Local input-to-state stability of production networks

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Abstract In this paper we analyze a given production network in view of stability, which means boundedness of the state of the network over time. From a mathematical point of view we model the network by differential equations. With help of local input-to-state stability (LISS) Lyapunov functions and a small gain condition we check, if the network is stable. This results in the derivation of conditions for the production rates for which stability of the production network is guaranteed.

1 Introduction

Production and supply networks or other modern logistic structures are typical examples of complex systems with a nonlinear and sometimes chaotic behavior. Their dynamics subject to many different perturbations due to changes on market, changes in customer behavior, information and transport congestions, unreliable elements of the network etc.

One approach to handle such complex systems is to shift from centralized to decentralized or autonomous control, i.e., to allow the entities of a network to make their own decisions based on some given rules and available local information. However a system emerging in this way may become unstable and hence be not effective in performance. Typical examples of unstable behaviour are unbounded growth of unsatisfied orders or unbounded growth of amount of workload to be processed by a machine and causes high inventory costs or loss of customers. To avoid instability it is worth to investigate its behavior in advance.

Mathematical methods can help to handle complex systems. In particular mathematical modelling and analysis provide helpful tools for investigation of such objects and can be used for design, optimization and control of such networks and for deeper understanding of their dynamical properties.

This paper focuses on the stability analysis of a production network, in order to identify stable parameter constellation. In particular cases stability means that the number of unsatisfied orders or/and amount of workload to be processed by a machine remain bounded over time in spite of disturbances. By application of the stability analysis to a logistical network we can draw conclusions of its behaviour and derive conditions to guarantee stability, which avoid negative outcomes mentioned above. The results of this analysis can be used to design logistical networks in order to have good properties to achieve economic goals. Obviously stability is decisive for the performance and vitality of a network.

In this paper we propose a model for a production logistic scenario comprising several autonomous production plants connected through transport routs. This network is modelled by ordinary differential equations. We show how its stability can be analyzed with help of small gain theorems recently developed for general type of dynamic networks. Explicit conditions of the production rates will be derived by application of mathematical systems theory of interconnected systems.

In Section 2 we describe the given production network with its conditions and model it mathematically by differential equations. A mathematical background is given in Section 3, which is used in Section 4 to derive stability conditions of the production network. In Section 5 some simulation results and their interpretations are given. Conclusions and outlines can be found in Section 6.

2 Model description

In this section we describe the given production network, which we model and analyze the system in view of stability with help of differential equations.

The production network in Figure 1 consists of six geographically distributed production locations, which are connected. In logistic there are many flows, e.g. material, information or worth flows. In Figure 1 the material flow is described by arrows and the information flow by dashed arrows. The *state* of each production location is denoted by $x_i(t) \in \mathbb{R}$ for $i = 1,\ldots,6$, where $t \in \mathbb{R}_+$ can be interpreted as time and $\mathbb{R}_+$ denotes all positive real values. In the rest of this paper we write *subsystem i* for the *i*-th production location. All six subsystems form the production network, which we name simply (whole) system.

We describe the production network by the information flow and interpret the state of the *i*-th subsystem as the number of unsatisfied orders within *i*-th production location. Subsystem 6 gets some orders of its product from the customers, denoted by $d(t) \in \mathbb{R}_+$. While processing the orders, subsystem 6 orders components, which it needs for production from subsystem 4 and 5. These two subsystems send orders for components, which they need to subsystem 2 and 3. Their orders will be sent to subsystem 1, which gets instantly its raw material from an external source.

The orders from subsystem 1 to subsystem 6 are interpreted as a kind of payment or the demand for its production of subsystem 1 of the final product of the given production network from subsystem 6.

