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Well-posedness of a quasi-static problem in thermo-elasto-plasticity with phase transitions in TRIP steels under mixed boundary conditions

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Well-posedness of a quasi-static problem in thermo-elasto-plasticity with phase transitions in TRIP steels under mixed boundary conditions

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Abstract

In this paper a model describing thermo-elasto-plasticity, phase transitions and transformation-induced plasticity (TRIP) is studied. The main objective is the analysis of the corresponding mathematical problem of TRIP and its interaction with classical plasticity.

Zusammenfassung

In dieser Arbeit wird ein komplexes Modell der Thermo-Elasto-Plastizität mit Phasenumwandlungen und Umwandlungsplastizität (TRIP) untersucht. Das Hauptziel ist die Analyse der entsprechenden mathematischen Aufgabe der Umwandlungsplastizität und deren Wechselwirkung mit der klassischen Plastizität.

Keywords

Coupled partial differential equations, existence and uniqueness, steel, material behavior, distortion engineering, heat treatment, phase transitions, TRIP, coupling of TRIP and plasticity, thermo-elasticity, thermo-mechanics.

2000 Math Subject Classification

34B60, 35Q74, 74A05, 74A15, 74C05, 74C10, 74F05, 74H20, 74H25, 74N99

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1. Introduction

1.1. Background

Heat treatment of steel (and some other metals) is a complex process in which heat conduction and thermo-elastic deformations are accompanied by classical plastic deformations and a change of the crystalline structure. The latter one induces a further kind of inelastic deformation – the transformation induced plasticity (TRIP). Such a process can be modeled by a system of elliptic, parabolic, ordinary and integral equations and a variational inequality, describing the displacement, heat conduction, phase transformations and TRIP and the classical-plasticity strains, resp. (cf. [Hau02, Šil97, Wil98], e.g. for a more general background. Moreover, there are numerous publications which refer to modeling of phase transformations and TRIP. Without claim to completeness we mention [DAA+99, Fis97, FSS03, HHR07, HHR10, LMD86a, LMD86b, Höm04, MSA09, MWSB12, TP06, WBH08, WBMS11], e.g.). In this note we show that the corresponding initial- and boundary value problem is well-posed. Models, like the one considered here, are the base for corresponding simulations aimed at forecasting the material behavior and small deformations under loading and cooling. The *point* of this paper is the *simultaneous* treatment of *all* the effects mentioned above.

1.2. Outline

We begin with some (mathematical) notation in section 1.3. The whole section 2 provides a summary of the model components. Section 3 lists the function spaces needed (cf. 3.1), gives a summary of the assumptions (cf. 3.2) and a short remark on modeling classical plasticity in section 3.3. Section 4 provides the main result which is proven in section 5.

1.3. Mathematical notation

The notations are standard, but for convenience of the reader we summarize them here. Let $k, m, n \in \mathbb{N}$, $p \in [1, \infty]$, $\lambda \in [0, 1]$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ – set of all positive and non-negative reals, resp. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with (at least) $C^{0,1}$-boundary, $\boldsymbol{\nu} : \Gamma \to S^{n-1}(0,1)$ – outward unit normal field to the boundary $\Gamma := \partial\Omega$, $\mathcal{C}^{k,p}$ – class of domains whose boundary is locally representable as a graph of a $C^{k,p}$-function, $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ (matrices, tensors), Id – identity tensor, $\mathbf{C} = (C_{ij}) : \Omega \to \mathbb{R}^{n\times n}$, $\mathbf{u} = (u_1, \ldots, u_n) : \Omega \to \mathbb{R}^n$. $\mathbf{u} \cdot \mathbf{v} := \sum_{i=1}^n u_i v_i$ and $\mathbf{A} : \mathbf{B} := \sum_{i,j=1}^n A_{ij} B_{ij}$ denote the usual scalar products, $\mathbf{u}^T$ and $\mathbf{A}^T$ are the transposed vectors and tensors, resp., $\operatorname{tr}(\mathbf{A})$ – trace of $\mathbf{A}$, $\mathbf{A}^* := \mathbf{A} - \frac{1}{n}\operatorname{tr}(\mathbf{A})\,\mathrm{Id}$ – deviator of $\mathbf{A}$, $\mathbb{R}^{n\times n}$ ($\mathbb{R}^{n\times n}_{\text{sym}}$) – set of all real (symmetric) $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\operatorname{meas}(K)$ – Lebesgue measure of a set K, χ_K – indicator function of a set K, $\frac{\partial}{\partial t}$ $\left(\text{resp. } \frac{\mathrm{d}}{\mathrm{d}t}\right)$ – partial (resp. total) derivative w.r.t. t, $\nabla\mathbf{u}$ – gradient (Jacobian) of the function $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^n$, $\operatorname{div}(\mathbf{q})$ – divergence of the vector field $\mathbf{q}$, $\operatorname{div}(\mathbf{A}) := \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} A_{ij}\right)_{i=1,\ldots,n}$ – divergence of the matrix field $\mathbf{A}$, $\partial\psi$ – sub-differential of the convex function $\psi : X \to \mathbb{R} \cup \{+\infty\}$. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ – normed spaces. Then: $(\cdot,\cdot)$ or $(\cdot,\cdot)_X$ – scalar product on X (if there is one); $(X^*, \|\cdot\|_*)$ – dual space, $X \times Y$, $X \cap Y$ and X^m – normed by the corresponding sum norms unless otherwise required, $\|\cdot\|_\infty$ – maximum norm on $\mathbb{R}^n$, $\|\cdot\|_X$ – norm on X, $\langle\cdot,\cdot\rangle_{X^*X}$ – dual pairing in $X^* \times X$, $T > 0$, $S := (0, T)$ – time interval, $\Omega_T := \Omega \times S$, $\Gamma_T := \Gamma \times S$, $C^k(\Omega)$ – set of all k times continuously differentiable $u : \Omega \to \mathbb{R}$, $C_0^k(\Omega)$ – subspace of $C^k(\Omega)$ of functions with compact support, $L^p(\Omega)$ – standard Lebesgue space over Ω, $W^{k,p}(\Omega)$ – standard Sobolev space over Ω, $W_0^{k,p}(\Omega)$ – subspace of $W^{k,p}(\Omega)$ of functions with zero boundary trace, $W_0^{1,p}(\Omega)$ – set of all $u \in W^{1,p}(\Omega; X)$ with zero boundary trace, normed by $\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p(\Omega)}$. $C^k(\overline{S}; X)$ – set of X-valued functions ($\in \overline{S} \to X$) with continuous derivatives up to order k, $L^p(S; X)$ – (standard) Bochner-Lebesgue spaces of function (classes) mapping $\Omega \to X$, $W^{k,p}(S; X)$ – (standard) Bochner-Sobolev space of (classes of) functions mapping $\Omega \to X$, $W^{1,p}(S; X, X^*)$ stands for the set of all (classes of) functions $u \in L^p(S; X)$ whose distributional derivative belongs to $L^p(S; X^*)$ (cf. [Zei90], e.g.). $W^{1,p}(S; X, X^*)$ is normed by $\|u\|_{W^{1,p}(S;X,X^*)} := \|u\|_{L^p(S;X)} + \|u'\|_{L^{p'}(S;X^*)}$. Let $(X, \|\cdot\|_X)$ be a Banach space, $\lambda \in \mathbb{R}$, $Y := C(\overline{S}; X)$ normed by $\|u\|_Y := \sup_{t\in\overline{S}} \|u(t)\|_X$ and $Z := L^p(S; X)$, $\|u\|_Z := \left(\int_S \|u(s)\|^p\,\mathrm{d}s\right)^{\frac{1}{p}}$. The notation Y_λ indicates that Y is to be equipped with the norm $\|u\|_{Y_\lambda} := \sup_{t\in\overline{S}} \{\exp(-\lambda t)\|u(t)\|_X\}$. In analogy: $\|u\|_{Z_\lambda} := \sup_{t\in S} \left\{\exp(-\lambda t) \int_0^t \|u(s)\|_X^p\,\mathrm{d}s\right\}^{\frac{1}{p}}$. For functions $u = u(x,t)$, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(x,t)$ etc. we use the following notion: $u(t) := u(\cdot,t)$, $\boldsymbol{\varepsilon}(t) = \boldsymbol{\varepsilon}(\cdot,t)$, $u'(t) := \frac{\partial}{\partial t}u(\cdot,t)$ etc. We abbreviate differences of functions by $\mathbf{D}\,u := u_2 - u_1$, $\mathbf{D}\,\widehat{\boldsymbol{\varepsilon}} := \widehat{\boldsymbol{\varepsilon}}_2 - \widehat{\boldsymbol{\varepsilon}}_1$ etc.

2. The model

2.1. Notation

Again, the notation is standard (references: [WBH08, WBMS11], e.g.). $\mathbf{u} = (u_1, u_2, u_3)^T$ – displacement, θ – absolute temperature, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$ – linearized Green strain tensor, $\boldsymbol{\sigma}$ – Cauchy stress tensor, $\mathbf{f}$ – external volume force density, r – external volume source density of heat supply, $\mathbf{q} = -\lambda_\theta\nabla\theta$ – heat flux (density) (we use Fourier's law), ρ_0 – bulk density w.r.t. the reference configuration, c_e – specific heat, λ_θ – heat conductivity, δ – heat-exchange coefficient, θ_Γ – temperature of the surrounding medium, $\boldsymbol{\varepsilon}_{te}$ – thermo-elastic strain (including (isotropic) density variations due to temperature changes and phase transformations), $\boldsymbol{\varepsilon}_{trip}$ – (non-isotropic) strain due to TRIP and $\boldsymbol{\varepsilon}_{cp}$ – strain due to (classical) plasticity, p_i – the phase (mass) fraction of the i^{th} phase ($i = 1, \ldots, m$), μ – shear modulus, λ – second Lamé coefficient,

$K := \lambda + \frac{2}{3}\mu$ – compression (bulk) modulus, $K_\alpha := K\alpha$ – modulus taking compression and linear heat-dilatation of the bulk material into account, $\rho_i(\theta_0)$ – density of the i^{th} phase at initial temperature θ_0, κ_i – Greenwood-Johnson parameter and ϕ_i – saturation function of the i^{th} phase, $\widehat{F}$ – yield function, R – yield stress/radius and Λ – the plastic multiplier.

