



# Zentrum für Technomathematik

Fachbereich 3 – Mathematik und Informatik

## Approximation of Reachable Sets by Direct Solution Methods of Optimal Control Problems

Robert Baier, Christof Büskens,  
Ilyes Aïssa Chama, Matthias Gerds

Report 04–11

Berichte aus der Technomathematik

Report 04–11

August 2004



# Approximation of Reachable Sets by Direct Solution Methods of Optimal Control Problems

Robert Baier\*, Christof Büskens†, Ilyes Aïssa Chahma‡, Matthias Gerds§

6th April 2004

## Abstract

A numerical method for the approximation of reachable sets of linear control systems is discussed. The method is based on the formulation of suitable optimal control problems with varying objective function, whose discretization by Runge-Kutta methods lead to finite dimensional convex optimization problems. It turns out that the order of approximation for the reachable set depends on the particular choice of the Runge-Kutta method in combination with the selection strategy used for control approximation. For an inappropriate combination the expected order of convergence can not be achieved in general. The method is illustrated by two examples using different Runge-Kutta methods and selection strategies and allows to estimate the order of convergence numerically.

**Keywords:** optimal control, approximation of reachable sets, direct solution methods, order of convergence

---

\*Department of Mathematics, University of Bayreuth, 95440 Bayreuth, Germany, Robert.Baier@uni-bayreuth.de

†Department of Mathematics, University of Bremen, 28344 Bremen, Germany

‡Cetelem Bank GmbH, Schwantalerstrasse 31, 80336 München, Germany, a.chahma@cetelembank.de

§Department of Mathematics, University of Bayreuth, 95440 Bayreuth, Germany, Matthias.Gerds@uni-bayreuth.de

# 1 Introduction

The subject of this paper is the description of an algorithm for the approximation of reachable sets of linear control problems. The problem of determining convex reachable sets can be equivalently described by infinitely many optimal control problems, where the objective function is adapted. By choosing only finitely many directions approximations of reachable sets can be obtained. The occurring optimal control problems are not solved theoretically by use of the Pontryagin's maximum principle as in [38] but numerically by suitable discretization methods. This allows to treat also time dependent linear problems and even nonlinear ones. Non-polyhedral control regions can be treated as nonlinear inequalities and equalities. Results concerning the convergence of discretized optimal control problems can be found in [30], [10] and the references stated therein.

In this context, the particular choice of the selection strategy used for control approximation turns out to be crucial for the order of convergence and depends on the choice of the Runge-Kutta scheme used for the discretization of the underlying differential equations. In order to illustrate this dependency several Runge-Kutta methods with different selection strategies (piecewise constant, piecewise linear, independent selection) are discussed in more detail for two illustrative examples.

By this approach cumbersome set operations (like Minkowski sums, unions of sets, ...) can be avoided and lead to comparatively fast methods, which in addition yield not only the endpoints of optimal trajectories, but the entire trajectory including the corresponding optimal control. However, the close connection between set-valued analysis and optimal control is shown in Section 3.

Methods for linear differential inclusions based on set-valued quadrature methods or set-valued Runge-Kutta methods are mentioned in [3] as well as other methods, e.g. estimation methods for reachable sets (cf. [15]) and ellipsoidal methods (cf. [23] for an overview). Newer developments of these methods achieve inner approximations ([24], [26]) and outer approximations [25] of the reachable set (see also [4]).

The problem of the approximation of reachable sets appears in several disciplines: control theory, ordinary differential equations with uncertainties or with discontinuities in the state, necessary conditions for a minimum in nonsmooth analysis, differential games and viability theory, cf. [5], [1], [33], [14]. The convexity of these reachable sets can be guaranteed for linear differential inclusions, but may also appear for nonlinear problems.

The paper is organized as follows. In Section 2 basic notations and properties of reachable sets are summarized. Basic facts on the description of convex sets and arithmetic set operations are introduced and form the basis for the results of Section 3. In particular, the Hausdorff and the Demyanov distances are defined, which are used to measure the speed of convergence w.r.t. the optimal value and the optimal trajectory, respectively. In Section 3 the problem of calculating the boundary of the reachable set is reformulated as infinitely many

optimal control problems which differ only in the objective function. These optimal control problems are discretized by use of explicit Runge-Kutta methods and suitable control approximations resulting in finite dimensional (linear/nonlinear) optimization problems. Herein, several approximation classes for the control lead to different selection strategies in the discretization. The section ends with a formulation of the proposed method for the approximation of reachable sets and its implementation. Several combinations of Runge-Kutta methods and selection strategies are discussed in Section 4 with illustrative examples. Tables with convergence results and visualizations of reachable sets are included. Finally, an outline for further research concludes the paper.