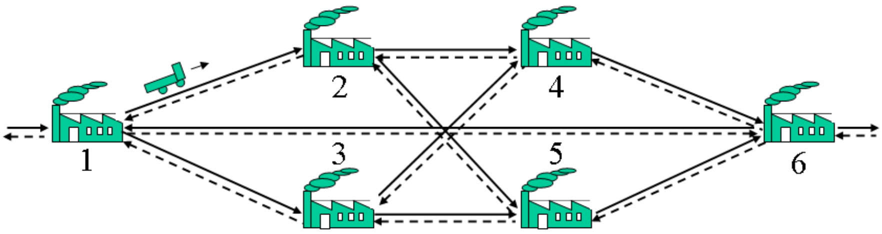

Fig. 1 The production network

We suppose all subsystems are autonomously controlled, it means the ability to adjust the production rate of the production location. This can be achieved by varying work times of the workers, transportation times of the products or the number of used machines for production. $\alpha_i \in \mathbb{R}_+$ denotes the (constant) maximum production rate of subsystem *i*. The actual production rate of subsystem *i* ($\tilde{f}_i$) converges to α_i, if the state of subsystem $x_i(t)$ is large and $\tilde{f}_i$ tends to zero, if the state of subsystem $x_i(t)$ tends to zero. This means, if there are many orders, the actual production rate is near to the maximum production rate and if there are no orders nothing will be produced. Therefor the actual production rate of each subsystem at time *t* is given by

$$\tilde{f}_i(x_i(t)) := \alpha_i(1-\exp(-x_i(t))), i=1,\dots,6.$$

With these considerations we can model the system presented in Figure 1 by differential equations for each subsystem, which are nothing but a description of changes of the state $x_i(t)$ of subsystem *i* along time $t \in \mathbb{R}_+$:

$$\begin{aligned}
\dot{x}_1(t) &= c_{12}\tilde{f}_2(x_2(t)) + c_{13}\tilde{f}_3(x_3(t)) - \tilde{f}_1(x_1(t)),\\
\dot{x}_2(t) &= c_{24}\tilde{f}_4(x_4(t)) + c_{25}\tilde{f}_5(x_5(t)) - \tilde{f}_2(x_2(t)),\\
\dot{x}_3(t) &= c_{34}\tilde{f}_4(x_4(t)) + c_{35}\tilde{f}_5(x_5(t)) - \tilde{f}_3(x_3(t)),\\
\dot{x}_4(t) &= c_{46}\tilde{f}_6(x_6(t)) - \tilde{f}_4(x_4(t)),\\
\dot{x}_5(t) &= c_{56}\tilde{f}_6(x_6(t)) - \tilde{f}_5(x_5(t)),\\
\dot{x}_6(t) &= d(t) + c_{61}\tilde{f}_1(x_1(t)) - \tilde{f}_6(x_6(t)),
\end{aligned} \tag{1}$$

where the constants $c_{ij} \in \mathbb{R}_+$ can be interpreted as the number of orders of components to subsystem *i* from subsystem *j*.

By definition of $f_i(x,d) := \dot{x}_i(t), i=1,\dots,6,\ x := (x_1,\dots,x_6)^T$ and $f(x,u) := (f_1(x,d),\dots,f_6(x,d))^T$ we can write the whole system as

$$\dot{x}_i(t) = f(x(t), d(t)),\ t \in \mathbb{R}_+. \tag{2}$$

Now the question arises, under which conditions the subsystems are stable, which means that the states of all subsystems will not increase to infinity. In other words, under which conditions all states of the subsystem and therefore of the whole system are bounded, which means stability of the production network?

3 Mathematical background

For investigation of the stability of system (1) and (2), respectively, we need some mathematical results. We present a stability property and a tool how to check, weather the system has the stability property.

We consider nonlinear dynamical system of the form

$$\dot{x}(t) = f(x(t), u(t)), \tag{3}$$

where $t \in \mathbb{R}_+$ is the time, $\dot{x}(t)$ the derivate of the state $x(t) \in \mathbb{R}^{\mathrm{N}}$ with the initial value x_0, input $u(t) \in \mathbb{R}^{\mathrm{m}}$, which is an essentially bounded measurable function and $f : \mathbb{R}^{\mathrm{N+m}} \to \mathbb{R}^{\mathrm{N}}$ nonlinear. To have existence and uniqueness of a solution of (3), function *f* has to be continuous and locally Lipschitz in *x* uniformly in *u*. The solution is denoted by $x(t; x_0, u)$ or $x(t)$ in short.

