The Application of Diffusion in Image Processing

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Outline

Diffusion

Diffusion is a physical process that equilibrates concentration differences without creating or destroying mass.

Fick's law

$$j = -D \cdot \nabla u,$$

where D is a positiv definite symmetric matrix.

Continuity equation

$$\partial_t u = -\mathrm{div} j.$$

Then we have the diffusion equation:

$$\partial_t u = \mathrm{div}(D \cdot \nabla u).$$

With different diffusion tensor we can classify the diffusion process in three cases:

homogeneous: where D is an identity matrix.

isotropic: where D is an identity matrix and multiplied by a scalar-valued factor.

anisotropic: where D is a positive definite symmetric matrix.

Homogeneous diffusion equation

If $x \in \Omega \subset \mathbb{R}^2$, $t \in [0, T]$, then the heat equation as follows:

$$\begin{cases} u_t = \Delta u \quad \text{in } \Omega \times [0, T] \\ u_n = 0 \quad \text{on } \partial\Omega \times [0, T] \\ u(x, 0) = u_0(x). \end{cases}$$

The solution is given by

$$u(x, t) = \int_\Omega K_{\sqrt{2t}}(x - y) u_0(y) \mathrm{d}y = \left(K_{\sqrt{2t}} * u_0 \right)(x),$$

where $K_\rho(x)$ with $\rho = \sqrt{2t}$ denotes the two-dim. Gaussian kernel.

Homogeneous diffusion equation

$$K_\rho := \frac{1}{2\pi\rho^2} \exp\left(-\frac{|x|^2}{2\rho^2}\right).$$

Remarks

The variance of the gaussian kernel depends on the diffusion time.

Representation in frequency space

The Fourier transformation of u_0:

$$F[u_0](\omega) := \int_{\Omega} u_0(x)\exp(-i\omega x)\mathrm{d}x, \quad \omega \in \mathbb{R}^2.$$

According to convolution theorem we have:

$$F[K_\rho * u_0](\omega) = F[K_\rho](\omega)F[u_o](\omega).$$

And

$$F[K_\rho](\omega) = \exp\left(-\frac{|\omega|^2}{2/\rho^2}\right).$$

From this it follows that

$$F[K_\rho * u_0](\omega) = \exp\left(-\frac{|\omega|^2}{2/\rho^2}\right) F[u_0](\omega).$$

We redefine the diffusion tensor D by g. So the homogeneous diffusion equation is

$$u_t = \Delta u = \mathrm{div}(g \nabla u), \quad \text{where } g := 1.$$

Question:
How can we smooth the image without blurring the edges?

Isotropic diffusion equation

$$u_t = \operatorname{div}\left(g\left(|\nabla u|^2\right)\nabla u\right),$$

g denotes:

$$g(s^2) := \frac{1}{\sqrt{1 + s^2/\lambda^2}}.$$

Property:

In area where $|\nabla u|$ is smaller(flat regions), the diffusion becomes bigger. Then the noise will be eliminated. And in area where $|\nabla u|$ is bigger(edges), the diffusion becomes smaller. Then the edges will be preserved. The parameter λ controls how strong is the diffusion.

Isotropic diffusion equation

Disadvantage of isotropic diffusion:
It's due to the direction independency of diffusion because g is scalar valued. e.g. In a noisy image made of uncontinuous structure diffusion can not be occured.

In order to overcome this disadvantage we can use the anisotropic diffusion.

Anisotropic diffusion equation

$$u_t = div\left(g\left(\nabla u \nabla u^T\right)\nabla u\right).$$

The matrix $A := \nabla u \nabla u^T$ is positive semidefinite and

$$\begin{aligned} \lambda_1 &= |\nabla u|^2 \qquad v_1 \| \nabla u \\ \lambda_2 &= 0 \qquad\qquad v_2 \perp v_1. \end{aligned}$$

And

$$A = S\mathrm{diag}(\lambda_i)S^T,$$

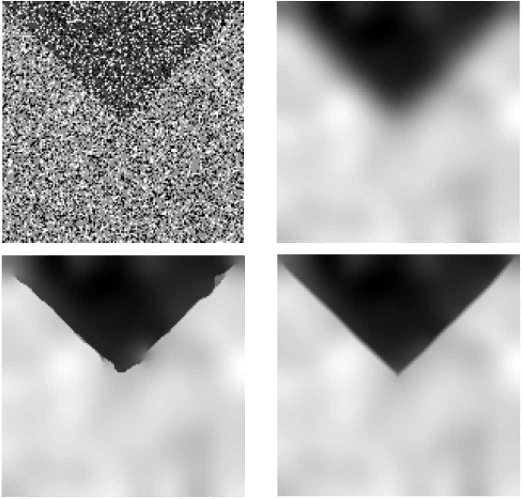
where $\mathrm{diag}(\lambda_i)$ is the diagonal matrix with diagonal elements λ_i and S is made up of the normalized eigenvectors.

Anisotropic diffusion equation

$$g(A) := S\mathrm{diag}(g(\lambda_i))S^T = \sum_{i=1}^{2} g(\lambda_i)s_i s_i^T.$$

Property:

- In flat regions $\lambda_1 = \lambda_2 = 0$, then follows $g(\lambda_1) = g(\lambda_2) = 1$. In this case the diffusion tensor $g(A) = I$.
- In edges region $\lambda_1 >> \lambda_2 = 0$, then follows $g(\lambda_1) \approx 0$ and $g(\lambda_2) = 1$. In this case the diffusion tensor $g(A) \approx s_2 s_2^T$. It means the smooth across edges is forbidden and the smooth along the edges is allowed.



Structuretensor

Definition

Structuretensor is a matrixfeld, which includes in each matrix element the information about orientation and intensity of surrounding structure.

If $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, the initial structuretensor of u is the tensorproduct

$$J_0 = \nabla_n u \nabla_n u^T,$$

where $\nabla_n := (\partial_{x_1}, \cdots, \partial_{x_n})^T$.

Linear structuretensor

The linear structuretensor is

$$J_\rho = K_\rho * \left(\nabla_n u \nabla_n u^T \right).$$

Property:

Linear structuretensor has some important information by convolved with surrounding structure. e.g. In two-dim. image is the eigenvector of the biggest eigenvalue the orientation with the highest grey value fluctuation. $\mathrm{tr} J_p$ determines the intensity. The coherence is determined by $(\lambda_1 - \lambda_2)^2$.

Linear structuretensor

According to construction of the homogeneous diffusion we are clear that J_ρ is equivalent to linear matrix valued homogeneous diffusion process:

$$\begin{cases} \partial_t u_{ij} = \Delta u_{ij} \\ (u_{ij})_n = 0 \\ u_{ij}(x,0) = \left(\nabla_n u_0 \nabla_n u_0^T\right)_{ij}(x), \end{cases}$$

in stoptime $t = \frac{1}{2}\rho^2$, where $i, j = 1, \cdots, n$.

Nonlinear structuretensor

We can have the nonlinear structuretensor by matrix valued nonlinear diffusion equations with the same boundary condition and initial value.

isotropic:

$$\partial_t u_{ij} = \operatorname{div}\left(g\left(\sum_{k,l=1}^{n} |\nabla u_{kl}|^2 \right) \nabla u_{ij} \right),$$

$$i, j = 1, \cdots, n.$$

anisotropic:

$$\partial_t u_{ij} = \operatorname{div}\left(g\left(\sum_{k,l=1}^{n} \nabla u_{kl} \nabla u_{kl}^T \right) \nabla u_{ij} \right),$$

$$i, j = 1, \cdots, n.$$