## 2 Notation

In this section, some introductory definitions and results are collected.

The basic underlying problem is the following control problem:

**Problem 2.1** *Let  $A(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  and  $B(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times n}$  be two  $L_1$ -integrable matrix functions.*

*Let  $U \subset \mathbb{R}^m$  be a nonempty, convex compact set and  $I := [t_0, t_f]$  be a real interval. For a given control function  $u : I \rightarrow \mathbb{R}^m$  with  $u(\cdot) \in L_\infty(I, \mathbb{R}^m)$  we are looking for a solution  $x(\cdot) \in W^{1,\infty}(I, \mathbb{R}^n)$  of the differential equation*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (\text{a.e. } t \in I), \quad (1a)$$

$$x(t_0) = x_0, \quad (1b)$$

$$u(t) \in U \quad (\text{a.e. } t \in I). \quad (1c)$$

**Definition 2.2** *Let us study Problem 2.1 and let  $t \in I$ . Then,*

$$\mathcal{R}(t, t_0, x_0) := \{y \in \mathbb{R}^n \mid \exists u(\cdot) \text{ control function and } \exists x(\cdot) \text{ corresponding solution of Problem 2.1 with } x(t) = y\}$$

*is called the reachable set of the corresponding control problem for the time  $t$ .*

In 1965, Aumann discovered the convexity of the set-valued integral in [2] which easily leads to the convexity of the reachable set for linear control problems.

**Proposition 2.3** *In Problem 2.1, the reachable set  $\mathcal{R}(t, t_0, x_0)$  is convex, compact and nonempty for every  $t \in I$ .*

**Proof:** see e.g. [37, Theorem 1] ■

Some notations from Convex Analysis are recalled which are necessary for the explanation of the algorithm described later.

**Definition 2.4** Denote by  $\mathcal{C}(\mathbb{R}^n)$  the set of all nonempty convex compact sets in  $\mathbb{R}^n$  and let  $C \in \mathcal{C}(\mathbb{R}^n)$  and  $l \in \mathbb{R}^n$ .

Then,

$$\delta^*(l, C) := \max_{c \in C} l^\top c$$

is the support function of  $C$  in direction  $l$  and

$$Y(l, C) := \{c \in C \mid l^\top c = \delta^*(l, C)\}$$

is the set of supporting points of  $C$  in direction  $l$ .

We need the following property of support functions:

**Lemma 2.5** Let  $C = C_1 \times C_2 \in \mathcal{C}(\mathbb{R}^n)$  with convex sets  $C_i \subset \mathbb{R}^{n_i}$ ,  $n_i \in \{1, \dots, n\}$ ,  $i = 1, 2$ , and  $n_1 + n_2 = n$ . Then, for given  $l = (l_1^\top, l_2^\top)^\top \in \mathbb{R}^n$  with  $l_i \in \mathbb{R}^{n_i}$ ,  $i = 1, 2$ , we have:

$$\delta^*(l, C) = \delta^*(l_1, C_1) + \delta^*(l_2, C_2).$$

**Proof:** see e.g. [19, §V, Discussion after Remark 3.3.6] ■

Support functions resp. supporting points describe fully a convex compact set.

**Proposition 2.6** Let  $C \in \mathcal{C}(\mathbb{R}^n)$ . Then,

$$C = \bigcap_{\|l\|_2=1} \{x \in \mathbb{R}^n \mid l^\top x \leq \delta^*(l, C)\}, \quad \partial C = \bigcup_{\|l\|_2=1} Y(l, C),$$

$$C = \text{co}\left(\bigcup_{\|l\|_2=1} \{y(l, C)\}\right) \text{ with arbitrary } y(l, C) \in Y(l, C),$$

where  $\partial C$  denotes the boundary of  $C$  and  $\text{co}(\cdot)$  denotes the convex hull of a set.

