FEM for Stokes Equations

Kanglin Chen

15. Dezember 2009

Outline

Stokes equations

Given (f, g). Find (u, p) s.t.

$$\begin{cases} -\Delta u + \nabla p &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= g \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Stokes equations

Given (f, g). Find (u, p) s.t.

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The Stokes equations describe two- or three-dimensional viscous flow.

If $g = 0$, the flow is incompressible.

Variational formulation

Find $(u,p) \in X \times Q$ s.t.

$$\begin{cases} a(u,v) + b(v,p) &= (f,v) \quad \forall v \in X, \\ b(u,\mu) &= (g,\mu) \quad \forall \mu \in Q, \end{cases} \tag{1}$$

where

$$a(u,v) := \int_\Omega \nabla u : \nabla v dxdy, \qquad b(v,p) := -\int_\Omega \mathrm{div}\, v p dxdy,$$

$$(f,v) := \int_\Omega f \cdot v dxdy, \qquad (g,\mu) := \int_\Omega g\mu dxdy.$$

Variational formulation

Find $(u,p) \in X \times Q$ s.t.

$$\begin{cases} a(u,v) + b(v,p) &= (f,v) \quad \forall v \in X, \\ b(u,\mu) &= (g,\mu) \quad \forall \mu \in Q, \end{cases} \tag{1}$$

where

$$a(u,v) := \int_\Omega \nabla u : \nabla v dxdy, \qquad b(v,p) := -\int_\Omega \mathrm{div}\, v p dxdy,$$

$$(f,v) := \int_\Omega f \cdot v dxdy, \qquad (g,\mu) := \int_\Omega g \mu dxdy.$$

We call this the saddle point problem. Why?

Operator equations

Define a linear mapping

$$\begin{aligned} L : X \times Q &\rightarrow X^{'} \times Q^{'} \\ (u,p) &\mapsto (f,g) \end{aligned}$$

where $X := H_0^1(\Omega),\ Q := L_0^2(\Omega) = \left\{ p \in L^2(\Omega) : \int_\Omega p dx dy = 0 \right\}.$

Operator equations

Define a linear mapping

$$\begin{aligned} L : X \times Q &\rightarrow X' \times Q' \\ (u, p) &\mapsto (f, g) \end{aligned}$$

where $X := H_0^1(\Omega)$, $Q := L_0^2(\Omega) = \left\{ p \in L^2(\Omega) : \int\limits_\Omega p dx dy = 0 \right\}$.

For the bilinear form a we can find operator A

$$\begin{aligned} A : X &\rightarrow X', \\ (Au, v) &= a(u, v) \quad \forall v \in X. \end{aligned}$$

Operator equations

Define a linear mapping

$$\begin{aligned} L : X \times Q &\rightarrow X' \times Q' \\ (u, p) &\mapsto (f, g) \end{aligned}$$

where $X := H_0^1(\Omega),\ Q := L_0^2(\Omega) = \left\{ p \in L^2(\Omega) : \int_\Omega p dx dy = 0 \right\}$.

For the bilinear form a we can find operator A

$$\begin{aligned} A : X &\rightarrow X', \\ (Au, v) &= a(u, v) \quad \forall v \in X. \end{aligned}$$

For the bilinear form b we can find operator B

$$\begin{aligned} B : X \rightarrow Q' \qquad\qquad & B^T : Q \rightarrow X', \\ (Bu, \mu) = b(u, \mu) \quad \forall \mu \in Q \qquad\qquad & (B^T p, v) = b(v, p) \quad \forall v \in X. \end{aligned}$$

Operator equations

The variational problem (1) is equivalent to

$$\begin{cases} Au + B^T p &= f, \\ Bu &= g. \end{cases}$$

Operator equations

The variational problem (1) is equivalent to

$$\begin{cases} Au + B^T p = f, \\ Bu = g. \end{cases}$$

Lemma

The following statements are equivalent

1. There exists some $\beta > 0$ with $\inf_{\mu \in Q} \sup_{v \in X} \frac{b(v,\mu)}{\|v\| \, \|\mu\|} \geq \beta$.
2. The operator $B : V^{\perp} \to Q'$ is an isomorphism.
   $V := \{v \in X : b(v,\mu) = 0 \quad \forall \mu \in Q\}$.
3. The operator $B^T : Q \to V^0$ is an isomorphism.
   $V^0 := \left\{ l \in X' : (l, v) = 0, v \in V \right\}$.

Brezzis Splitting Theorem

Brezzis Splitting Theorem

The solution operator

$$\begin{aligned} L: X \times Q &\rightarrow X' \times Q' \\ (u, p) &\mapsto (f, g) \end{aligned}$$

is an isomorphism iff the following conditions are fulfilled

1. the bilinear form a is coercive, i.e. $\exists \alpha > 0$ s.t. $a(v, v) \geq \alpha \|v\|^2, \forall v \in V$.
2. the bilinear form b fulfills the inf-sup condition.

