Input-to-state dynamical stability of interconnected systems

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***Abstract*— We consider networks of input-to-state dynamically stable (ISDS) systems and provide a small gain condition under which the entire network is again ISDS. A Lyapunov formulation of the nonlinear small gain theorem for two interconnected ISDS systems is proved. It provides a constructive method to find an ISDS Lyapunov function for such an interconnection.**

I. INTRODUCTION

In this paper we consider nonlinear systems of the form

$$\dot{x}(t) \;=\; f(x(t), u(t)), \tag{1}$$

where $t \in \mathbb{R}$ is the time, $\dot{x}(t)$ denotes the derivative of the state $x(t) \in \mathbb{R}^N$ with initial value x_0, input $u(t) \in \mathbb{R}^m$ is an essentially bounded measureable function and $f : \mathbb{R}^{N+m} \to \mathbb{R}^N$, $N, m \in \mathbb{N}$. To have existence and uniqueness of a solution of (1), let the function f be continuous and locally Lipschitz in x uniformly in u. The solution is denoted by $x(t; x_0, u)$ or $x(t)$ in short.

Stability analysis of such systems can be performed in different frameworks such as passivity, dissipativity [7], input-to-state (ISS) stability in [8] and others. We will use the notion of input-to-state dynamical (ISDS) stablility introduced in [4] and [3] respectively. This property is equivalent to ISS, however the advantage of ISDS over ISS is that the bound for the trajectories takes essentially only the recent values of the input u into account and in many cases it gives a better bound due to the *memory fading effect* of the disturbing input. Moreover the gains in the trajectory based definition of ISDS are the same as in the definition of the ISDS-Lyapunov function, which is in general not true for the ISS systems.

We are interested in interconnections of such systems

$$\dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), u(t)),\ i = 1, \ldots, n, \tag{2}$$

where $n \in \mathbb{N}$, $x_i(t) \in \mathbb{R}^{N_i}$, $f_i : \mathbb{R}^{\sum_{j=1}^n N_j + m} \to \mathbb{R}^{N_i}$, and each subsystem is ISDS. The question arises under which conditions the whole system of the form (1) with $x = \left(x_1^T, \ldots, x_n^T\right)^T$, $f(x, u) = \left(f_1(x, u)^T, \ldots, f_n(x, u)^T\right)^T$ is ISDS with respect to the state x and input u.

The stability condition for an interconnection of two ISS systems was developed in [5] and [6]. In [2] a small gain theorem for $n \in \mathbb{N}$ interconnected systems was proved. Since ISS Lyapunov functions are an important tool to verify the ISS property, a Lyapunov formulation of the small gain theorem was given for two interconnected systems in [6]. There, an explicit construction of the ISS Lyapunov function of the whole system was shown. In [1] an explicit construction of an ISS Lyapunov function for the overall system of n interconnected systems was derived under a sufficient small gain condition.

The equivalence of ISDS of system (1) and the existence of an ISDS Lyapunov function for system (1) was proved in [4]. Also an ISDS small gain theorem for two interconnected systems with no input u can be found in [4].

The purpose of this paper is to extend the mentioned results for ISS systems for the case of ISDS systems. In particular we present a small gain theorem for $n \in \mathbb{N}$ interconnected ISDS systems of the form (2) and provide a construction of an ISDS Lyapunov function for the entire system consisting of two interconnected ISDS systems under a small gain condition.

The organisation of this paper is the following: The next section introduces necessary notions. Section III contains the main results of the paper. The conclusions and future directions of research are in Section IV.

II. NOTATION AND DEFINITIONS

By x^T we denote the transposition of a vector $x \in \mathbb{R}^n$, $n \in \mathbb{N}$, furthermore $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_+^n$ denotes the positive orthant $\{x \in \mathbb{R}^n : x \geq 0\}$ where we use the standard partial order for $x, y \in \mathbb{R}^n$ given by

$$x \geq y \Leftrightarrow x_i \geq y_i,\ i = 1, \ldots, n \text{ and } x \not\geq y \Leftrightarrow \exists i : x_i < y_i.$$

We denote the standard euclidian norm in $\mathbb{R}^n$ by $|\cdot|$ and the essential supremum norm of a function f by $\|f\|$. ∇V denotes the gradient of a function V. For a function $v : \mathbb{R}_+ \to \mathbb{R}^m$ we define its restriction to the interval $[s_1, s_2]$ by

$$v_{[s_1,s_2]}(t) := \begin{cases} v(t) & \text{if } t \in [s_1, s_2], \\ 0 & \text{otherwise,} \end{cases} \qquad t,\ s_1,\ s_2 \in \mathbb{R}_+.$$

Definition 2.1: We define following classes of functions:

$$\begin{aligned}
\mathcal{P} &:= \{f : \mathbb{R}^n \to \mathbb{R}_+ \mid f(0) = 0,\ f(x) > 0,\ x \neq 0\} \\
\mathcal{K} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, } \gamma(0) = 0 \\
&\qquad \text{and strictly increasing}\} \\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\} \\
\mathcal{L} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly} \\
&\qquad \text{decreasing with } \lim_{t\to\infty} \gamma(t) = 0\} \\
\mathcal{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous,} \\
&\qquad \beta(\cdot, t) \in \mathcal{K},\ \beta(r, \cdot) \in \mathcal{L},\ \forall t, r \geq 0\} \\
\mathcal{KLD} &:= \{\mu \in \mathcal{KL} \mid \mu(r, t + s) = \mu(\mu(r, t), s), \forall r, t, s \geq 0\}
\end{aligned}$$

We will call functions of class $\mathcal{P}$ *positive definite*.

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Remark 2.2: Condition $\mu(r,t+s) = \mu(\mu(r,t),s)$ includes $\mu(r,0) = r, \forall\, r \geq 0$, which can be easily checked.

Note that, if $\gamma \in \mathcal{K}_\infty$, then there exist the inverse function $\gamma^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ with $\gamma^{-1} \in \mathcal{K}_\infty$.

Definition 2.3: System (1) is called *input-to-state stable (ISS)*, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$|x(t; x_0, u)| \leq \max\{\beta(|x_0|, t), \gamma^{\text{ISS}}(\|u\|)\} \tag{3}$$

$\forall\, x_0 \in \mathbb{R}^N,\ t \in \mathbb{R}_+$ and any input u. γ^{ISS} is called *gain*.

