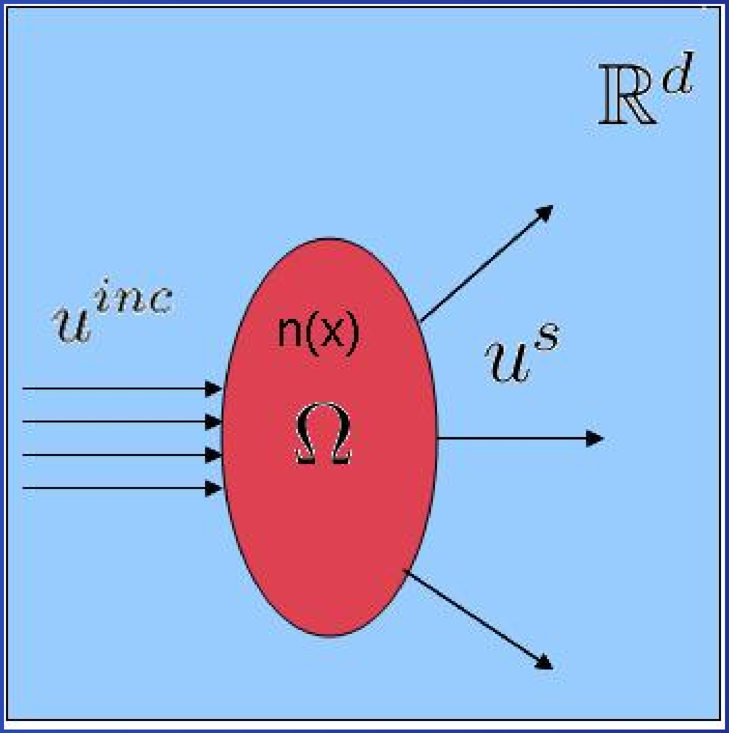
The MUSIC method and the factorization method in an inverse scattering problem

Pham Quy Muoi

Model of the problem



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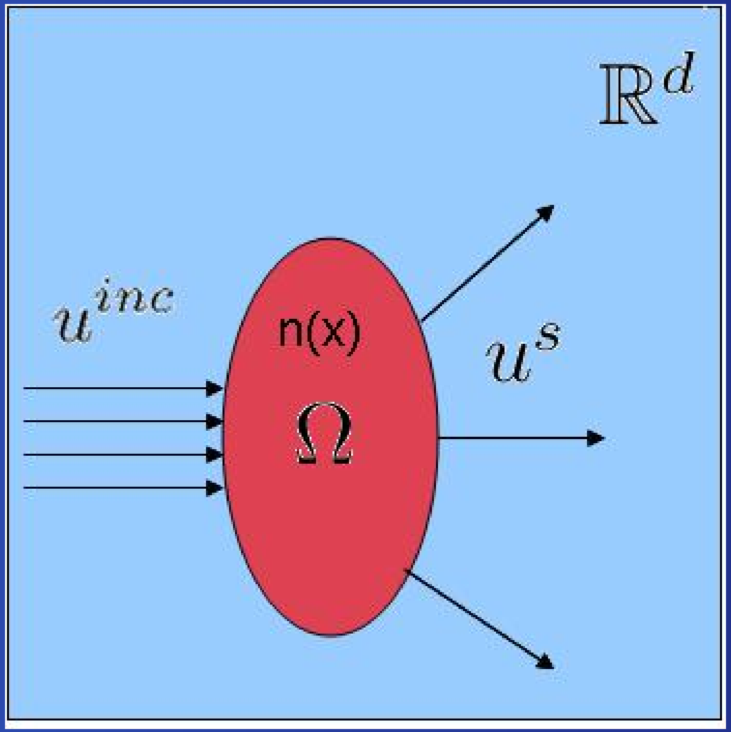

The func. n : Re $n \geqslant 0$, Im $n \geqslant 0$ and $n = 1$ in $\mathbb{R}^d \backslash \Omega, d = 2, 3.$

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Forward problem. Giving n, u^{inc}, we find the solution of (I), (II).

Inverse problem. Giving some information of the solution u (u^∞), determine Ω.