To describe the given production network we generalize (3) and consider $n \in \mathbb{N}$ interconnected systems. These are in general nonlinear dynamical systems of the form

$$\dot{x}(t) = f_i(x_1(t), \ldots, x_n(t), u_i(t)),\ i = 1, \ldots, n \tag{4}$$

where $t \in \mathbb{R}_+$, $x_i(t) \in \mathbb{R}^{N_i}$, $u_i(t) \in \mathbb{R}^{M_i}$, which are essentially bounded measurable functions, $f_i : \mathbb{R}^{\sum_j N_j + M_i} \to \mathbb{R}^{N_i}$, $i = 1, \ldots, n$, where f_i are continuous and locally Lipschitz in $x = (x_1^T, \ldots, x_n^T)^T$ uniformly in u_i. We consider x_j as internal input and u_i as external input of the *i*-th subsystem $i, j = 1, \ldots, n,\ i \neq j$. The solution is denoted by $x(t; x_i^0, x_j : j \neq i, u_i)$ or $x(t)$ in short.

If we define $N := \sum_{i=1}^n N_i$, $m := \sum_{i=1}^n M_i$, $x = (x_1^T, \ldots, x_n^T)^T$, $u := (u_1^T, \ldots, u_n^T)^T$ and $f := (f_1^T, \ldots, f_n^T)^T$, then (4) becomes

$$\dot{x}(t) = f(x(t), u(t)),\ t \in \mathbb{R}_+. \tag{5}$$

We denote the standard euclidian norm in $\mathbb{R}^{\mathrm{n}}$ by $\|\cdot\|$ and the essential supremum norm for essentially bounded functions *u* in $\mathbb{R}_+$ by $\|u\|_\infty$. We need some classes of functions to define the stability property, which we will use. A function $f : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}_+$ is said to be *positive definite*, if $f(0) = 0$ and $f(x) > 0, \forall x \in \mathbb{R}^{\mathrm{n}}$ holds. A class *K* function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $\gamma(0) = 0$ and strictly increasing. If it is additionally unbounded then it is of class K_∞. We call a function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ of class *KL* if β is continuous, $\beta(\cdot, t) \in K$ and $\beta(r, \cdot)$ strictly decreasing with $\lim_{t\to\infty} \beta(r, t) = 0, \forall t, r \geq 0$.

Now we define local input-to-state stability (LISS) and input-to-state stability (ISS), respectively, for each subsystem of (4). For system (3) the definition of LISS and ISS, respectively, can be found for example in [3] and [8], respectively.

Definition 1. The *i*-th subsystem of (4) is called LISS, if there exist constants $\rho_i, \rho_j^i, \rho_i^u > 0, \gamma_{ij}, \gamma_i \in K_\infty$ and $\beta_i \in KL$, such that for all initial values $\| x_i^0 \| \le \rho_i$, $\| x_i \|_\infty \le \rho_j^i$ and all inputs $\| u_i \|_\infty \le \rho_i^u$ the inequality

$$\| x_i(t; x_i^0, x_j : j \ne i, u_i) \| \le \max\{\beta_i(\| x_i^0 \|, t), \max_{j \ne i} \gamma_{ij}(\| x_j \|_\infty), \gamma_i(\| u_i \|_\infty)\} \quad (6)$$

is satisfied $\forall t \in \mathbb{R}_+$. γ_{ij} and γ_i are called (nonlinear) gains.

Note that, if $\rho_i, \rho_j^i, \rho_i^u = \infty$ then the *i*-th subsystem is ISS (see [1]). LISS and ISS, respectively, mean that the norm of the trajectories of each subsystem is bounded.

Furthermore we define the *gain matrix* $\Gamma := (\gamma_{ij}), i, j = 1,...,n, \gamma_{ii} = 0$, which defines a map $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$\Gamma(s) := \left(\max_j \gamma_{1j}(s_j), ..., \max_j \gamma_{nj}(s_j) \right)^T, \ s \in \mathbb{R}_+^n. \quad (7)$$

Previous investigations of two interconnected systems established a small gain condition to guarantee stability (see [5] and [6]). In [1] an ISS small gain theorem for general networks was proved, where the small gain condition is of the form

$$\Gamma(s) \not\ge s, \ \forall s \in \mathbb{R}_+^n \setminus \{0\} \quad (8)$$