This concept has been first introduced in [8], where an equivalent formulation with sum of the both terms instead of max in (3) has been used. It is known for ISS systems that if $\limsup_{t\to\infty} u(t) = 0$ then also $\lim_{t\to\infty} x(t) = 0$ holds. However (3) provides only a finite positive bound for $u \not\equiv 0$. The following stability concept was introduced in [4] and [3] respectively:

Definition 2.4: System (1) is called *input-to-state dynamically stable (ISDS)*, if there exist functions $\mu \in \mathcal{KLD}$, $\eta,\ \gamma^{\text{ISDS}} \in \mathcal{K}_\infty$ such that

$$|x(t; x_0, u)| \leq \max\{\mu(\eta(|x_0|), t), \nu(u,t)\} \tag{4}$$

$\forall\, t \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^N$ and any input $u \in \mathbb{R}^m$, where

$$\nu(u,t) := \operatorname*{ess\,sup}_{\tau\in[0,t]} \mu(\gamma^{\text{ISDS}}(|u(\tau)|), t-\tau),$$

μ is called *decay rate*, η *overshoot gain* and γ^{ISDS} *robustness gain*.

Note that for large t the bound (4) takes essentially only the recent values of the input u into account. And in particular it follows immediately from (4) that $\limsup_{t\to\infty} u(t) = 0 \Rightarrow \lim_{t\to\infty} x(t) = 0$.

Remark 2.5: Since $\mu(\eta(r),t)$ is a $\mathcal{KL}$-function from ISDS follows ISS with

$$\beta(r,t) := \mu(\eta(r),t),\ r,t \geq 0 \ \text{ and } \ \gamma^{\text{ISDS}} = \gamma^{\text{ISS}}.$$

Theorem 2.6: Assume system (1) is ISS with a $\mathcal{KL}$-function β and $\gamma^{\text{ISS}} \in \mathcal{K}_\infty$. Then for each $\mathcal{K}_\infty$-function γ^{ISDS} with $\gamma^{\text{ISDS}}(r) > \gamma^{\text{ISS}}(r),\ \forall r > 0$ there exists a $\mathcal{KLD}$-function μ such that system (1) is ISDS.

The proof can be found in [3]. Combining Remark 2.5 and Theorem 2.6, ISDS is equivalent to ISS.

In the rest of the paper we assume:

Assumption 2.7: Functions $\mu,\ \eta$ and γ^{ISDS} in Definition 2.4 are C^∞ in $\mathbb{R}_+ \times \mathbb{R}$ or $\mathbb{R}_+$ respectively.

Remark 2.8: For given nonsmooth rates and gains from Definition 2.4 one can find rates and gains arbitrarily close to the original ones, such that Assumption 2.7 and Definition 2.4 remains valid. Hence Assumption 2.7 is rather mild. (See [3] Appendix B for details.)

An important tool for the stability analysis of system (1) are Lyapunov functions. It is known that ISS implies the existence of an ISS Lyapunov function for system (1) (see [9]). A similar result for ISDS systems was proved in [4]:

Theorem 2.9: System (1) is ISDS with $\mu \in \mathcal{KLD}$ and $\eta,\ \gamma^{\text{ISDS}} \in \mathcal{K}_\infty$, which satisfy Assumption 2.7, if and only if for each $\varepsilon > 0$ there exists an ISDS Lyapunov function V, i.e., $V : \mathbb{R}^N \to \mathbb{R}_+$ is smooth on $\mathbb{R}^N \setminus \{0\}$ and satisfies for each $\varepsilon > 0$

$$(1-\varepsilon)\,|x| \leq V(x) \leq (1+\varepsilon)\eta\,(|x|) \tag{5}$$

$$\begin{aligned} &V(x) \geq \gamma^{\text{ISDS}}\left((1+\varepsilon)\,|u|\right) \\ \Rightarrow &\nabla V(x)\cdot f(x,u) \leq -\,(1-\varepsilon)\,g\,(V(x)) \end{aligned} \tag{6}$$

$\forall\ x \in \mathbb{R}^N \setminus \{0\}$ and all $u \in \mathbb{R}^m$, where function μ solves the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu(r,t) = -g\,(\mu\,(r,t))\,, \quad r,t > 0 \tag{7}$$

for a locally Lipschitz continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$.

Remark 2.10: An advantage of Theorem 2.9 to a corresponding theorem in the case of ISS is that the decay rate μ and gains $\eta,\ \gamma^{\text{ISDS}}$ in Definition 2.4 are exactly the same as in (5), (6) and (7) respectively, whereas in the case of ISS the gain defined in terms of trajectories (Definition 2.3) and the ISS Lyapunov gain are different in general.

For the main results we use locally Lipschitz continuous ISDS Lyapunov functions, which are differentiable almost everywhere (a.e.).

Definition 2.11: A function $V : \mathbb{R}^N \to \mathbb{R}_+$, which is locally Lipschitz continuous on $\mathbb{R}^N \setminus \{0\}$ is called *ISDS Lyapunov function of system (1)*, if there exist $\gamma^{\text{ISDS}}, \eta \in \mathcal{K}_\infty,\ \mu \in \mathcal{KLD}$ and V satisfies for each $\varepsilon > 0$

$$\frac{|x|}{1+\varepsilon} \leq V(x) \leq \eta\,(|x|)\,,\ \forall x \in \mathbb{R}^N \setminus \{0\}\,, \tag{8}$$

$$V(x) > \gamma^{\text{ISDS}}\,(|u|) \Rightarrow \nabla V(x)\cdot f(x,u) \leq -\,(1-\varepsilon)\,g\,(V(x)) \tag{9}$$

for almost all $x \in \mathbb{R}^N \setminus \{0\}$ and all $u \in \mathbb{R}^m$, where function g is locally Lipschitz continuous and μ solves (7).

The following theorem is a counterpart of Theorem 2.9 for the case of nonsmooth ISDS Lyapunov functions

Theorem 2.12: System (1) is ISDS with $\mu \in \mathcal{KLD}$ and $\eta,\ \gamma^{\text{ISDS}} \in \mathcal{K}_\infty$, which satisfy Assumption 2.7, if and only if there exists a locally Lipschitz continuous ISDS Lyapunov function V as in Definition 2.11 with $\mu \in \mathcal{KLD}$ and $\eta,\ \gamma^{\text{ISDS}} \in \mathcal{K}_\infty$.

