Stability in Logistic Networks

Sergey Dashkovskiy Björn S. Rüffer Fabian R. Wirth

Universität Bremen, Zentrum für Technomathematik,
Teilprojekt A5 im Sonderforschungsbereich 637 – Selbststeuerung logistischer Prozesse

Universität Bremen, 12. Januar 2007

Contents

Motivation
- Logistic Networks — An Example

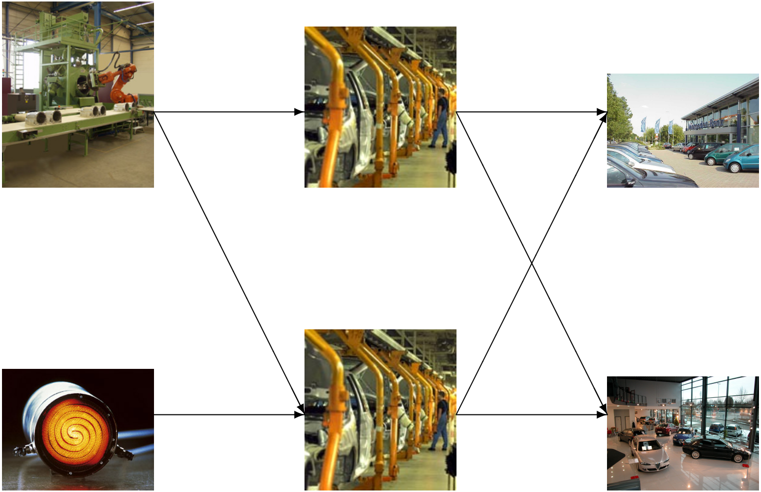
Problem statement
- Input-to-State Stability
- Graphs