2.2. The model and some specifications

Our references for generalities are [Hau02, Šil97, Wil98]. The contributions [DAA+99, Fis97, FSS03, HHR07, HHR10, LMD86a, LMD86b, Höm04, MSA09, MWSB12, TP06, WBH08, WBMS11], e.g., relate to the situation we are dealing with here. As usual in the theory of small deformations we employ the linearized Green strain tensor $\boldsymbol{\varepsilon}$ and assume the total strain to be given as as the sum of a thermo-elastic strain, a strain due to TRIP, and a strain due to (classical) plasticity:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{te} + \boldsymbol{\varepsilon}_{trip} + \boldsymbol{\varepsilon}_{cp}, \tag{1}$$

The material law (2) is a generalization of the Duhamel-Neumann's law (or generalized Hooke's law) of the classical (linear) thermo-elasticity for isotropic bodies, cf. e.g. [WBH08, WBMS11]. The stress tensor $\boldsymbol{\sigma}$ and the thermo-elastic part $\boldsymbol{\varepsilon}_{te}$ of the strain tensor are connected by the law of thermo-elasticity taking density changes due to phase transformations into account:

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}^*_{te} + K\,\mathrm{tr}(\boldsymbol{\varepsilon}_{te})\,\mathrm{Id} - 3K_\alpha\,(\theta-\theta_0)\,\mathrm{Id} - K\sum_{i=1}^{m}\left(\frac{\rho_0}{\rho_i(\theta_0)} - 1\right)p_i\,\mathrm{Id}\,. \tag{2}$$

The last term in (2) takes the density changes as a result of phase transitions into account. In order to separate this part from the thermal expansion, the phase densities appear at the initial temperature. The function we are looking for are the displacements $\mathbf{u} : \Omega_T \to \mathbb{R}^3$, the temperature $\theta : \Omega_T \to \mathbb{R}$, the strains $\boldsymbol{\varepsilon}_{trip}$, $\boldsymbol{\varepsilon}_{cp} : \Omega_T \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$, and the phase-fraction vector $\mathbf{p} = (p_1, \ldots, p_m) : \Omega_T \to \mathbb{R}^m$. For abbreviation we introduce

$$\mathbf{f}_0 = \mathbf{f}_0(\theta,\mathbf{p}), \quad \mathbf{f}_1 = \mathbf{f}_1(\boldsymbol{\varepsilon}_{trip},\boldsymbol{\varepsilon}_{cp}) \quad \text{and} \quad r_0 = r_0(\theta,\mathbf{p},\boldsymbol{\varepsilon}'_{trip},\boldsymbol{\varepsilon}'_{cp},\boldsymbol{\sigma}) \tag{3}$$

with

$$\mathbf{f}_0 := \mathbf{f}_{01}(\theta) + \mathbf{f}_{02}(\mathbf{p}) + \mathbf{f}_{03}(\boldsymbol{\varepsilon}_{trip}) + \mathbf{f}_{04}(\boldsymbol{\varepsilon}_{cp}), \tag{4}$$

$$\mathbf{f}_{01} := -\left(3K_\alpha\,(\theta-\theta_0)\right)\mathrm{Id}, \quad \mathbf{f}_{02} := -\left(K\sum_{i=1}^{m}\left(\frac{\rho_0}{\rho_i(\theta_0)} - 1\right)p_i\right)\mathrm{Id}, \tag{5}$$

$$\mathbf{f}_{03} := -2\left(\mu\boldsymbol{\varepsilon}_{trip}\right), \quad \mathbf{f}_{04} := -2\left(\mu\boldsymbol{\varepsilon}_{cp}\right), \tag{6}$$

$$r_0 := r_{01}(\boldsymbol{\varepsilon}'_{trip},\boldsymbol{\sigma}) + r_{02}(\boldsymbol{\varepsilon}'_{cp},\boldsymbol{\sigma}) + r_{03}(\theta,\boldsymbol{\sigma}) + r_{04}(\mathbf{p}) \quad (r_{01},\, r_{02} \text{ – intrinsic dissipation(s)}), \tag{7}$$

$$r_{01} := \boldsymbol{\sigma} : \boldsymbol{\varepsilon}'_{trip}, \quad r_{02} := \boldsymbol{\sigma} : \boldsymbol{\varepsilon}'_{cp}, \quad r_{03} := \theta\frac{\partial\boldsymbol{\sigma}}{\partial\theta} : \boldsymbol{\varepsilon}'_{te}, \quad r_{04} := \rho_0\sum\nolimits_{i=2}^{m} L_i p'_i. \tag{8}$$

Quasi-static[2] balance of momentum, energy balance, and phase evolution follow from the **general equations** in Ω_T as

$$-2\,\mathrm{div}(\mu\boldsymbol{\varepsilon}(\mathbf{u})) - \nabla(\lambda\,\mathrm{div}(\mathbf{u})) = \mathrm{div}(\mathbf{f}_0) + \mathbf{f}, \tag{9}$$

$$\rho_0 c_e\theta' - \mathrm{div}(\lambda_\theta\nabla\theta) = r_0 + r, \tag{10}$$

$$\mathbf{p}' = \boldsymbol{\gamma}\,(\mathbf{p},\theta)\,. \tag{11}$$

[2]Since the deformation is rather slow, it is justified to assume quasi-static conditions.

Moreover, one needs to require (balance and side conditions)

$$\sum_{i=1}^{m} p_i = 1, \quad p_i \geq 0 \quad \text{for} \quad i = 1, \ldots, m. \tag{12}$$

A **typical example for** γ_i: Let $\varepsilon \in (0,1)$, $\overline{p}_{ij} \in (0,1)$ be fixed, set $H_\varepsilon(s) := \begin{cases} 0, & s \leq 0, \\ s/\varepsilon, & 0 < s < \varepsilon, \\ 1, & s \geq \varepsilon, \end{cases}$

and let, for $i, j = 1, \ldots, m$, $\mathbf{p} \in [0,1]^m$, $\theta \in \mathbb{R}$

$$a_{ij} = a_{ij}(\mathbf{p}, \theta) \geq 0, \quad G_{ij} = G_{ij}(\theta) \geq 0 \tag{13}$$

s.t.

(14) a_{ij} is bounded and uniformly Lipschitz continuous w.r.t. both arguments,

(15) G_{ij} is bounded and uniformly Lipschitz continuous.

Set

$$\gamma_i := \gamma_i(\mathbf{p}, \theta) := -\sum_{j=1}^{m} a_{ij} H_\varepsilon(p_i) H_\varepsilon(\overline{p}_{ij} - p_j) G_{ij} + \sum_{j=1}^{m} a_{ji} H_\varepsilon(p_j) H_\varepsilon(\overline{p}_{ji} - p_i) G_{ji}. \tag{16}$$

The quantities a_{ij} are the proper transformation rates for the transformation $i \to j$. The functions G_{ij} and the (regularized) Heaviside function are controlling functions. Moreover, the latter one assures that the change from phase i to phase j stops, once p_j reaches the critical value $\overline{p}_{ij}$. And, clearly, the transformation $i \to j$ requires the presence of p_i. For more explanations, special cases in use and references we refer to [WBB07].

Classical plasticity and TRIP are modeled as follows: set

$$\widehat{F} = \widehat{F}(\boldsymbol{\sigma}, R) := \sqrt{\frac{3}{2} \boldsymbol{\sigma}^* : \boldsymbol{\sigma}^*} - R, \tag{17}$$

$$b_{trip} := \frac{3}{2} \boldsymbol{\sigma}^* \sum_{i=1}^{m} \kappa_i \frac{\partial \Phi_i}{\partial p_i}(p_i) \max\left\{p_i', 0\right\}. \tag{18}$$

Classical-plasticity and TRIP strains are given by

$$\boldsymbol{\varepsilon}_{cp}' = \Lambda\, \boldsymbol{\sigma}^*, \qquad \Lambda \geq 0 \quad \text{if} \quad \widehat{F} = 0 \quad \text{and} \quad \Lambda = 0 \quad \text{if} \quad \widehat{F} < 0, \tag{19}$$

$$\boldsymbol{\varepsilon}_{trip}' = b_{trip} \quad \text{in} \quad \Omega_T. \tag{20}$$

The relation (19) is equivalent to a variational inequality (cf. (38), (40)). The function ϕ_i describes the dependence of the transformed phase fraction p_i on the strain due to TRIP. There are various suggestions for **saturation functions** in the literature (cf. [WBS09] for discussion and further references), partially based on experiments, partially derived from theoretical considerations. For $p \in [0,1]$, they are:

(21)

$\Phi(p) = p$ (Tanaka),

(22)

$\Phi(p) = \frac{1}{2}\left\{1 + \frac{\sin(k(2p-1))}{\sin(k)}\right\}$, $k \in \left(0, \frac{\pi}{2}\right]$ (Böhm, Wolff, cf. [WBDH08]),

(23)

$$\Phi(p) = \frac{p}{k-1}\left(k - p^{k-1}\right), \; k = \tfrac{3}{2} \quad \text{(Abrassart)}, \quad k = 2 \quad \text{(Denis, Desalos)}, \quad k > 2 \quad \text{(Sjöström)}.$$

The model is complemented by **initial conditions**

(24) $$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{u}_1, \quad \theta(0) = \theta_0, \quad \boldsymbol{\varepsilon}_{trip}(0) = \mathbf{0}, \quad \boldsymbol{\varepsilon}_{cp}(0) = \mathbf{0}, \quad \mathbf{p}(0) = \mathbf{p}_0$$

in Ω with

(25) $$\sum_{i=1}^{m} p_{0i} = 1, \quad p_{0i} \geq 0 \quad \text{for} \quad i = 1, \ldots, m,$$

and by (mixed) **boundary conditions**

(26) $$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_1, \qquad \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Gamma_2, \qquad -\lambda_\theta \frac{\partial \theta}{\partial \boldsymbol{\nu}} = \delta\,(\theta - \theta_\Gamma) \quad \text{on} \quad \Gamma,$$

where Γ_1 and Γ_2 are mutually disjoint parts of the boundary Γ and Γ_1 is a closed subset of Γ with positive surface measure.

2.3. Summary of the model

Because of some (mathematical) difficulties we follow a common (modeling) procedure and replace the thermo-elastic dissipation r_{03} (cf. $(8)_3$) by its linearization $\widehat{r}_{03}$ around $\theta \approx \theta_0$

(27) $$\widehat{r}_{03} := -3K_\alpha \theta_0 \operatorname{div}(\mathbf{u}').$$

Finally, we summarize (9) – (12), (19) – (26) including the replacement (27) as **'problem $(\mathbf{P_{QS}})$'**. In theorem 1 we drop the intrinsic dissipation parts $r_{01} + r_{02}$.

3. Preparations

3.1. Function spaces

Let $\Omega \in \mathcal{C}^{0,1}$ and $r \in [1, \infty]$.
Spaces re. $\mathbf{u}$

$$\begin{aligned} &\mathbf{H_u} := [L^2(\Omega)]^3, && \mathcal{H}^r_{\mathbf{u}} := L^r(S; \mathbf{H_u}), && \mathfrak{H}^r_{\mathbf{u}} := W^{1,r}(S; \mathbf{H_u}), \\ &\mathbf{V_u} := \left\{\mathbf{u} \in [W^{1,2}(\Omega)]^3 : \mathbf{u}|_{\Gamma_1} = \mathbf{0}\right\}, && \mathcal{V}^r_{\mathbf{u}} := L^r(S; \mathbf{V_u}), && \mathfrak{V}^r_{\mathbf{u}} := W^{1,r}(S; \mathbf{V_u}), \\ &\mathcal{U}_{\mathbf{u}} := \left\{\mathbf{u} \in \mathcal{V}^2_{\mathbf{u}} : \mathbf{u}' \in \mathcal{H}^2_{\mathbf{u}}, \mathbf{u}'' \in \left(\mathcal{V}^2_{\mathbf{u}}\right)^*\right\}. \end{aligned}$$

Spaces re. θ

$$\begin{aligned} &H_\theta := L^2(\Omega), && \mathcal{H}^r_\theta := L^r(S; H_\theta), && \mathfrak{H}^r_\theta := W^{1,r}(S; H_\theta), \\ &V_\theta := W^{1,2}(\Omega), && \mathcal{V}^r_\theta := L^r(S; V_\theta), && \mathfrak{V}^r_\theta := W^{1,r}(S; V_\theta), \\ &\mathcal{U}_\theta := W^{1,2}(S; V_\theta, V_\theta^*). \end{aligned}$$

Spaces re. $\mathbf{p}$

$$\begin{aligned} &\mathbf{H_p} := [L^2(\Omega)]^m, && \mathcal{H}^r_{\mathbf{p}} := L^r(S; \mathbf{H_p}), && \mathfrak{H}^r_{\mathbf{p}} := W^{1,r}(S; \mathbf{H_p}), \\ &\mathbf{V_p} := [W^{1,2}(\Omega)]^m, && \mathcal{V}^r_{\mathbf{p}} := L^r(S; \mathbf{V_p}), && \mathfrak{V}^r_{\mathbf{p}} := W^{1,r}(S; \mathbf{V_p}), \end{aligned}$$

$$\mathbf{X_p} := [L^\infty(\Omega)]^m, \qquad \mathcal{X}^r_{\mathbf{p}} := L^r(S; \mathbf{X_p}), \qquad \mathfrak{X}^r_{\mathbf{p}} := [W^{1,r}(\Omega_T)]^m.$$