Proof: "$\Rightarrow$": This is Lemma 16 in [4].
"$\Leftarrow$": Fix $x \in \mathbb{R}^N$ and $t > 0$. Integrating (9), we obtain

$$\begin{aligned} &V(x(t,x,u)) \leq \mu(V(x),t),\ \text{for all } u \in \mathbb{R}^m \text{ with} \\ &\gamma^{\text{ISDS}}(|u(\tau)|) \leq \mu(V(x),t),\ \text{for almost all } \tau \in [0,t], \end{aligned} \tag{10}$$

where μ solves $\dot{\mu} = -g(\mu),\ \mu(r,0) = r$. Now (10) implies $V(x(t,x,u)) \leq \max \mu(V(x), t, \nu(u,t))$, which follows similarly to the proof of Lemma 15 and with Lemma 13 in [4]. By application of Theorem 4 in [4] the assertion follows. ■

In order to have ISDS Lyapunov functions with more regularity one can use Lemma 17 in [4], which shows that for a locally Lipschitz function V there exists a smooth function $\tilde{V}$ arbitrary close to V.

Now we consider interconnected systems of the form (2).

Definition 2.13: We call the *i-th subsystem of (2) ISS*, if there exists a $\mathcal{KL}$-function β_i and functions $\gamma_i^{\text{ISS}}, \gamma_{ij}^{\text{ISS}} \in \mathcal{K}_\infty \cup \{0\}\,,\ i,j = 1,\ldots,n$ with $\gamma_{ii}^{\text{ISS}} = 0$ such that the solution

$x_i(t;x_i^0,u) = x_i(t)$ of the i-th subsystem with any initial value $x_i(0)=x_i^0$ and any inputs x_j, u satisfies

$$|x_i(t)| \le \max[\beta_i\left(\left|x_i^0\right|,t\right), \max_j \gamma_{ij}^{\text{ISS}}(||x_{j\,[0,t]}||), \gamma_i^{\text{ISS}}(\|u\|)] \tag{11}$$

for all $t \in \mathbb{R}_+$. Functions γ_{ij}^{ISS} and γ_i^{ISS} are called (nonlinear) gains. Furthermore we define the gain matrix $\Gamma^{\text{ISS}} := (\gamma_{ij}^{\text{ISS}}),\ i,j=1,\dots,n$ and the map $\Gamma^{\text{ISS}}:\ \mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$\Gamma^{\text{ISS}}(s) := (\max_j \gamma_{1j}(s_j),\dots,\max_j \gamma_{nj}(s_j))^T,\ s\in\mathbb{R}_+^n. \tag{12}$$

Definition 2.14: For vector valued functions $x = (x_1^T,\dots,x_n^T)^T : \mathbb{R}_+ \to \mathbb{R}^{\sum_{i=1}^n N_i}$ with $x_i:\mathbb{R}_+\to\mathbb{R}^{N_i}$ and times $0\le t_1\le t_2,\ t\in\mathbb{R}_+$ we define

$$\begin{aligned}\|x_{[t_1,t_2]}\| &:= \left(\|x_{1,[t_1,t_2]}\|,\dots,\|x_{n,[t_1,t_2]}\|\right)^T \in \mathbb{R}_+^n,\\ |x(t)| &:= (|x_1(t)|,\dots,|x_n(t)|)^T \in\mathbb{R}_+^n.\end{aligned}$$

For $u\in\mathbb{R}^m$, $t\in\mathbb{R}_+$ and $s\in\mathbb{R}_+^n$ we define

$$\begin{aligned}\gamma^{\text{ISS}}(\|u\|) &:= (\gamma_1^{\text{ISS}}(\|u\|),\dots,\gamma_n^{\text{ISS}}(\|u\|))^T \in\mathbb{R}_+^n\\ \beta(s,t) &:= (\beta_1(s_1,t),\dots,\beta_n(s_n,t))^T \in\mathbb{R}_+^n,\end{aligned}$$

where $\gamma_i^{\text{ISS}},\beta_i,\ i=1,\dots,n$ are from (11).

Now we can rewrite (11) for $t\in\mathbb{R}_+$ and any initial value $x^0=x(0)=((x_1^0)^T,\dots,(x_n^0)^T)^T$ and any input u as

$$|x(t)| \le \max[\beta\left(|x^0|,t\right), \Gamma^{\text{ISS}}\left(\|x_{[0,t]}\|\right), \gamma^{\text{ISS}}(\|u\|)]. \tag{13}$$

Note that the maximum used in (13) for vectors is taken componentwise.

Definition 2.15: We call the *i-th subsystem of (2) ISDS*, if there exists a $\mathcal{KLD}$-function μ_i and functions η_i, γ_i^{ISDS} and $\gamma_{ij}^{\text{ISDS}} \in \mathcal{K}_\infty\cup\{0\},\ i,j=1,\dots,n$ with $\gamma_{ii}^{\text{ISDS}}=0$ such that the solution $x_i(t,x_i^0,u)=x_i(t)$ with any initial value $x_i(0)=x_i^0$ and any inputs $x_j,\ u$ satisfies

$$|x_i(t)| \le \max[\mu_i\left(\eta_i\left(\left|x_i^0\right|\right),t\right), \max_j\{\nu_{ij}(x_j,t)\}, \nu_i(u,t)] \tag{14}$$

for all $t\in\mathbb{R}_+$, where

$$\begin{aligned}\nu_i(u,t) &:= \operatorname*{ess\,sup}_{\tau\in[0,t]} \mu_i(\gamma_i^{\text{ISDS}}(|u(\tau)|),t-\tau)\\ \nu_{ij}(x_j,t) &:= \operatorname*{ess\,sup}_{\tau\in[0,t]} \mu_i(\gamma_{ij}^{\text{ISDS}}(|x_j(\tau)|),t-\tau)\end{aligned}$$

$i,j=1,\dots,n$. $\gamma_{ij}^{\text{ISDS}},\ \gamma_i^{\text{ISDS}}$ are called (nonlinear) robustness gains. The ISDS gain matrix Γ^{ISDS} is defined by $\Gamma^{\text{ISDS}} := \left(\gamma_{ij}^{\text{ISDS}}\right),\ i,j=1,\dots,n$ and the map $\Gamma^{\text{ISDS}}:\mathbb{R}_+^n\to\mathbb{R}_+^n$ by

$$\Gamma^{\text{ISDS}}(s) := (\max_j \gamma_{1j}^{\text{ISDS}}(s_j),\dots,\max_j \gamma_{nj}^{\text{ISDS}}(s_j))^T,\ s\in\mathbb{R}_+^n. \tag{15}$$

