mathbb{R}^n$, $\|\cdot\|_X$ – norm on X, $\langle\cdot,\cdot\rangle_{X^*X}$ – dual pairing in $X^* \times X$, $T > 0$, $S := (0, T)$ – time interval, $\Omega_T := \Omega \times S$, $\Gamma_T := \Gamma \times S$, $C^k(\Omega)$ – set of all k times continuously differentiable $u : \Omega \to \mathbb{R}$, $C_0^k(\Omega)$ – subspace of $C^k(\Omega)$ of functions with compact support, $L^p(\Omega)$ – standard Lebesgue space over Ω, $W^{k,p}(\Omega)$ – standard Sobolev space over Ω, $W_0^{k,p}(\Omega)$ – subspace of $W^{k,p}(\Omega)$ of functions with zero boundary trace, $W_0^{1,p}(\Omega)$ – set of all $u \in W^{1,p}(\Omega; X)$ with zero boundary trace, normed by $\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p(\Omega)}$.

$C^k(\overline{S}; X)$ – set of X-valued functions ($\in \overline{S} \to X$) with continuous derivatives up to order k, $L^p(S; X)$ – (standard) Bochner-Lebesgue spaces of function (classes) mapping $\Omega \to X$, $W^{k,p}(S; X)$ – (standard) Bochner-Sobolev space of (classes of) functions mapping $\Omega \to X$, $W^{1,p}(S; X, X^*)$ stands for the set of all (classes of) functions $u \in L^p(S; X)$ whose distributional derivative belongs to $L^p(S; X^*)$ (cf. [Zei90], e.g.). $W^{1,p}(S; X, X^*)$ is normed by $\|u\|_{W^{1,p}(S;X,X^*)} := \|u\|_{L^p(S;X)} + \|u'\|_{L^{p'}(S;X^*)}$.

Let $h > 0$, $u : \overline{S} \to X$, set

(1) $$\mathcal{S}_h(u)(t) := \begin{cases} \frac{1}{h}\int_t^{t+h} u(s)\,\mathrm{d}s & \text{for} \quad t \in [0, T-h] \\ 0 & \text{for} \quad t \in (T-h, T] \end{cases} .$$

Note: If $u \in C^1(\overline{S}; X)$, then

(2) $$\mathcal{S}_h(u')(t) = \begin{cases} \frac{1}{h}\big(u(t+h) - u(t)\big) & \text{for} \quad t \in [0, T-h] \\ 0 & \text{for} \quad t \in (T-h, T] \end{cases} .$$

If u' does not exist, take (2) as a definition.

For functions $u = u(x, t)$, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(x, t)$ etc. we use the following notion: $u(t) := u(\cdot, t)$, $\boldsymbol{\varepsilon}(t) = \boldsymbol{\varepsilon}(\cdot, t)$, $u'(t) := \frac{\partial}{\partial t}u(\cdot, t)$ etc.

1.3 General model

Again, the notation is standard (cf. [WBH08, WBMS11], e.g.). Our references for generalities are [Hau02, Šil97, Wil98].

The displacement $\mathbf{u} = (u_1, u_2, u_3)^T$, the Cauchy stress $\boldsymbol{\sigma}$ and the (absolute) temperature θ are governed by the balance of momentum and the balance law of internal energy

$$\rho_0 \mathbf{u}'' - \operatorname{div}(\boldsymbol{\sigma}) = \mathbf{f}, \tag{3}$$

$$\rho_0 c_e \theta' + \operatorname{div}(\mathbf{q}) = r. \tag{4}$$

Here, ρ_0 – bulk density w.r.t. the reference configuration, $\mathbf{f}$ – external volume force density, c_e – specific heat, r – external volume source density of heat supply, $\mathbf{q}$ – heat flux (density).

A simple law to describe phase transitions reads as

$$\mathbf{p}' = \boldsymbol{\gamma}\left(\mathbf{p}, \theta, \theta', I_1, I_2, I_3\right), \tag{5}$$

where $\mathbf{p} = (p_1, \ldots, p_m)^T$ – phase fraction vector, p_i – the phase (mass) fraction of the i^{th} phase ($i = 1, \ldots, m$), $\boldsymbol{\gamma}$ – transformation-rate vector and $I_1 := \operatorname{tr}(\boldsymbol{\sigma})$, $I_2 := \boldsymbol{\sigma}^* : \boldsymbol{\sigma}^*$, $I_3 := \det(\boldsymbol{\sigma})$ invariants of the stress.

Using Fourier's law and a generalization of the Duhamel-Neumann's law (or generalized Hooke's law) of the classical (linear) thermo-elasticity for isotropic bodies, cf. e.g. [WBH08, WBMS11]

$$\mathbf{q} = -\lambda_\theta \nabla\theta, \tag{6}$$

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}_{te}^* + K \operatorname{tr}(\boldsymbol{\varepsilon}_{te}) \operatorname{Id} - 3K_\alpha (\theta - \theta_0) \operatorname{Id} - K \sum_{i=1}^{m} \left(\frac{\rho_0}{\rho_i(\theta_0)} - 1 \right) p_i \operatorname{Id}. \tag{7}$$

Here, λ_θ – heat conductivity, μ – shear modulus, λ – second Lamé coefficient, $K := \lambda + \frac{2}{3}\mu$ – compression (bulk) modulus, $K_\alpha := K\alpha$ – modulus taking compression and linear heat-dilatation of the bulk material into account, $\rho_i(\theta_0)$ – density of the i^{th} phase at initial temperature θ_0.

Moreover, as usual in the theory of small deformations, we assume the additive decomposition of the (linearized) Green strain tensor $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{te} + \boldsymbol{\varepsilon}_{trip} + \boldsymbol{\varepsilon}_{cp}, \tag{8}$$

with $\boldsymbol{\varepsilon}_{te}$ – thermo-elastic strain (including (isotropic) density variations due to temperature changes and phase transformations), $\boldsymbol{\varepsilon}_{trip}$ – (non-isotropic) strain due to TRIP and $\boldsymbol{\varepsilon}_{cp}$ – strain due to (classical) plasticity as usual in the theory of small deformations.

As usual, the inelastic strains are assumed to be volume-preserving, i.e.

$$\operatorname{tr}(\boldsymbol{\varepsilon}_{trip}) = 0 \implies \boldsymbol{\varepsilon}_{trip} = \boldsymbol{\varepsilon}_{trip}^* \qquad \text{and} \qquad \operatorname{tr}(\boldsymbol{\varepsilon}_{cp}) = 0 \implies \boldsymbol{\varepsilon}_{cp} = \boldsymbol{\varepsilon}_{cp}^*. \tag{9}$$

Mathematical modeling of plastic material behavior leads to the description with the help of a plastic flow rule via a (parabolic) variational inequality or equivalently via a differential inclusion (cf. e.g. [WBH08]). A standard model to describe the classical plasticity and TRIP contribution is

$$\boldsymbol{\varepsilon}_{cp}' = \Lambda\left(\boldsymbol{\sigma}^* - \mathbf{X}_{cp}\right), \qquad \Lambda \geq 0 \quad \text{if} \quad \widehat{F} = 0 \quad \text{and} \quad \Lambda = 0 \quad \text{if} \quad \widehat{F} < 0, \tag{10}$$

$$\boldsymbol{\varepsilon}_{trip}' = b_{trip} \quad \text{in} \quad \Omega_T, \tag{11}$$

where the yield function $\widehat{F}$ and b_{trip} are given by

$$\widehat{F} = \widehat{F}(\boldsymbol{\sigma}, \mathbf{X}_{cp}, R, R_0) := \sqrt{\frac{3}{2}\left(\boldsymbol{\sigma}^* - \mathbf{X}_{cp}\right) : \left(\boldsymbol{\sigma}^* - \mathbf{X}_{cp}\right)} - (R_0 + R), \tag{12}$$

$$b_{trip} := \frac{3}{2}\left(\boldsymbol{\sigma}^* - \mathbf{X}_{trip}\right) \sum\nolimits_{i=1}^{m} \kappa_i \frac{\partial\Phi_i}{\partial p_i}(p_i) \max\left\{p_i', 0\right\}. \tag{13}$$

Here, κ_i – Greenwood-Johnson parameter, ϕ_i – saturation function of the i^{th} phase, R_0 – initial radius of the yield sphere in the stress space (initial yield stress/radius), R – its possible increment due to isotropic hardening and Λ – the plastic multiplier.

The relation (10) is equivalent to a differential inclusion or a variational inequality for $\boldsymbol{\sigma}^*$ (cf. (21), (44), following [DL76]).

This variational inequality can be solved via abstract results with the help of the Yosida approximation, cf. e.g. [Bar76, Bré73, HHNL88, Rou05, Sho97].

The following ansatz for the isotropic hardening variable R is suggested in [WBH08, WBMS11]

$$R' = \gamma_{cp} s'_{cp} - \beta_{cp} s'_{cp} R, \qquad R(0) = 0, \tag{14}$$

where the accumulated plastic and TRIP strain are given via

$$s_{cp} := \int_0^t \sqrt{\frac{2}{3}\boldsymbol{\varepsilon}'_{cp} : \boldsymbol{\varepsilon}'_{cp}}\, \mathrm{d}\tau \qquad \text{and} \qquad s_{trip} := \int_0^t \sqrt{\frac{2}{3}\boldsymbol{\varepsilon}'_{trip} : \boldsymbol{\varepsilon}'_{trip}}\, \mathrm{d}\tau \tag{15}$$

and β_{cp}, γ_{cp} parameters depending on θ, $\mathbf{p}$.