Spaces re. $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}_{te}$, $\boldsymbol{\varepsilon}_{trip}$ and $\boldsymbol{\varepsilon}_{cp}$

$$\begin{aligned}
&\mathbf{H}_{\boldsymbol{\sigma}} := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{3\times 3} : \boldsymbol{\tau}^T = \boldsymbol{\tau}\}, && \mathcal{H}^r_{\boldsymbol{\sigma}} := L^r(S; \mathbf{H}_{\boldsymbol{\sigma}}), && \mathfrak{H}^r_{\boldsymbol{\sigma}} := W^{1,r}(S; \mathbf{H}_{\boldsymbol{\sigma}}),\\
&\mathbf{V}_{\boldsymbol{\sigma}} := [W^{1,2}(\Omega)]^{3\times 3} \cap \mathbf{H}_{\boldsymbol{\sigma}}, && \mathcal{V}^r_{\boldsymbol{\sigma}} := L^r(S; \mathbf{V}_{\boldsymbol{\sigma}}), && \mathfrak{V}^r_{\boldsymbol{\sigma}} := W^{1,r}(S; \mathbf{V}_{\boldsymbol{\sigma}}),\\
&\mathbf{X}_{\boldsymbol{\sigma}} := [L^\infty(\Omega)]^{3\times 3}, && \mathcal{X}^r_{\boldsymbol{\sigma}} := L^r(S; \mathbf{X}_{\boldsymbol{\sigma}}), && \mathfrak{X}^r_{\boldsymbol{\sigma}} := W^{1,r}(S; \mathbf{X}_{\boldsymbol{\sigma}}).
\end{aligned}$$

Data and solution spaces

$$\begin{aligned}
\mathbf{Z}_D &:= \mathbf{V_u} \times \mathbf{H_u} \times V_\theta \times \mathbf{X_p} \times \mathcal{H}^2_{\mathbf{u}} \times \mathcal{H}^2_\theta,\\
\mathbf{Z}_S &:= \mathcal{U}_{\mathbf{u}} \cap \mathfrak{V}^2_{\mathbf{u}} \times \mathcal{U}_\theta \cap \mathfrak{H}^2_\theta \times C^1(\overline{S}; \mathbf{X_p}) \times \mathcal{H}^2_{\boldsymbol{\sigma}} \times C^1(\overline{S}; \mathbf{H}_{\boldsymbol{\sigma}}).
\end{aligned}$$

3.2. Summary of the assumptions

We require the following quantities to be constant and non-negative

(28) $$\rho_0,\ \rho_i,\ \mu,\ \lambda,\ \alpha,\ R,\ c_e,\ \lambda_\theta,\ \kappa_i,\ L_i > 0,\ \delta \geq 0, \qquad i = 1, \ldots, m.$$

Furthermore, we assume for the **initial conditions**

(29) $$\mathbf{u}_0 \in \mathbf{V_u}, \qquad \mathbf{u}_1 \in \mathbf{H_u}, \qquad \theta_0 \in V_\theta, \qquad \mathbf{p}_0 \in \mathbf{X_p}, \qquad \sum_{i=1}^m p_{0i} = 1, \qquad p_{0i} \geq 0 \quad \text{a.e.},$$

for the **right-hand sides**

(30) $$\mathbf{f} \in \mathfrak{H}^2_{\mathbf{u}}, \qquad r \in \mathcal{H}^2_\theta,$$

and for the **outside temperature**

(31) $$\theta_\Gamma \in W^{1,2}(S; L^2(\Gamma)).$$

For the **saturation functions** Φ_i we assume f.a. $\xi \in [0,1]$ and for $i = 1, \ldots, m$

(32) $$\Phi_i \in C^2([0,1]) \text{ with } \Phi_i(0) = 0,\ \Phi_i(1) = 1, \text{ and } 0 \leq \left|\frac{\partial \Phi_i}{\partial \xi}(\xi)\right|, \left|\frac{\partial^2 \Phi_i}{\partial \xi^2}(\xi)\right| \leq M_\Phi < \infty,$$

with some given $M_\Phi \geq 0$. For the **transformation-rate functions** we assume $\boldsymbol{\gamma} = (\gamma_1, ..., \gamma_m)$: $[0,1]^m \times \mathbb{R} \to \mathbb{R}^m$ is Lipschitz continuous and bounded, i.e. there is a constant $M_{\boldsymbol{\gamma}} \geq 0$ s.t.

(33) $$\|\boldsymbol{\gamma}(\mathbf{p}, \theta)\|_\infty \leq M_{\boldsymbol{\gamma}} \quad \text{f.a.} \quad \mathbf{p} \in [0,1]^m, \quad \theta \in \mathbb{R},$$

(34) $$\sum_{i=1}^m \gamma_i = 0,$$

and an implicit condition:

(35) For all $\theta \in C(\overline{S}; H_\theta)$ the initial value problem (11), $(24)_6$, (25) has a unique solution $\mathbf{p} \in C^1(\overline{S}; \mathbf{H_p})$ satisfying (12).

The typical example for γ_i given by (13) – (16) fulfills this condition, see [Hüß07] for the proof.

3.3. Classical plasticity via a variational inequality

Let $\widehat{F}:\mathbb{R}^{3\times 3}_{\text{sym}}\times\mathbb{R}^+\to\mathbb{R}$, $\widehat{F}(\boldsymbol{\tau};R):=\sqrt{\frac{3}{2}\boldsymbol{\tau}^*:\boldsymbol{\tau}^*}-R$, set

(36)
$$\widehat{\mathbf{K}}:=\left\{\boldsymbol{\tau}\in\mathbb{R}^{3\times 3}_{\text{sym}}:\operatorname{tr}(\boldsymbol{\tau})=0 \text{ and } \widehat{F}(\boldsymbol{\tau};R)\le 0\right\},\quad \mathbf{K}:=\left\{\boldsymbol{\tau}\in\mathbf{H}_{\boldsymbol{\sigma}}:\boldsymbol{\tau}(\mathbf{x})\in\widehat{\mathbf{K}} \text{ f.a.a. } \mathbf{x}\in\Omega\right\}$$

and introduce

$$F:\mathbf{K}\to\mathbb{R},\quad F(\boldsymbol{\tau}):=\widehat{F}(\boldsymbol{\tau}(\cdot);R). \tag{37}$$

Following [DL76], (19) and (36) imply for $\boldsymbol{\sigma}^*$

$$\left(\boldsymbol{\sigma}^*\right)'(t)+\partial\chi_{\mathbf{K}}(\boldsymbol{\sigma}^*(t))\ni g\left(\mathbf{u}'(t),\boldsymbol{\varepsilon}'_{trip}(t)\right)\quad\text{f.a.a.}\quad t\in S,\qquad \boldsymbol{\sigma}^*(0)=\boldsymbol{\sigma}^*_0:=2\mu\boldsymbol{\varepsilon}^*(\mathbf{u}_0) \tag{38}$$

with

$$g=g\left(\mathbf{u}',\boldsymbol{\varepsilon}'_{trip}\right):=2\mu\left(\boldsymbol{\varepsilon}^*(\mathbf{u}')-\boldsymbol{\varepsilon}'_{trip}\right). \tag{39}$$

The differential inclusion in (38) is equivalent to the variational inequality

(40)
$$\left(\left(\boldsymbol{\sigma}^*\right)'(t),\left(\boldsymbol{\tau}-\boldsymbol{\sigma}\right)\right)_{\mathbf{H}_{\boldsymbol{\sigma}}}-\left(g\left(\mathbf{u}'(t),\boldsymbol{\varepsilon}'_{trip}(t)\right),\left(\boldsymbol{\tau}-\boldsymbol{\sigma}\right)\right)_{\mathbf{H}_{\boldsymbol{\sigma}}}\ge 0\quad\text{f.a.a.}\quad t\in S,\quad\text{f.a.}\quad \boldsymbol{\tau}\in\mathbf{K}.$$

3.4. Related work re. mathematical analysis

There are numerous publications which refer to phase transformations, but in connection with (inelastic) deformation and temperature there exists very little *mathematical literature*. In particular coupled models for the material behavior of steel, which describe phase transformations in addition to the temperature and the deformation, have not drawn too much attention in a strict mathematical and numerical context so far. Closest to our approach seem to be [FP96, FDS85, FHP07, Höm95, Höm97, Hüß07, Pan10] (temperature and phase transformation, but no deformation), [AC02, Kam08, Kam09] (inelastic deformation without phase transformation and TRIP), [CR06, GH80] (thermo-plasticity, but no phase transformations and TRIP) and [Boe12b, CHK08, HK06, Ker11] (thermo-elasticity with phase transitions and TRIP, but no classical plasticity). In contrast to [Che03, CR06, CHK08, Ker11] we remain in the Hilbert Space setting and we deal additionally with mixed boundary conditions and classical plasticity following the argumentation in [BBW15, Boe12a].

4. Main result

Let $\Omega\in\mathcal{C}^{0,1}$ and assume (28) – (36) in all of section 4.

4.1. Weak solutions for problem $(\mathbf{P}_{QS})$

Let $\mathbf{A_u}:\mathbf{V_u}\to\mathbf{V_u^*}$ and $A_\theta:V_\theta\to V_\theta^*$ be defined by

$$\langle\mathbf{A_u}\mathbf{u},\mathbf{v}\rangle_{\mathbf{V_u^*}\mathbf{V_u}}:=2\int_\Omega\mu\,\boldsymbol{\varepsilon}(\mathbf{u}):\boldsymbol{\varepsilon}(\mathbf{v})\,\mathrm{d}\mathbf{x}+\int_\Omega\lambda\,\operatorname{div}(\mathbf{u})\,\operatorname{div}(\mathbf{v})\,\mathrm{d}\mathbf{x},\qquad\text{f.a. }\mathbf{v}\in\mathbf{V_u}, \tag{41}$$

$$\langle A_\theta\theta,\vartheta\rangle_{V_\theta^*V_\theta}:=\int_\Omega\lambda_\theta\,\nabla\theta\,\nabla\vartheta\,\mathrm{d}\mathbf{x}+\int_\Gamma\delta\,\theta\,\vartheta\,\mathrm{d}\sigma_{\mathbf{x}},\qquad\text{f.a. }\vartheta\in V_\theta. \tag{42}$$

A weak formulation for problem (P_{QS}) is given by

Definition 1. *The quintuple* $(\mathbf{u}, \theta, \mathbf{p}, \varepsilon_{cp}, \varepsilon_{trip}) \in \mathbf{Z}_S$ *is called a weak solution of problem* $(\mathbf{P}_{QS})$*, if* (24) *holds and if f.a.a.* $t \in S$ *and f.a.* $\mathbf{v} \in \mathbf{V_u}$*,* $\vartheta \in V_\theta$*,*

$$\langle \mathbf{A_u u}(t), \mathbf{v}\rangle_{\mathbf{V_u^* V_u}} = (\mathbf{f}_0(t), \nabla \mathbf{v})_{\mathbf{H_u}} + (\mathbf{f}(t), \mathbf{v})_{\mathbf{H_u}}, \tag{43}$$

$$\left\langle \rho_0\, c_e\, \theta'(t), \vartheta\right\rangle_{V_\theta^* V_\theta} + \langle A_\theta \theta(t), \vartheta\rangle_{V_\theta^* V_\theta} = (\widehat{r}_0(t) + r(t), \vartheta)_{H_\theta} + \int_\Gamma \delta\, \theta_\Gamma(t)\, \vartheta\, \mathrm{d}\sigma_{\mathbf{x}}, \tag{44}$$

$$\mathbf{p}'(t) = \boldsymbol{\gamma}(\mathbf{p}(t), \theta(t)) \quad in \quad \mathbf{X_p}, \tag{45}$$

$$\boldsymbol{\varepsilon}'_{trip}(t) = \frac{3}{2}\,\boldsymbol{\sigma}^*(t) \sum_{i=1}^m \kappa_i\, \frac{\partial \Phi_i}{\partial p_i}(p_i(t)) \max\{p_i'(t), 0\} \quad in \quad \mathbf{H}_{\boldsymbol{\sigma}}, \tag{46}$$

$$\boldsymbol{\varepsilon}_{cp}(t) = \boldsymbol{\varepsilon}^*\left(\mathbf{u}(t)\right) - \boldsymbol{\varepsilon}_{trip}(t) - \frac{1}{2\mu}\boldsymbol{\sigma}^*(t) \quad in \quad \mathbf{H}_{\boldsymbol{\sigma}}, \tag{47}$$

$$\left(\boldsymbol{\sigma}^*\right)'(t) + \partial \chi_{\mathbf{K}}(\boldsymbol{\sigma}^*(t)) \ni g(\mathbf{u}'(t), \boldsymbol{\varepsilon}'_{trip}(t)) \quad in \quad \mathbf{H}_{\boldsymbol{\sigma}} \tag{48}$$

hold.