Some well-known results

- The forward problem has unique solution and the solution of the problem is equivalent to the solution of the Lippmann - Schwinger integral equation:

$$u(x) - k^2 \int_\Omega q(y)u(y)\Phi(x,y)dy = u^{inc}(x), x \in \overline{\Omega}.$$

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- About inverse problem
  - In $\mathbb{R}^3$, Giving u^∞, Ω is determined uniquely.
  - There are some algorithms to determine Ω such as iterative methods, the linear sampling method and the factorization method.

Some well-known results

- The factorization method (FM)
  - In 1998, A. Kirsch introduce the FM to determine Ω in a scattering inverse problem.
  - In 2002, Grinberg applied this method for some scattering inverse problems.

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The MUSIC method

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- $\Phi(x, y) = \gamma_d \frac{exp(ikx)}{|x|^{(d-1)/2}} e^{-ik\hat{x}.y} + O\left(|x|^{-(d+1)/2}\right), |x| \to \infty$

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- **Inverse problem:** to determine the locations of scatterers $y_1,\ldots,y_M$ from $u^\infty(\hat{x},\hat{\theta}), \forall \hat{x},\hat{\theta} \in \mathbb{S}^{d-1}$ or $u^\infty(\hat{\theta}_i,\hat{\theta}_j), i,j = 1\ldots N$.

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- $F = STS^*$ and $R(S) = R(F)$. (1.1)
- For $z \in \mathbb{R}^d$, we define the vector $\Phi_z \in \mathbb{C}^N$ by $\Phi_z = (e^{-ik\hat{\theta}_1.z}, e^{-ik\hat{\theta}_2.z},\ldots, e^{-ik\hat{\theta}_N.z})$

The MUSIC method

- **Theorem 1.1.** Let $\{\hat{\theta}_n : n \in \mathbb{N}\} \subset \mathbb{S}^{d-1}$ with the property that any analytic function which vanishes in $\hat{\theta}_n, \forall n \in \mathbb{N}$ vanishes identically. Then there exists $N_0 \in \mathbb{N}$ such that for any $N \geqslant N_0$ the characterization holds
  $z \in \{y_1, y_2, \dots, y_M\} \Leftrightarrow \Phi_z \in R(S).$
  From (1.1) we have
  $z \in \{y_1, y_2, \dots, y_M\} \Leftrightarrow \Phi_z \in R(F) \Leftrightarrow P\Phi_z = 0$
  with $P : \mathbb{C}^N \to R(F)^\perp$ is the orthogonal projection.

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- **Example.** $d = 2, M = 2, N = 10, k = 2\pi$ and $\hat{\theta}_j, j = 1, \ldots, 10$, are equidistantly chosen directions. The values of t are $1 + i, 1.5 + i$ at $(-1, 1), (-1/2, -1)$,respectively. The plots of $W(z)$ give by

