Outcomes of the diploma thesis

Analysis of the mathematical problem of linear thermo-elasticity taking into account phase transitions and transformation-induced plasticity

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Outline

Problem

Unrequested distortion of steel workpieces during heat treatment

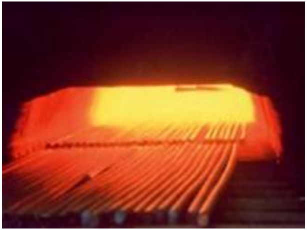


Figure: entering the furnace

Figure: after hardening

source: Krupp Edelstahl Profile, Siegen

Objective

General purpose

- Systematic analysis of the causes of distortion
- Prediction of distortion and required counteractions for compensation of distortion via simulations
- Minimization of distortion at the end of the production process

Aim of this work

- Mathematical modeling of the complex material behaviour of steel

$\Longrightarrow$ Coupled system of ordinary and partial differential equations

$\Longrightarrow$ Existence and uniqueness of solutions???

Steel

Basics

- Mostly used metallic material in various areas of application
  - High demands on quality
  - High complexity of the material behaviour
- Alloy consisting mostly of iron, with a carbon content between 0.2% and 2.14% by weight
- Significance of carbon:

  $\longrightarrow$ Steel properties

  $\longrightarrow$ Phase transitions in dependence of

  - surrounding temperature
  - carbon concentration

Formation of different crystal structures in steel

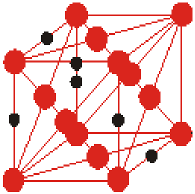


Figure: austenite

- γ-iron
- face-centered cubic (FCC) structure
- $\theta \geq 723^\circ \mathrm{C}$
- $a = 0.26\,\mathrm{nm}$

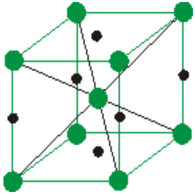


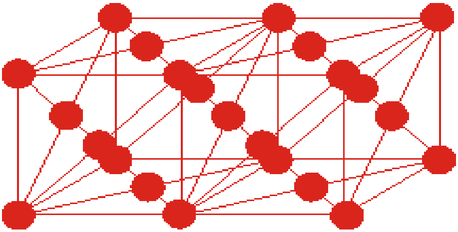
Figure: martensite

- α-iron
- body-centered cubic (BCC) structure
- $\theta \leq \theta_{M_S}$
- $a = 0.29\,\mathrm{nm}$

source: Frylunds Fagteori, Denmark

Steel structure

The heat treatment process for most steels involves heating the alloy until austenite forms, then cooling it so rapidly that the transformation of martensite occurs almost immediately



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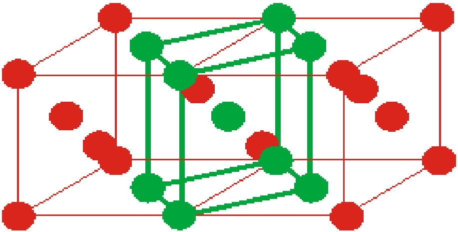


source: Frylunds Fagteori, Denmark

Transformation-induced plasticity

The presence of two phases due to the volume differences in the phases leads to plastic deformation

- Temperature

$\Longrightarrow$ phase transitions
$\Longrightarrow$ change of volume and shape of a crystal
$\Longrightarrow$ complex distribution of stress and deformation
$\Longrightarrow$ plasticity

- Occurance at
  - no stress
  - relatively low stress below yield stress
  - isothermal and continuous transformation in all phase transition levels

Idea

- Great challenge of modeling and simulation of material behaviour in order to predict
  - stress
  - deformation
  - phase transitions
- Little investigation of coupled models for the material behaviour which describe in addition to the temperature and the deformation as well the phase transitions

  $\longrightarrow$ integration of the material behaviour in general models of thermo-elasticity

System of equations

Momentum equation:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} - \mathrm{div}\,(S) = \rho_0 f \ \text{ in } \ \Omega_T$$

Energy equation:

$$\rho_0 \frac{\partial e}{\partial t} - \mathrm{div}(q) = S : \frac{\partial \varepsilon}{\partial t} + R \ \text{ in } \ \Omega_T$$