Notation $\not\ge$ means that there is at least one component $i \in \{1,...,n\}$ such that $\Gamma(s)_i < s_i$. Here we recall a local version of the small gain condition:

Definition 2. Γ satisfies the *local small gain condition* (LSGC) on $[0, w^*]$, provided that

$$\Gamma(w^*) < w^* \text{ and } \Gamma(s) \not\ge s, \ \forall s \in [0, w^*], s \ne 0. \quad (9)$$

Further details of (9) can be found in [3]. The small gain condition is equivalent to the compliance of the cycle condition (see [7], Lemma 2.3.14 for details). We quote the local version of the small gain theorem:

Theorem 1. Let all subsystems of (4) satisfy (6). Suppose Γ satisfies LSGC. Then there exist constants $\rho, \rho^u > 0, \beta \in KL$ and $\gamma \in K_\infty$, such that the whole system (5) is LISS.

The proof can be found in [3], Theorem 4.2. An important tool to verify LISS and ISS, respectively, are Lyapunov functions. For systems of the form (3) one can find the definition of Lyapunov functions for example in [6] and [3].

Definition 3. A smooth function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ is called LISS Lyapunov function of the *i*-th subsystem of system (4), if it satisfies the following two conditions:

1) There exist functions $\psi_{1i}, \psi_{2i} \in K_\infty$ such that

$$\psi_{1i}(\| x_i \|) \le V_i(x_i) \le \psi_{2i}(\| x_i \|), \ \forall x_i \in \mathbb{R}^{N_i} \quad (10)$$

2) There exist $\chi_{ij}, \chi_i \in K_\infty$, a positive function μ_i and constants $\rho_0^i, \rho_i^u > 0$ such

$$V_i(x_i) \geq \max\{\max_j \chi_{ij}(V_j(x_j)), \chi_i(\| u_i \|)\} \Rightarrow \nabla V_i(x_i) \cdot f_i(x,u) \leq -\mu_i(V_i(x_i)) \quad (11)$$

for all $x_i \in \mathbb{R}^{N_i}$, $\| x_i \|_\infty \leq \rho_0^i$, $u_i \in \mathbb{R}^{M_i}$, $\| u_i \|_\infty \leq \rho_i^u$, $\chi_{ii} = 0$, where ∇ denotes the gradient of V_i. Functions χ_{ij} and χ_i are called LISS Lyapunov gains.

Note that, if $\rho_0^i, \rho_i^u = \infty$ then the LISS Lyapunov function of the *i*-th subsystem becomes an ISS Lyapunov function of the *i*-th subsystem (see [4]).

To check if the whole system of the form (5) has the LISS or ISS property one can use LISS or ISS Lyapunov functions, respectively. If there exists a LISS or ISS Lyapunov function for a subsystem of (4) then the subsystem has the LISS or ISS property, respectively. Furthermore, if all subsystems have a LISS of ISS Lyapunov function and the LISS or ISS Lyapunov gains satisfy the small gain condition, then the whole system of the form (5) is LISS or ISS, respectively (see [2], [3] or [4]).

With this mathematical theory we can derive conditions, for which the subsystems and the whole system are stable. This will be presented in the next section.

4 Stability of the model

In this section we investigate all six subsystems of (1) to check if they have the LISS or ISS property, respectively. Therefor we choose a Lyapunov function candidate for each subsystem and check, weather conditions (10) and (11) are satisfied.

Remark 1. It can be shown that for any non-negative initial condition all subsystems of (1) are non-negative, since the term $f_i(x_i)$ is zero for $x_i = 0$ and $d = 0$, $i = 1,\ldots,n$.

We choose $V_i(x_i) = x_i$ as Lyapunov function candidate for *i=1,…,6*. V_i satisfies condition (10). For the investigation of the first subsystem we define

$$\chi_{1j}(x_j) := -\ln\left(1 - \frac{c_{12}\alpha_2 + c_{13}\alpha_3}{(1-\varepsilon_{1j})\alpha_1}(1-\exp(-x_j))\right) \leq x_1 = V_1(x_1),$$

$j = 2,3, \quad 1 > \varepsilon_{1j} > 0,$ which implies

$$c_{1j}\alpha_j(1-\exp(-x_j)) \leq \frac{c_{1j}\alpha_j}{c_{12}\alpha_2 + c_{13}\alpha_3}(1-\varepsilon_{1j})\,\alpha_1(1-\exp(-x_1)).$$