Discrete variational problem

Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$\begin{cases} a(u_h, v) + b(v, p_h) = (f, v) \quad \forall v \in V_h \subset X, \\ b(u_h, q) = 0, \quad \forall q \in Q_h \subset Q, \end{cases}$$

Discrete variational problem

Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$\begin{cases} a(u_h, v) + b(v, p_h) &= (f, v) \quad \forall v \in V_h \subset X, \\ b(u_h, q) &= 0, \quad \forall q \in Q_h \subset Q, \end{cases}$$

Regarding existence and uniqueness of solutions, we have to specify adequate finite element subspaces V_h and Q_h such that the discrete inf-sup condition

$$\exists \beta > 0 : \sup_{v \in V_h} \frac{b(v, p_h)}{\|v\| \, \|p_h\|} \geq \beta, \forall p_h \in Q_h$$

is fulfilled.

P2-P1 Triangulation

Not every finite element space satisfies the discrete inf-sup condition. The P2-P1 approximation, so-called Taylor and Hood elements in [1], works.

P2-P1 Triangulation

Not every finite element space satisfies the discrete inf-sup condition. The P2-P1 approximation, so-called Taylor and Hood elements in [1], works.

$$\begin{aligned} V_h &= \left\{ v \in C^0(\bar{\Omega})^2 \;\middle|\; v|_{\triangle_k} \in P_2(\Omega)^2, \quad \forall \triangle_k \in T_h \right\} \\ Q_h &= \left\{ q \in C^0(\bar{\Omega}) \;\middle|\; q|_{\triangle_k} \in P_1(\Omega), \quad \forall \triangle_k \in T_h \right\}. \end{aligned}$$

Linear system

Let $(\Phi_i), (\psi_i)$ be the basic functions of V_h and Q_h, then we have

$$\begin{aligned}
u_h &= \sum_{i=1}^{2N} \alpha_i \Phi_i, \quad p_h = \sum_{i=1}^{M} \gamma_i \psi_i \\
a(u_h, \Phi_j) &= \sum_{i=1}^{2N} \alpha_i a(\Phi_i, \Phi_j), \quad A_{ij} = a(\Phi_i, \Phi_j) \\
b(\Phi_i, p_h) &= \sum_{i=1}^{M} \gamma_i b(\Phi_i, \psi_j), \quad B_{ij}^T = b(\Phi_i, \psi_j) \\
f_{2N} &= (f_i), \quad f_i = (f, \Phi_i) \\
u_{2N} &= \{\alpha_i\}, \quad p_M = \{\gamma_i\}.
\end{aligned}$$

Then we derive

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u_{2N} \\ p_M \end{pmatrix} = \begin{pmatrix} f_{2N} \\ 0 \end{pmatrix}.$$

Linear system

The stiffness matrix A has the block form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix},$$

where $A_1 = \left(\int_\Omega \nabla\varphi_i \nabla\varphi_j dxdy\right)_{ij}, i,j = 1,\cdots,N$. The matrix B^T has also the block form

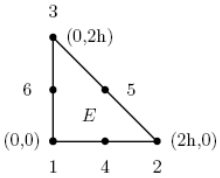
$$B^T = \begin{pmatrix} B_1^T \\ B_2^T \end{pmatrix},$$

$$\begin{aligned} B_1^T &= \left\{ -\int\limits_\Omega \frac{\partial\varphi_i}{\partial x}\psi_j dxdy, \quad i = 1,\cdots,N; j = 1,\cdots,M \right\} \\ B_2^T &= \left\{ -\int\limits_\Omega \frac{\partial\varphi_i}{\partial y}\psi_j dxdy, \quad i = N+1,\cdots,2N; j = 1,\cdots,M \right\}. \end{aligned}$$

Basic functions

In element E the basic functions of V_h are given by

$$
\begin{aligned}
\varphi_1(x,y) &= (1-\frac{x}{h}-\frac{y}{h})(1-\frac{x}{2h}-\frac{y}{2h}),\\
\varphi_2(x,y) &= \frac{x}{2h}(\frac{x}{h}-1),\\
\varphi_3(x,y) &= \frac{y}{2h}(\frac{y}{h}-1),\\
\varphi_4(x,y) &= \frac{x}{h}(2-\frac{x}{h}-\frac{y}{h}),\\
\varphi_5(x,y) &= \frac{xy}{h^2},\\
\varphi_6(x,y) &= \frac{y}{h}(2-\frac{x}{h}-\frac{y}{h}).
\end{aligned}
$$



satisfying $\varphi_i(x_j,y_j)=\delta_{ij}$. The basic functions of Q_h in element E are given by

$$
\begin{aligned}
\psi_1(x,y) &= 1-\frac{x}{2h}-\frac{y}{2h},\\
\psi_2(x,y) &= \frac{x}{2h},\\
\psi_3(x,y) &= \frac{y}{2h}.
\end{aligned}
$$

