# Spline Interpolation Based PMOR

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# Outline

1. Preliminaries
   - Statement of the problem: Parametric Model Order Reduction
   - Interpolation with B-Splines
2. Spline Interpolation Based PMOR
   - Approach
   - Linear Spline
   - Cubic Spline
3. Conclusion

# Statement of the problem

## Given a parameter-dependent systems of size $N$

$$\dot{x}(t) = A(p)x(t) + B(p)u(t),$$
$$y(t) = C(p)x(t), p \in [0, 1] \subset \mathbb{R}. \tag{1}$$

## Replace it with a parameter-dependent systems of size $n, n << N$

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# Requirements

- Reduction process independent the change of parameter on a fixed interval.
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Interpolation is the process of constructing a function which takes a given value-set, e.g. $\{y_1, \cdots, y_n\}$ at given data points, e.g. $\{x_1, \cdots, x_n\}$. This function is a linear combination of basis functions and **Basic Spline Curves** or **B-Splines** are especially suitable for computation. Let $\{f_i(x), i = 1, \cdots, n\}$ denote the basis functions, the interpolator $F(x)$ takes the form

$$F(x) = \sum_{i=1}^{n} c_i f_i(x)$$

and satisfies the **interpolation conditions**

$$F(x_i) = y_i, i = 1, \cdots, n.$$

Consider system (1) whose transfer function(TF) is denoted by $H(p, s)$. **What do we interpolate?**

We *can't* reduce the system with the change of $p$, but we *can* do it at each point $p_i \in [0, 1] \subset \mathbb{R}$.

- What we interpolate is the reduced TF $\widehat{H(p, s)}$.
- The *data points* is the set $0 = p_0, \cdots, p_k = 1$.
- The *given values* are the reduced TFs $H^r(p_j, s)$ of $H(p_j, s)$.

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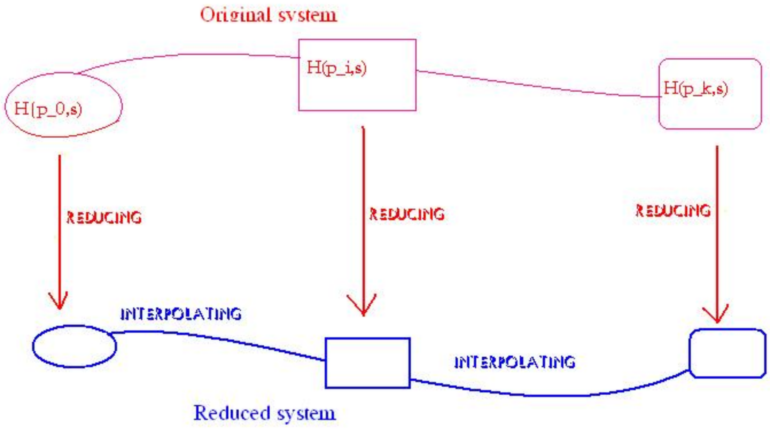
# PMOR process



# Difficulty

- TF depends also on frequency parameter $s$;
- The state space representation must be easily and properly recovered.

# Hypothesis

We consider the LTI system

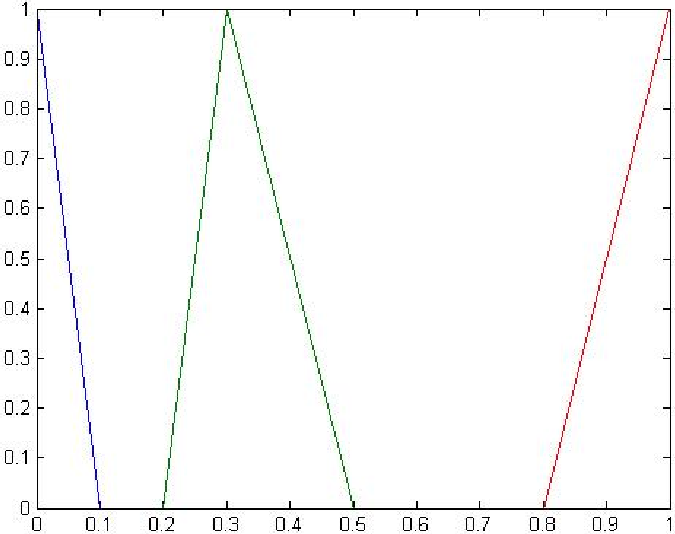
$$\Sigma(p) = \begin{cases} \dot{x} = A(p)x + B(p)u, \;\; x(0) = 0, \\ y = C(p)x \end{cases} \tag{3}$$

where $p \in [0, 1]$ and $A \in \mathbb{R}^{n\times n}, B(p) \in \mathbb{R}^{n\times m}, C(p) \in \mathbb{R}^{p\times n}$. We assume that for each $p$, $\Sigma(p)$ is reachable, observable and stable with transfer function $H(p, s) = C(p)(sI - A(p))^{-1}B(p)$.