To guarantee that χ_{1j} is well defined the condition

$$c_{12}\alpha_2 + c_{13}\alpha_3 < \alpha_1(1-\varepsilon_{1j}) < \alpha_1. \quad (12)$$

has to be satisfied. With this consideration it follows

$$
\begin{aligned}
&\nabla V_1(x_1(t)) f_1(x_1(t),\ldots,x_6(t),d(t)) \\
&= c_{12}\alpha_2(1-\exp(-x_2)) + c_{13}\alpha_3(1-\exp(-x_3)) - \alpha_1(1-\exp(-x_1)) \\
&\le \left(\frac{(1-\varepsilon_{12})\alpha_1 c_{12}\alpha_2}{c_{12}\alpha_2 + c_{13}\alpha_3} + \frac{(1-\varepsilon_{13})\alpha_1 c_{13}\alpha_3}{c_{12}\alpha_2 + c_{13}\alpha_3} - \alpha_1\right)(1-\exp(-x_1)) \\
&\le -\varepsilon_1\alpha_1(1-\exp(-x_1)) = -\mu_1(V_1(x_1(t)))
\end{aligned}
$$

where $\varepsilon_1 := \min\{\varepsilon_{12}, \varepsilon_{13}\}$ and $\mu_1(r) := \varepsilon_1\alpha_1(1-\exp(-r))$ is a positive definite function.

The reason of the introduction of the constant value ε_{1j} is to guarantee that μ_1 is positive definite. V_1 satisfies condition (11) and is the ISS Lyapunov function of the first subsystem from which we know that the first subsystem has the ISS property for all $x_j \in \mathbb{R}_+$, $j = 1,2,3,$ if condition (12) holds.

For subsystem 2 to 5 we do similar calculations and get the gains

$$
\begin{aligned}
\chi_{2j}(x_j) &:= -\ln\left(1 - \frac{c_{24}\alpha_4 + c_{25}\alpha_5}{(1-\varepsilon_{2j})\alpha_2}(1-\exp(-x_j))\right), 1 > \varepsilon_{2j} > 0,\ j = 4,5, \\
\chi_{3j}(x_j) &:= -\ln\left(1 - \frac{c_{34}\alpha_4 + c_{35}\alpha_5}{(1-\varepsilon_{3j})\alpha_3}(1-\exp(-x_j))\right), 1 > \varepsilon_{3j} > 0,\ j = 4,5, \\
\chi_{j6}(x_6) &:= -\ln\left(1 - \frac{c_{j6}\alpha_6}{(1-\varepsilon_{j6})\alpha_j}(1-\exp(-x_6))\right), 1 > \varepsilon_{j6} > 0,\ j = 4,5
\end{aligned}
$$

and conditions

$$
\alpha_2 > c_{24}\alpha_4 + c_{25}\alpha_5,\ \alpha_3 > c_{34}\alpha_4 + c_{35}\alpha_5,\ \alpha_4 > c_{46}\alpha_6,\ \alpha_5 > c_{56}\alpha_6 \tag{13}
$$

for which the subsystems 2 to 5 have the ISS property.

For subsystem 6 from

$$
\begin{aligned}
\chi_6(d(t)) &:= -\ln\left(1 - \frac{d(t)(\|d\|_\infty + c_{61}\alpha_1)}{\|d\|_\infty (1-\varepsilon_{6d})\alpha_6}\right) \le x_6 = V_6(x_6), \\
\chi_{61}(x_1) &:= -\ln\left(1 - \frac{\|d\|_\infty + c_{61}\alpha_1}{(1-\varepsilon_{61})\alpha_6}(1-\exp(-x_1))\right) \le x_6 = V_6(x_6),
\end{aligned} \tag{14}
$$

with $0 < \varepsilon_{61}, \varepsilon_{6d} < 1$ we get

$$
\begin{aligned}
&\nabla V_6(x_6(t)) f_6(x_1(t),\ldots,x_6(t),d(t)) \\
&= d(t) - \alpha_6(1-\exp(-x_6(t))) + c_{61}\alpha_1(1-\exp(-x_1(t))) \\
&\le -\varepsilon_6\alpha_6(1-\exp(-x_6(t))) = -\mu_6(V_6(x_6(t))),
\end{aligned}
$$

