Lyapunov-Razumikhin and Lyapunov-Krasovskii theorems for interconnected ISS time-delay systems

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***Abstract*— We consider an arbitrary number of interconnected nonlinear systems with time-delays and investigate them in view of input-to-state stability (ISS). The useful tools for single time-delay systems, the ISS Lyapunov-Razumikhin functions and ISS Lyapunov-Krasovskii functionals are redefined and applied to interconnected systems. By the help of a small-gain condition we prove that the whole system with time-delays has the ISS property, if each subsystem has an ISS Lyapunov-Razumikhin function or ISS Lyapunov-Krasovskii functional. Furthermore we construct the ISS Lyapunov-Razumikhin (-Krasovskii) function(al) and the corresponding gains of the whole system.**

I. INTRODUCTION

In this paper we study the input-to-state stability (ISS) property, introduced in [18], of systems with time-delays. ISS and its variants, for example input-to-state dynamical stability (ISDS) [6], local ISS (LISS) [21] and integral-ISS (iISS) [19] became important during the recent years for the stability analysis of dynamical systems and were applied in network control, engineering, biological or economical systems for example.

A useful tool to verify the ISS property for continuous systems are Lyapunov functions (see [20]) as well as for other variants of ISS. For time-delay systems the ISS property can be verified by ISS Lyapunov-Razumikhin functions ([22]) or ISS Lyapunov-Krasovskii functionals ([15]).

We are interested in the ISS property for interconnections of systems with time-delays. The first results on the ISS property for the delay-free case were given for two coupled continuous systems in [10] and for an arbitrarily large number ($n \in \mathbb{N}$) of coupled systems in [2], using a small-gain condition. Lyapunov versions of the ISS small-gain theorems were proved in [11] (two systems) and [3] (n systems), for the ISDS property in [4], for LISS in [5] and for iISS in [8] (two systems) and [9] (n systems), where Lyapunov functions for the overall system are constructed.

A general approach of the verification of the ISS property for interconnected systems can be found in [12].

In this paper we utilize on the one hand ISS Lyapunov-Razumikhin functions and on the other hand ISS Lyapunov-Krasovskii functionals to prove that a network of ISS systems with time-delays has the ISS property under a small-gain condition, provided that each subsystem has an ISS Lyapunov-Razumikhin function and an ISS Lyapunov-Krasovskii functional, respectively. To prove this we construct the ISS Lyapunov-Razumikhin function and ISS Lyapunov-Razumikhin functional, respectively, and the corresponding gains of the whole system.

The paper is organized as follows: In Section 2 we note some basic definitions. The main results, the ISS small-gain theorems for interconnected time-delay systems can be found in Section 3, where Subsection 3.1 contains the ISS Lyapunov-Razumikhin type theorem and Subsection 3.2 the ISS Lyapunov-Krasovskii type theorem. In Section 4 an example is given to illustrate the results. Finally Section 5 concludes this paper with a short summary.

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II. NOTATIONS AND DEFINITIONS

By x^T we denote the transposition of a vector $x \in \mathbb{R}^N$, $N \in \mathbb{N}$, furthermore $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}^N_+$ denotes the positive orthant $\left\{x \in \mathbb{R}^N : x \geq 0\right\}$ where we use the partial order for $x, y \in \mathbb{R}^N$ given by

$$x \geq y \Leftrightarrow x_i \geq y_i,\ i = 1, \ldots, N \text{ and } x \ngeq y \Leftrightarrow \exists i : x_i < y_i,$$
$$x > y \Leftrightarrow x_i > y_i,\ i = 1, \ldots, N.$$

We denote the Euclidean norm by $|\cdot|$. For $x = (x_1, \ldots, x_k)^T$ defined on an interval I, we let $\|x\|_{\mathrm{I}} = \max_{1 \leq i \leq k} \left\{\|x_i\|_{\mathrm{I}}\right\}$.

Let $\theta \in \mathbb{R}_+$. The function $x_t : [-\theta, 0] \to \mathbb{R}^N$ is given by $x_t(\tau) := x(t + \tau),\ \tau \in [-\theta, 0]$. For $a, b \in \mathbb{R},\ a < b$, let $C\left([a, b]; \mathbb{R}^N\right)$ denote the Banach space of continuous functions defined on $[a, b]$ equipped with the norm $\|\cdot\|_{[a,b]}$ and take values in $\mathbb{R}^N$. For functions x_t we define $|x_t(\tau)| := \max_{\tau - \theta \leq s \leq \tau} |x(s)|$.

Definition 2.1: We define following classes of functions:

$$\mathcal{K} := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, } \gamma(0) = 0 \text{ and strictly increasing}\}$$
$$\mathcal{K}_\infty := \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\}$$
$$\mathcal{L} := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t \to \infty} \gamma(t) = 0\}$$
$$\mathcal{KL} := \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K},\ \beta(r, \cdot) \in \mathcal{L},\ \forall t, r \geq 0\}$$

Note that for $\gamma \in \mathcal{K}_\infty$ the inverse function γ^{-1} always exists and $\gamma^{-1} \in \mathcal{K}_\infty$.

