Recent Approaches for PMOR

Nguyễn, Thanh Sơn

Universitaet Bremen
Zentrum fuer Technomathematik

27th May 2009

Outline

Statement of the problem

Given a parameters-dependent systems of size N

$$\begin{aligned} E(p)x'(t) &= A(p)x(t) + B(p)u(t), \\ y(t) &= C(p)x(t), p = (p_1, ..., p_k). \end{aligned} \quad (1)$$

Replace it with a parameters-dependent systems of size $n, n << N$

$$\begin{aligned} \hat{E}(p)x'(t) &= \hat{A}(p)x(t) + \hat{B}(p)u(t), \\ y(t) &= \hat{C}(p)x(t), p = (p_1, ..., p_k). \end{aligned} \quad (2)$$

Statement of the problem

Given a parameters-dependent systems of size N

$$\begin{aligned} E(p)x'(t) &= A(p)x(t) + B(p)u(t), \\ y(t) &= C(p)x(t), p = (p_1, ..., p_k). \end{aligned} \tag{1}$$

Replace it with a parameters-dependent systems of size $n, n << N$

$$\begin{aligned} \hat{E}(p)x'(t) &= \hat{A}(p)x(t) + \hat{B}(p)u(t), \\ y(t) &= \hat{C}(p)x(t), p = (p_1, ..., p_k). \end{aligned} \tag{2}$$

Requirements

- Reduction process independent of variation of parameter on a fixed range Ω.
- System (2) approximates system (1) in some sense.
- The reduction procedure should not cost so high.

Requirements

- Reduction process independent of variation of parameter on a fixed range Ω.
- System (2) approximates system (1) in some sense.
- The reduction procedure should not cost so high.

Requirements

- Reduction process independent of variation of parameter on a fixed range Ω.
- System (2) approximates system (1) in some sense.
- The reduction procedure should not cost so high.

Requirements

- Reduction process independent of variation of parameter on a fixed range Ω.
- System (2) approximates system (1) in some sense.
- The reduction procedure should not cost so high.

Recall the method

$$\begin{aligned} Ex'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{3}$$

$$\begin{aligned} H(s) =& C(sE - A)^{-1}B = -C((A - s_0E) - (s - s_0)E)^{-1}B \\ =& -C\sum_{i=0}^{\infty}((A - s_0E)^{-1}E)^i(A - s_0E)^{-1}B(s - s_0)^i \\ =& \sum_{i=0}^{\infty} M_i(s - s_0)^i. \end{aligned}$$

- Construct span$V_{s_0} = \mathcal{K}_{n_0}((A - s_0E)^{-1}E, (A - s_0E)^{-1}B)$, $V_{s_0}^T V_{s_0} = I$.
- Also, $V = \{V_{s_1}, ..., V_{s_k}\}$.

Recall the method

$$\begin{aligned} Ex'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{3}$$

$$\begin{aligned} H(s) =& C(sE - A)^{-1}B = -C((A - s_0E) - (s - s_0)E)^{-1}B \\ =& -C\sum_{i=0}^{\infty}((A - s_0E)^{-1}E)^i(A - s_0E)^{-1}B(s - s_0)^i \\ =& \sum_{i=0}^{\infty} M_i(s - s_0)^i. \end{aligned}$$

- Construct span$V_{s_0} = \mathcal{K}_{n_0}((A - s_0E)^{-1}E, (A - s_0E)^{-1}B)$, $V_{s_0}^T V_{s_0} = I$.
- Also, $V = \{V_{s_1}, ..., V_{s_k}\}$.

Recall the method

$$\begin{aligned} Ex'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{3}$$

$$\begin{aligned} H(s) =& C(sE - A)^{-1}B = -C((A - s_0E) - (s - s_0)E)^{-1}B \\ =& -C\sum_{i=0}^{\infty}((A - s_0E)^{-1}E)^i(A - s_0E)^{-1}B(s - s_0)^i \\ =& \sum_{i=0}^{\infty} M_i(s - s_0)^i. \end{aligned}$$

- Construct span$V_{s_0} = \mathcal{K}_{n_0}((A - s_0E)^{-1}E, (A - s_0E)^{-1}B)$, $V_{s_0}^T V_{s_0} = I$.
- Also, $V = \{V_{s_1}, ..., V_{s_k}\}$.