Results
- Main Result
- Monotone Systems
- A numerical Test

Application
- Automotive manufacturing revisited

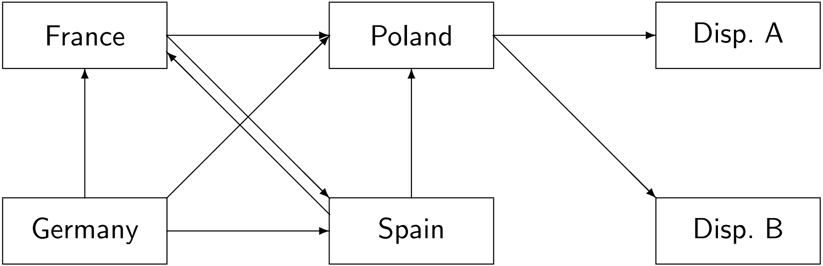
Automotive manufacturing network

A5



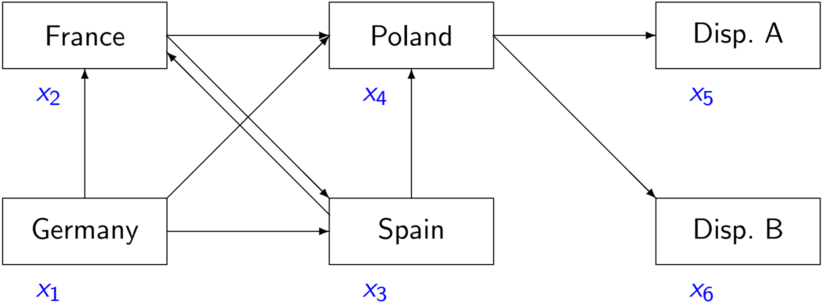
Automotive manufacturing network

A5

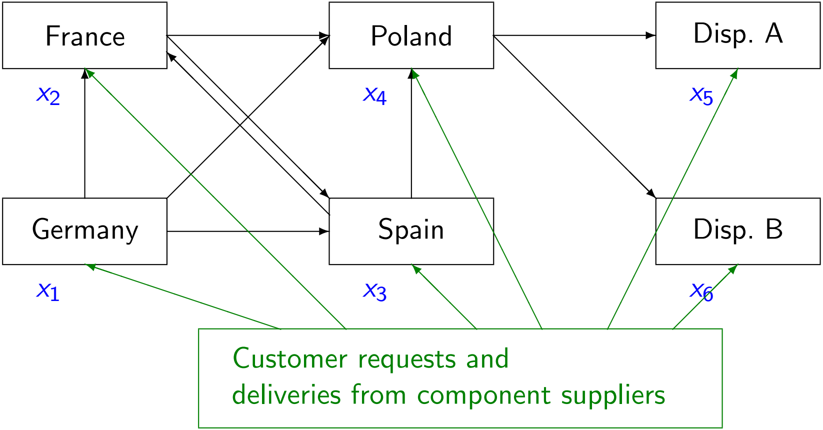


Automotive manufacturing network

A5



Automotive manufacturing network



Automotive manufacturing network

A5

State of a node

$$x_i = \begin{pmatrix} \text{number of jobs} \\ \text{quality} \\ \text{adherence to delivery dates} \\ \text{reliability} \\ \vdots \end{pmatrix}$$

State parameters of a node depend on state parameters of other nodes

Assume: $\dot{x}_i = f_i(x_i, u)$

Autonomous Control

A5

- Personnel in factories adapts service rates to parameter states, queues, etc:
  - Service rate is increased if own queue gets longer
  - Service rate is decreased if queues at subsequent nodes become longer
- Quality, adherence to delivery dates, and reliability of a node depend on parameters of preceding nodes

State equations

A5

$$\dot{x}_1 = u - \frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3}$$

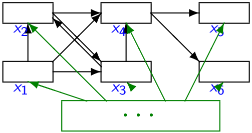
$$\dot{x}_2 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_3, c_3x_3\} - \min\{b_2, c_2x_2\}$$

$$\dot{x}_3 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_2, c_2x_2\} - \min\{b_3, c_3x_3\}$$

$$\dot{x}_4 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_2, c_2x_2\} + \min\{b_3, c_3x_3\} - \min\{b_4, c_4x_4\}$$

$$\dot{x}_5 = \frac{1}{2}\min\{b_4, c_4x_4\} - c_5x_5$$

$$\dot{x}_6 = \frac{1}{2}\min\{b_4, c_4x_4\} - c_6x_6$$



What does stability give us?

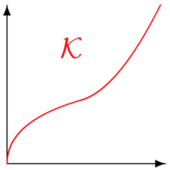
- boundedness of queues
- estimates for queues with respect to inputs
- hints on reliability of discrete event simulation
- predictability of the system

If a system is not stable, then a small disturbance of initial conditions of the input parameters may cause large fluctuations of state parameters/queues (see, e.g., Bramson94-Example)

Comparison functions

Definition

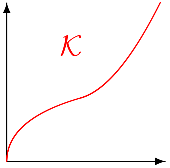
- $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is $\mathcal{K}$-function, if γ is continuous, strictly increasing with $\gamma(0) = 0$.
  γ is called $\mathcal{K}_\infty$-function, if it is unbounded.

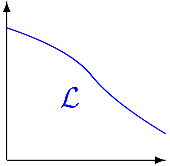


Comparison functions

Definition

- $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is $\mathcal{K}$-function, if γ is continuous, strictly increasing with $\gamma(0) = 0$.
  γ is called $\mathcal{K}_\infty$-function, if it is unbounded.
- $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called $\mathcal{KL}$-function, if
  - β is continuous
  - $\beta(\cdot, t)$ is a $\mathcal{K}$-function $\forall t \geq 0$ and
  - $\beta(s, t) \downarrow 0$ for $t \to \infty$ and all $s \geq 0$.