We recall the definition of ISS for single time-delay systems and note the main results of previous works. Single nonlinear time-delay systems are of the form

$$\begin{aligned} \dot{x}(t) &= f(x_t, u(t)),\ t \geq 0, \\ x(\tau) &= \xi_0(\tau),\ \tau \in [-\theta, 0], \end{aligned} \tag{1}$$

where $t \in \mathbb{R}_+,\ x(t) \in \mathbb{R}^N$, and $u(t) \in \mathbb{R}^M$ is an essentially bounded measurable input. θ is the maximum involved delay

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and $f : C\left([-\theta, 0]; \mathbb{R}^N\right) \times \mathbb{R}^M \to \mathbb{R}^N$ is a locally Lipschitz continuous functional on any bounded set to guarantee that the system (1) admits a unique solution $x(t)$ on a maximal interval $[-\theta, b)$, $0 < b \leq +\infty$, where $x(t)$ is locally absolutely continuous (see [7], [13], [15]). We denote the solution by $x(t, 0, \xi)$ or $x(t)$ for short, satisfying the initial condition $x_0 = \xi$ for any $\xi \in C([-\theta, 0], \mathbb{R}^N)$.

Definition 2.2: The system (1) is called *input-to-state stable* (ISS), if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $t \geq 0$ it holds

$$|x(t)| \leq \beta\left(\|\xi\|_{[-\theta,0]}, t\right) + \gamma\left(\|u\|_{[0,\infty)}\right).$$

Next we define ISS Lyapunov-Razumikhin functions, introduced in [22].

Definition 2.3: A locally Lipschitz function $V : \mathbb{R}^N \to \mathbb{R}_+$ is called an ***ISS Lyapunov-Razumikhin function*** for system (1), if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, χ_d, χ_u and $\alpha \in \mathcal{K}$ such that the following conditions hold:

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \tag{2}$$

$$\begin{aligned} &V(x) \geq \chi_d(|V_d(x)|)) + \chi_u(|u(t)|) \\ &\Rightarrow \mathrm{D}^+V(x) \leq -\alpha(V(x)), \end{aligned} \tag{3}$$

for all $x(t) \in \mathbb{R}^N$ and all essentially bounded measurable inputs $u(t) \in \mathbb{R}^M$, where $V_d(x(t)) := V(x(t+\tau))$, $\tau \in [-\theta, 0]$ and $\mathrm{D}^+V(x)$ denotes the upper right-hand derivative along the solution $x(t)$, which is defined as

$$\mathrm{D}^+V(x(t)) = \limsup_{h \to 0^+} \frac{V(x(t+h)) - V(x(t))}{h}.$$

With this definition we state the following:

Theorem 2.4: If there exists an ISS Lyapunov-Razumikhin function V for system (1) and $\chi_d(s) < s$, $s \in \mathbb{R}_+$, then the system (1) is ISS from u to x with gain $\gamma = \psi_1^{-1} \circ \chi_d$.

The proof can be found in [22].

Another approach to check if a system of the form (1) has the ISS property was introduced in [15]. There, ISS Lyapunov-Krasovskii functionals are used.

Given a locally Lipschitz continuous functional $V : C\left([-\theta, 0]; \mathbb{R}^N\right) \to \mathbb{R}_+$, the upper right-hand derivate D^+V of the functional V is defined for all $\phi \in C([-\theta, 0]; \mathbb{R}^N)$ (see [1], Definition 4.2.4, pp. 258) as follows

$$D^+V(\phi, u) := \limsup_{h \to 0^+} \frac{1}{h}\left(V\left(\phi_h^\star\right) - V(\phi)\right),$$

where $\phi_h^\star \in C\left([-\theta, 0]; \mathbb{R}^N\right)$ is given by

$$\phi_h^\star(s) = \begin{cases} \phi(s+h), & s \in [-\theta, -h], \\ \phi(0) + f(\phi, u)(h+s), & s \in [-h, 0]. \end{cases}$$

With the symbol $\|\cdot\|_a$ we indicate any norm in $C\left([-\theta, 0]; \mathbb{R}^N\right)$ such that for some positive reals b, c the following inequalities hold

$$b\,|\phi(0)| \leq \|\phi\|_a \leq c\,\|\phi\|_{[-\theta,0]}, \ \forall \phi \in C\left([-\theta, 0]; \mathbb{R}^N\right).$$

Definition 2.5: A locally Lipschitz continuous functional $V : C\left([-\theta, 0]; \mathbb{R}^N\right) \to \mathbb{R}_+$ is called an ***ISS Lyapunov-Krasovskii functional*** for system (1) if there exist functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ and functions $\chi, \alpha \in \mathcal{K}$ such that

$$\psi_1(|\phi(0)|) \leq V(\phi) \leq \psi_2(\|\phi\|_a), \tag{4}$$

$$V(\phi) \geq \chi(|u|) \ \Rightarrow \ D^+V(\phi, u) \leq -\alpha(V(\phi)), \tag{5}$$

$\forall \phi \in C\left([-\theta, 0]; \mathbb{R}^N\right), u \in \mathbb{R}^M$.