Note that by $\gamma_{ij}^{\text{ISDS}}\in\mathcal{K}_\infty\cup\{0\}$ and for $v,\ w\in\mathbb{R}_+^n$ we get

$$v\ge w \ \Rightarrow\ \Gamma^{\text{ISDS}}(v)\ge\Gamma^{\text{ISDS}}(w). \tag{16}$$

Definition 2.16: For $u\in\mathbb{R}^m$, $t\in\mathbb{R}_+$ and $s\in\mathbb{R}_+^n$ denote

$$\begin{aligned}\gamma^{\text{ISDS}}(|u(t)|) &:= (\gamma_1^{\text{ISDS}}(|u(t)|),\dots,\gamma_n^{\text{ISDS}}(|u(t)|))^T \in\mathbb{R}_+^n,\\ \mu(s,t) &:= (\mu_1(s_1,t),\dots,\mu_n(s_n,t))^T\in\mathbb{R}_+^n,\\ \eta(s) &:= (\eta_1(s_1),\dots,\eta_n(s_n))^T \in\mathbb{R}_+^n.\end{aligned}$$

Now we can rewrite condition (14) in the form

$$\begin{aligned}&|x(t)|\\ &\le \max[\mu\left(\eta\left(|x^0|\right),t\right), \operatorname*{ess\,sup}_{\tau\in[0,t]}\mu(\Gamma^{\text{ISDS}}(|x(\tau)|),t-\tau),\\ &\qquad \operatorname*{ess\,sup}_{\tau\in[0,t]}\mu(\gamma^{\text{ISDS}}(|u(\tau)|),t-\tau)]\end{aligned}$$

for all $t\in\mathbb{R}_+$.

If we define $N:=N_1+\ldots+N_n$, $x:=(x_1^T,\dots,\ x_n^T)^T$ and $f:=(f_1^T,\dots,\ f_n^T)^T$, then (2) becomes

$$\dot{x}(t) = f(x(t),u(t)),\ t\in\mathbb{R}_+. \tag{17}$$

Now we are interested in conditions guaranteeing that the whole system (17) is ISDS with respect to the input u and state x. The next section provides an ISDS small gain theorem for general networks and as the second result a Lyapunov formulation of the small gain theorem for two interconnected systems with an explicit construction of the ISDS Lyapunov function and corresponding gains and decay rate of the whole system.

III. MAIN RESULTS

A. ISDS small gain theorem

Recall that the small gain theorem for two interconnected ISS systems was proved in [5]. In [2] this result was extended for the case of $n\ge 2$ interconnected ISS systems:

Theorem 3.1: Consider system (2) and suppose each subsystem is ISS, i.e. condition (11) is satisfied for all $i=1,\dots,n,\ n\in\mathbb{N}$. Let Γ^{ISS} be given by (12). If

$$\Gamma^{\text{ISS}}(s)\not\ge s,\ \forall\ s\in\mathbb{R}_+^n\setminus\{0\} \tag{18}$$

then the whole system (17) is ISS from u to x.

The first main result of this paper is the small gain theorem for $n\ge 2$ interconnected ISDS systems:

Theorem 3.2: Consider system (2) and suppose each subsystem is ISDS, i.e., condition (14) holds for all $i=1,\dots,n$. Let Γ^{ISDS} be given by (15). If

$$\Gamma^{\text{ISDS}}(s)\not\ge s,\ \forall\ s\in\mathbb{R}_+^n\setminus\{0\}$$

holds then system (17) is ISDS from u to x.

Proof: Each subsystem of (2) is ISDS. Since $\mu\in\mathcal{KLD}$, $\gamma^{\text{ISDS}}\in\mathcal{K}_\infty$, and by application of Remark 2.2 and (16) we get

$$\begin{aligned}&|x(t)|\\ &\le \max[\mu\left(\eta\left(|x^0|\right),t\right), \operatorname*{ess\,sup}_{\tau\in[0,t]}\mu(\Gamma^{\text{ISDS}}(|x(\tau)|),t-\tau),\\ &\qquad \operatorname*{ess\,sup}_{\tau\in[0,t]}\mu(\gamma^{\text{ISDS}}(|u(\tau)|),t-\tau)]\\ &\le \max[\mu\left(\eta\left(|x^0|\right),t\right), \operatorname*{ess\,sup}_{\tau\in[0,t]}\mu(\Gamma^{\text{ISDS}}(|x(\tau)|),0),\\ &\qquad \operatorname*{ess\,sup}_{\tau\in[0,t]}\mu(\gamma^{\text{ISDS}}(|u(\tau)|),0)]\\ &\le \max[\beta\left(|x^0|,t\right), \Gamma^{\text{ISDS}}(\|x\|), \gamma^{\text{ISDS}}(\|u\|)],\end{aligned}$$

where $\beta\left(\left\| x^0\right\|, t\right) := \mu\left(\eta\left(\left\| x^0\right\|\right), t\right)$. Now set $\Gamma^{\text{ISS}} := \Gamma^{\text{ISDS}}$ and $\gamma^{\text{ISS}} := \gamma^{\text{ISDS}}$ and we obtain

$$\left\| x(t)\right\| \leq \max[\beta\left(\left\| x^0\right\|, t\right), \Gamma^{\text{ISS}}(\|x\|), \gamma^{\text{ISS}}(\|u\|)].$$

This is an ISS condition in the sense of (13) and in addition

$$\Gamma^{\text{ISDS}}(s) = \Gamma^{\text{ISS}}(s) \not\geq s,\ \forall\ s \in \mathbb{R}^n_+ \setminus \{0\}.$$

With application of Theorem 3.1, the whole system (17) is ISS with some $\bar{\beta}(r,t) \in \mathcal{KL}$ and $\bar{\gamma}^{\text{ISS}}(r) \in \mathcal{K}_\infty$. By Theorem 2.6 for each $\mathcal{K}_\infty$-function $\bar{\gamma}^{\text{ISDS}}(r) > \bar{\gamma}^{\text{ISS}}(r)$ for all $r > 0$ there exists a $\mathcal{KLD}$-function $\bar{\mu}$ such system (17) is ISDS. ■

Unfortunately we loose quantitative information of the ISDS gains of the whole system in the proof of Theorem 3.1. In order to conserve the quantitative information of the ISDS gains of the overall system we prove as the second main result an ISDS small gain theorem using ISDS Lyapunov functions in the following section.