Equations for the evolution of the phase fractions:

$$\frac{\partial p_i}{\partial t} = \gamma_i(\theta, p)\,,\ i = 1, \ldots, N \ \text{ in } \ \Omega_T$$

System of equations

Momentum equation:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} - \operatorname{div}\big(S(u, \theta, p)\big) = \rho_0 f \quad \text{in} \quad \Omega_T$$

Heat equation:

$$\rho_0 c_e \frac{\partial \theta}{\partial t} - \operatorname{div}(\kappa \nabla \theta) = -3 K_\alpha \theta_0 \operatorname{div}\left(\frac{\partial u}{\partial t}\right) + R(\theta, p) \quad \text{in} \quad \Omega_T$$

Equations for the evolution of the phase fractions:

$$\frac{\partial p_i}{\partial t} = \gamma_i(\theta, p)\,, \;\; i = 1, \ldots, N \quad \text{in} \quad \Omega_T$$

Linear thermo-elasticity with phase transitions

Strain tensor:

$$\varepsilon := \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$

Stress tensor:

$$S := 2\,\mu\,\varepsilon^* + \lambda\,\operatorname{tr}(\varepsilon)\,\mathrm{Id} - 3\,K_\alpha\,\left(\theta - \theta_0\right)\,\mathrm{Id} + - K\sum_{i=1}^{N}\left(\frac{\rho_0}{\rho(\theta_0)} - 1\right)\,p_i\,\mathrm{Id}$$

Right-hand side:

$$R := \rho_0\sum_{i=1}^{N} L_i\,\gamma_i(\theta, p) + r$$

Linear thermo-elasticity with phase transitions

Momentum equation:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} - 2\,\mathrm{div}\,(\mu\,\varepsilon) - \mathrm{grad}\,(\lambda\,\mathrm{div}(u)) +$$

$$+3\,\mathrm{grad}\,(K_\alpha\,(\theta - \theta_0)) + \mathrm{grad}\left(K \sum_{i=1}^{N} \left(\frac{\rho_0}{\rho(\theta_0)} - 1\right) p_i\right) = \rho_0\, f$$

Heat equation:

$$\rho_0\, c_e \frac{\mathrm{d}\theta}{\mathrm{d}t} - \mathrm{div}(\kappa\,\nabla\theta) =$$

$$= -3\,K\,\alpha\,\theta_0\,\mathrm{div}\left(\frac{\partial u}{\partial t}\right) + \rho_0 \sum_{i=1}^{N} L_i\,\gamma_i(\theta, p) + r$$

Boundary– and initial conditions

Boundary conditions:

$$u = 0 \quad \text{on} \quad \Gamma_0 \times]0, T[$$

$$S\,\nu = 0 \quad \text{on} \quad \Gamma_1 \times]0, T[$$

$$-\kappa\,\nabla\theta\,\nu = \delta\,(\theta - \theta_\Gamma) \quad \text{on} \quad \partial\Omega_T$$

Initial conditions:

$$u(x,0) = u_0\ ,\ \frac{\partial u}{\partial t}(x,0) = u_1\ ,\ \theta(x,0) = \theta_0 \quad \text{in} \quad \Omega$$

$$\sum_{i=1}^{N} p_{0i} = 1\ ,\ p_{0i} \geq 0\ ,\ i = 1, \ldots, N \quad \text{in} \quad \Omega$$

Linear thermo-elasticity with phase transitions and transformation-induced plasticity

Strain tensor:

$$\varepsilon = \varepsilon_{te} + \varepsilon_{trip}, \ \mathrm{tr}(\varepsilon_{trip}) = 0$$

$$\varepsilon_{trip} := -3\,\mu \sum_{i=1}^{N} \int_0^t G_i \frac{\partial \Phi_i}{\partial p_i} \max\left\{\frac{\partial p_i}{\partial s}, 0\right\} \left(S - \frac{1}{3}\,\mathrm{tr}(S)\ \mathrm{Id}\right)\,\mathrm{d}s$$

