# Introduction to the Finite Element Method

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# Outline

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- Finite Element Method (FEM)
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# Motivation

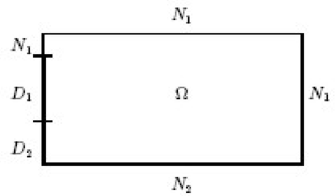


Figure: cross section of the room

(cf. A. Jüngel, Das kleine Finite-Elemente-Skript)

Situation:

- $\Omega \subset \mathbb{R}^2$ - room
- $D_1$ - window
- $D_2$ - heating
- $N_1$ - isolated walls, ceiling
- $N_2$ - totally isolated floor
- $\theta$ - temperature

# Motivation



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- Conservation of energy:

$$\int_0^T \int_\Omega \rho_0 c_e \frac{\partial \theta}{\partial t} \, dx \, dt = -\int_0^T \int_{\partial\Omega} \kappa \frac{\partial \theta}{\partial \nu} \, d\sigma_x \, dt + \int_0^T \int_\Omega f \, dx \, dt$$

- Heat equation:

$$\rho_0 c_e \frac{\partial \theta}{\partial t} - \mathrm{div}(\kappa \nabla \theta) = f \text{ in } \Omega \text{ for } t > 0$$

- Assumptions:
  - no time rate of change of the temperature, i.e. $\frac{\partial \theta}{\partial t} = 0$
  - no interior heat source/sink, i.e. $f = 0$
  - $\kappa = 1$

# Motivation

Model:

$$\begin{aligned} \Delta\theta &= 0 \text{ in } \Omega \\ \theta &= \theta_W \text{ on } D_1 \\ \theta &= \theta_H \text{ on } D_2 \\ \nabla\theta\cdot\nu &= 0 \text{ on } N_2 \\ \nabla\theta\cdot\nu + \alpha(\theta-\theta_W) &= 0 \text{ on } N_1 \end{aligned}$$

Example:

- $\theta_W = 10^\circ C$
- $\theta_H = 70^\circ C$
- $\alpha = 0.05$

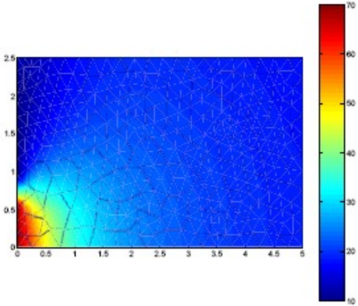


Figure: temperature distribution in the heated room

(cf. A. Jüngel, Das kleine Finite-Elemente-Skript)

# Partial Differential Equations (PDEs)



Second-order PDEs

- Elliptic PDE (stationary) e.g. Poisson Equation (scalar)

$$-\Delta u = f \text{ in } \Omega$$

or stationary elasticity (vector-valued)

$$-\operatorname{div}(\boldsymbol{\sigma}) = \mathbf{f} \text{ in } \Omega$$

- Parabolic PDE e.g. heat equation

$$\theta' - \operatorname{div}(\kappa \nabla \theta) = f \text{ in } \Omega \times (0, T)$$

- Hyperbolic PDE e.g. instationary elasticity

$$\mathbf{u}'' - \operatorname{div}(\boldsymbol{\sigma}) = \mathbf{f} \text{ in } \Omega \times (0, T)$$

# Classification of PDEs



- Linear PDE

$$\theta' - \Delta\theta = f \text{ in } \Omega \times (0, T)$$

- Semilinear PDE

$$\theta' - \Delta\theta = f(\theta) \text{ in } \Omega \times (0, T)$$

- Quasilinear PDE

$$\theta' - \operatorname{div}(\alpha(\nabla\theta)) = f \text{ in } \Omega \times (0, T)$$

- Fully nonlinear PDE

$$\theta' - g(\Delta\theta) = f \text{ in } \Omega \times (0, T)$$

# Boundary Conditions (BCs)

- Dirichlet BC (first kind, essential BC)

$$u = g \text{ on } \partial\Omega$$

- Neumann BC (second kind, natural BC)

$$\nabla u \cdot \nu = \frac{\partial u}{\partial \nu} = g \text{ on } \partial\Omega$$

- Robin BC (Cauchy BC, third kind)

$$\frac{\partial u}{\partial \nu} + \sigma u = g \text{ on } \partial\Omega$$

# Methods for solving PDEs



- Analytical Methods for PDEs / Existence and Uniqueness
  - Method of Separation of Variables
  - Method of Eigenfunction Expansion
  - Method of Diagonalisation (Fourier Transformation)
  - Method of Laplace Transformation
  - Method of Green's Functions
  - Method of Characteristics
  - Method of Semigroups
  - Variational Methods (e.g. Galerkin Approximation)