B. ISDS Lyapunov formulation of the small gain theorem

In this section we provide a Lyapunov version of the ISDS small gain theorem for two interconnected systems.

For the main result in this section we consider system (2) with $n = 2$ and define the ISDS Lyapunov functions of the subsystems:

Definition 3.3: A function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$, which is locally Lipschitz continuous on $\mathbb{R}^N_i \setminus \{0\}$ is called *ISDS Lyapunov function of the i-th subsystem of system (2)* for $i = 1, 2$, if it satisfies:

(i) There exist functions $\eta_i \in \mathcal{K}_\infty$ and constants ε_i such that

$$\frac{|x_i|}{1+\varepsilon_i} \leq V_i(x_i) \leq \eta_i\left(|x_i|\right) \tag{19}$$

for all $x_i \in \mathbb{R}^{N_i} \setminus \{0\}$.

(ii) There exist functions $\mu_i \in \mathcal{KLD}$, $\gamma_i^{\text{ISDS}} \in \mathcal{K}_\infty$ and $\gamma_{ij}^{\text{ISDS}} \in \mathcal{K}_\infty,\ i,j = 1,2,\ i \neq j$ such that

$$V_i(x_i) > \max[\gamma_i^{\text{ISDS}}\left(|u|\right), \gamma_{ij}^{\text{ISDS}}\left(V_j(x_j)\right)]$$

$$\Rightarrow\ \nabla V_i(x_i) f_i(x_i, x_j, u) \leq -\left(1-\varepsilon_i\right) g_i(V_i(x_i)), \tag{20}$$

$1 > \varepsilon_i > 0$, for almost all $x_i \in \mathbb{R}^{N_i} \setminus \{0\}$ and all $u \in \mathbb{R}^m$, where function $\mu_i \in \mathcal{KLD}$ solves the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_i(r,t) = -g_i\left(\mu_i\left(r,t\right)\right),\ r,t > 0$$

for locally Lipschitz functions $g_i : \mathbb{R}_+ \to \mathbb{R}_+,\ i = 1, 2$.

For the proof of the main result in this section we will need the following lemma.

Lemma 3.4: Let $\gamma_1 \in \mathcal{K}$ and $\gamma_2 \in \mathcal{K}_\infty$ such that $\gamma_1(r) < \gamma_2(r),\ \forall r > 0$. Then there exists a $\mathcal{K}_\infty$-function σ such that

(i) $\gamma_1(r) < \sigma(r) < \gamma_2(r)$ for all $r > 0$,

(ii) $\sigma(r)$ is continuous differentiable in $(0, \infty)$ and $\sigma'(r) > 0$ for all $r > 0$.

The proof can be found in [6].

The second main result gives an explicit construction of an locally Lipschitz ISDS Lyapunov function of two interconnected ISDS systems under a small gain condition.

Theorem 3.5: We consider system (2) for $n = 2$. Assume that for $i = 1, 2$ each subsystem of (2) is ISDS, functions $\mu_i,\ \eta_i, \gamma_i^{\text{ISDS}}$ and $\gamma_{ij}^{\text{ISDS}}$ satisfy Assumption 2.7 and for each subsystem there exists an ISDS Lyapunov function $V_i,\ i = 1, 2$, which satisfies (19) and (20). If

$$\gamma_{12}^{\text{ISDS}} \circ \gamma_{21}^{\text{ISDS}}\left(r\right) < r,\quad \forall\ r > 0, \tag{21}$$

then there exists an locally Lipschitz ISDS Lyapunov function for the whole system (17) of the form

$$V(x) = \psi^{-1}\left(\max\left\{\sigma\left(V_1(x_1)\right), V_2(x_2)\right\}\right),$$

i.e., conditions (8) and (9) hold true with $\varepsilon = \max\left\{\varepsilon_1, \varepsilon_2\right\}$, $\psi\left(t\right) = \min_i \sigma_i\left(t\right),\ t \in \mathbb{R}_+,\ \sigma_1(r) = \sigma(r),\ \sigma_2(r) = \mathrm{Id}(r) = r,\ r \in \mathbb{R}_+$, where σ is as in Lemma 3.4 for $\gamma_1(r) = \gamma_{21}^{\text{ISDS}}\left(r\right)$ and $\gamma_2(r) = \left(\gamma_{12}^{\text{ISDS}}\right)^{-1}(r),\ r > 0$. Furthermore the whole system (17) is ISDS with

$$\begin{aligned} g(r) &= (\psi^{-1})'\left(\psi(r)\right)\min\left\{\hat{g}_1(\psi(r)), g_2(\psi(r))\right\},\ r > 0,\\ \eta(r) &= \psi^{-1}\left(\max\left\{\sigma\left(\eta_1(r)\right), \eta_2(r)\right\}\right),\ r > 0, \\ \gamma^{\text{ISDS}}(r) &= \psi^{-1}\left(\max\left\{\sigma\left(\gamma_1^{\text{ISDS}}\left(r\right)\right), \gamma_2^{\text{ISDS}}(r)\right\}\right),\ r > 0. \end{aligned} \tag{22}$$

where $\hat{g}_1(r) = \sigma'\left(\sigma^{-1}(r)\right) g_1\left(\sigma^{-1}(r)\right)$ and μ solves the ordinary differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\mu(r,t) = -g\left(\mu(r,t)\right)$.

Remark 3.6: The small gain condition (21) we used here is without an operator D, which is necessary if the ISS property is defined in terms of sum over gains instead of the maximum (see [2], Section 4.3). Furthermore (21) is equivalent to

$$\gamma_{21}^{\text{ISDS}} \circ \gamma_{12}^{\text{ISDS}}\left(r\right) < r,\quad \forall\ r > 0,$$

which can be easily checked.

The proof of Theorem 3.5 follows the idea of the proof of Theorem 3.1 in [6] with corresponding changes to construct the gains and rate of the whole system as in (22).

Proof: First we define a function $\widetilde{V}$ for the whole system, which consists of ISDS Lyapunov functions of the subsystems. With this definition we construct a function V, which satisfies the conditions from Theorem 2.9 such that the whole system is ISDS.