Stress tensor:

$$S := 2\,\mu\,\varepsilon_{te}^{*} + \lambda\ \mathrm{tr}(\varepsilon_{te})\ \mathrm{Id} - 3\,K_\alpha\,(\theta - \theta_0)\ \mathrm{Id} + $$

$$- K \sum_{i=1}^{N} \left(\frac{\rho_0}{\rho(\theta_0)} - 1\right) p_i\ \mathrm{Id} +$$

$$-3\,\mu \sum_{i=1}^{N} \int_0^t G_i \frac{\partial \Phi_i}{\partial p_i} \max\left\{\frac{\partial p_i}{\partial s}, 0\right\} \left(S - \frac{1}{3}\,\mathrm{tr}(S)\ \mathrm{Id}\right)\,\mathrm{d}s$$

Linear thermo-elasticity with phase transitions and transformation-induced plasticity

Momentum equation:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} - 2\,\mathrm{div}\,(\mu\,\varepsilon) - \mathrm{grad}\,(\lambda\,\mathrm{div}(u)) +$$

$$+3\,\mathrm{grad}\,(K_\alpha\,(\theta - \theta_0)) + \mathrm{grad}\left(K \sum_{i=1}^{N} \left(\frac{\rho_0}{\rho(\theta_0)} - 1\right) p_i\right) +$$

$$+3\,\mathrm{div}\left(\mu \sum_{i=1}^{N} \int_0^t G_i\,\frac{\partial \Phi_i}{\partial p_i}\,\max\left\{\frac{\partial p_i}{\partial s}, 0\right\} \left(S - \frac{1}{3}\,\mathrm{tr}(S)\,\mathrm{Id}\right) \mathrm{d}s\right) = \rho_0\, f$$

Heat equation:

$$\rho_0\, c_e\, \frac{\partial \theta}{\partial t} - \mathrm{div}(\kappa\,\nabla\theta) =$$

$$= -3\,K\,\alpha\,\theta_0\,\mathrm{div}\left(\frac{\partial u}{\partial t}\right) + \rho_0 \sum_{i=1}^{N} L_i\,\gamma_i(\theta, p) + r$$

Particularities in the treatment of evolution equations

1. Idea: dealing with the space variable x and the time variable t in different ways
   - For fixed time t, think of the function $x \mapsto u(x,t)$ of the space variable x as an element of the Sobolev space V
   - Notation: $u(t) \in V$
   - If we now vary the time t in the interval $[0, T]$, then we obtain a function $t \mapsto u(t)$
   - Thus arises from the real function $(x,t) \mapsto u(x,t)$ the function $t \mapsto u(t)$ with values in the Banach space V
   - We are looking for the function $t \mapsto u(t)$ with $u(t) \in V$ for all $t \in [0, T]$

Particularities in the treatment of evolution equations

2. Two spaces H and V
   - H is obtained in connection with the time derivative, and V results from the elliptic term and the boundary condition
   - Concept of the evolution triple
3. We think of the time derivative as the generalized derivative on]0,T[:

$$\int_0^T u(t)\,\varphi'(t)\,\mathrm{d}t = -\int_0^T w(t)\,\varphi(t)\,\mathrm{d}t \quad \forall\,\varphi \in C_0^\infty(0,T)$$

4. Choice of the space for the solution: $L^2(0,T;V)$ consists of all those functions such that the following holds:

$$u(\cdot,t) \in V \quad \forall\, t \in (0,T)\,,\; F \in L^2(0,T) \text{ with } F(t) := \|u(\cdot,t)\|_V$$

Function spaces

Definition 1

$$
\begin{aligned}
V_u &:= \left\{ u \in \left[W^{1,2}(\Omega) \right]^3 \; : \; u = 0 \text{ auf } \Gamma_0 \right\} \\
H_u &:= \left[L^2(\Omega) \right]^3 \\
V_\theta &:= W^{1,2}(\Omega) \\
H_\theta &:= L^2(\Omega) \\
\mathcal{V}_p &:= \left\{ p \in \left[L^\infty(\Omega \times]0, T[) \right]^N \; : \; \frac{\partial p}{\partial t} \in \left[L^\infty(\Omega \times]0, T[) \right]^N \right\}
\end{aligned}
$$