The next theorem is a counterpart to Theorem 2.4 with according changes to Lyapunov-Krasovskii functionals.

Theorem 2.6: If there exists an ISS Lyapunov-Krasovskii functional $V : C\left([-\theta, 0]; \mathbb{R}^N\right) \to \mathbb{R}_+$ for system (1), then system (1) is ISS.

Proof: This follows by Theorem 3.1 in [15] by definition of $\rho := \psi_2^{-1} \circ \chi$ and

$$\mathrm{D}^+V(\phi, u) \leq -\alpha_3(||\phi||_a) \leq -\alpha(V(\phi)),$$

where $\alpha := \alpha_3 \circ \psi_2^{-1}$ and the functional is chosen locally Lipschitz continuous according to results in [14], [16]. ■

In the next section we consider interconnected time-delay systems and investigate under which conditions the network has the ISS property.

III. MAIN RESULTS

In this section we state our two main results, the ISS Lyapunov-Razumikhin and the ISS Lyapunov-Krasovskii small-gain theorem for general networks with time-delays.

We consider $n \in \mathbb{N}$ interconnected systems of the form

$$\dot{x}_i(t) = f_i\left(x_1^t, \ldots, x_n^t, u(t)\right), \ i = 1, \ldots, n, \tag{6}$$

where $x_i^t(\tau) := x_i(t+\tau)$, $\tau \in [-\theta, 0]$, $x_i \in \mathbb{R}^{N_i}$. θ denotes the maximal involved delay and x_i^t can be interpreted as the internal inputs of a subsystem. The functionals $f_i : C\left([-\theta, 0]; \mathbb{R}^{N_1}\right) \times \ldots \times C\left([-\theta, 0]; \mathbb{R}^{N_n}\right) \times \mathbb{R}^M \to \mathbb{R}^{N_i}$ are locally Lipschitz continuous on any bounded set. We denote the solution of a subsystem by $x_i(t, 0, \xi_i)$ or $x_i(t)$ for short, satisfying the initial condition $x_i^0 = \xi_i$ for any $\xi_i \in C([-\theta, 0], \mathbb{R}^{N_i})$.

Definition 3.1: The *i-th subsystem of* (6) *is called ISS*, if there exist $\beta_i \in \mathcal{KL}$ and $\gamma_{ij}^d, \gamma_i^u \in \mathcal{K}_\infty \cup \{0\}$, $j = 1, \ldots, n$, $j \neq i$ such that

$$|x_i(t)| \leq$$

$$\beta_i(\|\xi_i\|_{[-\theta,0]}, t) + \sum_{j=1}^{n} \gamma_{ij}^d(\|x_j\|_{[-\theta,t]}) + \gamma_i^u(\|u\|_{[0,\infty)}).$$

If we define $N := \sum N_i$, $x := (x_1^T, \ldots, x_n^T)^T$ and $f := (f_1^T, \ldots, f_n^T)^T$, then (6) becomes the system of the form (1), which we call the whole system. We investigate under which conditions the whole system has the ISS property and utilize Lyapunov-Razumikhin functions as well as Lyapunov-Krasovskii functionals.

A. Lyapunov-Razumikhin theorem for interconnected systems

In this subsection we state the first main result of this paper, the ISS Lyapunov-Razumikhin small-gain theorem for interconnected networks with time-delays.

Definition 3.2: A locally Lipschitz continuous function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ is called an *ISS Lyapunov-Razumikhin function for the i-th subsystem of* (6) if there exist functions $V_j,\ j = 1, \dots, n$, which are continuous, proper, positive definite and locally Lipschitz continuous on $\mathbb{R}^{N_j} \setminus \{0\}$, functions $\chi_i^u \in \mathcal{K} \cup \{0\}$, $\chi_{ij}^d \in \mathcal{K}_\infty \cup \{0\}$, $\alpha_i \in \mathcal{K}$, $j = 1, \dots, n$, such that the following condition holds:

$$V_i(x_i) \geq \sum_j \chi_{ij}^d(|V_j^d(x_j)|) + \chi_i^u(|u|) \\ \Rightarrow \mathrm{D}^+ V_i(x_i) \leq -\alpha_i(V_i(x_i)), \tag{7}$$

$\forall x_i \in \mathbb{R}^{N_i}$ and all essentially bounded measurable inputs $u \in \mathbb{R}^M$. The *gain-matrix* is defined by $\overline{\Gamma} := (\chi_{ij}^d)_{n \times n},\ i, j = 1, \dots, n$ and the map $\overline{\Gamma} : \mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$\overline{\Gamma}(s) := \left(\sum_j \chi_{1j}^d(s_j), \dots, \sum_j \chi_{nj}^d(s_j) \right)^T, \ s \in \mathbb{R}_+^n. \tag{8}$$

Note that we get for $v, w \in \mathbb{R}_+^n$: $v \geq w \Rightarrow \overline{\Gamma}(v) \geq \overline{\Gamma}(w)$.

