A Short Introduction to Reduced Basis Method for Parametrized Partial Differential Equations

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SCiE seminar, Bremen, 20. May 2010
Outline

1. Introduction

2. Reduced Basis Approach
   - Preliminaries
   - Approach

3. Summary and outlook
Outline

1 Introduction

2 Reduced Basis Approach
   - Preliminaries
   - Approach

3 Summary and outlook
Motivations

In many situations, one has to solve equations with the variation of one or some parameters.

1. The viscous Burgers equation:
   \[ u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial^2 x} = 0; \]

2. The Helmholtz equation:
   \[ \nabla^2 u + k^2 u = 0; \]

3. Steady convection-diffusion equations:
   \[ D \nabla^2 c - \vec{v} \nabla c = 0. \]

The shape of domain considered may also contribute parameter(s) to the formulation of the problem.
Motivations

- Discretize equations by FEM, usually high order basis due to the complexity of the geometry and the required accuracy,
- Want to know about the whole or a part of solution corresponding to many values of parameter(s),
- Such evaluations are too time-consuming since one has to work with very high order FE basis for each new value of parameter(s),
- Seek a way to reduce basis with which one works, hence reduce computing time.
Problem Statement

The following problem and the definitions and hypotheses mentioned later are considered in the FEM context. Let $f$ and $l$ be affine parametric linear form, $a$ an affine parametric bilinear form on $X$. Given $\mu \in \mathcal{D}$, find $u(\mu) \in X$ s.t.

$$a(u(\mu), v; \mu) = f(v; \mu), \ \forall v \in X$$  \hspace{1cm} (1)

and evaluate

$$s(\mu) = l(u(\mu); \mu).$$  \hspace{1cm} (2)
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Hypotheses

We only work with linear/bilinear form which are affine in parameter

\[ l(v; \mu) = \sum_{q=1}^{Q_l} \Theta^q_l(\mu) l^q(v), \]  
\[ f(v; \mu) = \sum_{q=1}^{Q_f} \Theta^q_f(\mu) f^q(v), \]  
\[ a(w, v; \mu) = \sum_{q=1}^{Q_a} \Theta^q_a(\mu) a^q(w, v); \]

in which, \( l = f, a \) are continuous on \( X \); \( a \) is symmetric and parametrically coercive. We call this problem compliant.
Definitions

Inner product and norm:

\[ ((w, v))_\mu = a(w, v; \mu), \quad \|w\|_\mu = \sqrt{a(w, w; \mu)}. \] (6)

Norm in \( X \): given a fixed \( \mu \in D \)

\[ (w, v)_X = ((((w, v))))_{\mu}, \quad \|w\|_X = \|w\|_{\mu}. \] (7)

Coercivity constant and continuity constant

\[ \alpha(\mu) = \inf_{w \in X} \frac{a(w, w; \mu)}{\|w\|_X^2} > 0 \quad \forall \mu \in D; \] (8)

\[ \gamma(\mu) = \sup_{w \in X} \sup_{v \in X} \frac{a(w, v; \mu)}{\|w\|_X \|v\|_X} < \infty \quad \forall \mu \in D. \] (9)
Idea and questions

Idea

Replace high order \((N)\) FE basis by a much lower order basis which consists of selected snapshots \(\{u(\mu^n), n = 1, \cdots, N\}\), so called Reduced Basis (RB) and decompose the whole process into Offline and Online stages.

Questions

- How to combine these snapshots to approximate solution?
- How to choose parameters points \(\mu^n\)?
- Online operation count and storage independent of \(N\)?
- How to estimate an error bound?
Projecting on Lagrange RB space

Given parameter set \( \{ \mu^n, n = 1, \cdots, N \} \), denote by \( u^n = u(\mu^n) \) snapshots and then define Lagrange RB space

\[
X_N = \text{span}\{u^n, 1 \leq n \leq N\}.
\]

The projected problem is: seek \( u_{X_N}(\mu) \in X_N \) s.t.

\[
a(u_{X_N}(\mu), v; \mu) = f(v; \mu), \quad \forall v \in X_N
\]

(10)

and then evaluate

\[
s_{X_N}(\mu) = f(u_{X_N}(\mu); \mu).
\]

(11)
Greedy algorithm

We actually work with a finite surrogate $\Xi_{\text{train}}$ of $\mathcal{D}$. Denote by $\Delta X_N(\mu)$ the RB error bound (specified later), $N_{\text{max}}$ the maximal size of RB space. Initiating with $N_0$, the initial size of the initial sample $S_{N_0}^* = \{\mu_{1*}, \cdots, \mu_{N_0*}\}$, and the tolerance $\varepsilon_{\text{tol,min}}$. 
Greedy\( (N_0, S^*_{N_0}, \Xi_{train}, \varepsilon_{tol, min}) \)

for \( N = N_0 + 1 : N_{max} \)

\[
\mu^{N*} = \arg \max_{\mu \in \Xi_{train}} \Delta x_{N-1}(\mu);
\]

\[
\varepsilon^*_{N-1} = \Delta x_{N-1}(\mu);
\]

if \( \varepsilon^*_{N-1} \leq \varepsilon_{tol, min} \)