**Proof:** see e.g. [19, §V., Theorem 2.2.2] and [19, §V., Proposition 3.1.5].

The last equation follows easily, if you estimate the support function of the right-hand side in direction  $\eta$  by  $\eta^\top y(\eta, C) = \delta^*(\eta, C)$  from below. ■

A common arithmetic operations on sets is the scalar multiplication and the Minkowski sum which are recalled here.

**Definition 2.7** Let  $C, D \in \mathcal{C}(\mathbb{R}^n)$ ,  $\lambda \in \mathbb{R}$  and  $A \in \mathbb{R}^{m \times n}$ . Then,

$$\lambda C := \{\lambda c \mid c \in C\}$$

defines the scalar multiplication,

$$AC := \{Ac \mid c \in C\}$$

the image of  $C$  under the linear map  $x \mapsto Ax$  and

$$C + D := \{c + d \mid c \in C, d \in D\}$$

the Minkowski sum.

We need the following theoretical result which states convexity and compactness of the set operations defined above.

**Lemma 2.8** *Let  $C, D \in \mathcal{C}(\mathbb{R}^n)$ ,  $\lambda \in \mathbb{R}$  and  $A \in \mathbb{R}^{m \times n}$ . Then,  $\lambda C$ ,  $AC$  and  $C + D$  are elements of  $\mathcal{C}(\mathbb{R}^n)$ . Furthermore,*

$$\begin{aligned} \delta^*(l, \lambda C) &= \lambda \delta^*(l, C), & Y(l, \lambda C) &= \lambda Y(l, C) & (\text{if } \lambda \geq 0), \\ \delta^*(l, AC) &= \delta^*(A^\top l, C), & Y(l, AC) &= AY(A^\top l, C), \\ \delta^*(l, C + D) &= \delta^*(l, C) + \delta^*(l, D), & Y(l, C + D) &= Y(l, C) + Y(l, D) \end{aligned}$$

for all  $l \in \mathbb{R}^n$ .

**Proof:** To guarantee that the operations give results in  $\mathcal{C}(\mathbb{R}^n)$  and the equations on the support functions see [19, §V, Theorem 3.3.3(i) and Proposition 3.3.4]. The equations on the supporting set follow immediately from calculus rules on the subdifferential in [19, §VI, Theorem 4.1.1 and equation (3.1)] and [32, Theorem 23.9], since [19, §VI, Proposition 2.1.5 and equation (3.1)] connects the subdifferential of the support function and the supporting set. ■

**Definition 2.9** *Let  $C, D \in \mathcal{C}(\mathbb{R}^n)$ . Then,*

$$\begin{aligned} d(C, D) &:= \max_{c \in C} \min_{d \in D} \|c - d\|_2, \\ d_H(C, D) &:= \max\{d(C, D), d(D, C)\} \end{aligned}$$

are defining the one-sided Hausdorff distance resp. the Hausdorff distance of the two sets.

The Demyanov distance between two sets is defined as

$$d_D(C, D) := \sup_{l \in T_C \cap T_D} \|y(l, C) - y(l, D)\|_2,$$

where  $T_C$  is defined as set of all normed directions in  $\mathbb{R}^n$  for which the supporting set  $Y(l, C)$  consists of only one point  $y(l, C)$  ( $T_D$  is defined analogously for the set  $D$ ).  $T_C$  and  $T_D$  are subsets of the unit sphere of full measure.

Well-known properties of the support function make it easy to prove the following result for the Hausdorff-distance:

**Proposition 2.10** *Let  $C, D \in \mathcal{C}(\mathbb{R}^n)$ . Then,*

$$d_H(C, D) := \max_{\|l\|_2=1} |\delta^*(l, C) - \delta^*(l, D)| \leq d_D(C, D).$$

**Proof:** see e.g. [19, §V, Theorem 3.3.8] and [9, Lemma 4.1] ■

## 3 New Method for the Approximation of Reachable Sets

### 3.1 Computation of the Reachable Set by Optimal Control

Since we know from Proposition 2.3 that the reachable set for problem 2.1 is convex, it is sufficient to calculate merely the boundary of the reachable set.