Reduced system

$$V^T E V x_r'(t) = V^T A V x_r(t) + V^T B u(t),$$
$$y_r(t) = C V x_r(t);$$

- n_0 moments of TF at s_0 matched,
- Easy to implement by Arnoldi algorithm,
- No guarantee of stability,
- Hard to derive error bound.

Reduced system

$$V^TEVx_r'(t) = V^TAVx_r(t) + V^TBu(t),$$
$$y_r(t) = CVx_r(t);$$

- n_0 moments of TF at s_0 matched,
- Easy to implement by Arnoldi algorithm,
- No guarantee of stability,
- Hard to derive error bound.

Multiparameter moment matching [4, 9]

Treat system parameters, frequency one in the same way

$$\begin{aligned} Ex'(t) &= (A + pA_1)x(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{4}$$

Denoting $s_1 = s, s_2 = p, E_1 = E, E_2 = -A_1$, expanding TF at (σ_1, σ_2) and denoting $S = A - \sigma_1 E_1 - \sigma_2 E_2$

$$H(s_1, s_2) = -C\sum_{i=0}^{\infty}((s_1 - \sigma_1)S^{-1}E_1 + (s_2 - \sigma_2)S^{-1}E_2)^i S^{-1}B$$

$$= -C\sum_{i=0}^{\infty}\sum_{j=0}^{i} F_j^i(S^{-1}E_1, S^{-1}E_2)S^{-1}B(s_1 - \sigma_1)^{i-j}(s_2 - \sigma_2)^j.$$

Multiparameter moment matching [4, 9]

Treat system parameters, frequency one in the same way

$$\begin{aligned} Ex'(t) &= (A + pA_1)x(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{4}$$

Denoting $s_1 = s, s_2 = p, E_1 = E, E_2 = -A_1$, expanding TF at (σ_1, σ_2) and denoting $S = A - \sigma_1 E_1 - \sigma_2 E_2$

$$H(s_1, s_2) = -C \sum_{i=0}^{\infty} ((s_1 - \sigma_1) S^{-1} E_1 + (s_2 - \sigma_2) S^{-1} E_2)^i S^{-1} B$$

$$= -C \sum_{i=0}^{\infty} \sum_{j=0}^{i} F_j^i (S^{-1} E_1, S^{-1} E_2) S^{-1} B (s_1 - \sigma_1)^{i-j} (s_2 - \sigma_2)^j.$$

- Defining the *generalized Krylov subspace*

$$\mathcal{G}_k(M_1, M_2, m) = \text{span} \cup_{i=0}^{k} \cup_{j=0}^{i} F_j^i(M_1, M_2)m.$$

- Construct span$V = \mathcal{G}_n(S^{-1}E_1, S^{-1}E_2, S^{-1}B)$,
- Reduced system

$$\begin{aligned} V^T E V x'(t) &= (V^T A V + p V^T A_1 V)x(t) + V^T B u(t), \\ y(t) &= CVx(t). \end{aligned} \tag{5}$$

- The first n "coefficients" of reduced system match ones of original one.
- Disadvantage: Difficult to implement stably since one cannot use Arnoldi algorithm.

- Defining the *generalized Krylov subspace*

$$\mathcal{G}_k(M_1, M_2, m) = \text{span} \cup_{i=0}^{k} \cup_{j=0}^{i} F_j^i(M_1, M_2)m.$$

- Construct span$V = \mathcal{G}_n(S^{-1}E_1, S^{-1}E_2, S^{-1}B)$,
- Reduced system

$$\begin{aligned} V^T E V x'(t) &= (V^T A V + p V^T A_1 V)x(t) + V^T B u(t), \\ y(t) &= C V x(t). \end{aligned} \tag{5}$$

- The first n "coefficients" of reduced system match ones of original one.
- Disadvantage: Difficult to implement stably since one cannot use Arnoldi algorithm.