The back-stresses $\mathbf{X}_{cp}$, $\mathbf{X}_{trip}$ associated with (classical) plasticity and TRIP, resp. (with $\mathrm{tr}(\mathbf{X}_{cp}) = 0$ and $\mathrm{tr}(\mathbf{X}_{trip}) = 0$, cf. e.g. [WBT10] for details) are given as generalizations of the well-known Armstrong-Frederick equations in plasticity (cf. [LC90, JK96, Hau02]) via the following coupled system of (parameter-dependent) ODEs

$$\mathbf{X}'_{cp} = c_{cp}\boldsymbol{\varepsilon}'_{cp} - a_{cp}\mathbf{X}_{cp}s'_{cp} + c_{int}\boldsymbol{\varepsilon}'_{trip} - \frac{c_{int}a_{trip}}{c_{trip}}\mathbf{X}_{trip}s'_{trip}, \tag{16}$$

$$\mathbf{X}'_{trip} = c_{int}\boldsymbol{\varepsilon}'_{cp} - \frac{c_{int}a_{cp}}{c_{cp}}\mathbf{X}_{cp}s'_{cp} + c_{trip}\boldsymbol{\varepsilon}'_{trip} - a_{trip}\mathbf{X}_{trip}s'_{trip} \tag{17}$$

with constant c_{cp}, c_{int}, c_{trip}, a_{cp}, a_{trip}.

1.4 Summary of the model equations

The function we are looking for are the displacements $\mathbf{u} : \Omega_T \to \mathbb{R}^3$, the temperature $\theta : \Omega_T \to \mathbb{R}$, the strains $\boldsymbol{\varepsilon}_{trip}$, $\boldsymbol{\varepsilon}_{cp} : \Omega_T \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$, and the phase-fraction vector $\mathbf{p} = (p_1, \dots, p_m) : \Omega_T \to \mathbb{R}^m$.

Let $a_k \in [0, \varepsilon, 1]$, $\varepsilon > 0$, $k = 1, 2$ and r_0, $\mathbf{f}_0$, g as specified in section 1.4.1. Then the model equations read as

$$a_1\rho_0\mathbf{u}'' - 2\,\mathrm{div}(\mu\boldsymbol{\varepsilon}(\mathbf{u})) - \nabla(\lambda\,\mathrm{div}(\mathbf{u})) - a_2\,\mathrm{div}(\mu\boldsymbol{\varepsilon}(\mathbf{u}')) = \mathbf{f}_0 \quad \text{in} \quad \Omega_T, \tag{18}$$

$$\rho_0 c_e\theta' - \mathrm{div}(\lambda_\theta\nabla\theta) = r_0 \quad \text{in} \quad \Omega_T, \tag{19}$$

$$\mathbf{p}' = \boldsymbol{\gamma}\left(\mathbf{p}, \theta, \theta', \mathrm{tr}(\boldsymbol{\sigma}), \boldsymbol{\sigma}^* : \boldsymbol{\sigma}^*, \det(\boldsymbol{\sigma})\right) \quad \text{in} \quad \Omega_T, \tag{20}$$

$$(\boldsymbol{\sigma}^*)'(t) + \partial\chi_{\mathbf{K}}(\boldsymbol{\sigma}^*(t)) \ni g\left(\mathbf{u}'(t), \boldsymbol{\varepsilon}'_{trip}(t)\right) \quad \text{f.a.a.} \quad t \in S, \tag{21}$$

$$\boldsymbol{\varepsilon}'_{trip} = \frac{3}{2}(\boldsymbol{\sigma}^* - \mathbf{X}_{trip})\sum_{i=1}^m \kappa_i\frac{\partial\Phi_i}{\partial p_i}(p_i)\max\left\{p'_i, 0\right\} \quad \text{in} \quad \Omega_T, \tag{22}$$

$$R' = \gamma_{cp}s'_{cp} - \beta_{cp}s'_{cp}R \quad \text{in} \quad \Omega_T, \tag{23}$$

$$\mathbf{X}'_{cp} = c_{cp}\boldsymbol{\varepsilon}'_{cp} - a_{cp}\mathbf{X}_{cp}s'_{cp} + c_{int}\boldsymbol{\varepsilon}'_{trip} - \frac{c_{int}a_{trip}}{c_{trip}}\mathbf{X}_{trip}s'_{trip} \quad \text{in} \quad \Omega_T, \tag{24}$$

$$\mathbf{X}'_{trip} = c_{int}\boldsymbol{\varepsilon}'_{cp} - \frac{c_{int}a_{cp}}{c_{cp}}\mathbf{X}_{cp}s'_{cp} + c_{trip}\boldsymbol{\varepsilon}'_{trip} - a_{trip}\mathbf{X}_{trip}s'_{trip} \quad \text{in} \quad \Omega_T. \tag{25}$$

The model is complemented by **initial conditions**

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \mathbf{u}'(0) = \mathbf{u}_1, \qquad \theta(0) = \theta_0, \tag{26}$$

$$\varepsilon_{trip}(0) = \mathbf{0}, \qquad \varepsilon_{cp}(0) = \mathbf{0}, \qquad \mathbf{p}(0) = \mathbf{p}_0 \tag{27}$$

$$\mathbf{X}_{trip}(0) = 0, \qquad \mathbf{X}_{cp}(0) = 0, \qquad R(0) = 0 \tag{28}$$

$$\boldsymbol{\sigma}^*(0) = \boldsymbol{\sigma}_0^* := 2\mu\varepsilon^*(\mathbf{u}_0) \tag{29}$$

in Ω with

$$\sum_{i=1}^{m} p_{0i} = 1, \quad p_{0i} \geq 0 \quad \text{for} \quad i = 1, \ldots, m, \tag{30}$$

and by (mixed) **boundary conditions**

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_1, \qquad \big(\boldsymbol{\sigma} + a_2\mu\varepsilon(\mathbf{u}')\big) \cdot \boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Gamma_2, \tag{31}$$

$$-\lambda_\theta \frac{\partial\theta}{\partial\boldsymbol{\nu}} = \delta\left(\theta - \theta_\Gamma\right) \quad \text{on} \quad \Gamma, \tag{32}$$

where Γ_1 and Γ_2 are mutually disjoint parts of the boundary Γ and Γ_1 is a closed subset of Γ with positive surface measure. Moreover, δ – heat-exchange coefficient, θ_Γ – temperature of the surrounding medium.

1.4.1 Some specifications of the general model

For abbreviation we introduce

$$\mathbf{f}_0 := \sum_{i=1}^{5} \mathbf{f}_i \quad \text{and} \quad r_0 = \sum_{i=1}^{7} r_i \tag{33}$$

with

$$\mathbf{f}_1 := \mathbf{f}, \qquad \mathbf{f}_2 := -\left(3K_\alpha\left(\theta - \theta_0\right)\right)\mathrm{Id}, \quad \mathbf{f}_3 := -\left(K\sum_{i=1}^{m}\left(\frac{\rho_0}{\rho_i(\theta_0)} - 1\right)p_i\right)\mathrm{Id}, \tag{34}$$

$$\mathbf{f}_4 := -2\left(\mu\varepsilon_{trip}\right), \quad \mathbf{f}_5 := -2\left(\mu\varepsilon_{cp}\right) \tag{35}$$

and

$$r_1 := r, \qquad r_2 := \left(\boldsymbol{\sigma} - \mathbf{X}_{trip}\right) : \varepsilon'_{trip}, \qquad r_3 := \left(\boldsymbol{\sigma} - \mathbf{X}_{cp}\right) : \varepsilon'_{cp}, \tag{36}$$

$$r_4 := \theta\frac{\partial\boldsymbol{\sigma}}{\partial\theta} : \varepsilon'_{te}, \qquad r_5 := \rho_0\sum_{i=2}^{m} L_i p'_i \qquad r_6 := \theta\frac{\partial\mathbf{X}_{cp}}{\partial\theta} : \varepsilon'_{cp}, \tag{37}$$

$$r_7 := \theta\frac{\partial\mathbf{X}_{trip}}{\partial\theta} : \varepsilon'_{trip}, \tag{38}$$

and

$$g = g\left(\mathbf{u}', \varepsilon'_{trip}\right) := 2\mu\left(\varepsilon^*(\mathbf{u}') - \varepsilon'_{trip}\right). \tag{39}$$

1.4.2 Classical plasticity via a variational inequality

Let $\widehat{F} : \mathbb{R}^{3\times3}_{\mathrm{sym}} \times \mathbb{R}^{3\times3}_{\mathrm{sym}} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ with

$$\widehat{F}(\boldsymbol{\tau}, \mathbf{X}, R, R_0) := \sqrt{\frac{3}{2}\left(\boldsymbol{\tau}^* - \mathbf{X}\right) : \left(\boldsymbol{\tau}^* - \mathbf{X}\right)} - \left(R_0 + R\right), \tag{40}$$

set

$$\widehat{\mathbf{K}} := \left\{ \boldsymbol{\tau} \in \mathbb{R}^{3\times 3}_{\text{sym}} : \operatorname{tr}(\boldsymbol{\tau}) = 0 \text{ and } \widehat{F}(\boldsymbol{\tau}, \mathbf{X}, R, R_0) \le 0 \right\}, \tag{41}$$

$$\mathbf{K} := \left\{ \boldsymbol{\tau} \in \mathbf{H}_{\boldsymbol{\sigma}} : \boldsymbol{\tau}(\mathbf{x}) \in \widehat{\mathbf{K}} \text{ f.a.a. } \mathbf{x} \in \Omega \right\} \tag{42}$$

and introduce

$$F : \mathbf{K} \to \mathbb{R}, \quad F(\boldsymbol{\tau}) := \widehat{F}(\boldsymbol{\tau}(\cdot); \mathbf{X}, R, R_0). \tag{43}$$

The differential inclusion in (21) is equivalent to the variational inequality

$$\left((\boldsymbol{\sigma}^*)'(t), (\boldsymbol{\tau} - \boldsymbol{\sigma}) \right)_{\mathbf{H}_{\boldsymbol{\sigma}}} - \left(g\left(\mathbf{u}'(t), \varepsilon'_{trip}(t)\right), (\boldsymbol{\tau} - \boldsymbol{\sigma}) \right)_{\mathbf{H}_{\boldsymbol{\sigma}}} \ge 0 \tag{44}$$

f.a.a. $t \in S$, f.a. $\boldsymbol{\tau} \in \mathbf{K}$.

