Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Computing the Value of Transshipment Flexibility in Distribution Networks

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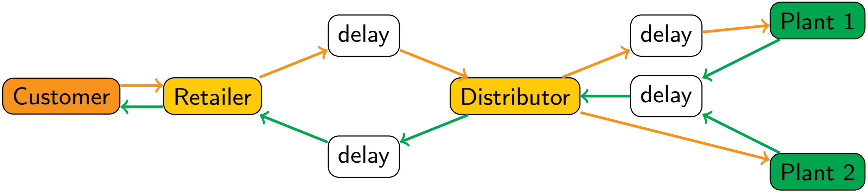
with F. Cassim, J. Chastenet and J. Uffer (EPF Lausanne)
and M. Reimann (Warwick Business School)

Bremen, 11 January 2008

Problem Setting

Given: Distribution network



Nodes: Facilities

Links: Flows

→ Information flows (orders)
← Material flows (shipments)

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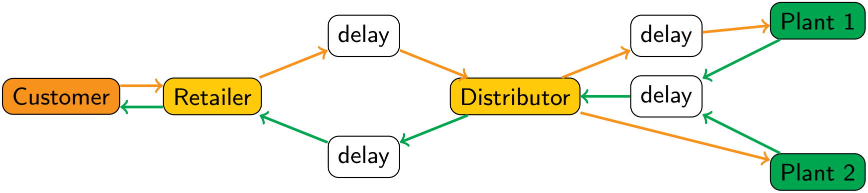


Nodes: Facilities

Links: Flows

⟶ Information flows (orders)

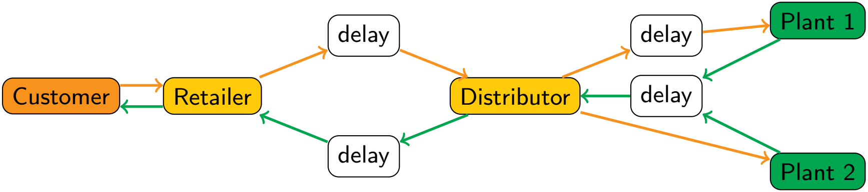
⟵ Material flows (shipments)

Goal: Determine optimal shipment policy

Overview

Overview

Discrete-time Supply Network Model



Nodes $v \in V$: Places where material is stored

Edges $e \in E$: Information and material flow

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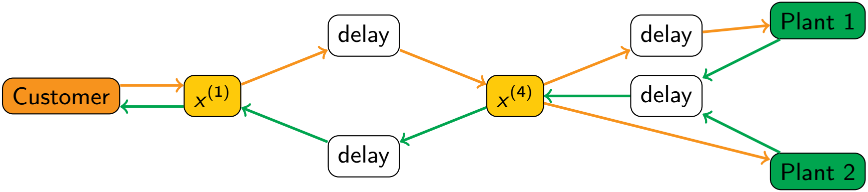
Discrete-time Supply Network Model



Nodes $v \in V$: Places where material is stored

- State variables $x^{(v)}$ for the inventories

Edges $e \in E$: Information and material flow

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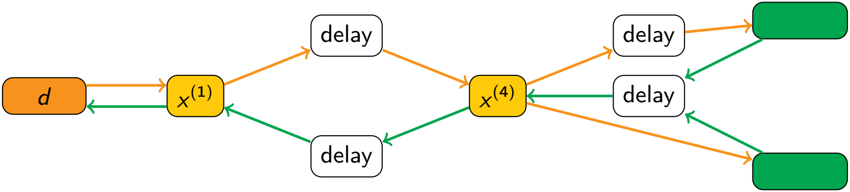

Nodes $v \in V$: Places where material is stored

- State variables $x^{(v)}$ for the inventories
- Sink node: generate demand d (stochastic)
- Source nodes: infinite supply

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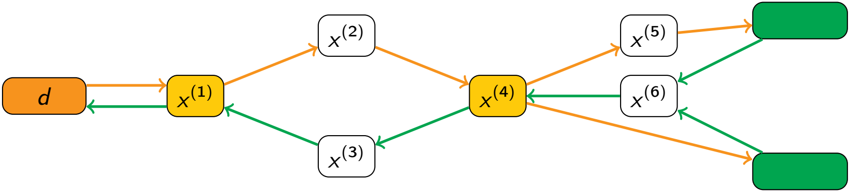

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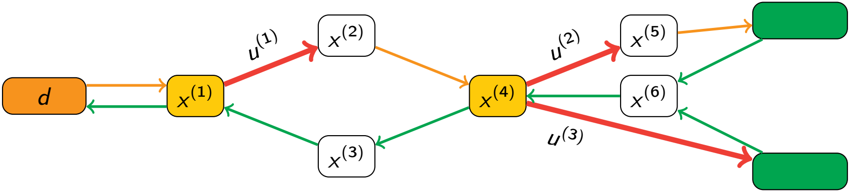