We say that for a *diagonal operator* $D : \mathbb{R}_+^n \to \mathbb{R}_+^n$, $d_{ii} = (\mathrm{Id} + \mu),\ \mu \in \mathcal{K}_\infty,\ d_{ij} = 0,\ i \neq j$ the matrix $\overline{\Gamma}$ satisfies the small-gain-condition if for all $s \in \mathbb{R}_+^n, s \neq 0$ we have

$$D \circ \overline{\Gamma}(s) := D(\overline{\Gamma}(s)) \not\geq s. \tag{9}$$

More information about the condition (9) can be found in [2], [17], [3].

For the proof of the results in this section we will need the following:

Definition 3.3: A continuous path $\sigma \in \mathcal{K}_\infty^n$ is called an *Ω-path* with respect to $D \circ \overline{\Gamma}$, where $\overline{\Gamma}$ is a gain-matrix and D a diagonal operator, if

(i) for each i, the function σ_i^{-1} is locally Lipschitz continuous on $(0, \infty)$;

(ii) for every compact set $K \subset (0, \infty)$ there are constants $0 < c < C$ such that for all points of differentiability of σ_i^{-1} and $i = 1, \dots, n$ we have

$$0 < c \leq (\sigma_i^{-1})'(r) \leq C,\ \forall r \in K;$$

(iii) it holds

$$D(\overline{\Gamma}(\sigma(r))) < \sigma(r),\ \forall r > 0. \tag{10}$$

If the gain-matrix $\overline{\Gamma}$ satisfies the small-gain condition (9), then there exists an Ω-path σ with respect to $D \circ \overline{\Gamma}$. This path can be chosen piecewise linear. This is Theorem 5.2 in [3].

We can now formulate our first main result:

Theorem 3.4: (ISS Lyapunov-Razumikhin theorem for general networks with time-delays)
Consider the interconnected system (6), where each subsystem has an ISS Lyapunov-Razumikhin function V_i. If the corresponding gain-matrix $\overline{\Gamma}$, given by (8) satisfies the small-gain condition (9), where D is a diagonal operator, then the function

$$V(x) = \max_i \{\sigma_i^{-1}(V_i(x_i))\}$$

is the ISS Lyapunov-Razumikhin function for the whole system of the form (1), which is ISS from u to x, where $\sigma = (\sigma_1, \dots, \sigma_n)^T$ is an Ω-path as in Definition 3.3. The gains are given by

$$\chi_d(r) := \max_{ij} \sigma_j^{-1}((\chi_{ij}^d)^{-1}((\mathrm{Id} + \frac{\mu}{2})^{-1})(\chi_{ij}^d(\sigma_j(r)))), \\ \chi_u(r) := \max_i \rho^{-1}(\chi_i^u(r))$$

for $r \geq 0$, where $\rho(r) := \min_k \rho_k(r)$, $\rho_k(r) := \frac{\mu}{2}(\sum \chi_{kj}^d(\sigma_k(r)))$.

Remark 3.5: The definition of ISS and ISS Lyapunov-Razumikhin functions given here in terms of sums is equivalent to the definition if one uses a maximum instead of a sum. Then the gains are given by

$$\chi_d(r) := \max_{i,j} \sigma_i^{-1}(\chi_{ij}^d(\sigma_j(r))), \\ \chi_u(r) := \max_i \sigma_i^{-1}(\chi_i^u(r)).$$

Proof: All subsystems of (6) have an ISS Lyapunov-Razumikhin function $V_i,\ i = 1, \dots, n$, i.e., V_i satisfies (7). From the small-gain condition (9) for $\overline{\Gamma}$, given by (8) and D is a diagonal operator, it follows by Theorem 5.2 in [3] that there exists an Ω-path $\sigma = (\sigma_1, \dots, \sigma_n)^T$ as in Definition 3.3. Note that $\sigma_i^{-1} \in \mathcal{K}_\infty,\ i = 1, \dots, n$.

Let $0 \neq x = (x_1^T, \dots, x_n^T)^T$. We define

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\}$$

as the ISS Lyapunov-Razumikhin function candidate for the overall system. Note that V is locally Lipschitz continuous. V satisfies (2), which can be easily checked. For any $i \in \{1, \dots, n\}$ consider open domains $M_i \in \mathbb{R}^N \setminus \{0\}$ defined by

$$M_i := \{ (x_1^T, \dots, x_n^T)^T \in \mathbb{R}^N \setminus \{0\} : \\ \sigma_i^{-1}(V_i(x_i)) > \max_{j \neq i} \{\sigma_j^{-1}(V_j(x_j))\}\}.$$

Now for any $\hat{x} = (\hat{x}_1^T, \dots, \hat{x}_n^T)^T \in \mathbb{R}^N \setminus \{0\}$ there is at least one $i \in \{1, \dots, n\}$ such that $\hat{x} \in M_i$ and it follows, that there is a neighborhood U of $\hat{x}$ such that $V(x) = \sigma_i^{-1}(V_i(x_i))$ holds for all $x \in U$.