\( N_{max} = N - 1; \)

exit;

end;

\[
S^*_N = S^*_{N-1} \cup \mu^{N*};
\]

\[
X^*_N = X^*_{N-1} + \text{span}\{u(\mu^{N*})\};
\]
Offline-Online decomposition

**Offline stage**

Compute all ingredients for stiffness matrix and load vector for online stage. The operation count and storage of these computations depend on $N$ and hence expensive.

**Online stage**

Given any $\mu \in \mathcal{D}$, we assemble the ingredients computed in offline stage to formulate stiffness matrix and load vector and then, solve (10) and evaluate (11). The operation count and storage of this process are independent of $N$ and hence are not expensive.
Error bound estimation

Lower bound for coercivity constant and upper bound for continuity constant.

\[
\alpha_{LB}(\mu) \equiv \Theta_{a}^{\min, \bar{\mu}}(\mu) = \min_{q \in \{1, \ldots, Q_a\}} \frac{\Theta_{a}^{q}(\mu)}{\Theta_{a}^{q}(\bar{\mu})},
\]
(12)

\[
\gamma_{UB}(\mu) \equiv \Theta_{a}^{\max, \bar{\mu}}(\mu) = \max_{q \in \{1, \ldots, Q_a\}} \frac{\Theta_{a}^{q}(\mu)}{\Theta_{a}^{q}(\bar{\mu})}.
\]
(13)

\[
\theta(\bar{\mu})(\mu) \equiv \frac{\gamma_{UB}(\mu)}{\alpha_{LB}(\mu)}.
\]
(14)

The residual and its Riesz representation:

\[
r(v; \mu) = f(v; \mu) - a(u_{X_N}(\mu), v; \mu).
\]
(15)

Define:

\[
(\hat{e}(\mu), v) = r(v; \mu);
\]
(16)
Error bound estimation

\[ \Delta_{N}^{en}(\mu) \equiv \frac{||\hat{e}(\mu)||_X}{\alpha_{LB}(\mu)^2}, \quad \eta_{N}^{en}(\mu) \equiv \frac{\Delta_{N}^{en}(\mu)}{||e(\mu)||_\mu}; \]  \hspace{1cm} (17)

\[ \Delta_{N}^{s}(\mu) \equiv \frac{||\hat{e}(\mu)||_X^2}{\alpha_{LB}(\mu)}, \quad \eta_{N}^{s}(\mu) \equiv \frac{\Delta_{N}^{s}(\mu)}{s(\mu) - s_{N}(\mu)}; \]  \hspace{1cm} (18)

\[ \Delta_{N}^{s,rel}(\mu) \equiv \frac{||\hat{e}(\mu)||_X^2}{\alpha_{LB}(\mu)s_{N}(\mu)}, \quad \eta_{N}^{s,rel}(\mu) \equiv \frac{\Delta_{N}^{s,rel}(\mu)}{(s(\mu) - s_{N}(\mu))/s(\mu)}; \]  \hspace{1cm} (19)

\( \Delta_{N}, \eta_{N} \) are error bounds and effectivities respectively.
## Error bound estimation

### Theorem

For given $N = 1, \ldots, N_{\text{max}}$,

\begin{align}
1 & \leq \eta^e_n(\mu) \leq \sqrt{\theta(\mu)}, \quad \forall \mu \in \mathcal{D}, \quad (20) \\
1 & \leq \eta^s_n(\mu) \leq \theta(\mu), \quad \forall \mu \in \mathcal{D}. \quad (21)
\end{align}

Furthermore, for $\Delta^s_{N,\text{rel}}(\mu) \leq 1$

\begin{align}
1 & \leq \eta^s_{N,\text{rel}}(\mu) \leq 2\theta(\mu), \quad \forall \mu \in \mathcal{D}, \quad (22)
\end{align}

where the left inequality of (22) is always valid.
Comments

- The error above bounds are some how local, i.e. depend on $\mu$,
- They also developed “global bounds” for solution $u(\mu)$ which is independent of $\mu$,
- The nonlinear and noncoercive problems require more general and thorough treatments, for more details, see the references.
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Summary and future work

- Reduced Basis method deals with Parametrized Partial Differential Equations: constructing reduced basis, offline-online decomposition and error bounds.
- Reduced basis method for incompressible, nonlinear and noncoercive problems: Navier-Stockes equations,
- Using this idea for parametric model order reduction.