Linear thermo-elasticity with phase transitions

Definition 2

Under suitable assumptions a triple $(u, \theta, p) \in L^2(0, T, V_u) \times L^2(0, T; V_\theta) \times \mathcal{V}_p$ with $u' \in L^2(0, T; V_u)$ is called weak solution if the following holds:

$$
\begin{aligned}
&- \int_0^T \int_\Omega \rho_0 \, u'(t) \, \varphi'(t) \, \mathrm{d}x \, \mathrm{d}t + 2 \int_0^T \int_\Omega \mu \, \varepsilon(u(t)) : \varepsilon(\varphi) \, \mathrm{d}x \, \mathrm{d}t + \\
&\quad + \int_0^T \int_\Omega \lambda(x) \, \operatorname{div}(u(t)) \, \operatorname{div}(\varphi) \, \mathrm{d}x \, \mathrm{d}t + 3 \int_0^T \int_\Omega K_\alpha \, (\theta(t) - \theta_0) \, \operatorname{div}(\varphi(t)) \, \mathrm{d}x \, \mathrm{d}t + \\
&\quad + \int_0^T \int_\Omega K \sum_{i=1}^N \left(\frac{\rho_0}{\rho_i(\theta_0)} - 1 \right) \, p_i(t) \, \operatorname{div}(\varphi(t)) \, \mathrm{d}x \, \mathrm{d}t = \\
&\quad = \int_0^T \int_\Omega f(t) \, \varphi(t) \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega \rho_0 \, u_1 \, \varphi(0) \, \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^T \int_\Omega \rho_0 \, c_e \, \theta(t) \, \psi'(t) \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega \kappa \, \nabla\theta(t) \, \nabla\psi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\partial\Omega} \delta \, \theta(t) \, \psi \, \mathrm{d}\sigma \, \mathrm{d}t = \\
&\quad = \int_0^T \int_{\partial\Omega} \delta \, \theta_\Gamma \, \psi(t) \, \mathrm{d}\sigma \, \mathrm{d}t + 3 \int_0^T \int_\Omega K_\alpha \, \theta_0 \, \operatorname{div}(u'(t)) \, \psi(t) \, \mathrm{d}x \, \mathrm{d}t + \\
&\quad + \int_0^T \int_\Omega \rho_0 \sum_{i=1}^N L_i \, \gamma_i(t) \, \psi(t) \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega r(t) \, \psi(t) \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega \rho_0 \, c_e \, \theta_0 \, \psi(0) \, \mathrm{d}x
\end{aligned}
$$

$$
p_i'(t) = \gamma_i(\theta(t), p(t)) \;, \;\; i = 1, \ldots, N
$$

Linear thermo-elasticity with phase transitions and transformation-induced plasticity

Definition 3

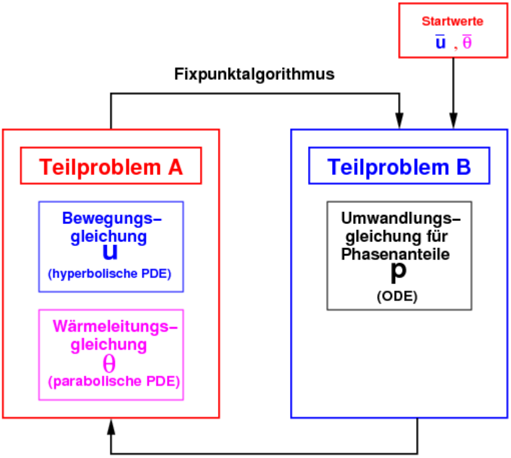
Under suitable assumptions a triple $(u, \theta, p) \in L^2(0, T, V_u) \times L^2(0, T; V_\theta) \times \mathcal{V}_p$ with $u' \in L^2(0, T; V_u)$ is called weak solution if the following holds:

$$
\begin{aligned}
&-\int_0^T \int_\Omega \rho_0\, u'(t)\, \varphi'(t)\,\mathrm{d}x\,\mathrm{d}t + 2\int_0^T \int_\Omega \mu\, \varepsilon(u(t)) : \varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}t + \\
&\quad + \int_0^T \int_\Omega \lambda(x)\ \mathrm{div}(u(t))\ \mathrm{div}(\varphi)\,\mathrm{d}x\,\mathrm{d}t + 3\int_0^T \int_\Omega K_\alpha\,(\theta(t)-\theta_0)\ \mathrm{div}(\varphi(t))\,\mathrm{d}x\,\mathrm{d}t + \\
&\quad + \int_0^T \int_\Omega K \sum_{i=1}^N \left(\frac{\rho_0}{\rho_i(\theta_0)} - 1\right)\ p_i(t)\ \mathrm{div}(\varphi(t))\,\mathrm{d}x\,\mathrm{d}t + \\
&\quad + 2\int_0^T \int_\Omega \mu \int_0^t b(s,t)\,\varepsilon^*(u(s))\,\mathrm{d}s\,\varepsilon^*(\varphi(t))\,\mathrm{d}x\,\mathrm{d}t + \\
&\quad = \int_0^T \int_\Omega f(t)\,\varphi(t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \rho_0\, u_1\,\varphi(0)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^T \int_\Omega \rho_0\, c_e\,\theta(t)\,\psi'(t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \kappa\,\nabla\theta(t)\,\nabla\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\partial\Omega} \delta\,\theta(t)\,\psi\,\mathrm{d}\sigma\,\mathrm{d}t = \\
&\quad = \int_0^T \int_{\partial\Omega} \delta\,\theta_\Gamma\,\psi(t)\,\mathrm{d}\sigma\,\mathrm{d}t + 3\int_0^T \int_\Omega K_\alpha\,\theta_0\ \mathrm{div}(u'(t))\,\psi(t)\,\mathrm{d}x\,\mathrm{d}t + \\
&\quad + \int_0^T \int_\Omega \rho_0 \sum_{i=1}^N L_i\,\gamma_i(t)\,\psi(t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega r(t)\,\psi(t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \rho_0\, c_e\,\theta_0\,\psi(0)\,\mathrm{d}x
\end{aligned}
$$

$$
p_i'(t) = \gamma_i(\theta(t), p(t))\ ,\ \ i = 1, \dots, N
$$

Solution strategy



Problem A

Theorem 5.1

Under suitable assumptions exists a unique weak solution $(u, \theta) \in L^2(0, T; V_u) \times L^2(0, T; V_\theta)$ *of*

Problem A
momentum equation
heat equation
bondary– and initial conditions

for fixed $p \in \mathcal{V}_p$. *Furthermore:*

$$u'' \in L^\infty(0, T; H_u), u' \in L^\infty(0, T; V_u), u \in L^\infty(0, T; V_u),$$

and $\quad \theta' \in L^\infty(0, T; H_\theta), \theta \in L^\infty(0, T; H_\theta).$

Outline of the Galerkin method

- Let P be the exact problem for which we intend to look for the existence of a solution in a space of functions constructed over a separable Hilbert space V
- Existence of the solution
    - Formulation of the approximate problem P_m in the finite dimensional space V_m having a unique solution u_m
    - A priori estimates on u which show that $\{u_m\}$ belongs to fixed balls of certain normed spaces
    - By using the results of weak compactness of the unit ball in a Banach space, it is possible to extract from $\{u_m\}$ a subsequence $\{u'_m\}$ which has a limit in the weak (weak*) topology of spaces which occur in the estimates
    - u is the solution of problem P

Sketch of the proof

Galerkin method

- *A priori estimates provide:*

$$\|u_m'\|_{L^\infty(0,T;H_u)} + \|u_m\|_{L^\infty(0,T;V_u)} + \\ + \|\theta_n\|_{L^\infty(0,T;H_\theta)} + \|\theta_n\|_{L^2(0,T;V_\theta)} < \infty$$