4.2. Main theorem

Theorem 1 (Well-posedness for problem (P_{QS}))**.** *Assume for the intrinsic (plastic) dissipation(s) (cf.* (8)*)*

$$r_{01} = r_{02} = 0. \tag{49}$$

Then problem (P_{QS}) *has a unique global solution (i.e. on all time intervals* $(0, T)$*,* $T > 0$*) and the solution map*

$$\mathbf{Z}_D \ni (\mathbf{u}_0, \mathbf{u}_1, \theta_0, \mathbf{p}_0, \mathbf{f}, r) \mapsto (\mathbf{u}, \theta, \mathbf{p}, \varepsilon_{cp}, \varepsilon_{trip}) \in \mathbf{Z}_S \tag{50}$$

is globally Lipschitz.

Remark 1. Taking into account the intrinsic (plastic) dissipation(s) r_{01} and r_{02} (cf. (8)) it is possible to show (at least) the existence of a global solution (i.e. on all time intervals $(0, T)$, $T > 0$) for problem (P_{QS}) using Schauder's fixed point theory (cf. [BBW15, CHK08, Ker11]).

5. Proof of Theorem 1

The proof of theorem 1 is organized as follows: In the next section an outline of the proof is given. Section 5.2 collects several auxiliary lemmas re. the proof of the main theorem. Section 5.3 provides the proofs of the statements in section 5.2. The proof of the main theorem is completed in section 5.4.
The constants c and $\widehat{c}$ appearing in the estimates below depend neither on t nor on the participating functions.

5.1. Outline

Figure 1 illustrates the idea of the proof: Initial and boundary values are fixed. We divide the whole problem in a succession of three subproblems $(P_{QS1}), \ldots, (P_{QS3})$. The 'outer' (fixed point) problem, (P_{QS3}), exploits Banach's fixed point theorem and goes as follows: Let $\widehat{\theta}$ be given. Then there is a quintuple $(\mathbf{u}, \boldsymbol{\sigma}, \varepsilon_{cp}, \mathbf{p}, \varepsilon_{trip})$ satisfying $(b10)$ – $(b16)$. Finally we plug these quantities in $(b18)$, obtain θ as in $(b17)$ and an operator $\widehat{\theta} \mapsto \theta$ which, employing an appropriate norm, is a contraction. In order to get $(\mathbf{u}, \boldsymbol{\sigma}, \varepsilon_{cp})$, we show that $(\overline{\mathbf{u}}, \overline{\boldsymbol{\sigma}}, \overline{\varepsilon}_{cp}) \mapsto (\mathbf{u}, \boldsymbol{\sigma}, \varepsilon_{cp})$ has a fixed point, too. This is subproblem (P_{QS2}). (P_{QS1}) yields $\boldsymbol{\varepsilon}_{trip}$ as a fixed point of $\tilde{\boldsymbol{\varepsilon}}_{trip} \mapsto \boldsymbol{\varepsilon}_{trip}$ (cf. $(b3)$ – $(b9)$). The same strategy is used in [BBW15, Boe12a].

(b1) $\widehat{\theta} \in \mathfrak{H}^2_\theta$

(b2) $(\overline{\mathbf{u}}, \overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\varepsilon}}_{cp}) \in \mathfrak{V}^2_{\mathbf{u}} \times \mathfrak{H}^2_{\boldsymbol{\sigma}} \times \mathfrak{H}^2_{\boldsymbol{\sigma}}$

(b3) $\tilde{\boldsymbol{\varepsilon}}_{trip} \in C^1(\overline{S}; \mathbf{H}_{\boldsymbol{\sigma}})$

(b4) $\boldsymbol{\sigma}^* \in \mathfrak{H}^2_{\boldsymbol{\sigma}} \cap \mathcal{X}^\infty_{\boldsymbol{\sigma}} \hookrightarrow C(\overline{S}; \mathbf{H}_{\boldsymbol{\sigma}}) \quad + \quad \boldsymbol{\sigma}^*(t) \in \mathbf{K}$ f.a.a. $t \in S$

(b5) $(\boldsymbol{\sigma}^*)' + \partial\chi_{\mathbf{K}}(\boldsymbol{\sigma}^*) \ni 2\mu\left(\boldsymbol{\varepsilon}^*(\overline{\mathbf{u}}') - \tilde{\boldsymbol{\varepsilon}}'_{trip}\right) \quad + \quad$ I.C. $(38)_2$

(b6) $\mathbf{p} \in C^1(\overline{S}; \mathbf{H}_{\mathbf{p}}) \cap \mathcal{X}^\infty_{\mathbf{p}}$

(b7) $\mathbf{p}' = \boldsymbol{\gamma}(\mathbf{p}, \widehat{\theta}) \quad + \quad \sum p_i = 1,\ p_i \geq 0 \quad + \quad$ I.C. $(24)_6$, (25)

(b8) $\boldsymbol{\varepsilon}_{trip} \in C^1(\overline{S}; \mathbf{H}_{\boldsymbol{\sigma}}) \cap \mathfrak{X}^\infty_\sigma$

(b9) $\boldsymbol{\varepsilon}'_{trip} = \frac{3}{2}\boldsymbol{\sigma}^* \sum \kappa_i \frac{\partial\Phi}{\partial p_i}(p_i) \max\{p'_i, 0\} \quad + \quad$ I.C. $(24)_4$

$(\mathbf{P}_{QS1})$

(b10) $\mathbf{u} \in \mathcal{U}_{\mathbf{u}} \cap \mathfrak{V}^2_{\mathbf{u}} \hookrightarrow \mathfrak{V}^2_{\mathbf{u}}$

(b11) $\mathbf{A}_{\mathbf{u}}\mathbf{u} = \widehat{f} \quad + \quad$ I.C. $(24)_{1,2} \quad + \quad$ B.C. $(26)_{1,2}$, $\quad \mathbf{A}_{\mathbf{u}}\mathbf{u} := -2\,\mathrm{div}\,(\mu\boldsymbol{\varepsilon}(\mathbf{u})) - \nabla(\lambda\,\mathrm{div}\,(\mathbf{u}))$,

(b12) $\widehat{f} := -3\nabla\left(K_\alpha\left(\widehat{\theta} - \theta_0\right)\right) - \nabla\left(K\sum\left(\frac{\rho_0}{\rho_i(\theta_0)} - 1\right)p_i\right) - 2\,\mathrm{div}\,(\mu(\overline{\boldsymbol{\varepsilon}}_{cp} + \boldsymbol{\varepsilon}_{trip})) + \mathbf{f}$

(b13) $(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{cp}) \in \mathfrak{H}^2_{\boldsymbol{\sigma}} \times \mathfrak{H}^2_{\boldsymbol{\sigma}}$

(b14) $\boldsymbol{\sigma} = 2\mu\left(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}_{cp} - \boldsymbol{\varepsilon}_{trip}\right) + \lambda\,\mathrm{div}\,(\mathbf{u})\,\mathrm{Id} - 3K_\alpha\left(\widehat{\theta} - \theta_0\right)\mathrm{Id} - K\sum\left(\frac{\rho_0}{\rho_i(\theta_0)} - 1\right)p_i\,\mathrm{Id} + h\boldsymbol{\varepsilon}(\mathbf{u}')$

(b15) $\boldsymbol{\varepsilon}_{cp} = \boldsymbol{\varepsilon}^*(\mathbf{u}) - \boldsymbol{\varepsilon}_{trip} - \frac{1}{2\mu}\boldsymbol{\sigma}^*$

(b16) $(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{cp}) \in \mathfrak{V}^2_{\mathbf{u}} \times \mathfrak{H}^2_{\boldsymbol{\sigma}} \times \mathfrak{H}^2_{\boldsymbol{\sigma}}$

$(\mathbf{P}_{QS2})$

(b17) $\theta \in \mathfrak{H}^2_\theta \cap \mathcal{U}_\theta \hookrightarrow \mathfrak{H}^2_\theta$

(b18) $\rho_0 c_e \theta' - \mathrm{div}\,(\lambda_\theta \nabla\theta) = -3K_\alpha\theta_0\,\mathrm{div}\,(\mathbf{u}') + \rho_0\sum L_i p'_i + r \quad + \quad$ I.C. $(24)_3 \quad + \quad$ B.C. $(26)_3$

$(\mathbf{P}_{QS3})$

Figure 1: Scheme for the proof of theorem 1 (I.C. and B.C. – initial and boundary condition).

5.2. Some auxiliary lemmas

In order to deal with the subproblems mentioned in the previous section, we follow the scheme described in figure 1.

Lemma 1. *Assume* (b1), (b2), (b3), $k = 1, 2$. *Then the following holds:*

1. *There is a unique* $\boldsymbol{\sigma}^*$ *satisfying* (b4), (b5) *and a constant* $c > 0$ *s.t.*

(51)
$$\|\boldsymbol{\sigma}^*(t) - \boldsymbol{\sigma}^*_0\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} \leq c\int_0^t \left\{\|\overline{\mathbf{u}}'(s)\|^2_{\mathbf{V}_{\mathbf{u}}} + \|\tilde{\boldsymbol{\varepsilon}}'_{trip}(s)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}}\right\} \mathrm{d}s, \quad f.a.\ t \in S,$$

(52)
$$\|(\boldsymbol{\sigma}^*)'(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} \leq c\left\{\|\overline{\mathbf{u}}'(t)\|^2_{\mathbf{V}_{\mathbf{u}}} + \|\tilde{\boldsymbol{\varepsilon}}'_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}}\right\}, \quad f.a.a.\ t \in S.$$

Moreover, let $\boldsymbol{\sigma}^*_k$ *be the solution of* (b5) *with* $(\mathbf{u}, \boldsymbol{\varepsilon}_{trip}) := (\mathbf{u}_k, \boldsymbol{\varepsilon}_{trip,k})$. *Then there is a constant* $\widehat{c} > 0$ *s.t.*

(53)
$$\|\boldsymbol{D}\boldsymbol{\sigma}^*(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} \leq \widehat{c}\left\{\|\boldsymbol{D}\boldsymbol{\sigma}^*_0\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \int_0^t \left[\|\boldsymbol{D}\overline{\mathbf{u}}'(s)\|^2_{\mathbf{V}_{\mathbf{u}}} + \|\boldsymbol{D}\boldsymbol{\varepsilon}'_{trip}(s)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}}\right] \mathrm{d}s\right\}, \quad f.a.\ t \in S.$$

2. *There is a unique solution* $\mathbf{p}$ *satisfying* (*b*6), (*b*7) *and a constant* $c>0$ *s.t. f.a.* $t\in S$

(54)
$$\|\mathbf{p}(t)\|_\infty \le 1, \quad \|\mathbf{p}(t)\|^2_{\mathbf{H_p}} \le \operatorname{meas}(\Omega), \quad \|\mathbf{p}'(t)\|_\infty \le M_{\boldsymbol{\gamma}}, \quad \|\mathbf{p}'(t)\|^2_{\mathbf{H_p}} \le M_{\boldsymbol{\gamma}} \operatorname{meas}(\Omega).$$

Moreover, let $\mathbf{p}$ *be the solution of* (*b*7) *with* $\mathbf{p}$ *replaced by* $\mathbf{p}_k$*. Then there is a constant* $\widehat{c}>0$ *s.t. f.a.* $t\in S$