where $\varepsilon_6 := \min\{\varepsilon_{61}, \varepsilon_{6d}\}$ and $\mu_6(r) := \varepsilon_6\alpha_6(1-\exp(-r))$ is positive definite, if

$$
\alpha_6 > \|d\|_\infty + c_{61}\alpha_1. \tag{15}
$$

holds true to guarantee that χ_6 and χ_{61} are well defined. Function χ_6 as defined in (14) is $\in K$, but we can find a continuation of χ_6 such that the composed function is K_∞. Hence V_6 satisfies condition (11) and from Section 3 we know that subsystem six has the LISS property for all $x_6^0 \in \mathbb{R}_+$ and $\| d \|_\infty < \alpha_6 - c_{61}\alpha_1 =: \rho^u$.

With $\exp(-r) < 1,\ r > 0 \Leftrightarrow (1-a)\exp(-r) < (1-a),\ 0 < a < 1$ $\Leftrightarrow \exp(-r) < 1 - a + a\exp(-r) \Leftrightarrow -\ln(1 - a + a\exp(-r)) < r$ it follows

$$
\begin{aligned}
&\chi_{12} \circ \chi_{24} \circ \chi_{46} \circ \chi_{61}(r) \\
&= -\ln\left(1 - \frac{c_{12}\alpha_2 + c_{13}\alpha_3}{(1-\varepsilon_{12})\alpha_1} \frac{c_{24}\alpha_4 + c_{25}\alpha_5}{(1-\varepsilon_{24})\alpha_2} \frac{c_{46}\alpha_6}{(1-\varepsilon_{46})\alpha_4} \frac{\| d \|_\infty + c_{61}\alpha_1}{(1-\varepsilon_{61})\alpha_6} (1 - \exp(-r))\right) \\
&< r, \quad r > 0.
\end{aligned}
$$

By similar calculations the following holds

$$
\begin{aligned}
&\chi_{12} \circ \chi_{24} \circ \chi_{46} \circ \chi_{61}(r) < r, \quad \chi_{13} \circ \chi_{34} \circ \chi_{46} \circ \chi_{61}(r) < r, \\
&\chi_{12} \circ \chi_{25} \circ \chi_{56} \circ \chi_{61}(r) < r, \quad \chi_{13} \circ \chi_{35} \circ \chi_{56} \circ \chi_{61}(r) < r,
\end{aligned}
\tag{16}
$$

for $r > 0$, such that the cycle condition and therefor the small gain condition is satisfied. We conclude that all subsystems are LISS or ISS, respectively, and we can apply Theorem 1, such that the whole system is LISS for all $x, x_0 \in \mathbb{R}_+^6$ and $\| d \|_\infty < \rho^u$ with additional conditions (12), (13) and (15).

5 Simulation results

To verify and demonstrate the results of the previous section we simulate all subsystems with help of Matlab.

At first we choose values for the parameters c_{ij}: $c_{61} = 0.0001, c_{12} = 4, c_{13} = 3,$ $c_{24} = 4, c_{34} = 9, c_{25} = 6, c_{35} = 2, c_{46} = 8, c_{56} = 4.$ Consider constant orders $d \equiv 20.$ Then the stability conditions (SC) (12), (13) and (15) become

$$
\begin{aligned}
&\alpha_1 > 4\,\alpha_2 + 3\,\alpha_3,\ \alpha_2 > 4\,\alpha_4 + 6\,\alpha_5,\ \alpha_3 > 9\,\alpha_4 + 2\,\alpha_5, \\
&\alpha_4 > 8\,\alpha_6, \qquad \alpha_5 > 4\,\alpha_6, \qquad \alpha_6 > 20 + 0.0001\,\alpha_1.
\end{aligned}
$$

By solving this system of linear inequalities we get the condition

$$\alpha := (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)^T > (9731.55, 1174.5, 1677.86, 167.79, 83.9, 20.98)^T$$

With the choice $\alpha = (9750, 1180, 1680, 169, 85, 21)^T$ and $x_0 = (1,1,1,1,1,1)^T$ the simulation results are presented in Figure 2, where the number of orders (No^i) of each subsystem for time t is displayed. We see, that all trajectories of the subsystems are bounded.