From (21) and application of Lemma 3.4 to $\gamma_{21}^{\text{ISDS}}$ and $\left(\gamma_{12}^{\text{ISDS}}\right)^{-1}$ we know that there exists a continuous differentiable in $(0, \infty)$ function $\sigma \in \mathcal{K}_\infty$ with $\sigma'(r) > 0$ for all $r > 0$ such that

$$\gamma_{21}^{\text{ISDS}}(r) < \sigma(r) < \left(\gamma_{12}^{\text{ISDS}}\right)^{-1}(r),\ \forall\ r > 0. \tag{23}$$

We define

$$\widetilde{V}(x_1, x_2) := \max\left\{\sigma\left(V_1(x_1)\right), V_2(x_2)\right\}.$$

At first we check condition (6) for $\widetilde{V}$. V_1 and V_2 are locally Lipschitz in $\mathbb{R}^{N_i} \setminus \{0\},\ i = 1, 2$ and $\sigma \in \mathcal{K}_\infty$. $\sigma\left(V_1(x_1)\right)$ is differentiable almost everywhere in $\mathbb{R}^{N_1} \setminus \{0\}$ and $\widetilde{V}$ is locally Lipschitz in $\mathbb{R}^N \setminus \{0\},\ N := N_1 + N_2$.

By Rademacher's Theorem $\widetilde{V}$ is differentiable almost everywhere in $\mathbb{R}^N \setminus \{0\}$. Now we define the following sets with $x = \left(x_1^T, x_2^T\right)^T \in \mathbb{R}^N$:

$$\begin{aligned} A &= \{x : V_2(x_2) < \sigma(V_1(x_1))\}, \\ B &= \{x : V_2(x_2) > \sigma(V_1(x_1))\}, \\ \Lambda &= \{x : V_2(x_2) = \sigma(V_1(x_1))\}. \end{aligned}$$

We fix a point $p = (p_1, p_2) \neq (0,0)$, $p \in \mathbb{R}^N$, an input $v \in \mathbb{R}^m$ and consider three cases.

Case 1: $p \in A$. It holds $V_2(x_2) < \sigma(V_1(x_1))$ hence $\widetilde{V}(x_1, x_2) = \sigma(V_1(x_1))$ in a neighborhood of p. Because $\widetilde{V}$ is differentiable almost everywhere in A we get

$$\nabla \widetilde{V}(p) f(p, v) = \sigma'(V_1(p_1)) \nabla V_1(p_1) f_1(p_1, p_2, v), \text{ a.e.} \tag{24}$$

From (23) follows

$$V_2(p_2) < \sigma(V_1(p_1)) < (\gamma_{12}^{\text{ISDS}})^{-1}(V_1(p_1))$$

and then $V_1(p_1) > \gamma_{12}^{\text{ISDS}}(V_2(p_2))$. Whenever $V_1(p_1) > \gamma_1^{\text{ISDS}}(|v|)$ we get from (20)

$$\nabla V_1(p_1) f_1(p_1, p_2, v) \leq -(1 - \varepsilon_1) g_1(V_1(p_1)), \text{ a.e,}$$

$1 > \varepsilon_1 > 0$. With (24) we have

$$\begin{aligned} &\nabla \widetilde{V}(p) f(p, v) \\ &\leq -(1 - \varepsilon_1) \sigma'(V_1(p_1)) g_1(V_1(p_1)) \\ &= -(1 - \varepsilon_1) \sigma'\left(\sigma^{-1}(\sigma(V_1(p_1)))\right) g_1\left(\sigma^{-1}(\sigma(V_1(p_1)))\right) \\ &= -(1 - \varepsilon_1) \sigma'(\sigma^{-1}(\widetilde{V}(p))) g_1(\sigma^{-1}(\widetilde{V}(p))) \\ &=: -(1 - \varepsilon_1) \hat{g}_1(\widetilde{V}(p)), \text{ a.e,} \end{aligned}$$

whenever $\widetilde{V}(p) > \sigma(\gamma_1^{\text{ISDS}}(|v|)) =: \hat{\gamma}_1^{\text{ISDS}}(|v|)$, where $\hat{g}_1$ is a positive definite and locally Lipschitz function, since g_1 is $\in \mathcal{P}$ and locally Lipschitz, $\sigma' > 0$ (see Lemma 3.4), σ^{-1} is again a $\mathcal{K}_\infty$ function for $\sigma \in \mathcal{K}_\infty$ and $\widetilde{V}$ maps $\mathbb{R}^N$ into $\mathbb{R}_+$.

Case 2: $p \in B$. It holds $V_2(x_2) > \sigma(V_1(x_1))$ so it is $\widetilde{V}(x_1, x_2) = V_2(x_2)$ in a neighborhood of p. As in case 1 we get

$$\nabla \widetilde{V}(p) f(p, v) = \nabla V_2(p_2) f_2(p_1, p_2, v), \text{ a.e}$$

and from (23) it follows $V_2(p_2) > \sigma(V_1(p_1)) > \gamma_{21}^{\text{ISDS}}(V_1(p_1))$. We have

$$\nabla V_2(p_2) f_2(p_1, p_2, v) \leq -(1 - \varepsilon_2) g_2(V_2(p_2)), \text{ a.e,}$$

$1 > \varepsilon_2 > 0$, whenever $V_2(p_2) > \gamma_2^{\text{ISDS}}(|v|)$. Hence we get

$$\begin{aligned} &\nabla \widetilde{V}(p) f(p, v) \\ &= \nabla V_2(p_2) f_2(p_1, p_2, v) \leq -(1 - \varepsilon_2) g_2(V_2(p_2)) \\ &= -(1 - \varepsilon_2) g_2(\widetilde{V}(p)), \text{ a.e} \end{aligned}$$

whenever $\widetilde{V}(p) > \gamma_2^{\text{ISDS}}(|v|)$, where g_2 is locally Lipschitz and positive definite.