(55)
$$\|\boldsymbol{D}\mathbf{p}(t)\|^2_{\mathbf{H_p}} \le \widehat{c}\left\{\|\boldsymbol{D}\mathbf{p}_0\|^2_{\mathbf{H_p}} + c\int_0^t \|\boldsymbol{D}\widehat{\theta}(s)\|^2_{H_\theta}\,\mathrm{d}s\right\},$$

(56)
$$\|\boldsymbol{D}\mathbf{p}'(t)\|^2_{\mathbf{H_p}} \le \widehat{c}\left\{\|\boldsymbol{D}\mathbf{p}_0\|^2_{\mathbf{H}_p} + \|\boldsymbol{D}\widehat{\theta}(s)\|^2_{H_\theta} + \int_0^t \|\boldsymbol{D}\widehat{\theta}(s)\|^2_{H_\theta}\,\mathrm{d}s\right\}.$$

3. *There is a unique solution* $\boldsymbol{\varepsilon}_{trip}$ *satisfying* (*b*8), (*b*9) *and f.a.* $t\in S$

(57)
$$\|\boldsymbol{\varepsilon}_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} \le c\left\{\|\boldsymbol{\varepsilon}_{trip}(0)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \int_0^t \|\boldsymbol{\sigma}^*(s)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}}\,\mathrm{d}s\right\}, \qquad \|\boldsymbol{\varepsilon}'_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} \le c\,\|\boldsymbol{\sigma}^*(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}}.$$

Moreover, let $\boldsymbol{\varepsilon}_{trip,k}$ *be the solution of* (*b*8), (*b*9) *where* $\mathbf{p}$ *and* $\boldsymbol{\sigma}^*$ *are replaced by* $\mathbf{p}_k$ *and* $\boldsymbol{\sigma}^*_k$*, resp. Then there is a constant* $\widehat{c}>0$ *s.t.*

(58)
$$\|\boldsymbol{D}\boldsymbol{\varepsilon}_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} \le \widehat{c}\left\{\|\boldsymbol{D}\boldsymbol{\varepsilon}_{trip}(0)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \int_0^t \left[\|\boldsymbol{D}\mathbf{p}(s)\|^2_{\mathbf{H_p}} + \|\boldsymbol{D}\mathbf{p}'(s)\|^2_{\mathbf{H_p}} + \|\boldsymbol{D}\boldsymbol{\sigma}^*(s)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}}\right]\mathrm{d}s\right\},$$

(59)
$$\|\boldsymbol{D}\boldsymbol{\varepsilon}'_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} \le \widehat{c}\left\{\|\boldsymbol{D}\mathbf{p}(t)\|^2_{\mathbf{H_p}} + \|\boldsymbol{D}\mathbf{p}'(t)\|^2_{\mathbf{H_p}} + \|\boldsymbol{D}\boldsymbol{\sigma}^*(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}}\right\} \quad \text{f.a. } t\in S.$$

Lemma 2. *Let* $\widehat{\theta}_k \in \mathfrak{H}^2_\theta$, $(\overline{\mathbf{u}}_k, \overline{\boldsymbol{\sigma}}_k, \overline{\varepsilon}_{cp,k}) \in \mathfrak{V}^2_{\mathbf{u}} \times \mathfrak{H}^2_{\boldsymbol{\sigma}} \times \mathfrak{H}^2_{\boldsymbol{\sigma}}$, $\tilde{\varepsilon}_{trip,k} \in \mathfrak{X}^\infty_{\boldsymbol{\sigma}}$ *and let* $\boldsymbol{\varepsilon}_{trip,k}$ *be the solution of* (*b*8), (*b*9) *obtained via* (*b*3) – (*b*7), $k=1,2$*. Then*

1. *There is a constant* $\widehat{c}>0$ *s.t. f.a.a.* $(\mathbf{x},t)\in\Omega_T$

(60)
$$|\boldsymbol{D}\boldsymbol{\varepsilon}'_{trip}(\mathbf{x},t)| \le \widehat{c}\left\{|\boldsymbol{\sigma}^*_2(\mathbf{x},t)|\left[|\boldsymbol{D}\widehat{\theta}(\mathbf{x},t)| + |\boldsymbol{D}\mathbf{p}(\mathbf{x},t)|\right] + |\boldsymbol{D}\boldsymbol{\sigma}^*(\mathbf{x},t)|\right\}.$$

2. *In particular, if* $\boldsymbol{D}\widehat{\theta} = \boldsymbol{D}\mathbf{p} = \boldsymbol{D}\overline{\mathbf{u}} = 0$*, then there is a constant* $\widehat{c}>0$ *s.t.*

(61)
$$\|\boldsymbol{D}\boldsymbol{\varepsilon}'_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} \le \widehat{c}\left\{\|\boldsymbol{D}\boldsymbol{\sigma}^*(0)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \int_0^t \|\boldsymbol{D}\tilde{\varepsilon}'_{trip}(s)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}}\,\mathrm{d}s\right\} \quad \text{f.a. } t\in S.$$

3. *Moreover, the map*

(62)
$$\mathbf{T}_{QS1} : C^1(\overline{S};\mathbf{H}_{\boldsymbol{\sigma}}) \to C^1(\overline{S};\mathbf{H}_{\boldsymbol{\sigma}}), \quad \tilde{\varepsilon}_{trip} \mapsto \mathbf{T}_{QS1}(\tilde{\varepsilon}_{trip}) =: \boldsymbol{\varepsilon}_{trip}$$

has a fixed point which is (also) denoted by $\boldsymbol{\varepsilon}_{trip}$.

Lemma 3. *Assume* (*b*1), (*b*2), (*b*3), $k=1,2$*. Then there is a unique solution* $\mathbf{u}$ *satisfying* (*b*10), (*b*11), (*b*12) *and a constant* $c>0$ *s.t. f.a.a.* $t\in S$

(63)
$$\|\mathbf{u}(t)\|^2_{\mathbf{V_u}} \le c\left\{\|\widehat{\theta}(t)\|^2_{H_\theta} + \|\mathbf{p}(t)\|^2_{\mathbf{H_p}} + \|\boldsymbol{\varepsilon}_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\overline{\varepsilon}_{cp}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\mathbf{f}(t)\|^2_{\mathbf{H_u}}\right\},$$

$$\|\mathbf{u}'(t)\|^2_{\mathbf{V}_\mathbf{u}} \le c\left\{\|\widehat{\theta}'(t)\|^2_{H_\theta} + \|\mathbf{p}'(t)\|^2_{\mathbf{H}_\mathbf{p}} + \|\boldsymbol{\varepsilon}'_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\overline{\boldsymbol{\varepsilon}}'_{cp}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\mathbf{f}'(t)\|^2_{\mathbf{H}_\mathbf{u}}\right\}. \tag{64}$$

Moreover, let $\mathbf{u}$ *be the solution of* $(b11)$, $(b12)$ *with* $\mathbf{u}$ *replaced by* $\mathbf{u}_k$. *Then there is a constant* $\widehat{c} > 0$ *s.t. f.a.a.* $t \in S$

$$\|\boldsymbol{D}\mathbf{u}(t)\|^2_{\mathbf{V}_\mathbf{u}} \le \widehat{c}\left\{\|\boldsymbol{D}\widehat{\theta}(t)\|^2_{H_\theta} + \|\boldsymbol{D}\mathbf{p}(t)\|^2_{\mathbf{H}_\mathbf{p}} + \|\boldsymbol{D}\boldsymbol{\varepsilon}_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\boldsymbol{D}\overline{\boldsymbol{\varepsilon}}_{cp}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\boldsymbol{D}\mathbf{f}(t)\|^2_{\mathbf{H}_\mathbf{u}}\right\}, \tag{65}$$

$$\|\boldsymbol{D}\mathbf{u}'(t)\|^2_{\mathbf{V}_\mathbf{u}} \le \widehat{c}\left\{\|\boldsymbol{D}\widehat{\theta}'(t)\|^2_{H_\theta} + \|\boldsymbol{D}\mathbf{p}'(t)\|^2_{\mathbf{H}_\mathbf{p}} + \|\boldsymbol{D}\boldsymbol{\varepsilon}'_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\boldsymbol{D}\overline{\boldsymbol{\varepsilon}}'_{cp}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\boldsymbol{D}\mathbf{f}'(t)\|^2_{\mathbf{H}_\mathbf{u}}\right\}. \tag{66}$$

Lemma 4. *Let* $\widehat{\theta}_k \in \mathfrak{H}^2_\theta$, $(\overline{\mathbf{u}}_k, \overline{\boldsymbol{\sigma}}_k, \overline{\boldsymbol{\varepsilon}}_{cp,k}) \in \mathfrak{V}^2_\mathbf{u} \times \mathfrak{H}^2_{\boldsymbol{\sigma}} \times \mathfrak{H}^2_{\boldsymbol{\sigma}}$ *and let* $(\mathbf{u}_k, \boldsymbol{\sigma}_k, \boldsymbol{\varepsilon}_{cp,k})$ *be the solution of* $(b10)$ – $(b16)$ *obtained via* $(b2)$ – $(b9)$, $k = 1, 2$. *Then*

1. *There is a constant* $\widehat{c} > 0$ *s.t. f.a.a.* $t \in S$

$$\|\boldsymbol{D}\mathbf{u}(t)\|^2_{\mathbf{V}_\mathbf{u}} \le \widehat{c}\left\{\|\boldsymbol{D}\widehat{\theta}(t)\|^2_{H_\theta} + \|\boldsymbol{D}\mathbf{p}(t)\|^2_{\mathbf{H}_\mathbf{p}} + \|\boldsymbol{D}\boldsymbol{\varepsilon}_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\boldsymbol{D}\overline{\boldsymbol{\varepsilon}}_{cp}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\boldsymbol{D}\mathbf{f}(t)\|^2_{\mathbf{H}_\mathbf{u}}\right\},$$

$$\|\boldsymbol{D}\mathbf{u}'(t)\|^2_{\mathbf{V}_\mathbf{u}} \le \widehat{c}\left\{\|\boldsymbol{D}\widehat{\theta}'(t)\|^2_{H_\theta} + \|\boldsymbol{D}\mathbf{p}'(t)\|^2_{\mathbf{H}_\mathbf{p}} + \|\boldsymbol{D}\boldsymbol{\varepsilon}'_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\boldsymbol{D}\overline{\boldsymbol{\varepsilon}}'_{cp}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\boldsymbol{D}\mathbf{f}'(t)\|^2_{\mathbf{H}_\mathbf{u}}\right\},$$

$$\|\boldsymbol{D}\boldsymbol{\sigma}(t)\|^2_{\mathbf{V}_\mathbf{u}} \le \widehat{c}\left\{\|\boldsymbol{D}\widehat{\theta}(t)\|^2_{H_\theta} + \|\boldsymbol{D}\mathbf{p}(t)\|^2_{\mathbf{H}_\mathbf{p}} + \|\boldsymbol{D}\boldsymbol{\varepsilon}_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\boldsymbol{D}\overline{\boldsymbol{\varepsilon}}_{cp}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\boldsymbol{D}\mathbf{u}(t)\|^2_{\mathbf{H}_\mathbf{u}}\right\},$$

$$\|\boldsymbol{D}\boldsymbol{\sigma}'(t)\|^2_{\mathbf{V}_\mathbf{u}} \le \widehat{c}\left\{\|\boldsymbol{D}\widehat{\theta}'(t)\|^2_{H_\theta} + \|\boldsymbol{D}\mathbf{p}'(t)\|^2_{\mathbf{H}_\mathbf{p}} + \|\boldsymbol{D}\boldsymbol{\varepsilon}'_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\boldsymbol{D}\overline{\boldsymbol{\varepsilon}}'_{cp}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\boldsymbol{D}\mathbf{u}'(t)\|^2_{\mathbf{H}_\mathbf{u}}\right\},$$

$$\|\boldsymbol{D}\boldsymbol{\varepsilon}_{cp}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} \le \widehat{c}\left\{\|\boldsymbol{D}\mathbf{u}(t)\|^2_{\mathbf{V}_\mathbf{u}} + \|\boldsymbol{D}\boldsymbol{\varepsilon}_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\boldsymbol{D}\boldsymbol{\sigma}^*(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}}\right\},$$

$$\|\boldsymbol{D}\boldsymbol{\varepsilon}'_{cp}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} \le \widehat{c}\left\{\|\boldsymbol{D}\mathbf{u}'(t)\|^2_{\mathbf{V}_\mathbf{u}} + \|\boldsymbol{D}\boldsymbol{\varepsilon}'_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\boldsymbol{D}(\boldsymbol{\sigma}^*)'(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}}\right\}. \tag{67}$$

2. *Moreover, the map*

$$\mathbf{T}_{QS2} : \mathfrak{V}^2_\mathbf{u} \times \mathfrak{H}^2_{\boldsymbol{\sigma}} \times \mathfrak{H}^2_{\boldsymbol{\sigma}} \to \mathfrak{V}^2_\mathbf{u} \times \mathfrak{H}^2_{\boldsymbol{\sigma}} \times \mathfrak{H}^2_{\boldsymbol{\sigma}}, \quad (\overline{\mathbf{u}}, \overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\varepsilon}}_{cp}) \mapsto \mathbf{T}_{QS2}(\overline{\mathbf{u}}, \overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\varepsilon}}_{cp}) =: (\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{cp}) \tag{68}$$

has a fixed point which is (also) denoted by $(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{cp})$.