Proposition 2.6 gives a motivation to calculate at least one support point (which lies automatically at the boundary) of the reachable set in direction  $l \in \mathbb{R}^n$  with  $\|l\|_2 = 1$ . Note that even in the case that the reachable set is not strictly convex and the set of supporting points is a  $(n - 1)$ -dimensional face, for a fixed direction  $l$ , one supporting point in this direction is sufficient to reconstruct the reachable set.

Thus, to calculate a supporting point  $x(t_f)$  on the boundary of the reachable set  $\mathcal{R}(t_f, t_0, x_0)$  in a fixed direction  $l$  we have to find an admissible control function  $u(t) \in U$  that maximizes the functional  $y \mapsto l^\top y$  (resulting in the support function  $\delta^*(l, \mathcal{R}(t_f, t_0, x_0))$  as optimal value). This constitutes the following special **optimal control problem** of Mayer type:

$$(\text{OCP}_1) \quad \begin{cases} \text{Maximize} & l^\top x(t_f) \\ \text{w.r.t.} & u \in L^\infty([t_0, t_f], \mathbb{R}^m), x \in W^{1,\infty}([t_0, t_f], \mathbb{R}^n) \\ & x(\cdot) \text{ corresponding solution to } u(\cdot) \text{ for (1a)–(1c).} \end{cases}$$

We denote the optimal solution of  $(\text{OCP}_l)$  by  $x^*(t; l)$  and  $u^*(t; l)$ , where the argument  $l$  indicates the dependency of the direction  $l$ .

As already mentioned in Proposition 2.6, the convexity and compactness of the reachable set guaranteed by Proposition 2.3 leads to the equivalent representation by considering supporting points in all directions  $l \in \mathbb{R}^n$ ,  $\|l\|_2 = 1$ :

$$\mathcal{R}(t_f, t_0, x_0) = \text{co}\{x^*(t_f; l) \mid l \in \mathbb{R}^n, \|l\|_2 = 1\}.$$

### 3.2 Approximation of Reachable Sets by Discretized Optimal Control Problems

In general, for complex problems neither we can compute a solution of  $(\text{OCP}_l)$  analytically nor for all directions  $l$ . Hence, we suggest to approximate  $(\text{OCP}_l)$  numerically and consider only a finite number of directions  $l_i$ ,  $i = 1, \dots, M := N_l$ . This yields an approximation

$$\mathcal{R}_M(t_f, t_0, x_0) \approx \mathcal{R}(t_f, t_0, x_0)$$

of the reachable set which will be specified hereafter.

For the moment let  $l$  be fixed with  $\|l\|_2 = 1$ .



For  $N_t \in \mathbb{N}$ ,  $N_t \geq 2$  we introduce a grid with grid points

$$t_i = t_0 + ih \in [t_0, t_f], i = 0, 1, \dots, N := N_t, h = \frac{t_f - t_0}{N_t}. \quad (2)$$

The control function  $u(t)$  is discretized on each subinterval  $[t_i, t_{i+1}]$  by the approximation

$$u_{app}^{(i)}(t; \hat{\mathbf{u}}), \quad t \in [t_i, t_{i+1}],$$

where  $\hat{\mathbf{u}} = (u_0, u_1, \dots, u_{P-1})^\top \in U^P$  is a finite dimensional vector parametrizing the selection strategy for the control in the following explicit Runge-Kutta scheme.

Let us first define explicit Runge-Kutta schemes before we will discuss particular strategies for the approximation of the control in more details. Each explicit Runge-Kutta scheme can be characterized by its Butcher array:

$$\begin{array}{c|cccc} \gamma_1 & 0 & \cdots & \cdots & 0 \\ \gamma_2 & \alpha_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \gamma_s & \alpha_{s1} & \cdots & \alpha_{s,s-1} & 0 \\ \hline & \beta_1 & \cdots & \beta_{s-1} & \beta_s \end{array}$$