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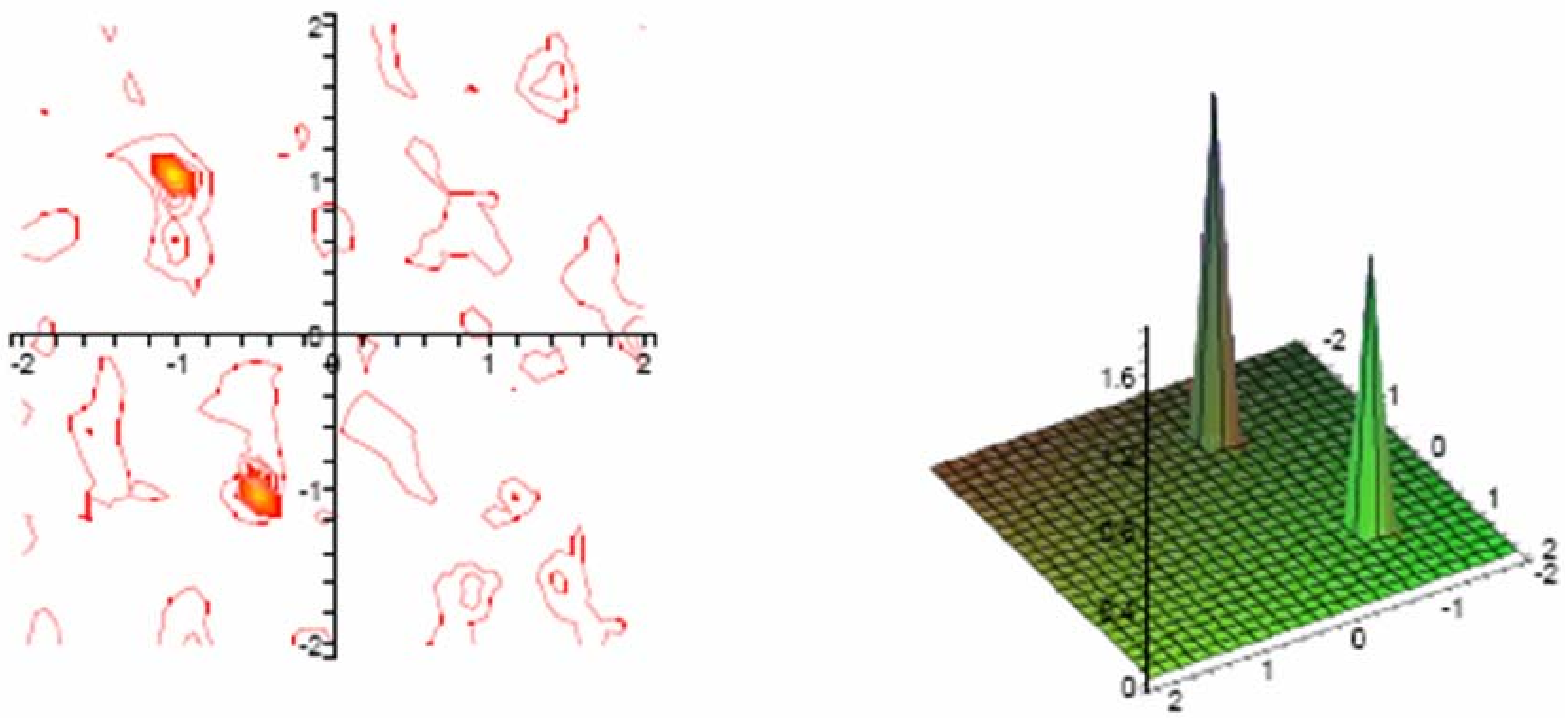

Hình 1.1:

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- Secondly, we define a function Φ_z such that $z \in \Omega \Leftrightarrow \Phi_z \in R(S)$.
- Finally, we find an operator F' that only depend on F such that $R(F') = R(S)$.

The factorization method

Forward Problem. Let $\Omega \subset \mathbb{R}^d$: bounded, open set and its complement is connected; $n = 1 + q, q \in L^\infty(\Omega)$, $u^{inc} = e^{ik\hat{\theta}.x}, x \in \mathbb{R}^d$.

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$$\frac{\partial u^s}{\partial n} - iku^s = O(r^{-(d+1)/2}), r = |x| \to \infty$$

The factorization method

- The problem is equivalent to the equation

$$u(x)-k^2\int_\Omega q(y)u(y)\Phi(x,y)dy = u^{inc}(x), x\in\overline{\Omega} \quad (2.2$$

or $u - Lu = u^{inc}$ with

$$Lu(x) = k^2\int_\Omega q(y)u(y)\Phi(x,y)dy, x\in\overline{\Omega}.$$

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- $u^s(x) = k^2 \int_\Omega q(y)u(y)\Phi(x,y)dy, x \in \mathbb{R}^d.$
- $u^\infty(\hat{x}, \hat{\theta}) = k^2 \int_\Omega q(y)u(y,\hat{\theta})e^{-ik\hat{x}.y}dy.$

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  $F\psi(\hat{x}) = \int_{\mathbb{S}^{d-1}} u^\infty(\hat{x},\hat{\theta})\psi(\hat{\theta})ds(\hat{\theta}), \hat{x} \in \mathbb{S}^{d-1}$
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  $S\phi(\hat{x}) = \int_\Omega e^{-ik\hat{x}.y}\phi(y)dy$
  $\Rightarrow S^*\psi(y) = \int_{\mathbb{S}^{d-1}} e^{ik\hat{x}.y}\psi(\hat{x})ds(\hat{x})$.