Combining with Arnoldi algorithm [5]

The TF of system (4) can be expanded at $(0,0)$ as

$$H(s,p) = -C(A + pA_1 - sE)^{-1}B$$
$$= -C\sum_{i=0}^{\infty}((A + pA_1)^{-1}E)^i(A + pA_1)^{-1}Bs^i = -C\sum_{i=0}^{\infty}M_is^i.$$

$$M_0 = (A + pA_1)^{-1}B = \sum_{i_0=0}^{\infty}(-A^{-1}A_1)^{i_0}A^{-1}Bp^{i_0}.$$

$$M_1 = (A + pA_1)^{-1}EM_0 = \sum_{i_0,i_1=0}^{\infty}(-A^{-1}A_1)^{i_1}A^{-1}E(-A^{-1}A_1)^{i_0}A^{-1}Bp^{i_1}p^{i_0},$$

$$M_j = \sum_{i_0,\ldots,i_j=0}^{\infty}(-A^{-1}A_1)^{i_j}A^{-1}E\ldots(-A^{-1}A_1)^{i_0}A^{-1}Bp^{i_j}\ldots p^{i_0}$$

- Constructing

$$\text{span}\{V_0\} = \mathcal{K}_{i_0}(A^{-1}A_1, A^{-1}B),\ B_1 = E\mathcal{K}_{i_0},$$
$$\text{span}\{V_1\} = \mathcal{K}_{i_1}(A^{-1}A_1, A^{-1}B_1),\ B_2 = E\mathcal{K}_{i_1}, ...$$
$$\text{span}\{V\} = \text{span}\{V_0, V-1, ..., V_j\}, V^TV = I.$$

- The reduced system is like (5)
- The first $i_0, ..., i_j$ elements in the moments $M_0, ..., M_j$ are matched.
- Disavantage: order of reduced system is large due to mixed moments.
- Another approach is Two-directional Arnoldi process [8].

- Constructing

$$\mathrm{span}\{V_0\} = \mathcal{K}_{i_0}(A^{-1}A_1, A^{-1}B),\ B_1 = E\mathcal{K}_{i_0},$$
$$\mathrm{span}\{V_1\} = \mathcal{K}_{i_1}(A^{-1}A_1, A^{-1}B_1),\ B_2 = E\mathcal{K}_{i_1}, \ldots$$
$$\mathrm{span}\{V\} = \mathrm{span}\{V_0, V-1, \ldots, V_j\}, V^TV = I.$$

- The reduced system is like (5)
- The first $i_0, \ldots, i_j$ elements in the moments $M_0, \ldots, M_j$ are matched.
- Disavantage: order of reduced system is large due to mixed moments.
- Another approach is Two-directional Arnoldi process [8].

Cutting off mixed moments [6]

Denoting TF as function of frequence and system parameters and then expading at $(0, ..., 0)$

$$H(s_1, ..., s_k) = -C(I - \sum_{j=0}^{k} s_j E_j)^{-1} A^{-1} B$$
$$= -C \sum_{i=0}^{\infty} (\sum_{j=0}^{k} s_j A^{-1} E_j)^i A^{-1} B$$

- Only keep *pure moments* $C(A^{-1}E_j)^i S^{-1} B$,
- Decreasing the size of reduced system significantly,
- No estimation of the loss of information due to discarding.

Cutting off mixed moments [6]

Denoting TF as function of frequence and system parameters and then expading at $(0, ..., 0)$

$$H(s_1, ..., s_k) = -C(I - \sum_{j=0}^{k} s_j E_j)^{-1} A^{-1} B$$
$$= -C \sum_{i=0}^{\infty} (\sum_{j=0}^{k} s_j A^{-1} E_j)^i A^{-1} B$$

- Only keep *pure moments* $C(A^{-1}E_j)^i S^{-1} B$,
- Decreasing the size of reduced system significantly,
- No estimation of the loss of information due to discarding.

Enlarging the range of interpolation [7]

- Transfer function

$$H(s_1, ..., s_k) = -C(I - \sum_{j=0}^{k} s_j A^{-1} E_j)^{-1} A^{-1} B$$

$$= -C \sum_{i=0}^{\infty} (\sum_{j=0}^{k} s_j A^{-1} E_j)^i A^{-1} B$$

$$= -C(I + \sum A^{-1} E_j s_j + \sum A^{-1} E_j A^{-1} E_k s_j s_k + \cdots) A^{-}$$

- Instead of matching high order moments, matching the first order moments at different points.
- Decrease the size of reduced system considerably,
- Capture the behavior of original system on wide range.