Case 3: $p \in \Lambda$. For the locally Lipschitz function $\widetilde{V}$

$$\nabla \widetilde{V}(p) f(p, v) = \frac{\mathrm{d}}{\mathrm{d}t} \widetilde{V}(\varphi(t)), \text{ a.e.,}$$

holds, where $\varphi(t) = (\varphi_1(t), \varphi_2(t))$ is the solution of the initial value problem

$$\dot{\varphi}(t) = f(\varphi(t), v), \ \varphi(0) = p.$$

Assume p is such that

$$\begin{aligned} &V_1(p_1) > \gamma_1^{\text{ISDS}}(|v|) \\ &\Rightarrow \nabla \sigma(V_1(p_1)) f_1(p_1, p_2, v) \leq -(1 - \varepsilon_1) \hat{g}_1(V(p)), \quad (25) \\ &V_2(p_2) > \gamma_2^{\text{ISDS}}(|v|) \\ &\Rightarrow \nabla V_2(p_2) f_2(p_1, p_2, v) \leq -(1 - \varepsilon_2) g_2(V(p)). \quad (26) \end{aligned}$$

Since $p_1, p_2 \neq 0$, σ is continuous differentiable, V_1, V_2 are locally Lipschitz and f is continuous there exist neighborhoods $\mathcal{U}_1$ of p_1 and $\mathcal{U}_2$ of p_2 such that

$$\begin{aligned} \nabla \sigma(V_1(x_1)) f_1(x_1, x_2, v) &\leq -(1 - \varepsilon_3)(1 - \varepsilon_1) \hat{g}_1(\widetilde{V}(p)) \\ &\leq -(1 - \bar{\varepsilon}_1) \hat{g}_1(\widetilde{V}(p)), \quad (27) \\ \nabla V_2(x_2) f_2(x_1, x_2, v) &\leq -(1 - \varepsilon_4)(1 - \varepsilon_2) g_2(\widetilde{V}(p)) \\ &\leq -(1 - \bar{\varepsilon}_2) g_2(\widetilde{V}(p)), \quad (28) \end{aligned}$$

hold almost everywhere, $\forall (x_1, x_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ and fixed $\varepsilon_3, \ \varepsilon_4 \in (0,1), 1 > \bar{\varepsilon}_1 = \varepsilon_1 + \varepsilon_3 > 0, \ 1 > \bar{\varepsilon}_2 = \varepsilon_2 + \varepsilon_4 > 0$.

Furthermore there exists $\delta > 0$ such that $\varphi(t) \in \mathcal{U}_1 \times \mathcal{U}_2, \ \forall \ 0 \leq t < \delta$. Now let $\Delta t \in (0, \delta)$. If $\varphi(\Delta t) \in A \cup \Lambda$, then with the mean value theorem (MVT)

$$\begin{aligned} \widetilde{V}(\varphi(\Delta t)) - \widetilde{V}(p) &= \sigma(V_1(\varphi_1(\Delta t))) - \sigma(V_1(p_1)) \\ = \nabla \sigma(V_1(\varphi_1(t))) \dot{\varphi}_1(t) \Delta t &\leq -(1 - \bar{\varepsilon}_1) \hat{g}_1(\widetilde{V}(p)) \Delta t \end{aligned}$$

holds true, where the last inequality follows from (27). If $\varphi(\Delta t) \in B \cup \Lambda$, then again with the MVT we get

$$\begin{aligned} \widetilde{V}(\varphi(\Delta t)) - \widetilde{V}(p) &= V_2(\varphi_2(\Delta t)) - V_2(p_2) \\ = \nabla V_2(\varphi_2(t)) \dot{\varphi}_2(t) \Delta t &\leq -(1 - \bar{\varepsilon}_2) g_2(\widetilde{V}(p)) \Delta t, \end{aligned}$$

where the last inequality follows from (28). Hence, if $\widetilde{V}$ is differentiable at p, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}p} \widetilde{V}(p) = \nabla \widetilde{V}(p) f(p, v) &= \lim_{\Delta t \to 0} \frac{\widetilde{V}(\varphi(\Delta t)) - \widetilde{V}(\varphi(0))}{\Delta t} \\ &\leq -(1 - \bar{\varepsilon}) \tilde{g}(\widetilde{V}(p)), \end{aligned}$$

where $\bar{\varepsilon} := \max\{\bar{\varepsilon}_1, \bar{\varepsilon}_2\}, \ \tilde{g}(r) := \min\{\hat{g}_1(r), g_2(r)\}$. Assumptions (25) and (26) hold true, if $\widetilde{V}(p) > \tilde{\gamma}^{\text{ISDS}}(|v|)$, with $\tilde{\gamma}^{\text{ISDS}}(r) := \max\{\hat{\gamma}_1^{\text{ISDS}}(r), \gamma_2^{\text{ISDS}}(r)\}$.

Now we combine all three cases and get, if $\widetilde{V}$ is differentiable at p

$$\nabla \widetilde{V}(p) f(p, v) \leq -(1 - \bar{\varepsilon}) \tilde{g}(\widetilde{V}(p)) \tag{29}$$

whenever $\widetilde{V}(p) > \tilde{\gamma}^{\text{ISDS}}(|v|)$.

Since $\widetilde{V}$ is differentiable a.e., (29) holds a.e., i.e. function $\widetilde{V}$ satisfies condition (9). Now we check condition (8). By definition of $\sigma_1(r) := \sigma(r)$, $\sigma_2(r) := \mathrm{Id}(r) = r$, $\varepsilon := \max\{\varepsilon_1, \varepsilon_2\}$ and with $|x_j|_\infty = |x|_\infty$ for some $j = 1, 2$ we have

$$\begin{aligned} \widetilde{V}(x) &\geq \max_i \sigma_i\left(\frac{|x_i|}{1 + \varepsilon_i}\right) \geq \max_i \sigma_i\left(\frac{|x_i|_\infty}{1 + \varepsilon}\right) \\ &\geq \sigma_j\left(\frac{|x_j|_\infty}{1 + \varepsilon}\right) \geq \min_i \sigma_i\left(\frac{|x|}{\sqrt{2}(1 + \varepsilon)}\right), \end{aligned}$$

By definition of $\tilde{\eta}(r) := \max\{\sigma(\eta_1(r)), \eta_2(r)\}$ we have

$$\begin{aligned}\widetilde{V}(x) &\le \max\{\sigma(\eta_1(|x_1|)), \eta_2(|x_2|)\}\\ &\le \max\{\sigma(\eta_1(|x|)), \eta_2(|x|)\} = \tilde{\eta}(|x|).\end{aligned}$$

By definition of $\psi(r) := \min_i \sigma_i(r),\ i = 1,2,\ r \ge 0$ it follows

$$\tfrac{|x|}{1+\varepsilon} \le \psi^{-1}(\widetilde{V}(x)) \le \psi^{-1}(\tilde{\eta}(|x|)).$$

