Optimal control governed by transport equations

Kanglin Chen

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Outline

What is the optimal control?

A control problem includes a cost functional that is a function of state and control variables. An optimal control is a set of differential equations describing the control variables that minimize the cost functional.

Purpose: find an optimality system of the control variables.

Optimal control for optical flow estimation

Given two images u_0, u_T and estimate the "best"optical flow b between them in the sense of the cost functional

$$\min_b J(b) = \frac{1}{2} \|S(u_0, b) - u_T\|^2 + \frac{\lambda}{2} \|b\|^2$$

governed by the transport equation

$$\begin{cases} \partial_t u(t,x) + b(t,x) \cdot \nabla_x u(t,x) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

where $S = E_T \circ G, \quad S : (u_0, b) \to u(T).$

Theoretical Analysis

- The function spaces of u, b.
- The solution theory of the transport equation.
- The wellposedness of the cost funcitonal.

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Maybe next time!...

Solution theory of the transport equation

Assume u_0, b smooth, then

$$u(t, \Phi(t, x)) = u_0(x).$$

It means that the intensity u is constant along the characteristic lines. The flow Φ fulfills the ODE

$$\begin{cases} \dfrac{\partial \Phi}{\partial t}(t, x) = b(t, \Phi(t, x)) \\ \\ \Phi(0, x) = x. \end{cases}$$

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Hence, $u(t, x) = u_0 \circ \Phi^{-1}(t, \cdot)(x)$.

Solution theory of the continuity equation

Assume p, b smooth.

$$\begin{cases} \partial_t p(t,x) + \nabla \cdot (b(t,x)p(t,x)) = 0 \\ p(0,x) = p_0(x) \end{cases}$$

If b is divergence free, then continuity equation is equal to the transport euqation. The solution is

$$p(t, \Phi(t,x)) \det(\nabla \Phi(t,x)) = p_0(x).$$

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Hence, $\quad p(t,x) = \dfrac{p_0 \circ \Phi^{-1}(t,\cdot)(x)}{\det(\nabla\Phi(t,x))}.$

The Jacobian determinant

The Jacobian $J(t,x) := \det(\nabla\Phi(t,x))$ satisfies the ODE

$$\begin{cases} \frac{\partial J}{\partial t}(t,x) = (\mathrm{div} b)(t,\Phi(t,x))J(t,x) \\ J(0,x) = 1. \end{cases}$$

The solution is given by

$$J(t,x) = exp(\int_0^t \nabla \cdot b(t,\Phi(t,s))ds).$$

It means that the Jacobian determinant is positive and equal to 1 if b is divergence free.

The necessary optimality conditions system

We use the Lagrangian technique to gain the system. Consider the functional

$$L(u, b, p) = \frac{1}{2}\|S(u_0, b) - u_T\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\nabla b\|^2_{L^2([0,T];L^2(\Omega))} + \int_0^T \int_\Omega (u_t + b \cdot \nabla u) p dx dt.$$

We call p the adjoint state. The necessary conditions of minimizing the functional are that the functional derivatives of L by u, b are equal to zeros.

Necessary optimality conditions system

$$
\begin{cases}
u_t + b \cdot \nabla u = 0 & u(0) = u_0 \\
p_t + \nabla \cdot (bp) = 0 & p(T) = -(u^T - u_T) \\
\lambda \Delta v = u_x p & \partial v / \partial n = 0 \text{ on } \partial\Omega \\
\lambda \Delta w = u_y p & \partial w / \partial n = 0 \text{ on } \partial\Omega
\end{cases}
$$

where $b = (v, w)$.

Total variation diminishing (TVD)

The TVD schemes are based on a concept aimed at preventing the generation of new extrema in the solution. Consider a scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0.$$

A numerical method is said to be total variation diminishing (TVD) if,

$$TV(u^{n+1}) \leq TV(u^n), \text{ where } TV(u^n) = \sum_i |u^n_{i+1} - u^n_i|.$$

It is proved that a difference scheme

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} + \frac{f_{i+1/2} - f_{i-1/2}}{\Delta x} = 0$$

Total variation diminishing (TVD)

is a TVD scheme provided that it should be in the form

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \underbrace{c_{i-1/2}}_{\geq 0}(u_{i-1} - u_i) + \underbrace{c_{i+1/2}}_{\geq 0}(u_{i+1} - u_i)$$

E.g. the first order upwind scheme for the transport equation is a TVD scheme.