Input-to-State Stability (ISS)

Definition (Sontag, 1989)

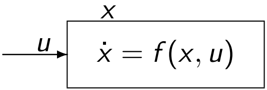
A system

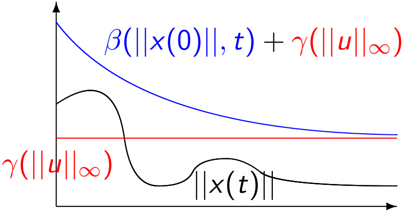
$$\dot{x}(t) = f(x(t), u(t))$$

is called ISS, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that

$$||x(t)|| \leq \beta(||x(0)||, t) + \gamma(||u||_\infty),$$

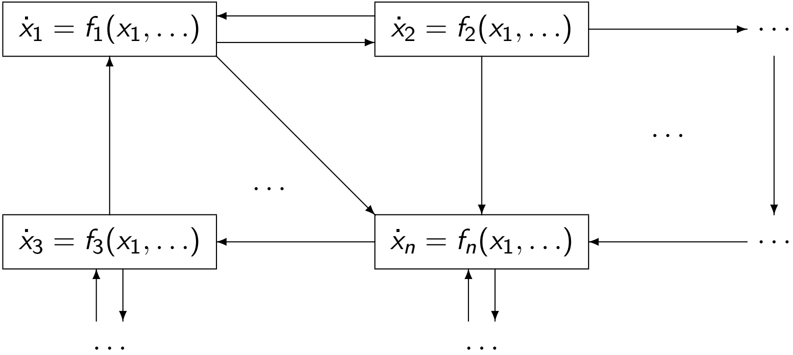
for all $x(0)$, $t \geq 0$, u ess. bounded.





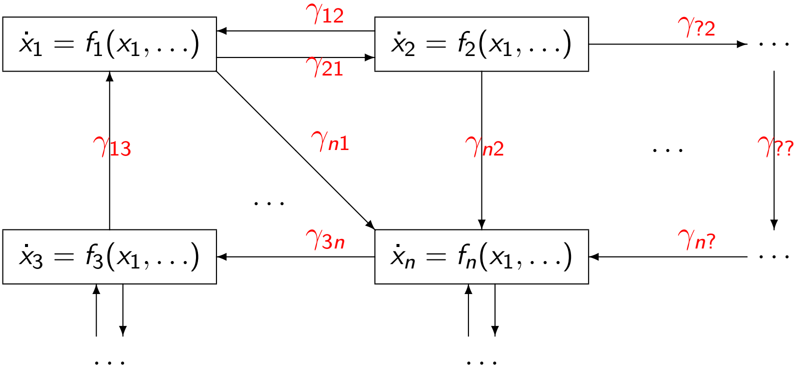
Large Networks

A5



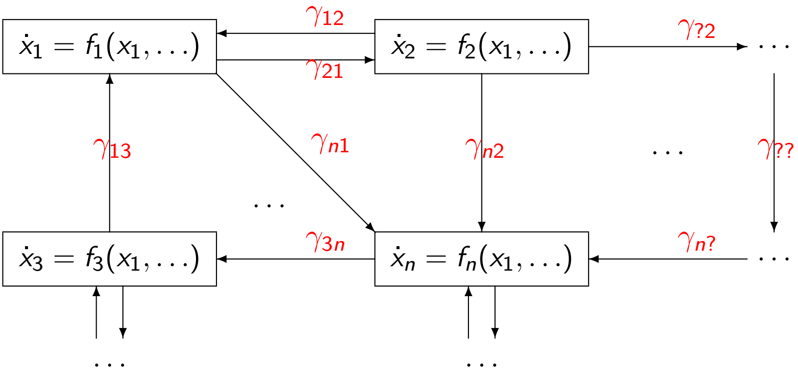
$$||x_i(t)|| \leq \beta(||x_i(0)||, t) + \sum_j \gamma_{ij}(||x_{ij}||_\infty) + \gamma(||u||_\infty)$$

Large Networks



$$||x_i(t)|| \leq \beta(||x_i(0)||, t) + \sum_j \gamma_{ij}(||x_{ij}||_\infty) + \gamma(||u||_\infty)$$

Large Networks



Definition: $\Gamma = (\gamma_{ij})$. Operator: $\Gamma(s)_i = \sum_{j=1}^{n} \gamma_{ij}(s_j)$ for $s \in \mathbb{R}^n_+$.