- *A priori estimates not satisfactory because of coupling terms*

Sketch of the proof

Problematical terms:

$$\rho_0 \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - 2\,\mathrm{div}\,(\mu\,\varepsilon) - \mathrm{grad}\,(\lambda\,\mathrm{div}(u)) +$$

$$+3\,\mathrm{grad}\left(K\,\alpha\,(\theta-\theta_0)\right) + \mathrm{grad}\left(K \sum_{i=1}^{N}\left(\frac{\rho_0}{\rho(\theta_0)}-1\right) p_i\right) +$$

$$+3\,\mathrm{div}\left(\mu \sum_{i=1}^{N}\int_0^t G_i\,\frac{\partial \Phi_i}{\partial p_i}\,\max\left\{\frac{\partial p_i}{\partial s},0\right\} S^*\,\mathrm{d}s\right) = \rho_0\, f$$

$$\rho_0\, c_e\,\frac{\partial \theta}{\partial t} - \mathrm{div}(\kappa\,\nabla\theta) = -3\,K\,\alpha\,\theta_0\,\mathrm{div}\left(\frac{\partial u}{\partial t}\right) + \rho_0 \sum_{i=1}^{N} L_i\,\gamma_i + r$$

Problem A

Sketch of the proof

- *A priori estimates because of coupling* *not* *satisfactory*
  - $\longrightarrow$ *Integration by parts in the space variable for simple boundary conditions [J. Gawinecki, 1986]*
  - $\longrightarrow$ *Differentiation der Galerkin equations with respect to time*
- *Additional a priori estimates provide:*

$$\|u_m''\|_{L^\infty(0,T;H_u)} + \|u_m'\|_{L^\infty(0,T;V_u)} + \\ + \|\theta_n'\|_{L^\infty(0,T;H_\theta)} + \|\theta_n'\|_{L^2(0,T;V_\theta)} < \infty$$

$\diamond$

Distinction of both cases

- Linear thermo-elasticity with phase transitions $\surd$
- Linear thermo-elasticity with phase transitions and transformation-induced plasticity
  - Regularization of the problem
  - Additional assumptions for uniqueness

Problem B

Theorem 5.2

Under suitable assumptions exists a unique solution $p \in \mathcal{V}_p$ of

Problem B
equations for evolution of phase fraction *initial condition*

for fixed $\bar{\theta} \in L^2(0, T; H_\theta)$.

Proof

I. Hüssler, 2007

$\diamond$

Full problem

Theorem 5.3

Under suitable assumptions exists a unique weak solution $(u, \theta, p) \in L^2(0, T; V_u) \times L^2(0, T; V_\theta)) \times \mathcal{V}_p$ *of*

full problem
momentum equation
heat equation
equations for the evolution of phase fractions
boundary– and initial conditions

Full problem

Sketch of the proof

Let $\mathbf{V} := V_u \times V_\theta$ *und* $\mathbf{H} := H_u \times H_\theta$. *Already proved:*

- *For given* $(\bar{u}, \bar{\theta}) \in L^2(0, T; \mathbf{H})$ *exists a unique solution* $p = p(\bar{u}, \bar{\theta}) \in \mathcal{V}_p$ *of problem B.*
- *For this given* $p \in \mathcal{V}_p$ *exists a unique solution* $(u, \theta) = \Big(u\big(p(\bar{u}, \bar{\theta})\big), \theta\big(p(\bar{u}, \bar{\theta})\big)\Big) \in L^2(0, T; \mathbf{V})$ *of problem A* *with* $\big\|(u, \theta)\big\|_{W^{1,2}(0,T;\mathbf{V},\mathbf{H})} < \infty$.

Thus defines the following fixed point operator:

$$\begin{aligned} T : L^2(0, T; \mathbf{H}) \to \quad & V_p \quad \to L^2(0, T; \mathbf{V}) \hookrightarrow L^2(0, T; \mathbf{H}) \\ (\bar{u}, \bar{\theta}) \mapsto \quad & p \quad \mapsto (u, \theta) \end{aligned}$$

Full problem

Sketch of the proof

Applying the Schauder fixed point theorem, we have to prove:

$$T : \overline{B_R(0)} \subset L^2(0, T; \mathbf{H}) \to \overline{B_R(0)} \quad \textit{is continuous and compact}$$