Lemma 5. *Assume* $(b1)$, $(b2)$, $(b3)$, $k = 1, 2$. *Then there is a unique solution* $\theta \in \mathcal{U}_\theta$ *satisfying* $(b18)$ *and a constant* $c > 0$ *s.t. f.a.a.* $t \in S$

$$\|\theta(t)\|^2_{\mathbf{H}_\theta} + \|\theta\|^2_{L^2((0,t);\mathbf{V}_\theta)} \le c\left\{\|\theta_0\|^2_{\mathbf{H}_\theta} + \int_0^t \left[\|\mathbf{u}'(s)\|^2_{\mathbf{V}_\mathbf{u}} + \|\mathbf{p}'(s)\|^2_{\mathbf{H}_\mathbf{p}} + \|r(s)\|^2_{\mathbf{H}_\theta} + \|\theta_\Gamma(s)\|^2_{L^2(\Gamma)}\right] \mathrm{d}s\right\}, \tag{69}$$

Furthermore, the solution θ *of problem* $(b18)$ *is an element of* $\mathcal{V}^\infty_\theta \cap \mathfrak{H}^2_\theta$ *and satisfies f.a.a.* $t \in S$ *the estimate*

$$\|\theta(t)\|^2_{\mathbf{V}_\theta} + \|\theta'\|^2_{L^2((0,t);\mathbf{H}_\theta)} \le c\left\{\|\theta_0\|^2_{\mathbf{V}_\theta} + \|\theta_\Gamma(t)\|^2_{L^2(\Gamma)} + \int_0^t \left[\|\mathbf{u}'(s)\|^2_{\mathbf{V}_\mathbf{u}} + \|\mathbf{p}'(s)\|^2_{\mathbf{H}_\mathbf{p}} + \|r(s)\|^2_{\mathbf{H}_\theta} + \|\theta'_\Gamma(s)\|^2_{L^2(\Gamma)}\right] \mathrm{d}s\right\}. \tag{70}$$

where c *is a positive constant. Moreover, let* θ *be the solution of* $(b18)$ *with* θ *replaced by* θ_k *and* θ_Γ *replaced by* $\theta_{\Gamma,k}$. *Then there is a constant* $\widehat{c} > 0$ *s.t. f.a.a.* $t \in S$

(71)

$$\|\boldsymbol{D}\theta(t)\|^2_{\mathbf{H}_\theta\cap\mathbf{V}_\theta} + \|\boldsymbol{D}\theta\|^2_{L^2((0,t);\mathbf{V}_\theta)} + \|\boldsymbol{D}\theta'\|^2_{L^2((0,t);\mathbf{H}_\theta)} \le \widehat{c}\Big\{\|\boldsymbol{D}\theta_0\|^2_{\mathbf{H}_\theta\cap\mathbf{V}_\theta} + \|\boldsymbol{D}\theta_\Gamma(t)\|^2_{L^2(\Gamma)} + \int_0^t \Big[\|\boldsymbol{D}\mathbf{u}'(s)\|^2_{\mathbf{V}_\mathbf{u}} + \|\boldsymbol{D}\mathbf{p}'(s)\|^2_{\mathbf{H}_\mathbf{p}} + \|\boldsymbol{D}r(s)\|^2_{\mathbf{H}_\theta} + \|\boldsymbol{D}\theta'_\Gamma(s)\|^2_{L^2(\Gamma)}\Big]\,\mathrm{d}s\Big\}.$$

Note: In the situation of the setting ($b1$) – ($b18$), θ_Γ *is fixed and therefore* $\boldsymbol{D}\theta_\Gamma = \boldsymbol{D}\theta'_\Gamma = 0$. (71) *is also an investment for the verification of the continuous dependence of the solution on the data.*

Lemma 6. *Let* $\widehat{\theta}_k \in \mathfrak{H}^2_\theta$ *and let* θ_k *be the solution of* ($b17$), ($b18$) *obtained via* ($b1$) – ($b16$), $k = 1, 2$. *Then there is a constant* $\widehat{c} > 0$ *s.t. f.a.a.* $t \in S$

(72)

$$\|\boldsymbol{D}\theta(t)\|^2_{\mathbf{H}_\theta\cap\mathbf{V}_\theta} + \|\boldsymbol{D}\theta\|^2_{L^2((0,t);\mathbf{V}_\theta)} + \|\boldsymbol{D}\theta'\|^2_{L^2((0,t);\mathbf{H}_\theta)} \le \widehat{c}\int_0^t \Big\{\|\boldsymbol{D}\mathbf{u}'(s)\|^2_{\mathbf{V}_\mathbf{u}} + \|\boldsymbol{D}\mathbf{p}'(s)\|^2_{\mathbf{H}_\mathbf{p}}\Big\}\,\mathrm{d}s.$$

5.3. Proofs of the statements in 5.2

Proof of Lemma 1. Re. (a) $\mathbf{K} \subset \mathbf{H}_{\boldsymbol{\sigma}}$ is closed, nonempty and convex, $\chi_\mathbf{K}$ is proper, convex and lower semi-continuous, $\partial\chi_\mathbf{K} : \mathbf{H}_{\boldsymbol{\sigma}} \to \mathcal{P}(\mathbf{H}_{\boldsymbol{\sigma}})$ is maximal monotone and $D(\chi_\mathbf{K}) = D(\partial\chi_\mathbf{K}) = \mathbf{K}$. The right-hand side of (38) belongs to $\mathcal{H}^2_{\boldsymbol{\sigma}}$. Therefore theorem 2 works. Using standard estimates on $g := g\left(\overline{\mathbf{u}}', \tilde{\boldsymbol{\varepsilon}}'_{trip}\right)$ (cf. (39))

$$\|g(t)\|_{\mathbf{H}_\sigma} \le c\left\{\left\|\overline{\mathbf{u}}'(t)\right\|_{\mathbf{V}_\mathbf{u}} + \left\|\tilde{\boldsymbol{\varepsilon}}'_{trip}(t)\right\|_{\mathbf{H}_{\boldsymbol{\sigma}}}\right\}, \qquad \text{f.a. } t \in S \tag{73}$$

one obtains (51) and (52); (53) follows by theorem 2 as well.
Re. (b) (54) follows directly from [Hüß07]. (55) follows by subtracting the equation $\mathbf{p}'_k = \boldsymbol{\gamma}(\mathbf{p}_k, \widehat{\theta}_k)$ for $k = 1$ from the one for $k = 2$, integrating $\mathbf{D}\,\mathbf{p} = \mathbf{D}\,\boldsymbol{\gamma}$, employing the Lipschitz-continuity assumptions on $\boldsymbol{\gamma}$ and Gronwall's inequality. (56) follows via $\boldsymbol{\gamma}$'s Lipschitz-continuity and (55). $t \mapsto \boldsymbol{\gamma}(\mathbf{p}(t), \widehat{\theta}(t)) \in \mathbf{H}_\mathbf{p}$ is continuous and $\mathbf{p} \mapsto \boldsymbol{\gamma}(\mathbf{p}, \cdot)$ is Lipschitz continuous. Therefore ($b6$), ($b7$) is uniquely solvable.
Re. (c) ($b4$) and ($b6$) imply $\boldsymbol{\varepsilon}'_{trip} \in C(\overline{S}; \mathbf{H}_{\boldsymbol{\sigma}})$ and integration yields $\boldsymbol{\varepsilon}_{trip} \in C^1(\overline{S}; \mathbf{H}_{\boldsymbol{\sigma}})$. The existence of $\boldsymbol{\varepsilon}_{trip}$ follows directly by integrating ($b9$) and observing ($b4$). $(57)_1$ follows from ($b9$) by integrating over $[0, t]$, squaring, integrating over Ω and observing $(54)_3$. In order to see (58), (59), fix for a moment $(\mathbf{x}, t)$, set, for $k = 1, 2$,

$$a_k := \boldsymbol{\sigma}^*_k(\mathbf{x}, t), \qquad b_k := \frac{3}{2}\sum_{i=1}^m \kappa_i \frac{\partial\Phi_{i,k}}{\partial p_{i,k}}(p_{i,k}(\mathbf{x}, t)), \qquad c_k := \max\{p'_{i,k}(\mathbf{x}, t), 0\}, \tag{74}$$

$$A := a_1\, b_1\, \mathbf{D}\, c, \qquad B := a_1\, c_2\, \mathbf{D}\, b, \qquad C := b_2\, c_2\, \mathbf{D}\, a, \tag{75}$$

and note

$$\mathbf{D}\,\boldsymbol{\varepsilon}'_{trip} = \mathbf{D}\,\boldsymbol{\gamma}\,\left(= \boldsymbol{\gamma}(\mathbf{p}_2, \widehat{\theta}_2) - \boldsymbol{\gamma}(\mathbf{p}_1, \widehat{\theta}_1)\right), \tag{76}$$

$$\mathbf{D}\,\boldsymbol{\varepsilon}_{trip}(t) = \mathbf{D}\,\boldsymbol{\varepsilon}_{trip}(0) + \int_0^t \mathbf{D}\,\boldsymbol{\gamma}\,\mathrm{d}s, \tag{77}$$

$$\mathbf{D}\,\boldsymbol{\gamma} = a_1 b_1 c_1 - a_2 b_2 c_2 = A + B + C. \tag{78}$$

By ($b4$) ($\boldsymbol{\sigma}^*(t) \in \mathbf{K}$)(!): $|a_k| = |\boldsymbol{\sigma}^*(\mathbf{x}, t)| \le \widehat{c}$ (uniformly in $(\mathbf{x}, t)$); by $(54)_1$ and (32): $|b_k| \le \widehat{c}$; by $(54)_3$: $|c_k| \le \widehat{c}$; by definition:

$$|\,\mathbf{D}\,a| = |\,\mathbf{D}\,\boldsymbol{\sigma}^*(\mathbf{x}, t)|, \tag{79}$$

by the mean-value theorem and by (32):

$$|\mathbf{D}\, b| \le \widehat{c}\,|\mathbf{D}\, p_i(\mathbf{x},t)| \le \widehat{c}\,\|\mathbf{D}\,\mathbf{p}(\mathbf{x},t)\|_\infty, \tag{80}$$

and by ($b7$), by the Lipschitz-continuity of the positive-part function and that of $\boldsymbol{\gamma}$:

$$|\mathbf{D}\, c| \le \widehat{c}\Big\{\|\mathbf{D}\,\mathbf{p}(\mathbf{x},t)\|_\infty + \|\mathbf{D}\,\widehat{\theta}(\mathbf{x},t)\|_\infty\Big\}. \tag{81}$$