Remark 1 (Set of admissible stresses). Due to the general time-dependence of R, the set of admissible stresses $\mathbf{K}$ varies in time, when considering a time-dependent process (cf. e.g. [HR99]).

In [BFM11, CR06] the yield function depends explicitly on the temperature. We assume in our application problem cooling or quenching processes and therefore assume a decreasing temperature, i.e. a growing yield radius (Ramberg-Osgood Model in [Suh10] or an alternative experimental based ansatz for isotropic hardening, cf. [Mac92]).

In real-life experience, $R = R(\theta)$, $\theta \mapsto R(\theta)$ is decreasing for these processes (this is not clear for problems with phase transitions because of the latent heat, the dissipation etc.). If $\theta = \theta(t)$, then $R = R(t) = R(\theta(t))$. For uniform cooling, the function $t \mapsto \theta(x; t)$ is decreasing a.e. implies $R = R(t) := R(\theta(t))$ and $K = K(t) := K(R(\theta(t)))$ are increasing in t. One would define $F : \mathbb{R}^{3\times 3} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ via

$$F(\boldsymbol{\sigma}, t, \mathbf{x}) := \sqrt{\frac{3}{2} \boldsymbol{\sigma}^* : \boldsymbol{\sigma}^*} - (R_0 + R(t, \mathbf{x})), \tag{45}$$

$$\mathbf{K}_F(t, \mathbf{x}) := \left\{ \boldsymbol{\tau} \in \mathbb{R}^{3\times 3}_{\text{sym}}, \operatorname{tr}(\boldsymbol{\tau}) = 0 : F(\boldsymbol{\tau}, t, \mathbf{x}) \le 0 \right\}, \tag{46}$$

$$\mathbf{K}(t) := \left\{ \boldsymbol{\tau} \in \mathbf{H}_{\boldsymbol{\sigma}} : \boldsymbol{\tau}(\mathbf{x}) \in \mathbf{K}_F(t, \mathbf{x}) \;\text{ f.a.a. } \mathbf{x} \in \Omega \right\}, \tag{47}$$

where R is defined, e.g., as in (14). Considering non-linear hardening, using internal variables, etc., there are plenty of relations for $R = R(t)$ thinkable.

Moreover, in real-life experience the composition of a material influences its plastic behavior, i.e. $R = R(\mathbf{p})$; hence, a parameter-dependent $\mathbf{K}$ would lead to

$$F(\boldsymbol{\tau}, \theta) := \sqrt{\frac{3}{2} \boldsymbol{\sigma}^* : \boldsymbol{\sigma}^*} - (R_0 + R(\theta)), \tag{48}$$

$$\mathbf{K}_F(\theta) := \left\{ \boldsymbol{\tau} \in \mathbb{R}^{3\times 3}_{\text{sym}}, \operatorname{tr}(\boldsymbol{\tau}) = 0 : F(\boldsymbol{\tau}, \theta) \le 0 \right\}, \tag{49}$$

$$\mathbf{K}(\theta) := \left\{ \boldsymbol{\tau} \in \mathbf{H}_{\boldsymbol{\sigma}} : \boldsymbol{\tau}(\mathbf{x}) \in \mathbf{K}_F(\theta) \text{ f.a.a. } \mathbf{x} \in \Omega \right\}. \tag{50}$$

1.4.3 Phase transitions

Additionally, one needs to require (balance and side conditions)

$$\sum_{i=1}^{m} p_i = 1, \quad p_i \ge 0 \quad \text{for} \quad i = 1, \dots, m. \tag{51}$$

A **typical example for** γ_i: Let $\varepsilon \in (0,1)$, $\overline{p}_{ij} \in (0,1)$ be fixed, set

$$H_\varepsilon(s) := \begin{cases} 0, & s \le 0, \\ s/\varepsilon, & 0 < s < \varepsilon, \\ 1, & s \ge \varepsilon, \end{cases} \tag{52}$$

and let, for $i, j = 1, \dots, m$, $\mathbf{p} \in [0,1]^m$, $\theta \in \mathbb{R}$

$$a_{ij} = a_{ij}(\mathbf{p}, \theta, \boldsymbol{\sigma}) \geq 0, \quad G_{ij} = G_{ij}(\theta) \geq 0 \tag{53}$$

s.t.

(54) a_{ij} is bounded and uniformly Lipschitz continuous w.r.t. all arguments,

(55) G_{ij} is bounded and uniformly Lipschitz continuous.

Set

$$\begin{aligned} \gamma_i &:= \gamma_i(\mathbf{p}, \theta, \theta', I_1, I_2, I_3) \\ &:= -\sum_{j=1}^{m} a_{ij} H_\varepsilon(p_i) H_\varepsilon(\overline{p}_{ij} - p_j) G_{ij} + \sum_{j=1}^{m} a_{ji} H_\varepsilon(p_j) H_\varepsilon(\overline{p}_{ji} - p_i) G_{ji}. \end{aligned} \tag{56}$$

The quantities a_{ij} are the proper transformation rates for the transformation $i \to j$. The functions G_{ij} and the (regularized) Heaviside function (52) are controlling functions.

Moreover, the latter one assures that the change from phase i to phase j stops, once p_j reaches the critical value $\overline{p}_{ij}$. And, clearly, the transformation $i \to j$ requires the presence of p_i.

For more explanations, special cases in use and references we refer to [WBB07].

1.4.4 TRIP

The function ϕ_i describes the dependence of the transformed phase fraction p_i on the strain due to TRIP. There are various suggestions for **saturation functions** in the literature (cf. [WBS09] for discussion and further references), partially based on experiments, partially derived from theoretical considerations. For $p \in [0,1]$, they are

$$\Phi(p) = p, \qquad \text{(Tanaka)}, \tag{57}$$

$$\Phi(p) = \frac{1}{2}\left\{1 + \frac{\sin(k(2p-1))}{\sin(k)}\right\}, \quad k \in \left(0, \frac{\pi}{2}\right], \quad \text{(Böhm, Wolff, cf. [WBDH08])}, \tag{58}$$

$$\Phi(p) = \frac{p}{k-1}\left(k - p^{k-1}\right), \qquad k > 1, \tag{59}$$

$$k = \frac{3}{2} \quad \text{(Abrassart)}, \quad k = 2 \quad \text{(Denis, Desalos)}, \quad k > 2 \quad \text{(Sjöström)}.$$

2 Review of some mathematical models

In this work we introduce and investigate a mathematical model for steel quenching. The idea is to give an overview of analytical results for such a models of linear thermo-elasto-plasticity with phase transitions and TRIP.

This work is based on [Boe12a, BBW15, Boeb, Boea], where the main results are the proofs of the unique existence via fixed-point argumentation of a (global-in-time) weak solution of the regularized IBVPs ($\mathbf{P}_A$), ($\mathbf{P}_{VE}$) and ($\mathbf{P}_{QS}$) (of the corresponding fully coupled problem ($\mathbf{P}$), cf. Sections 1.4) under suitable conditions and taking into account mixed BCs for different settings. In this work the following issues are covered:

- The Steklov regularization of the fully coupled problem is investigated in Section 2.4.2.
- In Section 2.4.3, a visco-elastic regularization of the fully coupled problem is studied.

- Finally, a quasi-static model for the displacement is considered in Section 2.4.4.

In particular coupled models for the material behavior of steel, which describe phase transformations in addition to the temperature and the deformation, have not drawn too much attention in a strict mathematical and numerical context so far. We collect some references in the following section.

2.1 Related work re. mathematical analysis

Our references for generalities are [DL90, DL76, HR99, SH98, Zei88].

There are numerous publications which refer to phase transformations (cf. [FP96, FHP07, CKRS04, CKRS07]), but in connection with (inelastic) deformation and temperature there exists very little *mathematical literature*. There are results in this direction which only take into account the temperature and the phase transitions, cf. e.g. [CHK07, FDS85, Höm95, Höm97, HK06, Hüß07, Mie07, Pan10].

For the problem of thermo-elasto-plasticity there exists literature in a smaller scale, cf. [DL76] for thermo-elasto-plasticity, [Bar11, BR08, MM10] for thermo-visco-plasticity and [CR06, GH80] for mathematical problems in thermo-plasticity.

In [CR06] thermo-plasticity with the Prandtl-Reuss flow rule and with a linear evolution equation for the kinematic hardening is studied. The yield function associated with the system under consideration depends explicitly on the temperature. To have a control on the temperature, the heat equation is slightly modified and it is proved that an approximation process, based on the Yosida approximation, converges to a global in time solution of the (modified) system of thermo-plasticity.

Within the framework of a diploma thesis in the field of industrial mathematics [Boe07], the mathematical problem of linear thermo-elasticity taking into account phase transitions and TRIP was investigated. Under suitable conditions, existence and uniqueness results for the weak solvability of the corresponding initial boundary value problem for the equations of linear elasticity as well as for the equations of classical linear thermo-elasticity were given. More references for the problem of thermo-elasticity with phase transitions and TRIP are [Boe07, CHK07, HK09, HK06, Ker11, Mie07, MM05].