# Steps

- Discretize $[0, 1]$ as $0 = p_0 < p_1 < \cdots < p_k = 1$ and denote $h = \max\{p_j - p_{j-1}, j = 1, \cdots, k\}$.
- For each $\Sigma(p_j), j = 0, \cdots, k$ compute a reduced order LTI system $\Sigma^{r_j}(p_j) = (A_j, B_j, C_j)$ of dimension $r_j$ with transfer function $H^r(p_j, s)$ by balanced truncation.
- Use **linear B-splines** $f_0, f_1, \cdots, f_k$ for the given grid.

# Linear spline basis



# Reduced system

- The transfer function

$$\widehat{H(p,s)} = \sum_{j=0}^{k} f_j(p) H^r(p_j, s). \quad (4)$$

- **State space representation** of $\widehat{\Sigma(p)}$ is $(\widehat{A}, \widehat{B}, \widehat{C})$, where $\widehat{A} := \mathrm{diag}(A_0, \cdots, A_k)$, $\widehat{B} = [B_0^T, \cdots, B_k^T]^T$ and $\widehat{C} = [f_0(p)C_0, \cdots, f_k(p)C_k]$.
  It is of the order $\sum_{j=0}^{k} r_j$.

# Results

- **Stability preservation**
- **Global error bound.**
  - **Norm for parameter-dependent systems**

$$\|\Sigma(p)\|_\infty = \|H(p,s)\|_\infty := \sup_{p\in[0,1]} \|H(p,s)\|_{\mathcal{H}_\infty}.$$

  - **Lipschitz condition** for $\Sigma(p)$:

$$\forall p_1, p_2 \in [0,1], \|H(p_1,s) - H(p_2,s)\|_{\mathcal{H}_\infty} \leq L|p_1 - p_2|. \quad (5)$$

**Theorem**

*Let $\Sigma(p)$ satisfy the Lipschitz condition (5) and let the reduced order system $\widehat{\Sigma(p)}$ be constructed as above. Then*

$$\|\Sigma(p) - \widehat{\Sigma(p)}\|_\infty \leq Lh + \mathcal{S}$$

*where* $\mathcal{S} = \max\{\|H(p_i,s) - H^r(p_i,s)\|_{\mathcal{H}_\infty}, i := 0,\ldots,k\}$.

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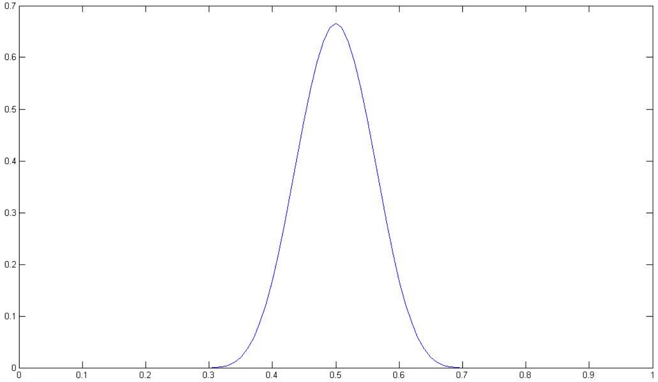
# Some remarks

- The interpolation process is quite easy thank to the simple structure of linear B-splines;
- The error bound is received by combining the error of MOR at each point and error of interpolation process;
- In the MIMO case, the proof has to resort to a Gerschgorin-type theorem, c.f Qi84.