Enlarging the range of interpolation [7]

- Transfer function

$$H(s_1, ..., s_k) = -C(I - \sum_{j=0}^{k} s_j A^{-1} E_j)^{-1} A^{-1} B$$

$$= -C \sum_{i=0}^{\infty} (\sum_{j=0}^{k} s_j A^{-1} E_j)^i A^{-1} B$$

$$= -C(I + \sum A^{-1} E_j s_j + \sum A^{-1} E_j A^{-1} E_k s_j s_k + \cdots) A^{-}$$

- Instead of matching high order moments, matching the first order moments at different points.
- Decrease the size of reduced system considerably,
- Capture the behavior of original system on wide range.

Enlarging the range of interpolation [7]

- Transfer function

$$H(s_1,...,s_k) = -C(I - \sum_{j=0}^{k} s_j A^{-1} E_j)^{-1} A^{-1} B$$

$$= -C \sum_{i=0}^{\infty} (\sum_{j=0}^{k} s_j A^{-1} E_j)^i A^{-1} B$$

$$= -C(I + \sum A^{-1} E_j s_j + \sum A^{-1} E_j A^{-1} E_k s_j s_k + \cdots) A^-$$

- Instead of matching high order moments, matching the first order moments at different points.
- Decrease the size of reduced system considerably,
- Capture the behavior of original system on wide range.

Recall the method

For the system (3) with $E = I$, first solve Lyapunov equations

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0, \; A^T\mathcal{Q} + \mathcal{Q}A + C^TC = 0.$$

The balancing transformation:

$$\tilde{\mathcal{P}} = \tilde{\mathcal{Q}} = \Sigma = diag(\sigma_1, ..., \sigma_k, ..., \sigma_n)$$

For an observable, reachable and stable system, there exists

$$T = \Sigma^{1/2}K^TU^{-1}, T^{-1} = UK\Sigma^{-1/2},$$

where $\tilde{\mathcal{P}} = UU^T, U^T\tilde{\mathcal{Q}}U = K\Sigma^2K^T$.

Very difficult to extend directly to parametric systems

Recall the method

For the system (3) with $E = I$, first solve Lyapunov equations

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0, \; A^T\mathcal{Q} + \mathcal{Q}A + C^TC = 0.$$

The balancing transformation:

$$\tilde{\mathcal{P}} = \tilde{\mathcal{Q}} = \Sigma = diag(\sigma_1, ..., \sigma_k, ..., \sigma_n)$$

For an observable, reachable and stable system, there exists

$$T = \Sigma^{1/2}K^TU^{-1}, T^{-1} = UK\Sigma^{-1/2},$$

where $\tilde{\mathcal{P}} = UU^T, U^T\tilde{\mathcal{Q}}U = K\Sigma^2K^T$.
Very difficult to extend directly to parametric systems

Combining with interpolation [2]

$$
\begin{aligned}
x'(t) &= A(p)x(t) + B(p)u(t),\\
y(t) &= C(p)x(t), p = (p_1, ..., p_d) \in \Omega \subset \mathbb{R}^d.
\end{aligned}
\tag{6}
$$

- Suppose (6) stable, observable, reachable and choosing interpolation points $p^{(1)}, ..., p^{(k)}$ and reducing (6) at $p^{(i)}$ by BT, yielding TF $H_r(p^{(i)}, s)$.
- Interpolating TF w.r.t parameter

$$\widehat{H(p,s)} = \sum_{i=0}^{k} f_i(p) H_r(p^{(i)}, s).$$

- Constructing a realization for TF $\widehat{H(p,s)}$.