In particular coupled models for the material behavior of steel, which describe phase transformations in addition to the temperature and the deformation, have not drawn too much attention in a strict mathematical and numerical context so far. Closest to our approach seem to be [FP96, FDS85, FHP07, Höm95, Höm97, Hüß07, Pan10] (temperature and phase transformation, but no deformation), [AC02, Kam08, Kam09] (inelastic deformation without phase transformation and TRIP), [CR06, GH80] (thermo-plasticity, but no phase transformations and TRIP), [Boe12b, CHK08, HK06, Ker11] (thermo-elasticity with phase transitions and TRIP, but no classical plasticity).

2.2 Function spaces

Let $\Omega \in \mathcal{C}^{0,1}$.

$$
\begin{aligned}
&\mathbf{H_u} := [L^2(\Omega)]^3, && \mathbf{V_u} := \left\{\mathbf{u} \in [W^{1,2}(\Omega)]^3 : \mathbf{u}|_{\Gamma_1} = \mathbf{0}\right\}, && \\
&H_\theta := L^2(\Omega), && V_\theta := W^{1,2}(\Omega), && U_\theta := W^{1,2}(S; V_\theta, V_\theta^*) \\
&\mathbf{H_p} := [L^2(\Omega)]^m, && \mathbf{V_p} := [W^{1,2}(\Omega)]^m, && \mathbf{X_p} := [L^\infty(\Omega)]^m, \\
&\mathbf{H}_{\boldsymbol{\sigma}} := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{3\times 3} : \boldsymbol{\tau}^T = \boldsymbol{\tau}\}, && \mathbf{V}_{\boldsymbol{\sigma}} := [W^{1,2}(\Omega)]^{3\times 3} \cap \mathbf{H}_{\boldsymbol{\sigma}}, && \mathbf{X}_{\boldsymbol{\sigma}} := [L^\infty(\Omega)]^{3\times 3}.
\end{aligned}
$$

2.3 Summary of the assumptions

We require the following quantities to be constant and non-negative

(60) $$\rho_0,\ \rho_i,\ \mu,\ \lambda,\ \alpha,\ R,\ c_e,\ \lambda_\theta,\ \kappa_i,\ L_i > 0,\ \delta \geq 0, \qquad i = 1, \ldots, m.$$

Furthermore, we assume for the **initial conditions**

(61) $$\mathbf{u}_0 \in \mathbf{V_u}, \qquad \mathbf{u}_1 \in \mathbf{H_u}, \qquad \theta_0 \in V_\theta, \qquad \mathbf{p}_0 \in \mathbf{X_p}, \qquad \sum_{i=1}^{m} p_{0i} = 1, \qquad p_{0i} \geq 0 \quad \text{a.e.},$$

for the **right-hand sides**

(62) $$\mathbf{f} \in L^2(S; \mathbf{H_u}), \qquad r \in L^2(S; \mathbf{H}_\theta),$$

and for the **outside temperature**

(63) $$\theta_\Gamma \in W^{1,2}(S; L^2(\Gamma)).$$

For the **saturation functions** Φ_i we assume f.a. $\xi \in [0,1]$ and for $i = 1, \ldots, m$

(64)
$$\Phi_i \in C^2([0,1]) \quad \text{with} \quad \Phi_i(0) = 0, \quad \Phi_i(1) = 1, \quad \text{and} \quad 0 \leq \left| \frac{\partial \Phi_i}{\partial \xi}(\xi) \right|, \ \left| \frac{\partial^2 \Phi_i}{\partial \xi^2}(\xi) \right| \leq M_\Phi < \infty,$$

with some given $M_\Phi \geq 0$. For the **transformation-rate functions** we assume $\boldsymbol{\gamma} = (\gamma_1, ..., \gamma_m) : [0,1]^m \times \mathbb{R} \to \mathbb{R}^m$ is Lipschitz continuous and bounded, i.e. there is a constant $M_{\boldsymbol{\gamma}} \geq 0$ s.t.

(65) $$\|\boldsymbol{\gamma}(\mathbf{p}, \theta)\|_\infty \leq M_{\boldsymbol{\gamma}} \quad \text{f.a.} \quad \mathbf{p} \in [0,1]^m, \quad \theta \in \mathbb{R},$$

(66) $$\sum_{i=1}^{m} \gamma_i = 0,$$

and an implicit condition:

(67) For all $\theta \in C(\overline{S}; H_\theta)$ the initial value problem (20), $(27)_3$, (30) has a unique solution $\mathbf{p} \in C^1(\overline{S}; \mathbf{H_p})$ satisfying (51).

The typical example for γ_i given by (53) – (56) fulfills this condition, see [Hüß07] for the proof.

2.4 Investigation of the fully coupled problem

As above-mentioned, three different settings are considered: In the first one, a Steklov regularization of the fully coupled problem is investigated (cf. Section 2.4.2). In the second one, a visco-elastic regularization of the fully coupled problem is studied (cf. Section 2.4.3 and in the third setting, a quasi-static model for the displacement is considered (cf. Section 2.4.4).

2.4.1 Technical modifications

Because of some (mathematical) difficulties we follow a common (modeling) procedure and replace the thermo-elastic dissipation r_4 (cf. $(37)_1$) by its linearization $\widehat{r}_4$ around $\theta \approx \theta_0$

$$\widehat{r}_4 := -3K_\alpha \theta_0 \operatorname{div}(\mathbf{u}'). \tag{68}$$

At some places we will substitute θ and $\mathbf{u}'$ by their averages (cf. (1), (2)). More precisely, the expression $(36)_3$ is replaced:

$$\widehat{r}_{03} \quad \text{by} \quad \widehat{r}_{03}^h \quad := -3K_\alpha \theta_0 \operatorname{div}\left(\mathcal{S}_h\left(\mathbf{u}'\right)\right), \tag{69}$$

$$\text{and} \quad g \quad \text{by} \quad g_h \quad := 2\mu\left(\boldsymbol{\varepsilon}^*(\mathcal{S}_h\left(\mathbf{u}'\right)) - \boldsymbol{\varepsilon}'_{trip}\right). \tag{70}$$

Additionally, the expression in $(34)_1$ is replaced:

$$\mathbf{f}_{01} \quad \text{by} \quad \mathbf{f}_{01}^h \quad := -(3K_a(\mathcal{S}_h(\theta) - \theta_0))\,\mathrm{Id}. \tag{71}$$

Definition 1 (Classification of the problems)**.** *We summarize* (18) – (51) *as the 'original problem* ($\mathbf{P}$)*' and* (18) – (51) *with* $a_1 = 1$, $a_2 = 0$ *including the replacements (by 'averages')* (69) – (70) *as 'problem* ($\mathbf{P}_A$)*'. Furthermore,* (18) – (51) *with* $a_1 = 1$, $a_2 = 1$ *is summarized as 'problem* ($\mathbf{P}_{VE}$)*' and* (18) – (51) *with* $a_1 = 0$, $a_2 = 0$ *as 'problem* ($\mathbf{P}_{QS}$)*'.*

Definition 2 (Definition of the operators)**.** *Let* $\mathbf{A_u} : \mathbf{V_u} \to \mathbf{V_u^*}$, $\mathbf{B_u^h} : \mathbf{V_u} \to \mathbf{V_u^*}$ *and* $A_\theta : V_\theta \to V_\theta^*$ *be defined by*

$$\langle \mathbf{A_u u}, \mathbf{v}\rangle_{\mathbf{V_u^*V_u}} := 2\int_\Omega \mu\, \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\,\mathrm{d}\mathbf{x} + \int_\Omega \lambda \operatorname{div}(\mathbf{u}) \operatorname{div}(\mathbf{v})\,\mathrm{d}\mathbf{x}, \qquad \textit{f.a. } \mathbf{v} \in \mathbf{V_u}, \tag{72}$$

$$\left\langle \mathbf{B_u^h u}, \mathbf{v}\right\rangle_{\mathbf{V_u^*V_u}} := h\int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\,\mathrm{d}\mathbf{x}, \qquad \textit{f.a. } \mathbf{v} \in \mathbf{V_u}, \tag{73}$$

$$\langle A_\theta \theta, \vartheta\rangle_{V_\theta^* V_\theta} := \int_\Omega \lambda_\theta \nabla\theta \nabla\vartheta\,\mathrm{d}\mathbf{x} + \int_\Gamma \delta\,\theta\,\vartheta\,\mathrm{d}\sigma_{\mathbf{x}}, \qquad \textit{f.a. } \vartheta \in V_\theta. \tag{74}$$

2.4.2 Regularized model

In this section, the Steklov regularization of the fully coupled problem is investigated, i.e. the first time derivative of the displacement vector is replaced by the difference quotient in the variational inequality (differential inclusion) for the plastic flow law as well as in the dissipation term in the heat equation in the full setting. Moreover, the temperature is replaced by the Steklov average in the law of thermo-elasticity in the balance equation of momentum.