Stability condition

Theorem (DRW 2005)

If $\exists\, D$, $D = \mathrm{diag}_n(id + \alpha)$ *for some* $\alpha \in \mathcal{K}_\infty$, *such that*

$$\Gamma \circ D(s) \not\geq s \quad \forall s \geq 0, s \neq 0$$

then the network is input/state stable.

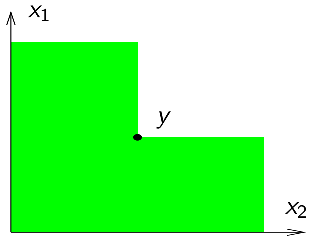
$\Gamma(s)_i = \sum_j \gamma_{ij}(s_j)$ and $\Gamma \circ D(s)_i = \sum_j \gamma_{ij} \circ (\mathrm{id} + \alpha)(s_j)$

Equivalent formulations

Theorem

The following are equivalent:

- $\exists D : \Gamma \circ D(s) \not\geq s \quad \forall s \geq 0, s \neq 0$
- $\exists D : D \circ \Gamma(s) \not\geq s \quad \forall s \geq 0, s \neq 0$



There is also a Lyapunov version of this theorem: The small gain condition is then stated in terms of Lyapunov gains and allows for an explicit construction of an ISS-Lyapunov function for the composite system.

Induced Dynamics

A5

Discrete systems

$$S: \quad s(k+1) := \Gamma(s(k))$$

Induced Dynamics

Discrete systems

$$S: \quad s(k+1) := \Gamma(s(k))$$

and

$$R: \quad r(k+1) := \Gamma \circ D(r(k)) \text{ on } \mathbb{R}^n_+.$$

Induced Dynamics

Discrete systems

$$S: \quad s(k+1) := \Gamma(s(k))$$

and

$$R: \quad r(k+1) := \Gamma \circ D(r(k)) \text{ on } \mathbb{R}^n_+.$$

Observation: Stability of S/R has something to do with stability condition of ISS network.

Linear Case

A5

Γ linear operator, $\Gamma \in \mathbb{R}_+{}^{n\times n}$, D can also taken to be linear, $D = \mathrm{diag}_n(1+\alpha)$, $\alpha > 0$

- $\Gamma \circ D \not\geq \mathrm{id} \iff \Gamma \not\geq \mathrm{id}$

Linear Case

Γ linear operator, $\Gamma \in \mathbb{R}_+{}^{n\times n}$, D can also taken to be linear, $D = \mathrm{diag}_n(1+\alpha)$, $\alpha > 0$

- $\Gamma \circ D \ngeq \mathrm{id} \quad \iff \quad \Gamma \ngeq \mathrm{id}$
- $\iff$ spectral radius $\rho(\Gamma) < 1$

Linear Case

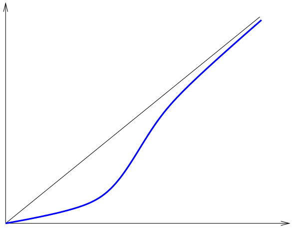
Γ linear operator, $\Gamma \in \mathbb{R}_+{}^{n\times n}$, D can also taken to be linear, $D = \mathrm{diag}_n(1+\alpha)$, $\alpha > 0$

- $\Gamma \circ D \ngeq \mathrm{id} \quad \Longleftrightarrow \quad \Gamma \ngeq \mathrm{id}$
- $\Longleftrightarrow$ spectral radius $\rho(\Gamma) < 1$
- $\Longleftrightarrow \quad S: \quad s(k+1) := \Gamma(s(k))$ is globally asymptotically stable (GAS)
- $\Longleftrightarrow \quad R: \quad r(k+1) := \Gamma \circ D(r(k))$ is globally asymptotically stable (GAS)

D is necessary

A5

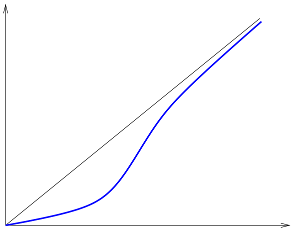
$$\Gamma = \begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix}$$



where $\gamma(t) = t \cdot (1 - e^{-t})$ (clings to id).