We define $\chi_d(r) := \max_{ij} \sigma_j^{-1}((\chi_{ij}^d)^{-1}((\mathrm{Id} + \frac{\mu}{2})^{-1})(\chi_{ij}^d(\sigma_j(r))))$, $\chi_u(r) := \max_i \rho^{-1}(\chi_i^u(r))$, $r > 0$, where $\rho(r) := \min_k \rho_k(r)$, $\rho_k(r) := \frac{\mu}{2}(\sum \chi_{kj}^d(\sigma_k(r)))$ and assume

$$V(x) \geq \chi_d(|V_d(x)|) + \chi_u(|u|).$$

Note that $\chi_d(r) < r$. It follows from (10)

$$V_i(x_i) = \sigma_i(V(x)) > (\mathrm{Id} + \mu) \sum_{j=1}^{n} \chi_{ij}^d(\sigma_j(V(x))) \geq \sum_{j=1}^{n} \chi_{ij}^d(\left|V_j^d(x_j)\right|) + \chi_i^u(|u|).$$

From (7) we obtain

$$\begin{aligned} \mathrm{D}^+ V(x) &= \mathrm{D}^+ \sigma_i^{-1}(V_i(x_i)) = (\sigma_i^{-1})'(V_i(x_i)) \mathrm{D}^+ V_i(x_i) \\ &\leq -(\sigma_i^{-1})'(V_i(x_i)) \alpha_i(V_i(x_i)) = -\tilde{\alpha}_i(V(x)), \end{aligned}$$

where $\tilde{\alpha}_i(r) := (\sigma_i^{-1})'(\sigma_i(r))\alpha_i(\sigma_i(r)),\ r > 0$. By definition of $\alpha := \min_i \tilde{\alpha}_i$ the function V satisfies (3).

All conditions of Definition 2.3 are satisfied and V is the ISS Lyapunov-Razumikhin function of the whole system of the form (1). By Theorem 2.4 the whole system is ISS from u to x. ■

B. Lyapunov-Krasovskii theorem for interconnected systems

In this subsection we provide a counterpart to Theorem 3.4, where we use ISS Lyapunov-Krasovskii functionals.

Definition 3.6: A locally Lipschitz continuous functional $V_i : C([-\theta, 0]; \mathbb{R}^{N_i}) \to \mathbb{R}_+$ is called an ***ISS Lyapunov-Krasovskii functional of the i-th subsystem*** of (6) if there exist functionals $V_j,\ j = 1, \ldots, n$, which are continuous, proper, positive definite and locally Lipschitz continuous on $C([-\theta, 0]; \mathbb{R}^{N_j}) \backslash \{0\}$, functions $\chi_{ij}, \chi_i \in \mathcal{K} \cup \{0\}$, $\alpha_i \in \mathcal{K}$, $j = 1, \ldots, n,\ i \neq j$ such that

$$\begin{aligned} V_i\left(\phi_i\right) &\geq \sum_{j=1}^{n} \chi_{ij}(V_j(|\phi_j|)) + \chi_i\left(|u|\right) \\ &\Rightarrow D^+ V_i\left(\phi_i, u\right) \leq -\alpha_i\left(V_i\left(\phi_i\right)\right), \end{aligned} \tag{11}$$

$\forall \phi_i \in C\left([-\theta, 0], \mathbb{R}^{N_i}\right),\ u \in \mathbb{R}^M,\ \chi_{ii} \equiv 0,\ i = 1, \ldots, n.$ The gain-matrix is defined by $\Gamma := (\chi_{ij})_{i,j=1}^n$ and the map $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$\Gamma(s) := \left(\sum_{j=1}^{n} \chi_{1j}(s_j), \ldots, \sum_{j=1}^{n} \chi_{nj}(s_j) \right)^T, \ s \in \mathbb{R}_+^n. \tag{12}$$

The next theorem is the second main result of this paper.

Theorem 3.7: (ISS Lyapunov-Krasovskii theorem for general networks with time-delays)
Consider the interconnected system (6). Assume that each subsystem has an ISS Lyapunov-Krasovskii functional V_i, which satisfies the conditions in Definition 3.6, $i = 1, \ldots, n$. If the corresponding gain-matrix Γ, given by (12) satisfies the small-gain condition (9), where D is a diagonal operator, then the functional

$$V(\phi) := \max_i \{\sigma_i^{-1}(V_i(\phi_i))\}$$

is the ISS Lyapunov-Krasovskii functional for the whole system of the form (1), which is ISS from u to x, where $\sigma = (\sigma_1, \ldots, \sigma_n)^T$ is an Ω-path as in Definition 3.3 and $\phi = (\phi_i, \ldots, \phi_n)^T \in C([-\theta, 0]; \mathbb{R}^N)$. The gain is given by $\chi(r) := \max_i \rho^{-1}(\chi_i(r))$ with $\rho := \min_{k=1,\ldots,n} \rho_k$, $\rho_k(r) := \mu \sum_{j=1, k \neq j}^{n} \chi_{kj}(\sigma_j(r))$.