(59) follows from (76) by squaring relation (76) f.(a.)a. $(\mathbf{x},t)$, integration over Ω and applying the estimates above. □

Proof of Lemma 2. Re. (a) (60) follows in a similar way as the last step in the proof of Lemma 1. Re. (b) By $(24)_4$: $\boldsymbol{\varepsilon}_{trip}(0) = \mathbf{0}$. Therefore the fixed-point operator $\mathbf{T}_{QS1} : \tilde{\boldsymbol{\varepsilon}}_{trip} \mapsto \boldsymbol{\varepsilon}_{trip}$ (cf. ($b3$),…,(integrated version of ($b9$))) *actually* acts on $\mathbf{Z}_{\boldsymbol{\sigma}0} := \{\boldsymbol{\tau} \in C^1(\overline{S};\mathbf{H}_{\boldsymbol{\sigma}}) : \boldsymbol{\tau}(0) = \mathbf{0}\}$ which can be equivalently normed by $\|\boldsymbol{\tau}\|_{\mathbf{Z}_{\boldsymbol{\sigma}0}} := \sup_{t\in\overline{S}} \|\boldsymbol{\tau}'(t)\|_{\mathbf{H}_{\boldsymbol{\sigma}}}$. $\lambda > 0$ will be specified below. Contractivity will be shown by means of the Bielicki-norm $\|\boldsymbol{\tau}\|_{\mathbf{Z}_{\boldsymbol{\sigma}0,\lambda}} := \sup_{t\in\overline{S}} \{\exp(-\lambda t)\|\boldsymbol{\tau}'(t)\|_{\mathbf{H}_{\boldsymbol{\sigma}}}\}$. First, note that the estimates (53), (55) and (55) simplify because $\mathbf{D}\,\boldsymbol{\sigma}_0^* = \mathbf{D}\,\mathbf{p}_0 = 0$. Estimating the right-hand side of (59) by the expressions provided by (53), (55) and (55) implies

$$\|\mathbf{D}\,\boldsymbol{\varepsilon}'_{trip}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} \le c\int_0^t \|\mathbf{D}\,\tilde{\boldsymbol{\varepsilon}}_{trip}(s)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}}\,\mathrm{d}s \quad \text{f.a. } t \in S. \tag{82}$$

Re. (c) (82) implies

$$\|\mathbf{D}\,\boldsymbol{\varepsilon}'_{trip}\|_{\mathfrak{H}_{\boldsymbol{\sigma}0,\lambda}} \le \sqrt{\frac{c}{\lambda}}\|\mathbf{D}\,\tilde{\boldsymbol{\varepsilon}}'_{trip}\|_{\mathfrak{H}_{\boldsymbol{\sigma}0,\lambda}}. \tag{83}$$

Choosing $\lambda > c$, we obtain a contraction estimate for $\mathbf{T}_{QS1}$, i.e. $\mathbf{T}_{QS1}$ has a fixed point. □

Proof of Lemma 3. Taking $(29)_1$, $(29)_2$, (4), ($b1$), ($b2$), ($b6$) and ($b8$) into account, one sees that the assumptions of theorem 3 are fulfilled, i.e. there is a solution $\mathbf{u}$ of ($b10$), ($b11$), ($b12$) and $\mathbf{u}$ satisfies the estimate (110). With

(84)
$$\mathbb{I}_k(t) := (f_{0k}(t), \mathrm{div}(\mathbf{u}(t)))_{\mathbf{H_u}},\ k = 1,2, \quad \mathbb{I}_k(t) := (f_{0k}(t), \nabla\mathbf{u}(t))_{\mathbf{H_u}},\ k = 3,4, \quad \mathbb{I}_5(t) := (\mathbf{f}(t),\mathbf{u}(t))_{\mathbf{H_u}}$$

one has

$$\Big\langle \widehat{\mathbf{f}}(t), \mathbf{u}(t)\Big\rangle_{\mathbf{V_u^*V_u}} = \sum_{k=1}^{5} \mathbb{I}_k(t). \tag{85}$$

Let ε be sufficiently small w.r.t. γ_0 (cf. theorem 3). First, we use observations like $\|\mathrm{div}(\mathbf{u}(t))\|_{\mathbf{H_u}} \le c\,\|\mathbf{u}(t)\|_{\mathbf{V_u}}$, and exploit Young's inequality

$$ab \le \varepsilon a^2 + C_\varepsilon b^2, \quad C_\varepsilon = \frac{1}{4\varepsilon} \tag{86}$$

and Hölder's inequality to arrive at

$$|\mathbb{I}_1(t)| \le \varepsilon\|\mathbf{u}(t)\|^2_{\mathbf{V_u}} + C_\varepsilon\Big\{1 + \|\widehat{\theta}(t)\|^2_{H_\theta}\Big\}, \tag{87}$$

$$|\mathbb{I}_2(t)| \le \varepsilon\|\mathbf{u}(t)\|^2_{\mathbf{V_u}} + C_\varepsilon\|\mathbf{p}(t)\|^2_{\mathbf{H_p}}, \tag{88}$$

$$|\mathbb{I}_3(t)| \le \varepsilon\|\mathbf{u}(t)\|^2_{\mathbf{V_u}} + C_\varepsilon\|\boldsymbol{\varepsilon}_{trip}\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}}, \tag{89}$$

$$|\mathbb{I}_4(t)| \le \varepsilon\|\mathbf{u}(t)\|^2_{\mathbf{V_u}} + C_\varepsilon\|\boldsymbol{\varepsilon}_{cp}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}}, \tag{90}$$

$$|\mathbb{I}_5(t)| \le \varepsilon\|\mathbf{u}(t)\|^2_{\mathbf{V_u}} + C_\varepsilon\|\mathbf{f}(t)\|^2_{\mathbf{H_u}} \tag{91}$$

for constant $c > 0$, arbitrary $\varepsilon > 0$ and corresponding C_ε. (65) follows analogously. □

Proof of Lemma 4. Re. (a) Estimates (67) follow in a similar way as in the proof of Lemma 3. Re. (b) The fixed-point operator $\mathbf{T}_{QS2} : (\overline{\mathbf{u}}, \overline{\boldsymbol{\sigma}}, \overline{\varepsilon}_{cp}) \mapsto (\mathbf{u}, \boldsymbol{\sigma}, \varepsilon_{cp})$ (cf. ($b2$),…,($b16$)) acts on $\mathbf{Z}_{\mathbf{u}\boldsymbol{\sigma}0} := \mathfrak{V}^2_{\mathbf{u}} \times \mathfrak{H}^2_{\boldsymbol{\sigma}} \times \mathfrak{H}^2_{\boldsymbol{\sigma}}$. $\lambda > 0$ will be specified below. Contractivity will be shown by means of the Bielicki-norm $\|\boldsymbol{\tau}\|_{\mathbf{Z}_{\mathbf{u}\boldsymbol{\sigma}0,\lambda}}$ in a similar way as in the proof of lemma 2. Estimating the right-hand side of (63) by the expressions provided by (61) and (65) implies

(92)
$$\begin{aligned} &\|\,\mathbf{D}\,\mathbf{u}(t)\|^2_{\mathbf{V_u}} + \|\,\mathbf{D}\,\mathbf{u}'(t)\|^2_{\mathbf{V_u}} + \|\,\mathbf{D}\,\boldsymbol{\sigma}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\,\mathbf{D}\,\boldsymbol{\sigma}'(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\,\mathbf{D}\,\varepsilon_{cp}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\,\mathbf{D}\,\varepsilon'_{cp}(t)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} \\ &\le c \int_0^t \Big\{ \|\,\mathbf{D}\,\overline{\mathbf{u}}(s)\|^2_{\mathbf{V_u}} + \|\,\mathbf{D}\,\overline{\mathbf{u}}'(s)\|^2_{\mathbf{V_u}} + \|\,\mathbf{D}\,\overline{\boldsymbol{\sigma}}(s)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \\ &\qquad + \|\,\mathbf{D}\,\overline{\boldsymbol{\sigma}}'(s)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\,\mathbf{D}\,\overline{\varepsilon}_{cp}(s)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} + \|\,\mathbf{D}\,\overline{\varepsilon}'_{cp}(s)\|^2_{\mathbf{H}_{\boldsymbol{\sigma}}} \Big\}\,\mathrm{d}s \end{aligned}$$

f.a. $t \in S$.
Re. (c) (92) implies

$$\|\,\mathbf{D}(\mathbf{u}, \boldsymbol{\sigma}, \varepsilon_{cp})\|_{\mathbf{Z}_{\mathbf{u}\boldsymbol{\sigma}0,\lambda}} \le \sqrt{\frac{c}{\lambda}}\|\,\mathbf{D}(\overline{\mathbf{u}}, \overline{\boldsymbol{\sigma}}, \overline{\varepsilon}_{cp})\|_{\mathbf{Z}_{\mathbf{u}\boldsymbol{\sigma}0,\lambda}}. \tag{93}$$

Choosing $\lambda > c$, we obtain a contraction estimate for $\mathbf{T}_{QS2}$, i.e. $\mathbf{T}_{QS2}$ has a fixed point. □

Proof of Lemma 5. Taking $(29)_3$, (7), (27), ($b1$), ($b2$), ($b6$) and ($b10$) into account, one sees that the assumptions of theorem 4 are fulfilled, i.e. there is a solution θ of ($b17$), ($b18$) and θ satisfies the estimate (116). We exploit Young's inequality and Hölder's inequality to arrive at

$$\int_0^t (\widehat{r}_{03}(s), \theta(s))_{H_\theta}\,\mathrm{d}s \le c \int_0^t \{\|\mathbf{u}'(s)\|^2_{\mathbf{V_u}} + \|\theta(t)\|^2_{H_\theta}\}\,\mathrm{d}s, \tag{94}$$

$$\int_0^t (r_{04}(s), \theta(s))_{H_\theta}\,\mathrm{d}s \le c \int_0^t \Big\{\|\mathbf{p}'(s)\|^2_{\mathbf{H_p}} + \|\theta(s)\|^2_{H_\theta}\Big\}\,\mathrm{d}s, \tag{95}$$

$$\int_0^t (r(s), \theta(s))_{H_\theta} \le c \int_0^t \{\|r(s)\|^2_{H_\theta} + \|\theta(s)\|^2_{H_\theta}\}\,\mathrm{d}s, \tag{96}$$

$$\int_0^t \delta \int_\Gamma \theta_\Gamma(s)\theta(s)\,\mathrm{d}\sigma_{\mathbf{x}}\mathrm{d}s \le \varepsilon \int_0^t \|\theta(s)\|^2_{V_\theta}\,\mathrm{d}s + C_\varepsilon \int_0^t \|\theta_\Gamma(s)\|^2_{L^2(\Gamma)}\,\mathrm{d}s. \tag{97}$$

for constant $c > 0$, arbitrary $\varepsilon > 0$ and corresponding C_ε. Therefore the Gronwall's inequality yields (69). Taking $(29)_3$, (7), (27), ($b1$), ($b2$), ($b6$) and ($b10$) into account, the necessary a-priori estimate (70) can be obtained by formal multiplication of ($b18$) with θ' and using similar estimates as above. We refer to the literature for a justification for this approach (cf. [LM73], e.g.). Subtracting the equations ($b18$) for θ_1 and θ_2 from each other and testing with the difference $\mathbf{D}\,\theta$ and $\mathbf{D}\,\theta'$ resp., one gets f.a.a. $t \in S$ the estimate (71). □

Proof of Lemma 6. (72) follows in a similar way as in the proof of Lemma 5.