The factorization method

- **Theorem 2.1.** We have

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- **Theorem 2.2.** For any $z \in \mathbb{R}^d$, we have

$$z \in \Omega \Leftrightarrow \Phi_z \in R(S).$$

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The factorization method

- **Theorem 2.3.** Let $q \in L^\infty(\Omega)$ such that there exist $q_0 > 0$ with $Req(x) \geqslant q_0$ and $Imq(x) \geqslant 0$ for all most $x \in \Omega$. Furthermore, let k^2 be not eigenvalue of interior transmission problem. Then for any $z \in \mathbb{R}^d$:

$$z \in \Omega \Leftrightarrow \Phi_z \in R(F_\sharp^{1/2})$$

and $F_\sharp = |ReF| + ImF$ is positive op..

The factorization method

- **Theorem 2.4.** Let $q \in L^\infty(\Omega)$ such that there exists $q_0 > 0$ with $Imq(x) \geqslant q_0$ for all most $x \in \Omega$. Then for any $z \in \mathbb{R}^d$

$$z \in \Omega \Leftrightarrow \Phi_z \in R(F_\sharp^{1/2})$$

with $F_\sharp = ImF$.

Some examples

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- solving the forward problem by the integral equation method (presented by Vainiko) with $G = [-2, -2] \times [-2, 2]$. Then computing $u^\infty(x_i, x_j), x_i \in \mathbb{S}^{d-1}, i, j = 1, \ldots, 16$ corresponding to $M = 16$ equidistantly chosen points on unit circle and $k = 1$.

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- computing $F = [u^{\infty}(x_i, x_j)], F_{\sharp} = ImF$ and $F_{\sharp} = |ReF| + ImF$.

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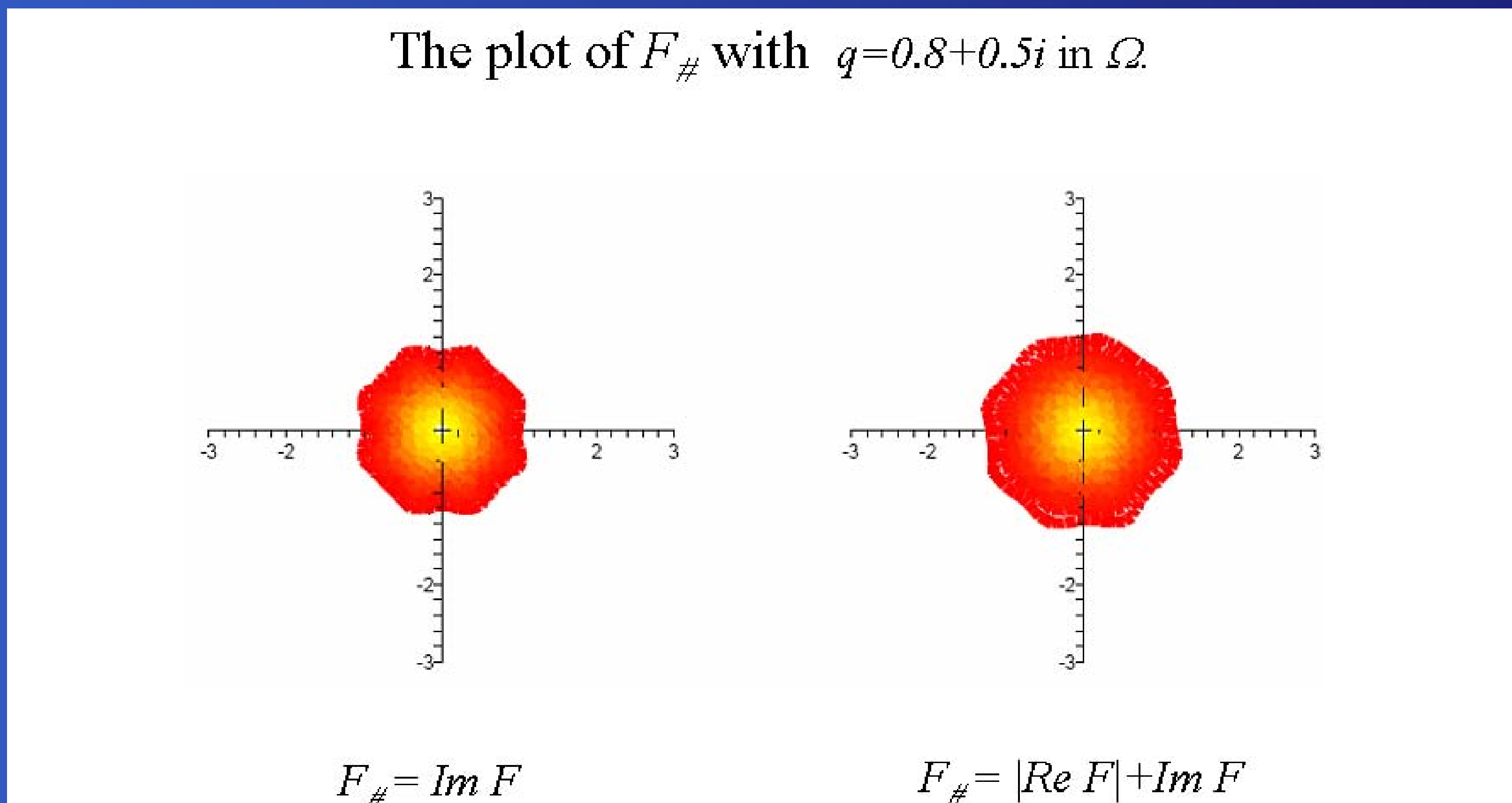
- Computing an eigensystem $\{(\lambda_i, U_i) : i = 1, \ldots, M\}$ of $F_\sharp$.
- Defining the function