- *Mapping of $\overline{B_R(0)}$ into itself*
  - *A priori estimates*
- *Continuity of the operator:*
  - $p_n \to p$ *in* $\mathcal{V}_p$ *for* $(\bar{u}_n, \bar{\theta}_n) \to (\bar{u}, \bar{\theta})$ *in* $L^2(0, T; \mathbf{H})$
  - $(u_n, \theta_n) \to (u, \theta)$ *in* $L^2(0, T; \mathbf{H})$ *for* $p_n \to p$ *in* $\mathcal{V}_p$
- *Precompactness of the image*
  - *Lions-Aubin theorem provides the compact imbedding* $W^{1,2}(0, T; \mathbf{V}, \mathbf{H}) \hookrightarrow L^2(0, T; \mathbf{H})$ ◇

Phase transitions in steel 100Cr6

- Presentation of the solution of the coupled system consisting of the heat equation and the equation for the evolution of the phase fractions
- Assumptions:
  - Homogeneous temperature and phase distribution
  - Stress-free phase transition
- Material data from IWT Bremen
- Calculations via MATLAB®

System of equations

Heat equation:

$$\theta' = \frac{r}{\rho_0\, c_e} + \sum_{i=2}^{4} \frac{L_i}{c_e}\, p_i' - \frac{4\,\delta}{D\,\rho_0\, c_e}(\theta - \theta_\Gamma) \quad \text{in }]0, T[$$

Equations for the evolution of the phase fractions:

$$p_1' = -\, p_2' - p_3' - p_4' \quad \text{in }]0, T[$$
$$p_2' = H_2(\theta, \theta_1, \theta_2)\,(e_2(\theta) + p_2)^{r_2(\theta)} \, \max\{\bar{p}_2 - p_2, 0\}^{s_2(\theta)}\, g_2(\theta) \quad \text{in }]0, T[$$
$$p_3' = H_3(\theta, \theta_2, \theta_3)\,(e_3(\theta) + p_3)^{r_3(\theta)} \, \max\{\bar{p}_3 - p_3, 0\}^{s_3(\theta)}\, g_3(\theta) \quad \text{in }]0, T[$$
$$p_4' = H_4(\theta, \theta_3)\, \max\{\bar{p}_4 - p_4, 0\}\, \mu \quad \text{in }]0, T[$$

Initial conditions:

$$p_1(0) = 1\,,\ p_2(0) = 0\,,\ p_3(0) = 0\,,\ p_4(0) = 0\,,\ \theta(0) = \theta_0$$

Results

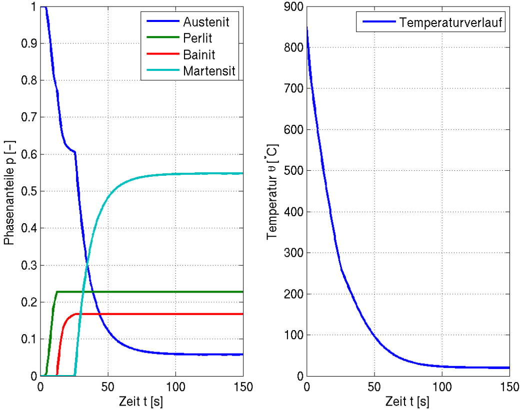


Figure: cooling from the austenitic state

Summary

- Insight into the problem
- Formulation of the mathematical problem of linear thermo-elasticity taking into account phase transitions and transformation-induced plasticity
- Existence and uniqueness of a weak solution
- Numerical analysis of a coupled system consisting of the heat equation and the equation for the evolution of the phase fractions

Outlook

- Further investigation of the uniqueness and the continuity with respect to the data of the full problem
- Application of further proof techniques
  - Regularization methods, e.g. special Galerkin-Bases
  - Time-disrecte or fully discrete approximations
- Consideration of
  - Temperature– and phase-dependent parameters
  - Stress-dependent phase transitions
  - Nitriding, nitrocarburization and carbon diffusion
  - Classical plasticity
- Meso-macro-investigations
- Non-dimensionalization
- Numerical analysis

Thank you for your attention.