D is necessary

$$\Gamma = \begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix}$$



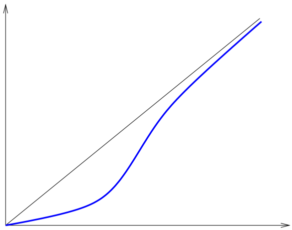
where $\gamma(t) = t \cdot (1 - e^{-t})$ (clings to id).

$\Gamma \not\geq \text{id}$ and $S: \quad s(k+1) = \Gamma(s(k))$ is GAS.

D is necessary

A5

$$\Gamma = \begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix}$$



where $\gamma(t) = t \cdot (1 - e^{-t})$ (clings to id).

$\Gamma \not\geq \text{id}$ and $S: \quad s(k+1) = \Gamma(s(k))$ is GAS.

Can't consider R, since no $D = diag(\text{id} + \alpha)$ exists, such that $\Gamma \circ D \not\geq \text{id}$.

Instability for non-matrix operators

A5

Fix some real constants $\lambda \in]0,1[$ and $\mu \geq 0$.

- $$\Gamma\left(\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}\right) = \begin{bmatrix} \lambda s_1 + s_1^2 s_2 + \mu s_2 \\ \lambda s_2 \end{bmatrix}$$

  for all $s = (s_1, s_2)^T \in \mathbb{R}_+^2$.

Instability for non-matrix operators

A5

Fix some real constants $\lambda \in]0, 1[$ and $\mu \geq 0$.

- $$\Gamma\left(\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}\right) = \begin{bmatrix} \lambda s_1 + s_1^2 s_2 + \mu s_2 \\ \lambda s_2 \end{bmatrix}$$
  for all $s = (s_1, s_2)^T \in \mathbb{R}^2_+$.
- $\Gamma \not\geq \mathrm{id}$

Instability for non-matrix operators

A5

Fix some real constants $\lambda \in]0,1[$ and $\mu \geq 0$.

- $$\Gamma\left(\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}\right) = \begin{bmatrix} \lambda s_1 + s_1^2 s_2 + \mu s_2 \\ \lambda s_2 \end{bmatrix}$$
  for all $s = (s_1, s_2)^T \in \mathbb{R}^2_+$.
- $\Gamma \not\geq \mathrm{id}$
- $D = (1 + \frac{1}{2\lambda}) \cdot \mathrm{id}_{\mathbb{R}^n}$ even gives $\Gamma \circ D \not\geq \mathrm{id}$

Instability for non-matrix operators

A5

Fix some real constants $\lambda \in]0,1[$ and $\mu \geq 0$.

- $$\Gamma\left(\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}\right) = \begin{bmatrix} \lambda s_1 + s_1^2 s_2 + \mu s_2 \\ \lambda s_2 \end{bmatrix}$$
  for all $s = (s_1, s_2)^T \in \mathbb{R}^2_+$.
- $\Gamma \not\geq \text{id}$
- $D = (1 + \frac{1}{2\lambda}) \cdot \text{id}_{\mathbb{R}^n}$ even gives $\Gamma \circ D \not\geq \text{id}$
- but neither R nor S are GAS

Micro and macro dynamics correspond

A5

Theorem

Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$*. Then the following are equivalent:*

1. *There exists a* $\rho \in \mathcal{K}_\infty$ *such that for* $D = \mathrm{diag}_n(id + \rho)$ *we have* $\Gamma \circ D \not\geq id$*.*
2. *There exists a* $\delta \in \mathcal{K}_\infty$ *such that for* $D = \mathrm{diag}_n(id + \delta)$ *the discrete dynamical system defined by*

$$R : \quad r(0) \in \mathbb{R}^n_+, \; r(k+1) := \Gamma \circ D(r(k)), \; k \in \mathbb{N}_0,$$

is globally asymptotically stable in 0*.*

Some related sets

A5

$$\Omega_i = \left\{ s \in \mathbb{R}^n_+ : s_i > \sum_{j \neq i} \gamma_{ij}(s_j) \right\}.$$

$$\Psi_i = \left\{ s \in \mathbb{R}^n_+ : s_i \geq \sum_{j \neq i} \gamma_{ij}(s_j) \right\}.$$