Proof: All subsystems of (6) have an ISS Lyapunov-Krasovskii functional $V_i,\ i = 1, \ldots, n$, i.e. V_i satisfies (11). From the small-gain condition (9) for Γ there exists an Ω-path $\sigma = (\sigma_1, \ldots, \sigma_n)^T$.

Let $0 \neq x_t = ((x_1^t)^T, \ldots, (x_n^t)^T)^T \in C([-\theta, 0]; \mathbb{R}^N)$. We define

$$V(x_t) := \max_i \{\sigma_i^{-1}(V_i(x_i^t))\}$$

as the ISS Lyapunov-Krasovskii functional candidate. Note that V is locally Lipschitz. V satisfies (4), which can be easily checked. For any $i \in \{1, \ldots, n\}$ consider open domains $M_i \in \mathbb{R}^N \backslash \{0\}$ defined by

$$\begin{aligned} M_i := \{ \left((x_1^t)^T, \ldots, (x_n^t)^T\right)^T \in \mathbb{R}^N \backslash \{0\} : \\ \sigma_i^{-1}(V_i(x_i^t)) > \max_{j \neq i} \left\{\sigma_j^{-1}(V_j(x_j^t))\right\}\}. \end{aligned}$$

Now for any $\hat{x}_t = ((\hat{x}_1^t)^T, \ldots, (\hat{\phi}_n^t)^T)^T \in \mathbb{R}^N \backslash \{0\}$ there is at least one $i \in \{1, \ldots, n\}$ such that $\hat{x}_t \in M_i$ and it follows, that there is a neighborhood U of $\hat{x}_t$ such that $V(x_t) = \sigma_i^{-1}(V_i(x_i^t))$ holds for all $x_t \in U$. From (10) we get

$$\begin{aligned} &\sigma_i(r) > (\mathrm{Id} + \mu) \sum_{j=1, i \neq j}^{n} \chi_{ij}(\sigma_j(r)),\ r > 0 \\ \Leftrightarrow\ &\sigma_i(r) - \sum_{j=1, i \neq j}^{n} \chi_{ij}(\sigma_j(r)) \\ &> \mu \sum_{j=1, i \neq j}^{n} \chi_{ij}(\sigma_j(r)) =: \rho_i(r),\ r > 0. \end{aligned}$$

If we define $\rho := \min_i \rho_i$ and assume $V(x_t) \geq \rho^{-1}(\chi_i(|u|))$, it follows

$$\begin{aligned} &\rho(V(x_t)) \geq \chi_i(|u|) \\ \Rightarrow\ &\sigma_i(V(x_t)) - \sum_{j=1, i \neq j}^{n} \chi_{ij}(\sigma_j(V(x_t))) > \chi_i(|u|) \end{aligned}$$

and we get

$$\begin{aligned} V_i(x_i^t) = \sigma_i(V(x_t)) &> \chi_i(|u|) + \sum_{j=1, i \neq j}^{n} \chi_{ij}(\sigma_j(V(x_t))) \\ &= \chi_i(|u|) + \sum_{j=1, i \neq j}^{n} \chi_{ij}(V_j(x_j^t)). \end{aligned}$$

From (11) we obtain

$$\begin{aligned} \mathrm{D}^+ V(x_t, u) &= \mathrm{D}^+ \sigma_i^{-1} V_i(x_i^t, u) \\ &\leq -(\sigma_i^{-1})'(V_i(x_i^t)) \alpha_i(V_i(x_i^t)) = -\tilde{\alpha}_i(V(x_t)), \end{aligned}$$

where $\tilde{\alpha}_i(r) := (\sigma_i^{-1})'(\sigma_i(r))\alpha_i(\sigma_i(r)),\ r > 0$. By definition of $\chi := \max_i \rho^{-1}\chi_i$ and $\alpha := \min_i \tilde{\alpha}_i$, the function V satisfies (5).

All conditions of Definition 2.5 are satisfied and V is the Lyapunov-Krasovskii functional of the whole system of the form (1). By Theorem 2.6 the whole system is ISS from u to x. ■

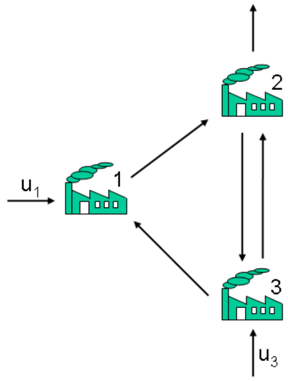

Fig. 1. The given production network

IV. EXAMPLE

In this section we provide an example to apply the main results of this paper.