Let

$$Z_U^A := \left\{\mathbf{u} \in L^2(S;\mathbf{V_u}) : \mathbf{u}' \in L^2(S;\mathbf{H_u}), \mathbf{u}'' \in \left(L^2(S;\mathbf{V_u})\right)^*\right\}, \tag{75}$$

$$Z_D^A := \mathbf{V_u} \times \mathbf{H_u} \times V_\theta \times \mathbf{X_p} \times L^2(S;\mathbf{H_u}) \times L^2(S;\mathbf{H}_\theta), \tag{76}$$

$$Z_S^A := Z_U^A \times U_\theta \cap W^{1,2}(S;\mathbf{H}_\theta) \times C^1(\overline{S};\mathbf{X_p}) \times W^{1,2}(S;\mathbf{H}_{\boldsymbol{\sigma}}) \times C^1(\overline{S};\mathbf{H}_{\boldsymbol{\sigma}}). \tag{77}$$

Theorem 1 (Well-posedness for problem ($\mathbf{P}_A$))**.** *Let* $\Omega \in \mathcal{C}^{0,1}$ *and assume* (60) – (67)*.*

1. *Problem* ($\mathbf{P}_A$) *has (at least) a global solution (i.e. on all time intervals* $(0,T)$, $T > 0$*)* $(\mathbf{u}, \theta, \mathbf{p}, \boldsymbol{\varepsilon}_{cp}, \boldsymbol{\varepsilon}_{trip}) \in Z_S^A$ *in the sense that* (26)–(32) *holds and f.a.a.* $t \in S$ *and f.a.* $\mathbf{v} \in \mathbf{V_u}$, $\vartheta \in V_\theta$*,*

$$\left\langle \rho_0\, \mathbf{u}''(t), \mathbf{v}\right\rangle_{\mathbf{V_u^*V_u}} + \langle \mathbf{A_u u}(t), \mathbf{v}\rangle_{\mathbf{V_u^*V_u}} = (\mathbf{f}_0(t), \nabla\mathbf{v})_{\mathbf{H_u}} + (\mathbf{f}(t), \mathbf{v})_{\mathbf{H_u}}, \tag{78}$$

(79) $\quad \left\langle \rho_0\, c_e\, \theta'(t), \vartheta \right\rangle_{V_\theta^* V_\theta} + \langle A_\theta \theta(t), \vartheta \rangle_{V_\theta^* V_\theta} = \left(\widehat{r}_0^h(t) + r(t), \vartheta \right)_{H_\theta} + \int_\Gamma \delta\, \theta_\Gamma(t)\, \vartheta \, \mathrm{d}\sigma_{\mathbf{x}},$

(80) $\quad \mathbf{p}'(t) = \boldsymbol{\gamma}(\mathbf{p}(t), \theta(t)) \quad in \quad \mathbf{X}_{\mathbf{p}},$

(81) $\quad \boldsymbol{\varepsilon}'_{trip}(t) = \dfrac{3}{2}\, \boldsymbol{\sigma}^*(t) \sum_{i=1}^{m} \kappa_i \, \dfrac{\partial \Phi_i}{\partial p_i}(p_i(t)) \max\{p_i'(t), 0\} \quad in \quad \mathbf{H}_{\boldsymbol{\sigma}},$

(82) $\quad \boldsymbol{\varepsilon}_{cp}(t) = \boldsymbol{\varepsilon}^* \left(\mathcal{S}_h \left(\mathbf{u} \right) (t) \right) - \boldsymbol{\varepsilon}_{trip}(t) - \dfrac{1}{2\mu} \boldsymbol{\sigma}^*(t) \quad in \quad \mathbf{H}_{\boldsymbol{\sigma}},$

(83) $\quad \left(\boldsymbol{\sigma}^* \right)'(t) + \partial \chi_{\mathbf{K}}(\boldsymbol{\sigma}^*(t)) \ni g_h(\mathbf{u}'(t), \boldsymbol{\varepsilon}'_{trip}(t)) \quad in \quad \mathbf{H}_{\boldsymbol{\sigma}}$

holds.

2. *Assume for the intrinsic (plastic) dissipation(s) (cf.* $(36)_2$*,* $(36)_3$*)*

(84) $$r_{01} = r_{02} = 0.$$

Then problem $(\mathbf{P}_A)$ *has a unique global solution (i.e. on all time intervals* $(0, T)$*,* $T > 0$*)* $(\mathbf{u}, \theta, \mathbf{p}, \boldsymbol{\varepsilon}_{cp}, \boldsymbol{\varepsilon}_{trip}) \in Z_S^A$ *in the sense that* (26)–(32) *holds and f.a.a.* $t \in S$ *and f.a.* $\mathbf{v} \in \mathbf{V}_{\mathbf{u}}$*,* $\vartheta \in V_\theta$*,* (78)–(83) *holds and the solution map*

(85) $$Z_D^A \ni (\mathbf{u}_0, \mathbf{u}_1, \theta_0, \mathbf{p}_0, \mathbf{f}, r) \mapsto (\mathbf{u}, \theta, \mathbf{p}, \boldsymbol{\varepsilon}_{cp}, \boldsymbol{\varepsilon}_{trip}) \in Z_S^A$$

is globally Lipschitz.

Proof. The proof is obtained utilizing fixed point arguments applied for a series of subproblems until finally the complete original equation system is solved and can be found in [Boe12a, BBW15]. □

Here, only a (slightly) regularized modification of the problem is studied. Non-linearity in some terms involving derivatives, global-in-time solutions (heat treatment might stretch over *longer* time intervals!) without any (artificial) smallness conditions and mixed boundary conditions restrict our results. From a modeling point of view, averaging makes sense, because changes of quantities cannot be measured exactly at a particular point at a certain time. In the numerical realization also difference quotients are used instead of the derivatives. From the point of mathematical analysis this regularization is indeed a (small) loss, but to our best knowledge this case has not been dealt with by other authors, who usually neglect classical plasticity and mixed boundary conditions (cf. section 2.1). For some particular modified case we are able to prove well-posedness of a weak solution. This refers to the following situation:

Remark 2 (Well-posedness for problem $(\mathbf{P})$). We summarize (18) – (51), (10) – (30) including (simplified) boundary conditions

(86) $$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_T \qquad \text{and} \qquad \frac{\partial \theta}{\partial \boldsymbol{\nu}} = 0 \quad \text{on} \quad \Gamma_T,$$

the replacements $(58)_1$, (69)–(71) and vanishing intrinsic dissipation $r_{01} = r_{02} = 0$ (cf. $(36)_3$) as 'problem $(\mathbf{P}_0)$'. Then, for higher regularity of the domain Ω, the right-hand sides (62) and the initial values (61), problem $(\mathbf{P})$ has a unique global solution (i.e. on all time intervals $(0, T)$, $T > 0$).

2.4.3 Visco-elastic model

The idea of visco-plastic regularization is used in [AL87, DL76].

The thermo-visco-elastic problem without additional dissipation is investigated in [BB03, EJK05, Rou05] by using the Galerkin approximation and a fixed-point argumentation. [Rou05] proves the existence of a weak solution by Schauder's fixed-point technique involving a mapping from $W^{2,2}(S;W^{1,2}(\Omega))\cap W^{1,\infty}(S;W^{2,2}(\Omega))$. A-priori estimate can be obtained by differentiation of the equation w.r.t. time and by testing it with the acceleration $\mathbf{u}''$. Moreover, Green's formula, embedding inequalities, Gronwall's and Korn's inequality are used in the proof.

An additional viscous dissipation in the heat equation, given by $\mu\boldsymbol{\varepsilon}(\mathbf{u}') : \boldsymbol{\varepsilon}(\mathbf{u}')^2$ is considered in [BB05, BB08] and leads to local (in time) solutions. Again, the used methods are the simultaneous Galerkin method and the usage of a (general) Gronwall-Bihari inequality.

In Problem $(\mathbf{P}_{VE})$ we follow a (mathematical) approach to solve the fully coupled problem $(\mathbf{P})$ and consider a slightly different medium with a small viscosity (or rather damping force in the context of friction). In order to describe visco-elastic material behavior, i.e. a material with a (small) viscosity h, we must add a regularizing term $h\boldsymbol{\varepsilon}(\mathbf{u}')$ in the stress tensor $\boldsymbol{\sigma}$ and an additional dissipation term $h\,|\boldsymbol{\varepsilon}(\mathbf{u}')|^2$ to the right-hand side of the heat equation. The (mathematical) analysis of this thermo-visco-elastic material behavior leads to local (in time) solutions (using a fixed-point argument, the Galerkin method and a non-linear Gronwall inequality in order to obtain the necessary a-priori estimates, cf. [BB05, BB08, Rou05] for details). But in this work we do not cover visco-elastic material behavior, since steel is not a visco-elastic material. The passage to the limit in the regularization parameter was not subject of the reference.

Let

$$Z_D^{VE} := \mathbf{V_u}\times\mathbf{H_u}\times V_\theta\times\mathbf{X_p}\times L^2(S;\mathbf{H_u})\times L^2(S;\mathcal{H}_\theta), \tag{87}$$

$$Z_S^{VE} := W^{1,2}(S;\mathbf{V_u})\times U_\theta\cap W^{1,2}(S;H_\theta)\times C^1(\overline{S};\mathbf{X_p})\times W^{1,2}(S;\mathbf{H}_{\boldsymbol{\sigma}})\times C^1(\overline{S};\mathbf{H}_{\boldsymbol{\sigma}}), \tag{88}$$

Theorem 2 (Well-posedness for problem $(\mathbf{P}_{VE})$)**.** *Let $\Omega\in\mathcal{C}^{0,1}$ and assume* (60) – (67)*.*