Some related sets

$$\Omega_i = \left\{ s \in \mathbb{R}^n_+ : s_i > \sum_{j \neq i} \gamma_{ij}(s_j) \right\}.$$

$$\Psi_i = \left\{ s \in \mathbb{R}^n_+ : s_i \geq \sum_{j \neq i} \gamma_{ij}(s_j) \right\}.$$

$$\Omega = \bigcap_i \Omega_i$$

$$\Psi = \bigcap_i \Psi_i$$

Some related sets

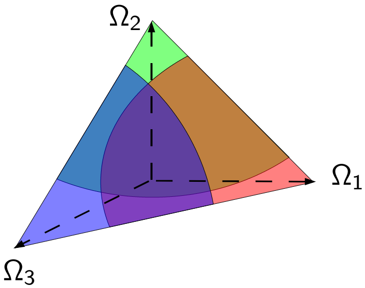
A5

$$\Omega_i = \left\{ s \in \mathbb{R}^n_+ : s_i > \sum_{j \neq i} \gamma_{ij}(s_j) \right\}.$$

$$\Psi_i = \left\{ s \in \mathbb{R}^n_+ : s_i \geq \sum_{j \neq i} \gamma_{ij}(s_j) \right\}.$$

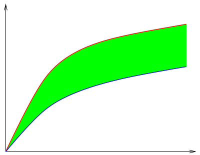
$$\Omega = \bigcap_i \Omega_i$$

$$\Psi = \bigcap_i \Psi_i$$



Examples

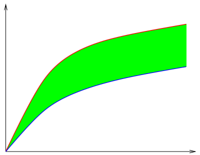
- $\Gamma = \begin{bmatrix} 0 & \frac{1}{2}(\cdot)^2 \\ \sqrt{\cdot} & 0 \end{bmatrix}$



Examples

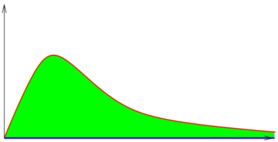
A5

- $\Gamma = \begin{bmatrix} 0 & \frac{1}{2}(\cdot)^2 \\ \sqrt{\cdot} & 0 \end{bmatrix}$



- $\gamma(t) = t \cdot (1 - e^{-t})$

$$\Gamma = \begin{bmatrix} \gamma & id \\ 0 & \gamma \end{bmatrix}$$



Radial unboundedness of Ω

Theorem

$\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *monotone, continuous,* $\Gamma(0) = 0$.
Then $\Gamma \not\geq id$ *implies* $\Omega \cap S_r \neq \emptyset$ *for all* $r > 0$, S_r *denoting sphere around the origin in* $\mathbb{R}^n_+$ *of radius* $r > 0$ *with respect to the 1-norm,* $S_r = \{s \in \mathbb{R}^n_+ : \sum_{i=1}^n s_i = r\}$.

Radial unboundedness of Ω

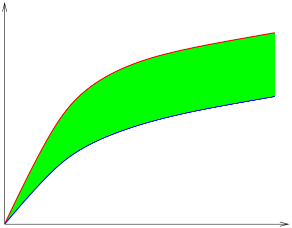
A5

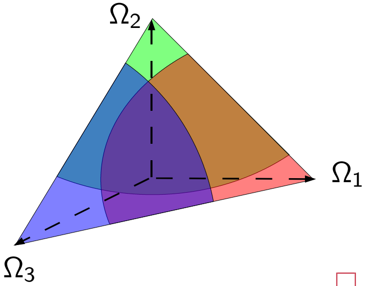
Theorem

$\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *monotone, continuous,* $\Gamma(0) = 0$.
Then $\Gamma \not\geq id$ *implies* $\Omega \cap S_r \neq \emptyset$ *for all* $r > 0$, S_r *denoting sphere around the origin in* $\mathbb{R}^n_+$ *of radius* $r > 0$ *with respect to the 1-norm,* $S_r = \{s \in \mathbb{R}^n_+ : \sum_{i=1}^n s_i = r\}$.