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}[\max(a,0)(u_i^n - u_{i-1}^n) + \min(a,0)(u_{i+1}^n - u_i^n)]$$

Drawback: linear TVD scheme are at most first-order accurate.
Way out: use the flux limiter for the non-monotone part in the higher-order schemes.

Total variation diminishing (TVD)

Let us consider a second order difference for $u_t + bu_x = 0$ with $b > 0$

$$\begin{aligned}
\frac{\mathrm{d}u_i}{\mathrm{d}t} &= -\frac{b}{2\Delta x}(3u_i - 4u_{i-1} + u_{i-2}) \\
&= \underbrace{-\frac{b}{\Delta x}(u_i - u_{i-1})}_{\text{First order monotone upwind scheme}} \quad -\frac{b}{\Delta x}\left[\frac{1}{2}(u_i - u_{i-1}) - \frac{1}{2}(u_{i-1} - u_{i-2})\right]
\end{aligned}$$

Multiply the two non-monotone terms by $\Psi(r_i)$ and $\Psi(r_{i-1})$, where

$$r_{i-1} = \frac{u_i - u_{i-1}}{u_i - u_{i-1}} \quad r_i = \frac{u_{i+1} - u_i}{u_i - u_{i-1}}$$

leading to

Total variation diminishing (TVD)

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{b}{\Delta x}\left[1 + \frac{1}{2}\Psi(r_i) - \frac{1}{2}\frac{\Psi(r_{i-1})}{r_{i-1}}\right](u_i - u_{i-1})$$

where the limiter defined by

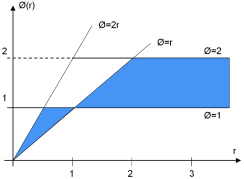
$$\Psi(r) = (r + |r|)/(1 + r)$$

called Van Leer's limiter or

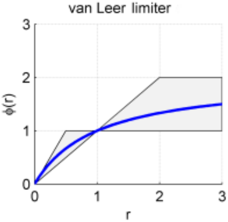
$$\Psi(r) = \max[0, \min(2r, 1), \min(r, 2)]$$

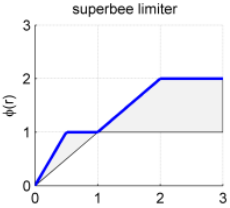
under the nickname of superbee and has excellent resolution properties for jump discontinuities.

Total variation diminishing (TVD)



Admissible limiter region for second-order TVD schemes
(Sweby, 1984)





Runge-Kutta 4th order for ODE

$$\Phi^{'} = b(t, \Phi) \text{ with } \Phi(t_0, x) = x$$

The numerical scheme of RK4

$$\begin{aligned}
&\Phi_{n+1} = \Phi_n + \tfrac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \\
&t_{n+1} = t_n + h \\
&k_1 = b(t_n, \Phi_n) \\
&k_2 = b(t_n + \tfrac{1}{2}h, \Phi_n + \tfrac{1}{2}hk_1) \\
&k_3 = b(t_n + \tfrac{1}{2}h, \Phi_n + \tfrac{1}{2}hk_2) \\
&k_4 = b(t_n + h, \Phi_n + hk_3)
\end{aligned}$$

Solver for poission equations

$$\Delta u = f, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

We use the 2D-Laplace filter to discretize the Laplace operator:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 \\ 2 & -12 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Fitting on the Neumann's boundary condition we can use the mirror reflections of itself on the boundary elements of the image u.

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- Gauss-Seidel method
  initialise the $(u_{ij}^0) = 0$.

$$u_{ij}^{n+1} = (u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j-1}^n + u_{i,j+1}^n - f_{ij}h^2)/4$$

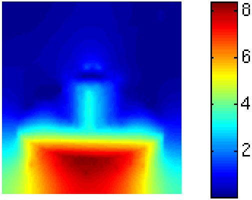
for $1 \leq i \leq M, 1 \leq j \leq N$.

Filter for Laplace operator

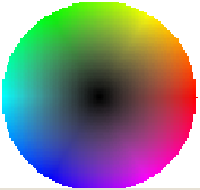
Numerical Results

The color plot of the velocity field.

The absolute value of the flow.



The reference color map.