1. *Problem $(\mathbf{P}_{VE})$ has (at least) a global solution (i.e. on all time intervals $(0,T)$, $T>0$) $(\mathbf{u},\theta,\mathbf{p},\boldsymbol{\varepsilon}_{cp},\boldsymbol{\varepsilon}_{trip})\in Z_S^{VE}$ in the sense that* (26)–(32) *holds and f.a.a. $t\in S$ and f.a. $\mathbf{v}\in\mathbf{V_u}$, $\vartheta\in V_\theta$,*

$$\begin{aligned}\left\langle\rho_0\,\mathbf{u}''(t),\mathbf{v}\right\rangle_{\mathbf{V_u^*V_u}}+\left\langle\mathbf{B}^h_{\mathbf{u}}\mathbf{u}'(t),\mathbf{v}\right\rangle_{\mathbf{V_u^*V_u}}+\left\langle\mathbf{A_u}\mathbf{u}(t),\mathbf{v}\right\rangle_{\mathbf{V_u^*V_u}}=\\=(\mathbf{f}_0(t),\nabla\mathbf{v})_{\mathbf{H_u}}+(\mathbf{f}(t),\mathbf{v})_{\mathbf{H_u}}\,,\end{aligned}\tag{89}$$

$$\left\langle\rho_0\,c_e\,\theta'(t),\vartheta\right\rangle_{V_\theta^*V_\theta}+\left\langle A_\theta\theta(t),\vartheta\right\rangle_{V_\theta^*V_\theta}=(\widehat{r}_0(t)+r(t),\vartheta)_{H_\theta}+\int_\Gamma\delta\,\theta_\Gamma(t)\,\vartheta\,\mathrm{d}\sigma_{\mathbf{x}},\tag{90}$$

$$\mathbf{p}'(t)=\boldsymbol{\gamma}(\mathbf{p}(t),\theta(t))\quad in\quad\mathbf{X_p},\tag{91}$$

$$\boldsymbol{\varepsilon}'_{trip}(t)=\frac{3}{2}\,\boldsymbol{\sigma}^*(t)\sum_{i=1}^m\kappa_i\,\frac{\partial\Phi_i}{\partial p_i}(p_i(t))\max\{p_i'(t),0\}\quad in\quad\mathbf{H}_{\boldsymbol{\sigma}},\tag{92}$$

$$\boldsymbol{\varepsilon}_{cp}(t)=\boldsymbol{\varepsilon}^*\left(\mathbf{u}(t)\right)-\boldsymbol{\varepsilon}_{trip}(t)-\frac{1}{2\mu}\boldsymbol{\sigma}^*(t)\quad in\quad\mathbf{H}_{\boldsymbol{\sigma}},\tag{93}$$

$$\left(\boldsymbol{\sigma}^*\right)'(t)+\partial\chi_{\mathbf{K}}(\boldsymbol{\sigma}^*(t))\ni g(\mathbf{u}'(t),\boldsymbol{\varepsilon}'_{trip}(t))\quad in\quad\mathbf{H}_{\boldsymbol{\sigma}}\tag{94}$$

holds.

2. *Assume for the intrinsic (plastic) dissipation(s) (cf.* $(36)_2$, $(36)_3$*)*

$$r_{01}=r_{02}=0.\tag{95}$$

Then problem $(\mathbf{P}_{VE})$ *has a unique global solution (i.e. on all time intervals* $(0,T)$, $T>0$*)* $(\mathbf{u},\theta,\mathbf{p},\boldsymbol{\varepsilon}_{cp},\boldsymbol{\varepsilon}_{trip}) \in Z_S^{VE}$ *in the sense that* (26)–(32) *holds and f.a.a.* $t \in S$ *and f.a.* $\mathbf{v} \in \mathbf{V_u}$, $\vartheta \in V_\theta$, (89)–(94) *holds and the solution map*

$$Z_D^{VE} \ni (\mathbf{u}_0,\mathbf{u}_1,\theta_0,\mathbf{p}_0,\mathbf{f},r) \mapsto (\mathbf{u},\theta,\mathbf{p},\boldsymbol{\varepsilon}_{cp},\boldsymbol{\varepsilon}_{trip}) \in Z_S^{VE} \tag{96}$$

is globally Lipschitz.

Proof. The proof is obtained utilizing fixed point arguments applied for a series of subproblems until finally the complete original equation system is solved and can be found in [Boe12a, Boea]. □

2.4.4 Quasi-static model

In this approach the displacement $\mathbf{u}$ (and therefore the stress $\boldsymbol{\sigma}$) is governed by the quasi-static momentum balance, i.e. the hyperbolic equation turns into an elliptic equation for the balance of momentum. The background to this setting is that in most cases it is sensible to neglect the inertial term $\mathbf{u}''$ because we are usually not discussing situations, where stresses appear and vanish abruptly (cf. comparative simulations in [Suh10] and a similar approach in [Ker11]). Of course, it is worth discussing if it makes sense to use the dissipation term $\operatorname{div}(\mathbf{u}')$ in this context.

The quasi-static situation without phase transitions and TRIP is considered in [Bar11, CR06]. The problem is rewritten as

$$\frac{1}{\rho}\operatorname{div}(D(\boldsymbol{\varepsilon}-B\mathbf{z})) - c\operatorname{grad}\theta = \mathbf{f} \tag{97}$$

$$\theta' = \kappa\Delta\theta - \gamma\operatorname{div}(\mathbf{u}') \tag{98}$$

$$\boldsymbol{\varepsilon}_{cp}' \in \partial\chi_{K(\theta)}(T-\alpha\boldsymbol{\varepsilon}_{cp}) \tag{99}$$

in Ω_T and then, the theory of monotone operators is applied.

The situation without mechanical dissipation is treated in [Suh10]. The emphasis of that work is the numerical simulation including hardening.

Let

$$Z_D^{QS} := \mathbf{V_u}\times\mathbf{H_u}\times V_\theta\times\mathbf{X_p}\times L^2(S;\mathbf{H_u})\times L^2(S;\mathbf{H}_\theta), \tag{100}$$

$$Z_S^{QS} := W^{1,2}(S;\mathbf{V_u})\times U_\theta\cap W^{1,2}(S;H_\theta)\times C^1(\overline{S};\mathbf{X_p})\times W^{1,2}(S;\mathbf{H}_{\boldsymbol{\sigma}})\times C^1(\overline{S};\mathbf{H}_{\boldsymbol{\sigma}}), \tag{101}$$

Theorem 3 (Well-posedness for problem $(\mathbf{P}_{QS})$)**.** *Let* $\Omega \in \mathcal{C}^{0,1}$ *and assume* (60) – (67).

1. *Problem* $(\mathbf{P}_{QS})$ *has (at least) a global solution (i.e. on all time intervals* $(0,T)$, $T>0$*)* $(\mathbf{u},\theta,\mathbf{p},\boldsymbol{\varepsilon}_{cp},\boldsymbol{\varepsilon}_{trip}) \in Z_S^{QS}$ *in the sense that* (26)–(32) *holds and f.a.a.* $t\in S$ *and f.a.* $\mathbf{v}\in\mathbf{V_u}$, $\vartheta\in V_\theta$,

$$\langle \mathbf{A_u}\mathbf{u}(t),\mathbf{v}\rangle_{\mathbf{V_u^*}\mathbf{V_u}} = (\mathbf{f}_0(t),\nabla\mathbf{v})_{\mathbf{H_u}} + (\mathbf{f}(t),\mathbf{v})_{\mathbf{H_u}}, \tag{102}$$

$$\left\langle \rho_0\, c_e\, \theta'(t),\vartheta\right\rangle_{V_\theta^* V_\theta} + \langle A_\theta\theta(t),\vartheta\rangle_{V_\theta^* V_\theta} = (\widehat{r}_0(t)+r(t),\vartheta)_{H_\theta} + \int_\Gamma \delta\,\theta_\Gamma(t)\,\vartheta\,\mathrm{d}\sigma_{\mathbf{x}}, \tag{103}$$

$$\mathbf{p}'(t) = \boldsymbol{\gamma}(\mathbf{p}(t),\theta(t)) \quad in \quad \mathbf{X_p}, \tag{104}$$

$$\boldsymbol{\varepsilon}_{trip}'(t) = \frac{3}{2}\boldsymbol{\sigma}^*(t)\sum_{i=1}^m \kappa_i\frac{\partial\Phi_i}{\partial p_i}(p_i(t))\max\{p_i'(t),0\} \quad in \quad \mathbf{H}_{\boldsymbol{\sigma}}, \tag{105}$$

$$\boldsymbol{\varepsilon}_{cp}(t) = \boldsymbol{\varepsilon}^*(\mathbf{u}(t)) - \boldsymbol{\varepsilon}_{trip}(t) - \frac{1}{2\mu}\boldsymbol{\sigma}^*(t) \quad in \quad \mathbf{H}_{\boldsymbol{\sigma}}, \tag{106}$$

(107) $\quad (\boldsymbol{\sigma}^*)'(t) + \partial\chi_{\mathbf{K}}(\boldsymbol{\sigma}^*(t)) \ni g(\mathbf{u}'(t), \boldsymbol{\varepsilon}'_{trip}(t)) \quad in \quad \mathbf{H}_{\boldsymbol{\sigma}}$

hold.

2. *Assume for the intrinsic (plastic) dissipation(s) (cf.* $(36)_2$, $(36)_3$*)*

   (108)
   $$r_{01} = r_{02} = 0.$$

   Then problem $(\mathbf{P}_{QS})$ *has a unique global solution (i.e. on all time intervals* $(0,T)$, $T > 0$*)* $(\mathbf{u}, \theta, \mathbf{p}, \boldsymbol{\varepsilon}_{cp}, \boldsymbol{\varepsilon}_{trip}) \in Z_S^{QS}$ *n the sense that* (26)–(32) *holds and f.a.a.* $t \in S$ *and f.a.* $\mathbf{v} \in \mathbf{V_u}$, $\vartheta \in V_\theta$, (102)–(107) *hold and the solution map*

   (109)
   $$Z_D^{QS} \ni (\mathbf{u}_0, \mathbf{u}_1, \theta_0, \mathbf{p}_0, \mathbf{f}, r) \mapsto (\mathbf{u}, \theta, \mathbf{p}, \boldsymbol{\varepsilon}_{cp}, \boldsymbol{\varepsilon}_{trip}) \in Z_S^{QS}$$

   is globally Lipschitz.

Proof. The proof is obtained utilizing fixed point arguments applied for a series of subproblems until finally the complete original equation system is solved and can be found in [Boe12a, Boeb]. □

Related work in the literature In the following paragraph we summarize some existence and uniqueness results found in the literature and close to our approach.