# Cubic spline basis



# Cubic spline = More efforts

- Wider support of basis function ⇒ Instead of (4), solving a linear system
- Solving system + getting error bound ⇒ Bounding norm of inverse matrices
- State space representation ⇒ How to choose the end conditions

- LTI SISO system

$$\begin{aligned} \dot{x} &= A(p)x + b(p)u, \\ y &= c(p)x, A \in \mathbb{R}^{n\times n} \end{aligned} \tag{6}$$

- Let the reduced TF take the form

$$\widehat{H(p,s)} = \sum_{i=-1}^{k+1} c_i^r f_i(p), \tag{7}$$

- Interpolation conditions, using reduced TF at knots

$$\sum_{i=j-1}^{j+1} c_i^r f_i(p_j) = H^r(p_j, s), j = 0, \ldots, k.$$

- Natural end conditions

$$\frac{\partial^2 \widehat{H(p_0,s)}}{\partial p^2} = \frac{\partial^2 \widehat{H(p_k,s)}}{\partial p^2} = 0.$$

The system determines $c_i^r$ in (7): $FC^r = H^r$. The collocation matrix

$$F = \frac{1}{6}\begin{bmatrix} \frac{6}{h^2} & -\frac{12}{h^2} & \frac{6}{h^2} & \cdots & 0 & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \ddots & \ddots & \cdots & \cdots & \cdots \\ \\ 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & \cdots & \frac{6}{h^2} & -\frac{12}{h^2} & \frac{6}{h^2} \end{bmatrix}.$$

# A short deviation: Bound for norm of matrix inverse

- Strictly diagonally dominant(SDD) matrices, $S$-SDD matrices, Varah75, Varga76, Moraca07/08,...
- PM-matrices, PH-matrices, Kolotilina95/08/09,...
- Problem: Matrix $F$ is neither SDD, $S$-SDD nor PH-matrix.
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# Procedure

- Scale $F$ by $D_1 = \text{diag}(1, \frac{1}{3}, \cdots, \frac{1}{3}, 1)$, turns $F$ into $S$-SDD/PH-matrix.
- Scale $F_1 = FD$ by $D_2 = \text{diag}(1, 1, \gamma, \cdots, \gamma, 1, 1)$ where $\gamma \in (\frac{1}{3}, 1)$
- Repeat **discretize + choose the smallest interval** until the step size quite small.

**Theorem**

*The inverse of collocation satisfies*

$$\|F^{-1}\|_\infty \leq 6 \max\{\frac{64}{17}, \frac{288 + 48h^2}{77}\}$$

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# Remark

The above bound is not the best but good enough. Our verification: compute the real infinity norm of the inverse of the comparision matrix $\mathcal{M}(F)^{-1}$:

- the size $7x7$: $6(\frac{31}{10} + \frac{13}{30}h^2)$
- the size $10x10$: $6(\frac{61}{20} + \frac{17}{40}h^2)$

# Remark

If we use Hermite end condition, the collocation matrix

$$F = \frac{1}{6}\begin{bmatrix} -\frac{3}{h} & 0 & \frac{3}{h} & \cdots & 0 & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 & 0 \\ \\ \cdots & \cdots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \\ 0 & 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & 0 & \cdots & -\frac{3}{h} & 0 & \frac{3}{h} \end{bmatrix}$$

and $\|F^{-1}\| \leq 6\frac{1+h}{2}$ but no state space representation.

# Results with cubic spline interpolation

- **Stability preservation**.
- **State space representation:** Denote $[f(p)]$ the column of cubic B-spline, and $\mathcal{F}(p) = [\mathcal{F}_i(p)]^T = [f(p)]^T F^{-1}$. Then, the reduced system $\widehat{\Sigma(p)}$ is $(\widehat{A}, \widehat{b}, \widehat{c})$, where $\widehat{A} = \text{diag}(A_0, \cdots, A_k), \widehat{b} = [\mathcal{F}_0(p) b_0^T, \cdots, \mathcal{F}_k(p) b_k^T]^T$ and $\widehat{c} = [c_0, \cdots, c_k]$, of the order $(k+1)r$.

# Results with cubic spline interpolation

## Theorem

*Assume system (6) be observable, controllable, stable and moreover,* $\forall s \in \mathbb{C}^+, A(p), b(p), c(p)$ *are* $C^4$ *over* $[0,1]$, *then*

$$\|\Sigma(p) - \widehat{\Sigma(p)}\|_\infty \leq \frac{5}{384}\|\frac{\partial^4 H}{\partial p^4}\|_\infty h^4 + 6\max\{\frac{64}{17}, \frac{288+48h^2}{77}\}\mathcal{S}$$

*where* $\mathcal{S} = \max\{\|H(p_i, s) - H^r(p_i, s)\|_{\mathcal{H}_\infty}, i := 0, \ldots, k\}.$

# Numerical Experiment

UPCOMING TIME!

- Combining Spline interpolation and Balanced truncation to PMOR
- Results for both linear and cubic spline interpolation:
    - Formally preserve parameter
    - Preserve stability
    - Proper state space representation
    - Error bound

# End

THANK YOU!