# Methods for solving PDEs



- Numerical Methods for PDEs
  - Finite Difference Method (FDM)
    - pointwise approximation of the differential equation
    - geometry is divided into an orthogonal grid
  - Finite Element Method (FEM)
    - powerful computational technique for the solution of differential and integral equations that arise in various fields of engineering and applied sciences
    - differential equations will be solved with an equivalent variation problem
    - geometry must be divided into small elements
    - problem is solved by choosing basis functions which are supposed to approximate the problem

# FDM



- Consider

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

- Idea:
  - approximate differential quotients by difference quotients
  - reduce differential equation to algebraic system
- Assumptions:
  - $\Omega = (0,1)^2$
  - equidistant nodes $(x_i, y_j) \in \Omega$ $(i, j = 0, \dots, N)$ with $h = x_{i+1} - x_i = y_{i+1} - y_i$
- Taylor Expansion

$$u(x_{i+1}, y_j) = u(x_i, y_j) + \frac{\partial u}{\partial x}(x_i, y_j)h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x_i, y_j)h^2 + \mathcal{O}(h^3)$$

$$u(x_{i-1}, y_j) = u(x_i, y_j) - \frac{\partial u}{\partial x}(x_i, y_j)h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x_i, y_j)h^2 + \mathcal{O}(h^3)$$

# FDM



- Second-order centered difference

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}\left(u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)\right) + \mathcal{O}(h)$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{h^2}\left(u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})\right) + \mathcal{O}(h)$$

- Approximation of $\Delta u$

$$\Delta u(x_i, y_j) \approx \frac{1}{h^2}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}\right)$$

- Find $u_{ij} = u(x_i, y_j)$ s.t.

$$-u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + 4u_{ij} = h^2 f_{ij},\ (x_i, y_j) \in \Omega$$
$$u_{ij} = 0,\ \ (x_i, y_j) \in \partial\Omega$$

whereas $f_{ij} = f(x_i, y_j)$

# FDM



- Linear system of equations:

$$U_k := u_{ij},\; F_k := f_{ij} \text{ with } k = iN + j \text{ leads to } AU = F$$

$$A = \frac{1}{h^2}\begin{pmatrix} A_0 & I & & & 0 \\ I & A_0 & I & & \\ & \ddots & \ddots & \ddots & \\ & & I & A_0 & I \\ 0 & & & I & A_0 \end{pmatrix}, \quad A_0 = \frac{1}{h^2}\begin{pmatrix} 4 & -1 & & & 0 \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ 0 & & & -1 & 4 \end{pmatrix}$$

- Disadvantages of FDM
  - complex (or changing) geometries and BCs
  - existence of third derivatives
  - $f$ not continuous

# FEM



- Consider

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

- Trick: transform PDE into equivalent variational form
- Multiplication with arbitrary $v \in X$ and integration over $\Omega$

$$\begin{aligned}\int_\Omega fv\,dx &= -\int_\Omega \mathrm{div}(\nabla u)v\,dx \\ &= -\underbrace{\int_{\partial\Omega} v\nabla u \cdot \nu\,dx}_{=0} + \int_\Omega \nabla u \nabla v\,dx\end{aligned}$$

- Find $u \in X$: $\forall\, v \in X$

$$\int_\Omega \nabla u \nabla v\,dx = \int_\Omega fv\,dx$$

# FEM



- Approximation: Find solution of a finite dimensional problem
- Let $(X_h)_{h\geq 0}$ a sequence of finite dimensional spaces with $X_h \to X$ $(h \to 0)$ and elements of $X_h$ vanish on $\partial\Omega$
- Find $u_h \in X_h$ s.t. $\forall\, v \in X_h$

$$\int_\Omega \nabla u_h \nabla v\, dx = \int_\Omega fv\, dx$$

- Let $\{\varphi_i\}_{i=1,\dots,N}$ a basis of $X_h$. The ansatz $u_h(x) = \sum_{i=1}^N y_i \varphi_i(x)$ and the choice $v = \varphi_j$ lead to

$$\sum_{i=1}^N y_i \underbrace{\int_\Omega \nabla\varphi_i \nabla\varphi_j\, dx}_{=:A_{ij}} = \underbrace{\int_\Omega f\varphi_j\, dx}_{=:F_j},\ j = 1, \dots, N$$