We consider a logistic network, consisting of three production locations, which are connected by transport routes as shown in Figure 1. In the following we call a production location only subsystem. Subsystems one and three get some raw material from an external source, denoted by u_1 and $u_3 \in \mathbb{R}_+$. Subsystem three produces the material with some production rate $p_3(x_3(t))$, where $x_i(t) \in \mathbb{R}_+,\ i = 1, 2, 3$, denotes the amount of unprocessed parts within subsystem i. 50% of the production will be send to subsystem one and two in each case. There the parts enter the subsystems with the time-delay T_{31} and T_{32}, which denotes the transportation time from subsystem three to one and two, respectively.

Subsystem one processes the parts with the rate $p_1(x_1(t))$ and sends the processed parts to subsystem two, where they arrive with the time-delay T_{12} and will be processed with the rate $p_2(x_2(t))$. 50% of the processed parts of subsystem two will be send to subsystem three (time-delay T_{23}) and 50% will leave the system. This can be interpreted as customer supply.

The production rates are given by $p_i(x_i) := x_i^2$ and we have

$$\begin{aligned}\dot{x}_1(t) &= u_1(t) + \tfrac{1}{2}p_3(x_3(t - T_{31})) - p_1(x_1(t)),\\ \dot{x}_2(t) &= p_1(x_1(t - T_{12})) + \tfrac{1}{2}p_3(x_3(t - T_{32})) - p_2(x_2(t)),\\ \dot{x}_3(t) &= u_3(t) + \tfrac{1}{2}p_2(x_2(t - T_{23})) - p_3(x_3(t)).\end{aligned}$$

It is easy to check that for $\xi_i(\tau) \geq 0,\ \tau \in [-\theta, 0]$, where $\theta := \max T_{ij}$, it holds $x_i(t) \geq 0,\ \forall t \in \mathbb{R}_+$.

At first we use Lyapunov-Razumikhin functions to investigate the network in view of stability. We choose $V_i(x_i) := x_i^2,\ i = 1, 2, 3$ as ISS Lyapunov-Razumikhin function candidates of the subsystems, which are continuous, positive definite and proper and locally Lipschitz continuous. At first we investigate subsystem one and choose the gains

$$\chi_1^u(|u_1|) := \tfrac{|u_1|}{1-\frac{\varepsilon_1}{2}},\ \chi_{13}^d(|(x_3(t+\tau))^2|) := \tfrac{|(x_3(t+\tau))^2|}{2(1-\frac{\varepsilon_1}{2})},$$

where $1 > \varepsilon_1 > 0$ and $\tau \in [-T_{31}, 0]$. By the assumption $V_1(x_1(t)) \geq \chi_{13}^d(|(x_3(t+\tau))^2|) + \chi_1(|u_1(t)|)$ and the definition of the gains it follows

$$\begin{aligned}\mathrm{D}^+V_1(x_1(t)) &= 2(u_1(t) + \tfrac{1}{2}(x_3(t - T_{31}))^2 - x_1^2(t))\\ &\leq -\alpha_1(V_1(x_1(t))),\end{aligned}$$

where $\alpha_1(r) := \varepsilon_1 r,\ r \geq 0$. Therefor V_1 satisfies the condition (7) and we conclude that V_1 is the ISS Lyapunov-Razumikhin function for subsystem one.

By definition of the gains

$$\begin{aligned}\chi_{21}^d(|(x_1(t+\tau))^2|) &:= \tfrac{|(x_1(t+\tau))^2|}{1-\frac{\varepsilon_2}{2}},\ \tau \in [-T_{12}, 0],\\ \chi_{23}^d(|(x_3(t+\tau))^2|) &:= \tfrac{|(x_3(t+\tau))^2|}{2(1-\frac{\varepsilon_2}{2})},\ \tau \in [-T_{32}, 0],\\ \chi_3^u(|u_3(t)|) &:= \tfrac{|u_3(t)|}{1-\frac{\varepsilon_3}{2}},\\ \chi_{32}^d((|x_2(t+\tau)|)^2) &:= \tfrac{|(x_2(t+\tau))^2|}{2(1-\frac{\varepsilon_3}{2})},\ \tau \in [-T_{23}, 0],\end{aligned}$$

$1 > \varepsilon_2 > 0,\ 1 > \varepsilon_3 > 0$, we can prove that V_2 and V_3 are the ISS Lyapunov-Razumikhin functions of the subsystems two and three. Now we check if the small-gain condition is satisfied, where

$$\overline{\Gamma} := \begin{pmatrix} 0 & 0 & \chi_{13}^d \\ \chi_{21}^d & 0 & \chi_{23}^d \\ 0 & \chi_{32}^d & 0 \end{pmatrix}.$$