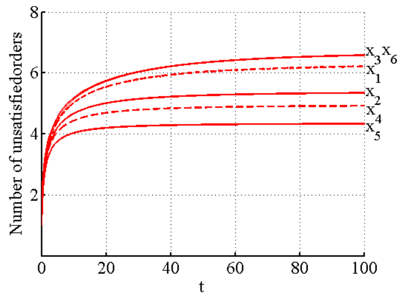

Fig. 2 No^i, if (SC) are satisfied

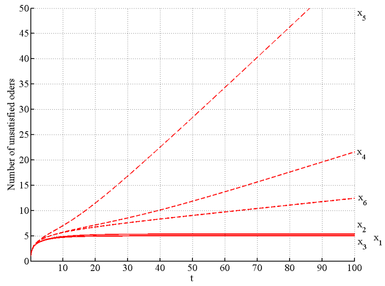

Fig. 3 No^i, if (SC) are not satisfied

Now we choose the maximum production rates only a bit smaller:

$$\alpha = (9730, 1174, 1677, 167, 83, 20.9)^T$$

The simulation results are displayed in Figure 3. We see that the trajectories of the subsystems 1 to 3 are bounded, but the trajectories of the subsystems 4 to 6 are unbounded, which means that the whole system is not stable.

By further simulations of the system we discover that for other inputs where $\| d \|_\infty < \rho^u$ is not satisfied, the system can be stable. We consider all values c_{ij} as before, choose the maximum production rates $\alpha = (9750, 1180, 1680, 169, 85, 21)^T$ such that conditions (12) and (13) are satisfied and replace d by $d(t) = 20 \cdot (\sin(t) + 1)$. It is $\| d \|_\infty = 40 > \rho^u$, but by simulation results, which are presented in Figures 4 and 5, all subsystems and therefor the whole system are stable.

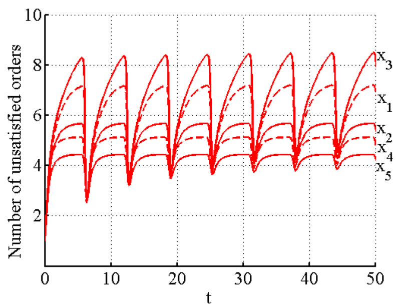

Fig. 4 Simulation results for x_1 to x_5 with d(t) = 20 (sin(t) + 1)

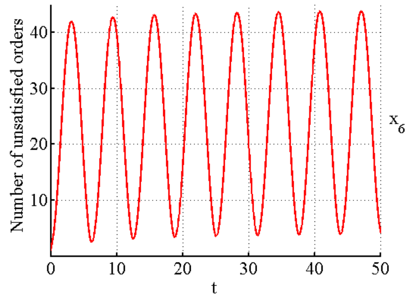

Fig. 5 Simulation results for x_6 with d(t) = 20 (sin(t) + 1)

This result is caused by the usage of the “worst case” within the mathematical theory, namely the supremum norm $\| \cdot \|_\infty$. In particular for oscillating inputs (e.g. seasonal changes of demand) the maximum value is used for all the time to derive stability conditions, such that lower inputs will not be considered over the time. Whereas in the Matlab simulation the actual input for time t is used, which is not the maximum value for all the time for an oscillating input and therefore lower stability conditions can be obtained. By mathematical theory used in this paper it is not possible to cover all inputs for which the system is stable, in particular oscillating inputs. This is an actual mathematical problem to find the domain of stability as large as possible.

6 Conclusions and outline

In this paper we have described a model for networks of autonomous production plants. This model was investigated on stability. In particular necessary conditions for its stable behavior were provided. This paper illustrates an approach for modelling and analysis of autonomous logistic systems, which can be transferred to other more complex logistical networks equivalently. By application of the stability analysis as presented here one can derive stability conditions to guarantee stability of the network and they help to design the network to avoid negative outcomes and to achieve economic goals.

For validation of the provided methods a comparison of the obtained results with simulations provided by discrete event simulation is of interest and is planned for the future research.

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