Function $V(x) := \psi^{-1}(\widetilde{V}(x))$ is the ISDS Lyapunov function candidate and satisfies condition (8) with $\eta(r) := \psi^{-1}(\tilde{\eta}(r))$ as seen before. Note that $\psi^{-1} \in \mathcal{K}_\infty$ and $V(x)$ is locally Lipschitz continuous. To check condition (9) for function V it follows from (29)

$$\begin{aligned}&V(x) > \psi^{-1}(\tilde{\gamma}^{\text{ISDS}}(|u|)) =: \gamma^{\text{ISDS}}(|u|)\\ \Rightarrow &\tfrac{\mathrm{d}}{\mathrm{d}t}V(x) = (\psi^{-1})'(\widetilde{V}(x)) \cdot \nabla\widetilde{V}(x) \cdot f(x,u)\\ &\le -(1-\bar{\varepsilon})(\psi^{-1})'(\widetilde{V}(x))\tilde{g}(\widetilde{V}(x))\\ &= -(1-\bar{\varepsilon})g(V(x)),\ \text{a.e.},\end{aligned}$$

where $g(r) := (\psi^{-1})'(\psi(r))\,\tilde{g}(\psi(r))$ is locally Lipschitz continuous and positive definite. This means $V(x)$ satisfies (9), where μ solves the ordinary differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\mu(r,t) = -g(\mu(r,t))$. Hence $V(x)$ is the locally Lipschitz ISDS Lyapunov function of the whole system and by application of Theorem 2.12 the whole system is ISDS, i.e., it holds

$$|x(t)| \le \max\{\mu(\eta(|x^0|),t), \operatorname*{ess\,sup}_{\tau\in[0,t]} \mu(\gamma^{\text{ISDS}}(|u(\tau)|), t-\tau)\}.$$

■

The following example illustrates the application of the last theorem for a construction of an ISDS Lyapunov function.

Example 3.7: Consider two interconnected systems

$$\begin{aligned}\dot{x}_1 &= -2x_1|x_1| + x_2|x_2| + u\\ \dot{x}_2 &= x_1 - 3x_2 + u\end{aligned}$$

$x_1, x_2 \in \mathbb{R}$ and any input $u \in \mathbb{R}$. We choose $V_1(x_1) = |x_1|$ and $V_2(x_2) = |x_2|^2$ as Lyapunov function candidates for the subsystems. Whenever $\gamma_1^{\text{ISDS}}(|u|) := \sqrt{2|u|} \le |x_1|$ and $\gamma_{12}^{\text{ISDS}}(V_2(x_2)) := \sqrt{|x_2|^2} \le |x_1|$ holds, we get

$$\nabla V_1(x_1) f_1(x_1,x_2,u) \le -2|x_1|^2 + |x_2|^2 + |u| \le -\tfrac{1}{2}|x_1|^2$$

and whenever $\gamma_2^{\text{ISDS}}(|u|) := \frac{1}{2}|u|^2 \le |x_2|^2$ and $\gamma_{21}^{\text{ISDS}}(V_1(x_1)) := \frac{1}{2}|x_1|^2 \le |x_2|^2$ holds, we get

$$\nabla V_2(x_2) f_2(x_1,x_2,u) \le -(6-4\sqrt{2})|x_2|^2.$$

$g_1(r) := \frac{1}{2}r^2$ and $g_2(r) := (6-4\sqrt{2})r$ are positive definite differentiable functions $\forall\ r \in \mathbb{R}_+$. We conclude that V_i are the ISDS Lyapunov functions of the subsystems, hence both subsystems are ISDS with

$$\mu_1(r,t) = \frac{1}{\frac{1}{2}t + \frac{1}{r}},\ \mu_2(r,t) = \exp(-\tfrac{t}{6-4\sqrt{2}})r,$$

$\eta_1(r) = \mathrm{Id}(r),\ \eta_2(r) = r^2, r > 0$ and $\gamma_{12}^{\text{ISDS}},\ \gamma_{21}^{\text{ISDS}},\ \gamma_1^{\text{ISDS}},\ \gamma_2^{\text{ISDS}}$ as defined before. The small gain condition is satisfied, since

$$\gamma_{12}^{\text{ISDS}} \circ \gamma_{21}^{\text{ISDS}}(r) = \tfrac{1}{2}\left(\sqrt{r}\right)^2 = \tfrac{1}{2}r < r,\ r > 0.$$

We choose $\sigma(r) = \sqrt{\frac{3}{2}r}$, which satisfies the conditions of Lemma 3.4. Then we have $\psi(r) = \min\left\{\sqrt{\frac{3}{2}r}, r\right\}$ and $\psi^{-1}(r) = \max\left\{\frac{2}{3}r^2, r\right\}$. Now we apply Theorem 3.5 from which it follows that the whole system is ISDS with

$$\begin{aligned}V(x) &= \max\{|x_1|, \tfrac{2}{3}|x_2|^4, \sqrt{\tfrac{3}{2}|x_1|}, |x_2|^2\}\\ g(r) &= \max\{\tfrac{4}{3}\psi(r), 1\}\min\{\tfrac{1}{6}\psi^3(r), (6-4\sqrt{2})\psi(r)\},\\ \eta(r) &= \max\{r, \tfrac{2}{3}r^4, \sqrt{\tfrac{3}{2}r}, r^2\},\\ \gamma^{\text{ISDS}} &= \max\{\sqrt{2r}, \tfrac{1}{6}r^4, \sqrt{\tfrac{3}{2}\sqrt{2r}}, \tfrac{1}{2}r^2\},\end{aligned}$$

where g is locally Lipschitz and positive definite. The decay rate of the whole system μ can be obtained by solving (7).

IV. CONCLUSIONS AND FUTURE WORKS

We have proved that a system consisting of n interconnected ISDS systems is again ISDS under the small gain condition (18). For two interconnected ISDS systems we provide an explicit ISDS Lyapunov construction of the entire system. The advantage here is, that the decay rate and gains of the whole system can be immediately used to obtain a bound for the trajectories of the solutions.

In a future work we are going to extend the given construction of an ISDS Lyapunov function for the case of $n > 2$ interconnected ISDS systems under a small gain condition, such that the decay rates and gains will be calculated by the rates and gains of the n subsystems.

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