We choose $\mu(r) := \tilde{\varepsilon}r,\ r > 0$,where $\tilde{\varepsilon} > 0$ is arbitrarily small and the diagonal operator is then given with its diagonal elements $d_{ii}(r) = (1+\tilde{\varepsilon})r$. The Ω-path candidate $\sigma(r) = (\sigma_1(r), \sigma_2(r), \sigma_3(r))^T$ is chosen as $\sigma_1(r) = \sigma_3(r) := r$ and $\sigma_2(r) := \frac{7}{4}r$. Note that the conditions (i) and (ii) of Definition 3.3 are satisfied. Let us check the condition (iii):

$$D \circ \overline{\Gamma}(\sigma(s)) = \begin{pmatrix} (1+\tilde{\varepsilon})(\frac{1}{2(1-\varepsilon_3)}\sigma_3(s)) \\ \frac{(1+\tilde{\varepsilon})}{1-\varepsilon_1}\sigma_1(s) + (1+\tilde{\varepsilon})\frac{1}{2(1-\varepsilon_3)}\sigma_3(s) \\ (1+\tilde{\varepsilon})\frac{1}{2(1-\varepsilon_2)}\sigma_2(s) \end{pmatrix}$$

and by the choice of the Ω-path candidate above we have $D \circ \overline{\Gamma}(\sigma(s)) < \sigma(s),\ s > 0$ for sufficient small $\tilde{\varepsilon},\ \varepsilon_i,\ i = 1, 2, 3$, such that σ is the Ω-path, which is equivalent to the satisfaction of the small-gain condition. By the application of Theorem 3.4 the whole network is ISS, where the ISS Lyapunov-Razumikhin function is given by

$$V(x) = \max\{x_1^2, \tfrac{4}{7}x_2^2, x_3^2\}.$$

We now utilize Lyapunov-Krasovskii functionals to investigate the network in view of stability. We choose $V_i(x_i^t) = x_i^2(t),\ i = 1, 2, 3$ as the ISS Lyapunov-Krasovskii functional candidates.

By

$$\begin{aligned}\chi_1(|u_1(t)|) &:= \tfrac{|u_1(t)|}{1-\frac{\varepsilon_1}{2}},\\ \chi_{13}(V_3(x_3^t)) &:= \tfrac{(||x_3||_{[t-T_{31},t]})^2}{2(1-\frac{\varepsilon_1}{2})},\end{aligned}$$

where $1 > \varepsilon_1 > 0$ and the assumption $V_1(x_1^t) \geq \chi_{13}(V_3(x_3^t)) + \chi_1(|u_1(t)|)$ we get for the first subsystem

$$\begin{aligned}\mathrm{D}^+V_1(x_1^t) &= 2(u_1(t) + \tfrac{1}{2}(x_3(t - T_{31}))^2 - x_1^2(t))\\ &\leq -\alpha_1(V_1(x_1^t)),\end{aligned}$$

where $\alpha_1(r) := \varepsilon_1 r \in \mathcal{K},\ r \geq 0$. By

$$\begin{aligned}
\chi_{21}(V_1(x_1^t)) &:= \tfrac{(||x_1||_{[t-T_{12},t]})^2}{1-\frac{\varepsilon_2}{2}},\\
\chi_{23}(V_3(x_3^t)) &:= \tfrac{(||x_3||_{[t-T_{32},t]})^2}{2(1-\frac{\varepsilon_2}{2})},\\
\chi_3(|u_3(t)|) &:= \tfrac{|u_3(t)|}{1-\frac{\varepsilon_3}{2}},\\
\chi_{32}(V_2(x_2^t)) &:= \tfrac{(||x_2||_{[t-T_{23},t]})^2}{2(1-\frac{\varepsilon_3}{2})},\\
& 1 > \varepsilon_2 > 0,\ 1 > \varepsilon_3 > 0,
\end{aligned}$$

and similar calculations for the other subsystems as for the first subsystem, we conclude that $V_i(x_i^t) = x_i^2(t),\ i = 1,2,3$ are the ISS Lyapunov-Krasovskii functionals for the subsystems. The small-gain condition is satisfied (see above) and by application of Theorem 3.7 for the ISS property the whole network is ISS, where the Lyapunov-Krasovskii functional of the whole system is given by

$$V(x_t) = \max\{x_1^2, \tfrac{4}{7}x_2^2, x_3^2\}.$$

V. CONCLUSIONS

We have proved two theorems: an ISS Lyapunov-Razumikhin and an ISS Lyapunov-Krasovskii small-gain theorem. They state that a network of time-delay systems has the ISS property, provided that the small-gain condition is satisfied and that each subsystem has an ISS Lyapunov-Razumikhin function and ISS Lyapunov-Krasovskii functional, respectively. Furthermore we showed how to construct the ISS Lyapunov-Razumikhin function, the ISS Lyapunov-Krasovskii functional and the corresponding gains of the whole system. This was illustrated by a short example from the logistics.

VI. ACKNOWLEDGMENTS

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