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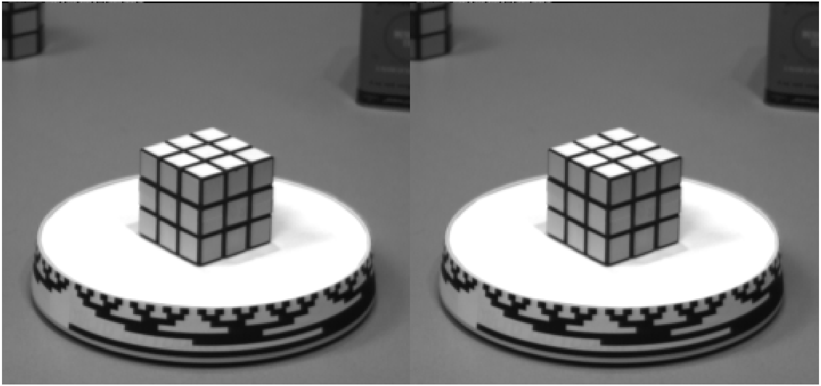

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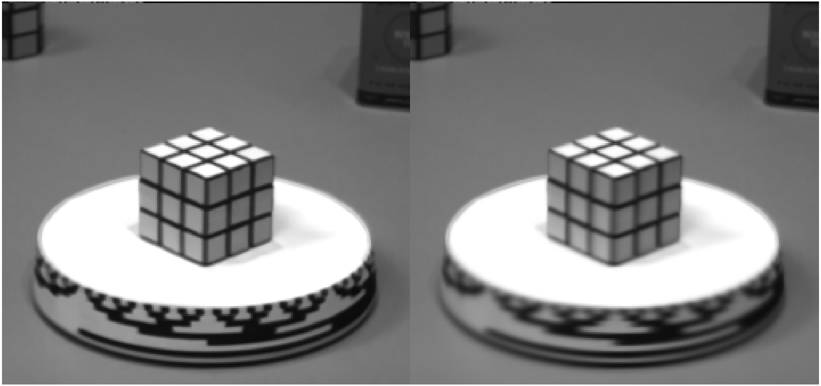
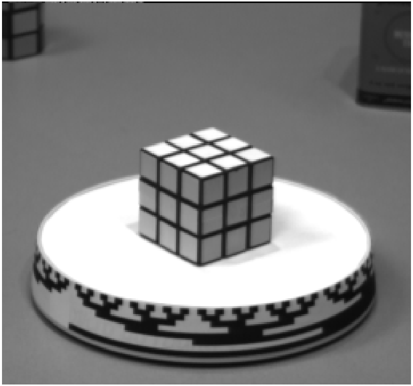


Optical flow estimated by the optimal control



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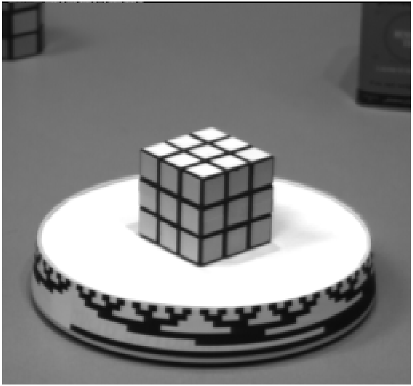

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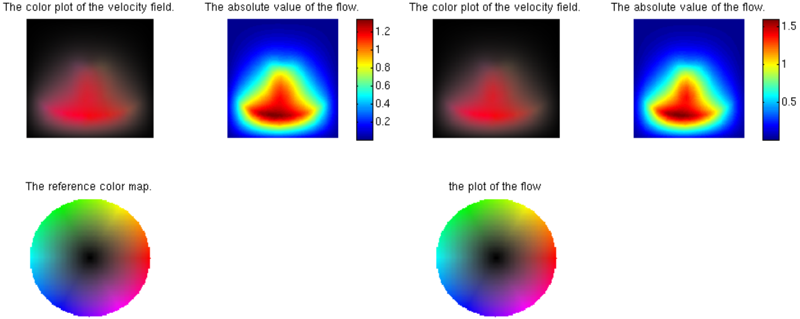

- Left: Results of the optimality system using TVD for transport & continuity equations.
- Right: Results of the optimality system using characteristics for transport & continuity equations.

Thank you for your attention!