Example

The velocity field:

$$\begin{aligned} u_1 &= x^2(1-x)^2 2y(1-y)(2y-1), \\ u_2 &= y^2(1-y)^2 2x(1-x)(1-2x). \end{aligned}$$

The pressures:

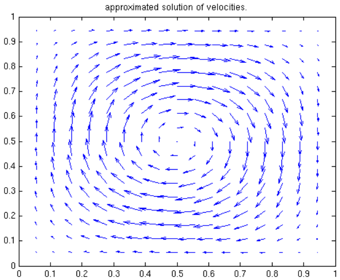
$$p(x,y) = x(1-x)(1-y)$$

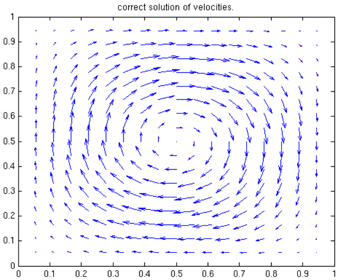
The right-hand side:

$$\begin{aligned} f_1(x,y) &= 4(x^3(6-12y) + x^4(-3+6y) + y(1-3y+2y^2) \\ &\quad -6xy(1-3y+2y^2) + 3x^2(-1+4y-6y^2+4y^3)) \\ &\quad +y(1-y)(1-2x), \\ f_2(x,y) &= -4(-3(-1+y)^2y^2 - 3x^2(1-6y+6y^2) \\ &\quad +2x^3(1-6y+6y^2) + x(1-6y+12y^2-12y^3+6y^4) \\ &\quad +x(1-x)(1-2y)). \end{aligned}$$

Results

Case: $h = \frac{1}{18}$.





	$h = \frac{1}{8}$	$h = \frac{1}{18}$	$h = \frac{1}{28}$
L^2-Errors of u	$2.2e-03$	$1.0e-03$	$1.1e-03$
Iterations of MINRES: TOL=$1e-03$	31	75	120

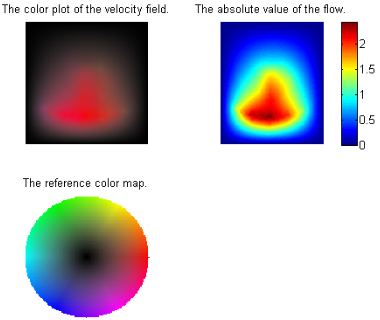
Optical flow estimation

$$
\begin{cases}
u_t + b \cdot \nabla u = 0 & u(0) = \bar{u} \\
p_t + b \cdot \nabla p = 0 & p(T) = -(u^T - u_T) \\
\operatorname{div} b = 0 & \\
\lambda \Delta b + \nabla q = p \nabla u & b = 0 \text{ on } \partial\Omega
\end{cases}
$$

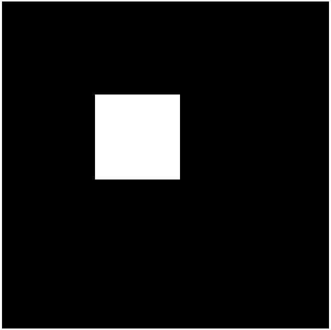
Test1

Test1

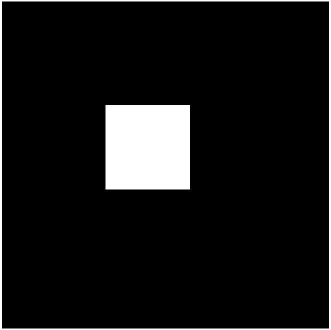
Test1


The color plot of the velocity field.
The absolute value of the flow.
2
1.5
1
0.5
0
The reference color map.

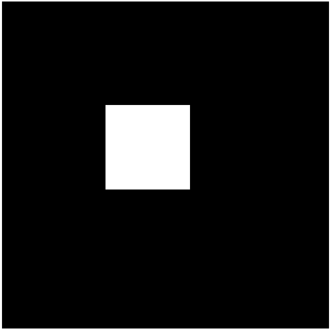
Test2

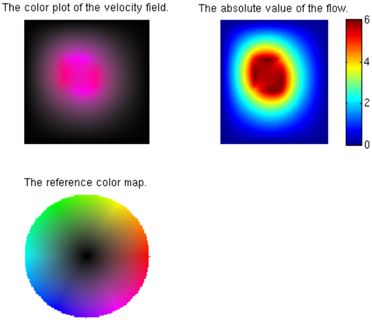


Test2



Test2




The color plot of the velocity field.
The absolute value of the flow.
6
4
2
0
The reference color map.

Franco Brezzi and Michel Fortin.
Mixed and Hybrid Finite Element Methods.
Springer-Verlag, 1991.

Thank you for your attention!