□

5.4. Completion of the proof

Proof of Theorem 1. According to section 5.1, the general idea of the proof of the main theorem is to use Banach's fixed point theorem. We construct an operator

$$\mathbf{T}_{QS3} : \mathfrak{H}^2_\theta \to \mathfrak{H}^2_\theta, \quad \widehat{\theta} \mapsto \mathbf{T}_{QS3}(\widehat{\theta}) =: \theta \tag{98}$$

which has a (unique) fixed point: Take a function ($b1$) (cf. figure 1) and find a solution of the problem ($b10$) – ($b16$). In order to get ($b16$), we construct an operator (68) which has a (unique)

fixed point: Take a function ($b2$) and find a solution of the problem ($b3$) – ($b9$). In order to get ($b8$), we construct an operator (62) which has a (unique) fixed point. Lemma 2 provides the (unique) existence of ($b8$). Using lemma 4 we get the (unique) existence of ($b16$). Finally, we have to show that the map (98) has a fixed point which is (also) denoted by θ in order to complete the proof.
The fixed-point operator $\mathbf{T}_{QS3} : \widehat{\theta} \mapsto \theta$ (cf. ($b1$),…,($b18$)) acts on $Z_{\theta 0} := \mathfrak{H}^2_{\theta}$. $L > 0$ will be specified below. Contractivity will be shown by means of the Bielicki-norm $\|\tau\|_{Z_{\theta 0,L}}$ in a similar way as in the proof of lemma 2. Estimating the right-hand side of (72) by the expressions provided by (56) and $(67)_1$ implies f.a. $t \in S$

$$\int_0^t \left\{ \| \mathbf{D}\,\mathbf{u}(s)\|^2_{\mathbf{V_u}} + \| \mathbf{D}\,\mathbf{p}'(s)\|^2_{\mathbf{H_p}} \right\} \mathrm{d}s \le c \int_0^t \left\{ \| \mathbf{D}\,\widehat{\theta}(s)\|^2_{H_\theta} + \| \mathbf{D}\,\widehat{\theta}'(s)\|^2_{H_\theta} \right\} \mathrm{d}s. \tag{99}$$

Therefore

(100)

$$\| \mathbf{D}\,\theta(t)\|^2_{\mathbf{H}_\theta \cap \mathbf{V}_\theta} + \| \mathbf{D}\,\theta\|^2_{L^2((0,t);\mathbf{V}_\theta)} + \| \mathbf{D}\,\theta'\|^2_{L^2((0,t);\mathbf{H}_\theta)} \le c \int_0^t \left\{ \| \mathbf{D}\,\widehat{\theta}(s)\|^2_{H_\theta} + \| \mathbf{D}\,\widehat{\theta}'(s)\|^2_{H_\theta} \right\} \mathrm{d}s$$

and hence

$$\| \mathbf{D}\,\theta\|_{Z_{\theta 0,L}} \le \sqrt{\frac{c}{L}} \| \mathbf{D}\,\widehat{\theta}\|_{Z_{\theta 0,L}} \tag{101}$$

by similar arguments as in the proof of lemma 2. Choosing $L > c$, we obtain a contraction estimate for $\mathbf{T}_{QS3}$, i.e. $\mathbf{T}_{QS3}$ has a fixed point. □

A. Appendix

For the convenience of the reader we summarize some facts.

A.1. Variational inequalities

Theorem 2. *Let $(X, \|\cdot\|)$ be a Hilbert space, $\psi : X \to \mathbb{R} \cup \{\infty\}$ – proper, lower semi-continuous and convex, $f \in L^2(S;X)$, $u_0 \in D(\psi)$. Then the problem*

$$u'(t) + \partial\psi(u(t)) \ni f(t), \quad u(0) = u_0 \tag{102}$$

has a unique solution $u = u(f, u_0)$. u satisfies $u \in C(\overline{S};X) \cap W^{1,2}(S;H)$, $u(t) \in D(\psi)$ f.a.a. $t \in S$ and

$$\|u(t) - u(0)\|^2_X \le \int_0^t \|f(\xi)\|^2_X \,\mathrm{d}\xi \quad \text{f.a. } t \in S. \tag{103}$$

Moreover, if $(f_i, u_{0i}) \in L^2(S;X) \times D(\psi)$, $i = 1, 2$, then

$$\| \boldsymbol{D} u(t)\|_X \le \| \boldsymbol{D} u(s)\|_X + \int_0^t \| \boldsymbol{D} f(\xi)\|_X \,\mathrm{d}\xi, \qquad \text{f.a. } 0 \le s \le t \le T. \tag{104}$$

Proof. [Bar76, Bré73, Rou05, Sho97, Zei85], e.g. □

A.2. Calculation rules

Relation (2) implies

(105)

$$\mathrm{tr}(\boldsymbol{\sigma}) = 2\mu\ \mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u})) = 2\mu\ \mathrm{div}(\mathbf{u}), \qquad \boldsymbol{\sigma}^* = 2\mu\, \boldsymbol{\varepsilon}^*_{te} = 2\mu\ (\boldsymbol{\varepsilon}^*(\mathbf{u}) - \boldsymbol{\varepsilon}_{trip} - \boldsymbol{\varepsilon}_{cp}),$$

(106)

$$\mathrm{tr}(\boldsymbol{\sigma}) = (2\mu + 3\lambda)\,\mathrm{div}(\mathbf{u}) - 9K_\alpha\,(\theta - \theta_0) - 3K\sum_{i=1}^{m}\left(\frac{\rho_0}{\rho_i(\theta_0)} - 1\right)p_i, \quad \boldsymbol{\sigma}^*(0) = \boldsymbol{\sigma}^*_0 := 2\mu\boldsymbol{\varepsilon}^*(\mathbf{u}_0).$$

A.3. Basics on some evolution equations

Theorem 3. *Let* $\mathbf{u}_0 \in \mathbf{V}_\mathbf{u}$, $\mathbf{u}_1 \in \mathbf{H}_\mathbf{u}$, $\widehat{\mathbf{f}} \in W^{1,2}(S;\mathbf{H}_\mathbf{u})$ *and* $\mathbf{A}_\mathbf{u} : \mathbf{V}_\mathbf{u} \to \mathbf{V}^*_\mathbf{u}$ *defined by*

$$(107)\qquad \langle \mathbf{A}_\mathbf{u}\mathbf{u}, \mathbf{v}\rangle_{\mathbf{V}^*_\mathbf{u}\mathbf{V}_\mathbf{u}} := 2\int_\Omega \mu\,\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\,\mathrm{d}\mathbf{x} + \int_\Omega \lambda\ \mathrm{div}(\mathbf{u})\ \mathrm{div}(\mathbf{v})\,\mathrm{d}\mathbf{x}, \qquad \textit{f.a. } \mathbf{u},\mathbf{v} \in \mathbf{V}_\mathbf{u},$$

then $\mathbf{A}_\mathbf{u}$ *is linear, symmetric, continuous and*

1. *there is some* γ_0 *s.t.*

   $$(108)\qquad \langle \mathbf{A}_\mathbf{u}\mathbf{v}, \mathbf{v}\rangle_{\mathbf{V}^*_\mathbf{u}\mathbf{V}_\mathbf{u}} \geq \gamma_0\,\|\mathbf{v}\|^2_{\mathbf{V}_\mathbf{u}},$$

2. *there is a unique weak solution* $\mathbf{u} \in L^2(S;\mathbf{V}_\mathbf{u})$ *with*

   (109)

   $$\mathbf{A}_\mathbf{u}\mathbf{u} = \widehat{\mathbf{f}} \textit{ in } \Omega_T, \quad \mathbf{u}(0) = \mathbf{u}_0 \textit{ in } \Omega, \quad \mathbf{u}'(0) = \mathbf{u}_1 \textit{ in } \Omega, \quad \mathbf{u} = \mathbf{0} \textit{ on } \Gamma_1, \quad \boldsymbol{\sigma}\cdot\boldsymbol{\nu} = \mathbf{0} \textit{ on } \Gamma_2.$$

3. $\mathbf{u}$ *satisfies the (energy-) estimate*

   $$(110)\qquad \|\mathbf{u}(t)\|^2_{[W^{2,2}(\Omega)]^3\cap\mathbf{V}_\mathbf{u}} \leq c\,\|\mathbf{f}(t)\|^2_{\mathbf{H}_\mathbf{u}}$$

   with some $c = c(\gamma_0, T) = \mathrm{const.} > 0$*, independent of* $t \in [0,T]$.

4. *Moreover,* $\mathbf{u}' \in L^2(S;\mathbf{V}_\mathbf{u})$ *and the estimate*

   $$(111)\qquad \|\mathbf{u}'(t)\|^2_{\mathbf{V}_\mathbf{u}} \leq c\,\|\mathbf{f}'(t)\|^2_{\mathbf{H}_\mathbf{u}}$$

   holds with some $c = c(\gamma_0, T) = \mathrm{const.} > 0$*, independent of* $t \in [0,T]$.

Proof. [ADN64, BF02, CWH98, Wlo87, WRL95], e.g. □

Theorem 4. *Let* $\theta_0 \in H_\theta$, $\rho_0, c_e = \mathrm{const.} > 0$, $\widehat{r} \in L^2(S;V^*_\theta)$, $\theta_\Gamma \in L^2(\Gamma_T)$, $A_\theta : V_\theta \to V^*_\theta$ *defined by*

$$(112)\qquad \langle A_\theta\theta, \vartheta\rangle_{V^*_\theta V_\theta} := \int_\Omega \lambda_\theta\,\nabla\theta\,\nabla\vartheta\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \delta\,\theta\,\vartheta\,\mathrm{d}\sigma_\mathbf{x}, \qquad \textit{f.a. } \theta,\vartheta \in V_\theta,$$

then A_θ *is linear, symmetric, continuous and*

1. *there is some* $\gamma_1 > 0$ *s.t.*

   $$(113)\qquad \langle A_\theta\vartheta, \vartheta\rangle_{V^*_\theta V_\theta} \geq \gamma_1\,\|\vartheta\|^2_{V_\theta}, \qquad \textit{f.a. } \vartheta \in V_\theta,$$

2. *there is a unique weak solution*

$$\theta \in C(\overline{S}; H_\theta) \cap L^2(S; V_\theta), \quad \theta' \in L^2(S; V_\theta^*) \tag{114}$$

with

$$\rho_0 c_e \theta' + A_\theta \theta = \widehat{r} \text{ in } \Omega_T, \quad \theta(0) = \theta_0 \text{ in } \Omega, \quad -\lambda_\theta \frac{\partial \theta}{\partial \boldsymbol{\nu}} = \delta\,(\theta - \theta_\Gamma) \text{ on } \Gamma_T. \tag{115}$$

3. *θ satisfies the (energy-) estimate*

(116)

$$\|\theta(t)\|^2_{H_\theta} + \|\theta\|^2_{L^2((0,t);V_\theta)} + \|\theta'\|^2_{L^2((0,t);V_\theta^*)} \le c \left\{ \|\theta_0\|^2_{H_\theta} + \int_0^t \left[\|\widehat{r}(s)\|^2_{V_\theta^*} + \|\theta_\Gamma(s)\|^2_{L^2(\Gamma)} \right] \mathrm{d}s \right\}$$

with some $c = c(\rho_0, c_e, \gamma_1, T) = \text{const.} > 0$, independent of $t \in [0, T]$.

4. *Assume in addition $\theta_0 \in V_\theta$ and $\theta_\Gamma \in W^{1,2}(\Gamma_T)$. Then the solution θ of* (114), (115) *is an element of $W^{1,2}(S; H_\theta)$ and satisfies the estimate*

(117)

$$\|\theta(t)\|^2_{V_\theta} + \|\theta'\|^2_{L^2((0,t);H_\theta)} \le c \left\{ \|\theta_0\|^2_{V_\theta} + \|\theta_\Gamma(t)\|^2_{L^2(\Gamma)} + \int_0^t \left[\|\widehat{r}(s)\|^2_{V_\theta^*} + \|\theta'_\Gamma(s)\|^2_{L^2(\Gamma)} \right] \mathrm{d}s \right\}$$

Proof. [LM73, Zei90, Wlo87, DL92], e.g. □

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