Theorem 4 (Well-posedness of a quasi-static problem, [Bar11])**.** *Let* $\Omega \in \mathcal{C}^{0,1}$ *and assume* (60) – (67)*. The problem*

$$-\operatorname{div}(\boldsymbol{\sigma}) = \mathbf{f} \tag{110}$$
$$\boldsymbol{\sigma} = D(\boldsymbol{\varepsilon}(\nabla\mathbf{u}) - \boldsymbol{\varepsilon}_p) - c\,\mathrm{Id}(\theta - \theta_0) \tag{111}$$
$$\boldsymbol{\varepsilon}'_p = G(\boldsymbol{\sigma}^*, \theta, y)\frac{\boldsymbol{\sigma}^*}{|\boldsymbol{\sigma}^*|} \tag{112}$$
$$y' = \gamma(y)G(\boldsymbol{\sigma}^*, \theta, y)\boldsymbol{\sigma}^* - A\delta(y) \tag{113}$$
$$\theta' = \kappa\Delta\theta - \alpha\operatorname{div}(\mathbf{u}') + \boldsymbol{\varepsilon}'_p : \boldsymbol{\sigma} \tag{114}$$

with initial and boundary conditions

$$\mathbf{u} = g_D, \qquad \theta = g_\theta, \qquad \boldsymbol{\varepsilon}_p(0) = \boldsymbol{\varepsilon}_{p,0}, \qquad y(0) = y_0, \qquad \theta(0) = \theta_0 \tag{115}$$

yields a solution

$$\mathbf{u} \in L^\infty(S; H^2(\Omega)), \qquad \mathbf{u}' \in L^2(S; H^1(\Omega)), \tag{116}$$
$$\boldsymbol{\varepsilon}_p \in L^\infty(S; H^1(\Omega)), \qquad \boldsymbol{\varepsilon}'_p \in L^\infty(S; L^\infty(\Omega)), \tag{117}$$
$$y \in L^\infty(S; H^1(\Omega)), \qquad y' \in L^\infty(S; L^2(\Omega)), \tag{118}$$
$$\theta \in L^2(S; H^2(\Omega)) \cap L^2(S; H_0^1(\Omega)), \qquad \theta' \in L^2(S; L^2(\Omega)). \tag{119}$$

Proof. The theorem and the corresponding proof can be found in [Bar11]. □

Theorem 5 (Well-posedness of a quasi-static problem, [CR06])**.** *Let* $\Omega \in \mathcal{C}^{0,1}$ *and assume* (60) – (67)*,* $\mathbf{f} \in L^\infty(S; L^2(\Omega))$*,* $\mathbf{f}' \in L^2(S; L^2(\Omega))$*,* g_θ*,* θ_0 *bounded by a critical temperature,* $\theta_0 \in W^{1,2}(\Omega)$*,* $\boldsymbol{\varepsilon}_{p,0} \in L^2(\Omega)$*. Moreover,*

$$g_D \in L^\infty(S, W^{\frac{1}{2},2}(\Gamma_1)), \qquad g'_D \in L^2(S, W^{\frac{1}{2},2}(\Gamma_1)), \tag{120}$$
$$g_N \in L^\infty(S, W^{-\frac{1}{2},2}(\Gamma_2)), \qquad g'_N \in L^2(S, W^{-\frac{1}{2},2}(\Gamma_2)), \tag{121}$$

$$(122) \qquad g_\theta \in L^\infty(S, W^{\frac{3}{2},2}(\partial\Omega)), \qquad g'_\theta \in L^2(S, W^{-\frac{1}{2},2}(\partial\Omega)).$$

Then the problem

$$\operatorname{div}(\boldsymbol{\sigma}) = -\mathbf{f} \tag{123}$$
$$\theta' = \kappa\Delta\theta - \gamma \operatorname{div}(\mathbf{u}')\beta(\theta) \tag{124}$$
$$\boldsymbol{\sigma} = D(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p) - c\theta \operatorname{Id} \tag{125}$$
$$\boldsymbol{\varepsilon}'_p \in \partial I_{K(\theta)}(\boldsymbol{\sigma} - \alpha\boldsymbol{\varepsilon}'_p) \tag{126}$$

with initial and boundary conditions

$$(127) \qquad \mathbf{u} = g_D, \qquad \boldsymbol{\sigma}\cdot\nu = g_N, \qquad \theta = g_\theta, \qquad \theta(0) = \theta_0, \qquad \boldsymbol{\varepsilon}_p(0) = \boldsymbol{\varepsilon}_{p,0}$$

yields a solution

$$(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_p, \theta) \in L^\infty(S; W^{1,2}(\Omega) \times L^2(\Omega)^2 \times W^{1,2}(\Omega)), \tag{128}$$
$$(\mathbf{u}', \boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}'_p, \theta') \in L^2(S; L^2(\Omega) \times L^2(\Omega)^2 \times L^2(\Omega)). \tag{129}$$

Proof. The theorem and the corresponding proof using a semi-group approach can be found in [CR06]. Moreover, the authors also provide a maximum principle for the temperature. □

Theorem 6 (Well-posedness of a quasi-static problem, [HK06])**.** *Let* $\Omega \in \mathcal{C}^{0,1}$ *and assume* (60) – (67), $\mathbf{f} \in W^{1,2}(S; L^2(\Omega))^3$, $g \in W^{1,2}(S; L^2(\Gamma))$, $\theta_0 \in W^{1,2}(\Omega)$ $q(\varphi) = \delta_1\varphi + \delta_2(1-\varphi)$. *Then the problem*

$$-\operatorname{div}(\boldsymbol{\sigma}) = \mathbf{f} \tag{130}$$
$$\boldsymbol{\varepsilon}(\mathbf{u}) = c\boldsymbol{\sigma} + q(\varphi)B\theta \tag{131}$$
$$\theta' - \Delta\theta + q(\varphi)\operatorname{div}(\mathbf{u}') = \varphi' \tag{132}$$
$$\varphi' = h(\theta, \varphi) \tag{133}$$

with initial and boundary conditions

$$(134) \qquad \theta(0) = \theta_0, \qquad \varphi(0) = 0, \qquad \theta = 0, \qquad \mathbf{u} = 0, \qquad \frac{\partial\theta}{\partial\nu} = g$$

yields a solution

$$(135) \qquad u \in W^{1,2}(S; W^{1,2}_{\Gamma_0}(\Omega)^3), \qquad \boldsymbol{\sigma} \in W^{1,2}(S; L^2(\Omega)^{3\times 3}),$$
$$(136) \qquad \theta \in L^2(S; W^{1,2}_{\Gamma_0}(\Omega)), \qquad \theta' \in L^2(\Omega_T),$$
$$(137) \qquad \varphi, \varphi' \in L^\infty(\Omega_T),\ 0 < \varphi < 1,\ |\varphi'| \le c.$$

Proof. The theorem and the corresponding proof can be found in [HK06]. The main idea of the proof is the use of a fixed-point argument using Schauder's fixed-point theorem. The uniqueness result is shown for the one-dimensional case. □

Theorem 7 (Well-posedness of a quasi-static problem, [CHK08])**.** *Let* $\Omega \in \mathcal{C}^{0,1}$ *and assume* (60) – (67), $\Omega \in \mathbb{R}^3$, $\partial\Omega$ *smooth,* γ *bounded and Lipschitz-continuous,* $\mathbf{f} \in W^{1,p}(S; L^p(\Omega)) \cap L^{2p}(\Omega_T)$, $\theta_0 \in W^{1,p}(\Omega)$, $h \in L^p(\Omega_T)$. *Then the problem*

$$\operatorname{div}(\boldsymbol{\sigma}) = -\mathbf{f} \tag{138}$$
$$\boldsymbol{\sigma} = K\boldsymbol{\varepsilon}(\mathbf{u}) - \beta(\theta, p)\operatorname{Id} - \int_0^t \gamma(\theta, p, p')\boldsymbol{\sigma}^*\, \mathrm{d}\tau \tag{139}$$

$$\mathbf{u}|_{\partial\Omega} = 0 \tag{140}$$
$$p' = f(\theta, p, \boldsymbol{\sigma}) \tag{141}$$
$$p(0) = 0 \tag{142}$$
$$\begin{aligned}(\rho c_e - \operatorname{tr}(\boldsymbol{\sigma})q(p) - 9\kappa q(p)^2\theta)\theta' - \kappa\Delta\theta + 3Kq(p)\theta\operatorname{div}(\mathbf{u}') = \\ = h + (\rho L + \operatorname{tr}(\boldsymbol{\sigma})\theta\bar{q} + 9\kappa\theta^2 q(p)\bar{q})p' + \gamma(\theta, p, p')|\boldsymbol{\sigma}^*|^2\end{aligned} \tag{143}$$
$$\frac{\partial\theta}{\partial\nu} = 0, \theta(0) = \theta_0 \tag{144}$$

yields a solution for $p \geq 4$

$$\mathbf{u} \in L^p(S; W^{2,p}(\Omega)), \qquad \mathbf{u}' \in L^p(S; W^{1,p}(\Omega)), \tag{145}$$
$$\theta \in W_p^{2,1}(\Omega_T), \qquad p_i \in W^{1,p}(S; W^{1,p}(\Omega)), \qquad \sum p_i \in [0,1]. \tag{146}$$

Proof. The theorem and the corresponding proof can be found in [CHK08]. The proof is obtained utilizing fixed point arguments applied for a series of subproblems until finally the complete original equation system is solved. □

3 Discussion and outlook

This work focuses on a model of linear thermo-elasto-plasticity with phase transitions and TRIP describing the material behavior of steel in the context of macroscopic continuum mechanics.Due to the possible interaction (coupling) of transformation-induced and classical plasticity, the usual approach in plasticity without phase transformations has to be modified substantially.

The main objective is the mathematical analysis of this initial-boundary value problem or rather of three modified problems discussed in section 2.