Proof.

Based on famous theorem by Knaster-Kuratowski-Mazurkiewicz, 1929.





□

Numerical stability test

A5

Question: When does $\Gamma \ngeq \mathrm{id}$ hold?

For $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$, $\Gamma \ngeq \mathrm{id}$, by KKM-Theorem can find $x \in \Omega$.

Numerical stability test

Question: When does $\Gamma \not\geq \mathrm{id}$ hold?

For $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$, $\Gamma \not\geq \mathrm{id}$, by KKM-Theorem can find $x \in \Omega$.

If Γ has no zero rows, then $\{\Gamma^k(x)\}_{k=0}^\infty \subset \Omega$, also

$$(1-\lambda)\Gamma^{k+1} + \lambda\Gamma^k(x) \in \Omega, \quad k \geq 0.$$

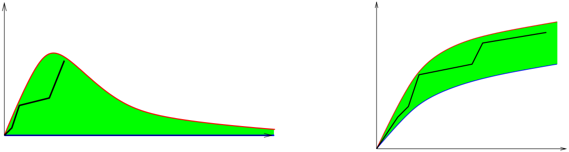
Numerical stability test

Question: When does $\Gamma \not\geq \text{id}$ hold?

For $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$, $\Gamma \not\geq \text{id}$, by KKM-Theorem can find $x \in \Omega$.

If Γ has no zero rows, then $\{\Gamma^k(x)\}_{k=0}^\infty \subset \Omega$, also

$$(1-\lambda)\Gamma^{k+1} + \lambda\Gamma^k(x) \in \Omega, \quad k \geq 0.$$

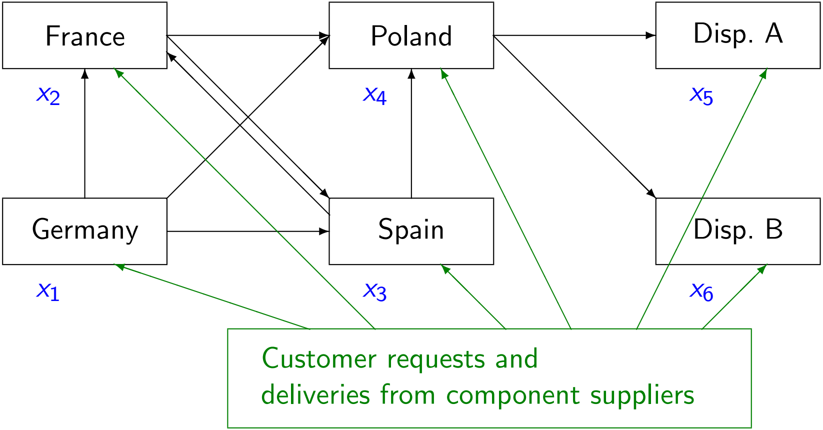


This implies $\Gamma \not\geq \text{id}$ on $[0, x] \subset \mathbb{R}_+^n$.

Similar for Γ with zero rows.