# FEM



- Linear system of equations:

$$\sum_{i=1}^{N} A_{ij} y_i = F_j, \; j = 1, \ldots, N$$

- Notation:
  - $A$ - stiffness matrix
  - $F$ - force vector
  - $y_i = u_h(x_i)$ - solution vector
- Questions: What about $X$, $X_h$, $\{\varphi_i\}_{i=1,\ldots,N}$?
- Hint: choose basis s.t. as much as possible $A_{ij} = 0$!
  ($A$ less costly to form, $Ay = F$ can be solved more efficiently)

# FEM



- Idea: discretise the domain into finite elements and define basis functions which vansih on most of these elements
  - 1D: interval
  - 2D: triangular/quadrilateral shape
  - 3D: tetrahedral, hexahedral forms
- Ansatz functions:
  - support of basis functions as small as possible and number of basis functions whose supports intersect as small as possible
  - use of piecewise (images of) polynomials

# FEM



- Example:
  - $\Omega \subset \mathbb{R}^2$ bounded Lipschitz domain, $f \in L^2(\Omega)$
  - $X = H_0^1(\Omega)$
  - Triangulation of $\Omega$ by subdividing $\Omega$ into a set $\mathcal{T}_h = \{K_1, \dots, K_n\}$ of non-overlapping triangles $K_i$ s.t. no vertex of a triangle lies on the edge of another triangle
  - $\Omega = \cup_{K \in \mathcal{T}_h} K$
  - Mesh parameter $h = \max_{K \in \mathcal{T}_h} \operatorname{diam}(K)$
  - $X_h := \{u_h \in C(\overline{\Omega}, \mathbb{R}) \colon\ u_h \text{ piecewise linear, } u_h = 0 \text{ on } \partial\Omega\}$
  - Linear elements in 1D:

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & , \quad x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & , \quad x_{i-1} \leq x \leq x_i \\ 0 & , \quad \text{otherwise} \end{cases} \quad , \; i = 1, \dots, N-1$$

# FEM



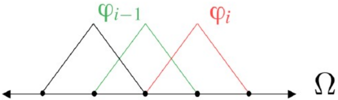


Figure: basis of 1D linear finite elements (cf. T. M. Wagner, A very short introduction to the FEM)

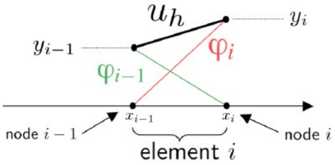


Figure: linear finite elements in 1D (cf. T. M. Wagner, A very short introduction to the FEM)

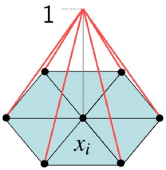


Figure: basis function of 2D linear finite elements (cf. T. M. Wagner, A very short introduction to the FEM)

- Linear or high-order elements?
  - Advantages: small error, better approximation, fast error convergence, less computing time for same error
  - Disadvantages: larger matrix for same grid, no conservation of algebraic sign

# FEM



- Matrix $A$ is large, but sparse: only a few matrix elements are not equal to zero
  (intersection of the support of basis function is mostly empty)
- $A$ symmetric, positive definit $\rightsquigarrow$ unique solution
- Linear system of equations: many methods in numerical linear algebra exist to solve linear systems of equations
  - direct solvers (Gaussian elemination, LU decomposition, Cholesky decomposition): for $N \times N$ matrix $\approx N^3$ operations
  - iterative solvers (CG, GMRES, ...): $\approx N$ operations for each iteration
- Runge-Kutta methods (ODEs for unsteady problems)

# FEM



- Standard Error Estimation ($\mathbb{P}_k$-elements, $u$ sufficiently smooth):

$$\left(\int_\Omega |u - u_h|^2 \, dx\right)^{\frac{1}{2}} \leq c\, h^{k+1} \left(\int_\Omega |D^{k+1} u|^2 \, dx\right)^{\frac{1}{2}}$$

$$\left(\int_\Omega |\nabla u - \nabla u_h|^2 \, dx\right)^{\frac{1}{2}} \leq c\, h^{k} \left(\int_\Omega |D^{k+1} u|^2 \, dx\right)^{\frac{1}{2}}$$

- Consistency: exact solution solves approximate problem but for error that vanishes as $h \longrightarrow 0$
- Stability: errors remain bounded as $h \longrightarrow 0$
- Convergence: approximate solution must converge to a solution of the original problem for $h \longrightarrow 0$
- suitable for adaptive method

# Model Algorithm of the FEM



1. Transformation of the given PDE via the variational principle
2. Selection of a finite element type
3. Discretization of the domain of interest into elements
4. Derivation of the basis from the discretisation and the chosen ansatz function
5. Calculation of the stiffness matrix and the right-hand side
6. Solution of the linear system of equations
7. Obtainment (and visualisation) of the approximation

- Software: ALBERTA, COMSOL, MATLAB, SYSWELD

# References




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Thank you for your attention.