$$W(z) = \left(\sum_{\lambda_i \geqslant 0.001} \frac{| < \Phi_z, U_i > |^2}{\lambda_i} \right)^{-1}$$

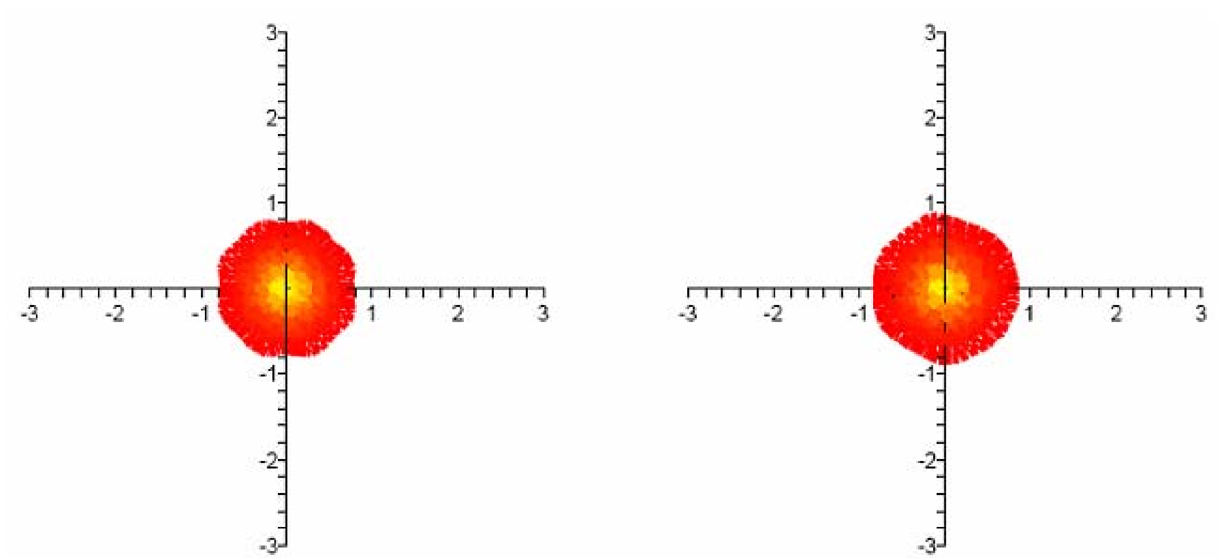
with $\Phi_z = (e^{-ikx_1.z}, \ldots, e^{-ikx_M.z})$.
Then we expect that the value of $W(z)$ is much greater for $z \in \Omega$ than for $z \notin \Omega$.

Example 1: The graph of $W(z)$



Example 2: The plots of $W(z)$

The plot of $F_{\#}$ with q=0.8 in Ω.



$F_{\#} = Im\ F$ $\quad$ $F_{\#} = |Re\ F| + Im\ F$


Thank you for your
attention