After briefly qualifying the research context, an overview of the function spaces, the (mathematical) assumptions, the main results and the related literature are given. Existence and uniqueness results are presented for the following three different settings:

- The Steklov regularization of the fully coupled problem is investigated in section 2.4.2. In the full setting, the regularization replaces the first time derivative of the displacement vector by the difference quotient in the variational inequality (differential inclusion) for the plastic flow law as well as in the dissipation term in the heat equation. Moreover, the temperature is replaced by the Steklov average in the law of thermo-elasticity in the balance equation of momentum.

- In section 2.4.3, a visco-elastic regularization of the fully coupled problem is studied. Now the regularized problem means that an additional term including the first time derivative of the displacement vector with a small prefactor appears in the law of thermo-elasticity in the balance equation of momentum. In the literature this approach is called parabolic regularization, visco-elastic regularization or visco-plastic regularization if an additional regularization is used in the plastic flow law.

- Finally, a quasi-static model for the displacement is considered in section 2.4.4. Now the quasi-static problem means that the second time derivative of the displacement is neglected in the equation of momentum.

An important objective of one possible continuation of this work is to obtain better regularity results for the solution of the full problem for mixed boundary conditions in order to get rid of the various regularization terms used in the different settings. A restriction for better regularity results might be simplified boundary conditions or local solvability of the full problem.

In the mathematical and analytical part of this work we used basically energy methods (Galerkin approximation) and Banach and/or Schauder fixed-point argumentation in the L^2-setting for the investigation of the differential equations. The application of the Rothe method, semi-group methods or various fixed-point principles in a different L^p-setting, $p \in]1; \infty[$, might provide new insights.

Tracking other approaches (cf. e.g. [Mielke 2007]) might also work in special cases and need further attention.

3.1 Outlook

In this section we collect some different ideas from the literature in order to get existence and uniqueness results for the mathematical problem of linear thermo-elasto-plasticity with phase transitions and TRIP.

The ansatz in [AC02, Che03, BF96, Kam08, Kam09] is to rewrite the problem as a system of parabolic equations, like

$$\mathbf{v}' + \mathbf{A}\mathbf{v} = \mathbf{g}, \qquad \mathbf{v}(0) = \mathbf{v}_0 \tag{147}$$

or in the case of elasto-plasticity as a system of parabolic inclusions and apply the theory of monotone operators in addition with fixed-point arguments.

The idea for an analytical investigation in order to provide an existence and uniqueness result would be the following: Because there is no dissipation, the equations for $\mathbf{u}$ and (θ, p) are decoupled. We can prescribe a fixed temperature and use a fixed-point scheme (either Banach's or Schauder's fixed point theorem).

$$T_1 : L^2(\Omega_T) \to L^2(S; W^{2,2}(\Omega)) \cap W^{1,2}(S; H_\theta) \hookrightarrow L^2(\Omega_T), \tag{148}$$
$$\overline{\theta} \mapsto \mathbf{p} \mapsto \theta,$$

$$T_2 : W^{1,2}(S; \mathbf{V_u}) \times W^{1,2}(S; \mathbf{H}_{\boldsymbol{\sigma}}) \to W^{1,2}(S; \mathbf{V_u}) \times W^{1,2}(S; \mathbf{H}_{\boldsymbol{\sigma}}), \tag{149}$$
$$(\overline{\mathbf{u}}, \overline{\boldsymbol{\varepsilon}}_{cp}) \mapsto \boldsymbol{\varepsilon}_{trip} \mapsto \boldsymbol{\sigma}^* \mapsto \boldsymbol{\varepsilon}_{cp} \mapsto \mathbf{u}.$$

The situation without temperature, phase transitions and TRIP is investigated in [AC02, Kam08, Kam09]: Let Ω be an open, bounded domain with Lipschitz boundary and $g : \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$ maximal monotone with $0 \in g(0)$. Consider

$$\mathbf{u}' = \mathbf{v}, \qquad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \tag{150}$$

$$\mathbf{v}' = \frac{1}{\rho}\,\mathrm{div}(D(\boldsymbol{\varepsilon} - B\mathbf{z})) + \frac{1}{\rho}\mathbf{f}, \qquad \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \tag{151}$$

$$\boldsymbol{\varepsilon}' = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T), \qquad \boldsymbol{\varepsilon}(\mathbf{x}, 0) = \boldsymbol{\varepsilon}_0(\mathbf{x}), \tag{152}$$

$$\mathbf{z}' \in g(-\rho\nabla_{\mathbf{z}}\Psi(\boldsymbol{\varepsilon}, \mathbf{z})), \qquad \mathbf{z}(\mathbf{x}, 0) = \mathbf{z}_0(\mathbf{x}) \tag{153}$$

including boundary conditions $\mathbf{v}|_{\Gamma_1} = 0$ and $\rho\nabla_{\boldsymbol{\varepsilon}}\Psi(\boldsymbol{\varepsilon}, \mathbf{z})\mathbf{u}|_{\Gamma_2} = 0$.

In order to get a solution $(\mathbf{v}, \boldsymbol{\varepsilon}, \mathbf{z}) : \Omega_T \to \mathbb{R}^3 \times \mathbb{R}^{3\times3}_{sym} \times \mathbb{R}^N$ of this problem, the idea is to introduce an operator $A : L^2(\Omega, \mathbb{R}^3 \times \mathbb{R}^{3\times3}_{sym} \times \mathbb{R}^N) \to \mathcal{P}(L^2(\Omega, \mathbb{R}^3 \times \mathbb{R}^{3\times3}_{sym} \times \mathbb{R}^N))$, $D(A) := \{(\mathbf{v}, \boldsymbol{\varepsilon}, \mathbf{z}) \in \mathbf{V_u} \times \mathbf{H}_{\boldsymbol{\sigma}} \times [L^2(\Omega)]^N : A((\mathbf{v}, \boldsymbol{\varepsilon}, \mathbf{z})) \neq \emptyset\}$ and show that A is a maximal monotone operator regarding the scalar product

$$\langle(\mathbf{v}, \boldsymbol{\varepsilon}, \mathbf{z}), (\overline{\mathbf{v}}, \overline{\boldsymbol{\varepsilon}}, \overline{\mathbf{z}})\rangle := \int_\Omega [\rho\mathbf{v} \cdot \overline{\mathbf{v}} + (D(\boldsymbol{\varepsilon} - B\mathbf{z})) \cdot (\overline{\boldsymbol{\varepsilon}} - B\overline{\mathbf{z}}) + (L\mathbf{z}) \cdot \overline{\mathbf{z}}]\, \mathrm{d}\mathbf{x} \tag{154}$$

in $L^2(\Omega, \mathbb{R}^3 \times \mathbb{R}^{3\times3}_{sym} \times \mathbb{R}^N)$ and that $\rho\Psi(\boldsymbol{\varepsilon}, \mathbf{z}) = \frac{1}{2}(D\boldsymbol{\varepsilon} - B\mathbf{z}) \cdot (\boldsymbol{\varepsilon} - B\mathbf{z}) + \frac{1}{2}L\mathbf{z} \cdot \mathbf{z}$ is quadratic and positive definite on $\mathbb{R}^{3\times3}_{sym} \times \mathbb{R}^N$. Applying the general theory of monotone operators (the standard literature for such problems are e.g. [Bar76, Bré71, Sho97, Zei85]) yields the existence of a solution.

In order to adapt this approach to our setting, we set $\mathbf{z} := (\varepsilon_{trip}, \varepsilon_{cp})$ (or possibly in case of hardening $\mathbf{z} := (\varepsilon_{trip}, \varepsilon_{cp}, \mathbf{X}_{trip}, \mathbf{X}_{cp})$) and

$$\Psi = \frac{1}{\rho}\left(\mu\varepsilon_{te}^{*} : \varepsilon_{te}^{*} + \frac{K}{2}\operatorname{tr}(\varepsilon_{te})\right)^2 - 3K_\alpha(\theta-\theta_0)\operatorname{tr}(\varepsilon_{te}) - K\sum_{i=1}^{m}\left(\frac{\rho_0}{\rho_i(\theta_0)} - 1\right)p_i\operatorname{tr}(\varepsilon_{te}) + \frac{1}{\rho_0}\sum_i p_i + \frac{1}{2\rho_0}c_{cp}\varepsilon_{cp} : \varepsilon_{cp} + 2c_{int}\varepsilon_{cp} : \varepsilon_{trip} + c_{trip}\varepsilon_{trip} : \varepsilon_{trip} \tag{155}$$

and rewrite our system as

$$\mathbf{u}' = \mathbf{v} \tag{156}$$

$$\mathbf{v}' = \frac{1}{\rho}\operatorname{div}(\boldsymbol{\sigma}) + \frac{1}{\rho}\mathbf{f} \tag{157}$$

$$\varepsilon' = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T) \tag{158}$$

$$\mathbf{p}' = \gamma(\mathbf{p}, \theta) \tag{159}$$

$$\varepsilon_{trip}' = b(\theta, \mathbf{p}, \mathbf{p}')\boldsymbol{\sigma}^* \tag{160}$$

$$\theta' = \frac{\lambda_\theta}{c_e\rho}\Delta\theta + \frac{3K_\alpha}{c_e\rho}\operatorname{div}(\mathbf{v}) + \frac{\rho_0}{c_e\rho}\sum_{i=2}^{m}L_i p_i' + \frac{1}{c_e\rho}r \tag{161}$$

$$(\boldsymbol{\sigma}^*)' \in \partial\chi_{K(\theta)} + \varepsilon^*(\mathbf{v}) - b(\theta, \mathbf{p}, \mathbf{p}')\boldsymbol{\sigma}^* \tag{162}$$

in Ω_T including initial and boundary conditions.

Unfortunately, the treatment of the fully coupled problem as a differential inclusion does not work. The phase transitions do not fit into this scheme, because the integrability conditions are not fulfilled, i.e. there is no representation as a potential of a conservative vector field for the evolution equations of the phase fractions.

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