Combining with interpolation [2]

$$x'(t) = A(p)x(t) + B(p)u(t),$$
$$y(t) = C(p)x(t), p = (p_1, ..., p_d) \in \Omega \subset \mathbb{R}^d. \quad (6)$$

- Suppose (6) stable, observable, reachable and choosing interpolation points $p^{(1)}, ..., p^{(k)}$ and reducing (6) at $p^{(i)}$ by BT, yielding TF $H_r(p^{(i)}, s)$.
- Interpolating TF w.r.t parameter

$$\widehat{H(p,s)} = \sum_{i=0}^{k} f_i(p) H_r(p^{(i)}, s).$$

- Constructing a realization for TF $\widehat{H(p,s)}$.

More on theoretical aspects [3]

Problems

- Stability preservation: Is the reduced system is stable for all $p = (p_1, ..., p_d) \in \Omega$?
- Does there exist an error bound?
- Order of reduced system remains high $\sum n_i$

More on theoretical aspects [3]

Problems

- Stability preservation: Is the reduced system is stable for all $p = (p_1, ..., p_d) \in \Omega$?
- Does there exist an error bound?
- Order of reduced system remains high $\sum n_i$

More on theoretical aspects [3]

Problems

- Stability preservation: Is the reduced system is stable for all $p = (p_1, ..., p_d) \in \Omega$?
- Does there exist an error bound?
- Order of reduced system remains high $\sum n_i$

More on theoretical aspects [3]

Problems

- Stability preservation: Is the reduced system is stable for all $p = (p_1, ..., p_d) \in \Omega$?
- Does there exist an error bound?
- Order of reduced system remains high $\sum n_i$

More on theoretical aspects [3]

- Preserving stability,
- Existing an error bound w.r.t. supnorm,

$$\|H(p,s)\|_{\infty} = \sup_{p\in\Omega} \|H(p,s)\|_{\mathcal{H}_{\infty}}.$$

More on theoretical aspects [3]

- Preserving stability,
- Existing an error bound w.r.t. supnorm,

$$\|H(p,s)\|_{\infty} = \sup_{p \in \Omega} \|H(p,s)\|_{\mathcal{H}_{\infty}}.$$

Remarks

- Moment matching based are preferable,
- Linearize the nonlinear-dependtdence before using;
- Despite of high cost of computation, BT based suitable for estimating an error bound,
- Problem of a priori error bound.

End

Thank you!

- [Antoulas05] A. C. Antoulas, *Approximation of Large-scale Dynamical Systems,* Advances in Design and Control DC-06, SIAM, Philadenphia, 2005.
- [BB08] U. Baur, P. Benner, "Parametric model reduction with sparse grids", GMA FA Workshop, 2008.
- [BS09] A. Bunse-Gerstner, N.T. Son, "On theoretical aspects of parametric model order reduction with balanced truncation", *to appear*.
- [Daniel04] L. Daniel, O. C. Siong, L. S. Chay, K. H. Lee, J. White, "A multiparameter moment-matching model-reduction approach for generating geometrically parameterized interconnect performance models", *IEEE Tran. computer-aided design of integrated circuits and systems*, Vol. 23, No. 5, pp. 678-693 2004.

- [Feng05] L. H. Feng, E. B. Rudnyi, J. G. Korvink, "Preserving the film coefficient as a parameter in the compact thermal model for fast electrothermal simulation", *IEEE Tran. computer-aided design of integrated circuits and systems*, Vol. 24, No. 12, pp. 1838-1847, 2005.
- [Moosman07] C. Moosmann, *ParaMOR- Model Order Reduction for Parameterized MEMS applications,* PhD Thesis, IMTEK, Albert-Ludwigs-University of Freiburg, 2007.
- [Li05] P. Li, F. Liu, X. Li, L. T. Pileggi, S. R. Nassif, "Modeling interconnect using efficient parametric model order reduction", Proc. Desi. Auto. Test Euro. Conf. Exhi, 2005.
- [Li08] Y. T. Li, Z. Bai, Y. Su, X. Zeng, "Model order reduction of parameterized interconnect networks via a

two-directional Arnoldi process", IEEE Trans. Com.-Aid Des. Int. Cir. Sys., Vol. 27, No. 9, pp. 1571-1582, 2008.

[Weile99] D. S. Weile, E. Michielssen, E. J. Grimme, K. Gallivan, "A method for generating rational interpolant reduced order models of two-parameter linear systems", *Appl. Math. Lett.*, Vol. 12, pp. 93-102, 1999.