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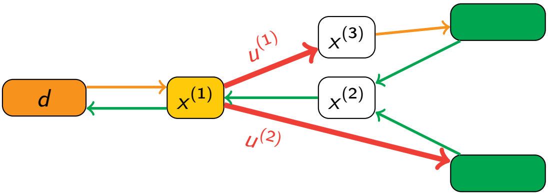
Control: Orders $u^{(e)}$ of facilities

Example



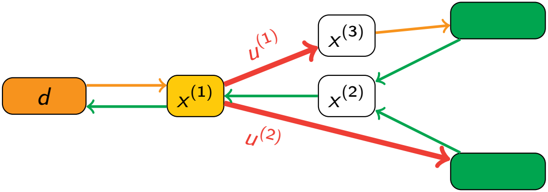
Dynamics of inventories

$$x_{k+1}^{(1)} = x_k^{(1)} + x_k^{(2)} - d_k \quad (1)$$

$$x_{k+1}^{(2)} = x_k^{(3)} + u_k^{(1)} \quad (2)$$

$$x_{k+1}^{(3)} = u_k^{(2)} \quad (3)$$

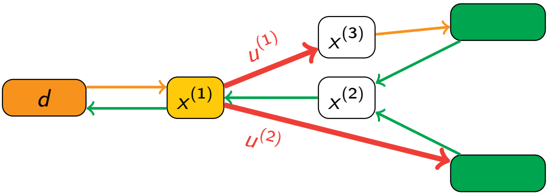
Example



Dynamics of inventories

$$\mathbf{x}_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{:=\mathbf{A}} \mathbf{x}_k + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{:=\mathbf{B}} \mathbf{u}_k + \mathbf{d}_k$$

Example



Dynamics of inventories

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{d}_k$$

Overview

1 Supply Networks as Controlled Dynamical Systems

2 Stochastic Robust Optimal Control

3 Examples
- Dual Suppliers
- Lateral Inventory Transshipments

Finite Horizon Robust Optimal Control

Given

- Dynamics $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{d}_k$
- Linear constraints on state and controls $\mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u} \leq \mathbf{g}$
- Linear costs on state $(\mathbf{p}^T\mathbf{x})$ and control $(\mathbf{q}^T\mathbf{u})$
- Stochastic disturbances $\mathbf{d}_k \in \mathbf{D} := \{\mathbf{d}^1, \mathbf{d}^2, \ldots, \mathbf{d}^\ell\}$
- Fixed time horizon N

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$$\bar{J}(\mathbf{x}_0, \pi) := \mathbb{E}\left(\mathbf{p}^T\mathbf{x}_N + \sum_{i=0}^{N-1} \mathbf{p}^T\mathbf{x}_i + \mathbf{q}^T\mathbf{u}_i(\mathbf{x}_i)\right)$$

where $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k(\mathbf{x}_k) + \mathbf{d}_k$.

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Let $\mathbf{u}_k(\mathbf{x}_k)$ denote the control input when system is in state $\mathbf{x}$ at time k and $\pi = (\mathbf{u}_0, \ldots, \mathbf{u}_{N-1})$. The worst-case cost is

$$\hat{J}(\mathbf{x}_0, \pi) := \max_{(\mathbf{d_k})_{k=0}^{N-1}} \mathbf{p}^T\mathbf{x}_N \sum_{i=0}^{N-1} [\mathbf{p}^T\mathbf{x}_i + \mathbf{q}^T\mathbf{u}_i(\mathbf{x}_i)]$$

where $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k(\mathbf{x}_k) + \mathbf{d}_k$.

Stochastic Dynamic Programming

Let $J_k^* : \mathcal{X}_k \longrightarrow \mathbb{R}$ the optimal value function at time k.

Terminal condition: $J_N^*(\mathbf{x}) = \mathbf{p}_N^T \mathbf{x}$ and $\mathcal{X}_N = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{H}_N \mathbf{x} \leq \mathbf{h}_N\}$.