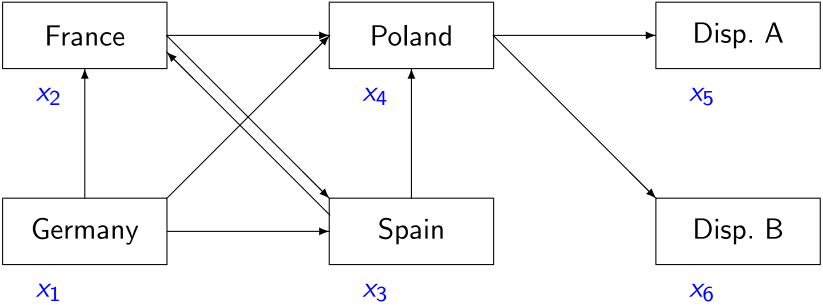
Automotive manufacturing network

A5



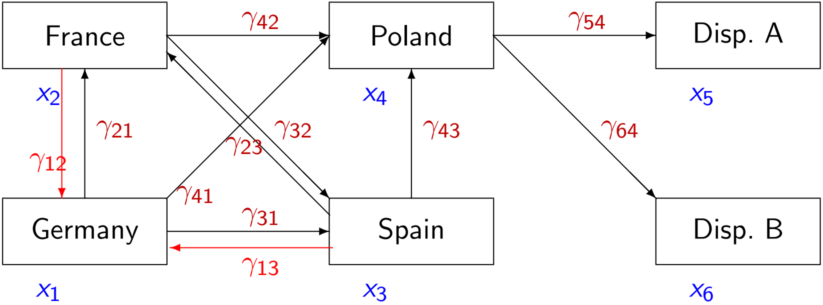
Automotive manufacturing network

A5



Automotive manufacturing network

A5



State equations

$$\dot{x}_1 = u - \frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3}$$

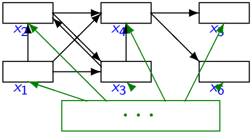
$$\dot{x}_2 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_3, c_3x_3\} - \min\{b_2, c_2x_2\}$$

$$\dot{x}_3 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_2, c_2x_2\} - \min\{b_3, c_3x_3\}$$

$$\dot{x}_4 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_2, c_2x_2\} + \min\{b_3, c_3x_3\} - \min\{b_4, c_4x_4\}$$

$$\dot{x}_5 = \frac{1}{2}\min\{b_4, c_4x_4\} - c_5x_5$$

$$\dot{x}_6 = \frac{1}{2}\min\{b_4, c_4x_4\} - c_6x_6$$



Gain matrix

A5

$$\Gamma = (\gamma_{ij}) = \begin{bmatrix} 0 & \gamma_{12} & \gamma_{13} & 0 & 0 & 0 \\ \gamma_{21} & 0 & \gamma_{23} & 0 & 0 & 0 \\ \gamma_{31} & \gamma_{32} & 0 & 0 & 0 & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_{54} & 0 & 0 \\ 0 & 0 & 0 & \gamma_{64} & 0 & 0 \end{bmatrix}$$

For example:

$$\gamma_{21}(x_1) = \max\left\{ \sqrt{\frac{ax_1 + b\sqrt{x_1}}{3c_2}}, \frac{ax_1 + b\sqrt{x_1}}{3\min\{b_2, c_2, b_2 - \frac{1}{2}b_3\}} \right\}$$

Additional constraints: $c_2 > c_3 > b_2 > \frac{1}{2}b_3 \geq 0$.

Numerical stability test

Choose $r \gg 0$ and use an efficient algorithm to find $s \in \Omega \cap S_r$ (see, e.g. Scarf, Eaves, ...),

$$\Omega = \{s \in \mathbb{R}^n_+ : \Gamma(D \cdot s) < s\}$$

for some $D = (1 + \varepsilon) \cdot \mathrm{id}$ or similar.

Numerical stability test

A5

Choose $r \gg 0$ and use an efficient algorithm to find $s \in \Omega \cap S_r$ (see, e.g. Scarf, Eaves, ...),

$$\Omega = \{s \in \mathbb{R}^n_+ : \Gamma(D \cdot s) < s\}$$

for some $D = (1 + \varepsilon) \cdot \mathrm{id}$ or similar.

If such an s can be found, deduce stability on $[0, s] \in \mathbb{R}^n_+$ by monotonicity of Γ.

In our example this yields a condition on the constants a, b, b_2, b_3, c_2, c_3.

Conclusions

- Stability is an important concept for logistic networks
- A stability criterion for arbitrary logistic networks has been derived
- The criterion is applicable for networks incorporating autonomous control
- Using an explicit example it was shown how to verify this condition