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& \quad \text{s.t.} \quad \mathbf{F}\mathbf{x}_k + \mathbf{G}\mathbf{u}_k \leq \mathbf{q} \\
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Remark: Problem is a parametric LP (with state $\mathbf{x}$ as paramter) if

- J_{k+1}^* is piecewise linear and convex
- $\mathcal{X}_{k+1}$ is a polyhedron

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Properties

Theorem

- *Value function* $J_k^*(\mathbf{x}_k)$ *is piecewise affine and convex in* x.
- *Optimal solution* $\mathbf{u}_k^*(\mathbf{x}_k)$ *is piecewise affine and continuous in* $\mathbf{x}_k$.

$$\begin{cases} J_k^*(\mathbf{x}) = \mathbf{V}_k^{(i)}\mathbf{x} + \mathbf{W}_k^{(i)} \\ \mathbf{u}_k^*(\mathbf{x}) = \mathbf{R}_k^{(i)}\mathbf{x} + \mathbf{S}_k^{(i)} \end{cases} \quad \text{for } \mathbf{x} \in \mathcal{R}_k^{(i)}.$$

where $\mathcal{R}_k = \left\{\mathcal{R}_k^{(i)}\right\}$ *is a partition of* $\mathcal{X}_k$ *with* $\mathrm{cl}(\mathcal{R}_k^{(i)})$ *being polyhedra.*

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Consequence: Solve N pLPs to recursively obtain $J_0^*(\mathbf{x}_0)$ and $\mathbf{u}_0^*(\mathbf{x}_0)$.

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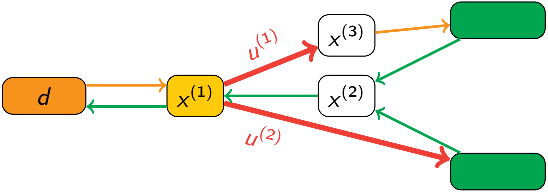
Goal: We would like to approximate $\mathbf{u}^* = \lim_{N\to\infty} \mathbf{u}_0^*$.

Overview

Example: One Retailer With Two Suppliers



$$\mathbf{x}_{k+1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}_k + \mathbf{d}_k$$

$u^{(1)}$: orders at supplier 1 with lead time 1 and unit cost 4
$u^{(2)}$: orders at supplier 2 with lead time 2 and unit cost 1
$u^{(1)}, u^{(2)} \leq 8$
$\mathbf{d}_k = (-\delta, 0, 0)^T$ where $\delta \in [0, 8]$

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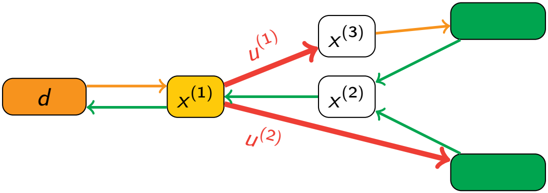

Resulting optimal control law:

(w.r.t. worst-case cost)

$$u^{(1)*} = \min\{\max\{20 - x_1 - x_2 - x_3, 0\}, 4\}$$
$$u^{(2)*} = \max\{16 - x_1 - x_2 - x_3, 0\}$$

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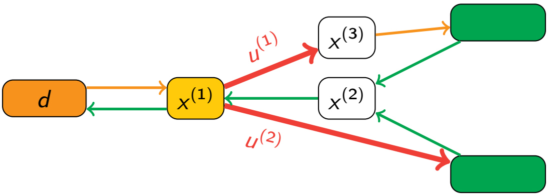

Resulting optimal control law:

(w.r.t. average cost if $\delta \sim \mathcal{U}_{\{0,1,\ldots,8\}}$)

$$u^{(1)*} = \min\{\max\{22 - x_1 - x_2 - x_3 - x_4, 0\}, 4\}$$
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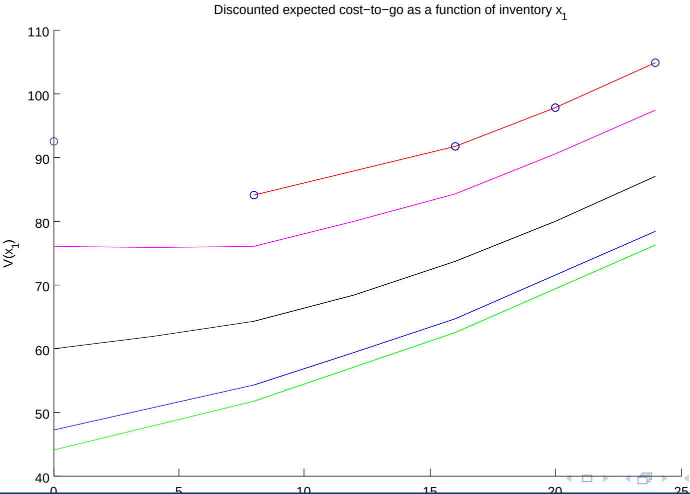
Results for Transshipment Problem



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  - cost and constraints are piecewise linear and convex

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- Able to compute explicit state-feedback control policies for general networks if
  - the dynamics of the nodes is linear
  - cost and constraints are piecewise linear and convex
- Use of the model here:
  - Determine the value function (expected discounted infinite-time cost when starting at a given state) for different scenarios (flexibility options)
  - Valuate flexibility options by cost decrease versus base case
    - (i) using typical/nominal states (here)
    - (ii) via Markov chain: cost of stationary distribution (in progress)