For a given control approximation  $u_{app}^{(i)}(t; \hat{\mathbf{u}})$  on  $[t_i, t_{i+1}]$  a state approximation  $x_{app}(t; \hat{\mathbf{u}})$  is obtained via an explicit  $s$ -step Runge-Kutta discretization scheme:

$$\begin{aligned} x_{app}(t_{i+1}; \hat{\mathbf{u}}) &= x_{app}(t_i; \hat{\mathbf{u}}) + h\Phi(x_{app}(t_i; \hat{\mathbf{u}}), \hat{\mathbf{u}}, h), \quad i = 0, 1, \dots, N_t - 1, \\ x_{app}(t_0; \hat{\mathbf{u}}) &= x_0 \end{aligned} \quad (3)$$

and

$$\begin{aligned} \Phi(x_{app}(t_i; \hat{\mathbf{u}}), \hat{\mathbf{u}}, h) &:= \sum_{j=1}^s \beta_j \left( A(t_i + \gamma_j h) \eta_{i+1}^{(j)} + B(t_i + \gamma_j h) u_{app}^{(i)}(t_i + \gamma_j h; \hat{\mathbf{u}}) \right), \\ \eta_{i+1}^{(j)} &:= x_{app}(t_i; \hat{\mathbf{u}}) + h \sum_{k=1}^{j-1} \alpha_{jk} \left( A(t_i + \gamma_k h) \eta_{i+1}^{(k)} + B(t_i + \gamma_k h) u_{app}^{(i)}(t_i + \gamma_k h; \hat{\mathbf{u}}) \right). \end{aligned}$$

Suitable values for the coefficients  $\alpha_{jk}$ ,  $\beta_j$  and  $\gamma_j$ ,  $1 \leq j, k \leq s$  can be found in [7].

Let us now consider examples for selection strategies used in Section 4.

(i) Continuous and piecewise linear approximation:

$$u_{app}^{(i)}(t; \hat{\mathbf{u}}) := u_i + \frac{t - t_i}{h} (u_{i+1} - u_i) \quad \text{for } t \in [t_i, t_{i+1}], i = 0, 1, \dots, N - 1,$$

with  $P = N + 1$ .

(ii) Piecewise constant approximation:

$$u_{app}^{(i)}(t; \hat{\mathbf{u}}) := u_i \quad \text{for } t \in [t_i, t_{i+1}], i = 0, 1, \dots, N-1,$$

with  $P = N$ .

(iii) Independent selections at intermediate grid points  $t_i + \gamma_j h$ :

$$u_{app}^{(i)}(t_i + \gamma_j h; \hat{\mathbf{u}}) := u_{i \cdot s + j - 1}, \quad i = 0, 1, \dots, N-1, j = 1, \dots, s,$$

with  $P = s \cdot N$ .

Please notice, that further selection strategies are possible, e.g. independent selections with additional continuity constraints at the inner grid points  $t_i$ ,  $i = 1, \dots, N-1$ , or additional equality constraints at those intermediate grid points  $t_i + \gamma_j h$  where different indices  $j$  produce the same intermediate grid point (i.e., points where  $\gamma_j = \gamma_k$  with  $j \neq k$ ).

Thus, by this discretization the infinite dimensional optimal control problem ( $OCP_l$ ) is approximated by the *finite dimensional convex programming problem*

$$(\mathbf{CP}_l^1) \quad \begin{cases} \text{Maximize} & l^\top x_{app}(t_{N_t}; \hat{\mathbf{u}}) \\ \text{w.r.t.} & \hat{\mathbf{u}} \in U^P \\ \text{subject to} & x_{app}(t_{i+1}; \hat{\mathbf{u}}) = x_{app}(t_i; \hat{\mathbf{u}}) + h\Phi(x_{app}(t_i; \hat{\mathbf{u}}), \hat{\mathbf{u}}, h), \\ & \quad \quad \quad i = 0, 1, \dots, N_t - 1 \\ & x_{app}(t_0; \hat{\mathbf{u}}) = x_0, \\ & \hat{\mathbf{u}} \in U^P. \end{cases} \quad (\star)$$

Notice, that  $\hat{\mathbf{u}}$  implicitly defines a control approximation  $u_{app}^{(i)}(\cdot; \hat{\mathbf{u}})$  on each subinterval  $[t_i, t_{i+1}]$ , compare the examples (i)-(iii).

We denote the optimal solution of ( $CP_l^1$ ) by  $\hat{\mathbf{u}}^*$ .

If the conditions ( $\star$ ) can be written with a finite number of affine inequalities, ( $CP_l^1$ ) is a linear programming problem and called ( $LP_l^1$ ), otherwise a nonlinear (convex) programming problem.

In the sequel, we investigate the simplest case, the Euler's method with piecewise constant control approximation, since it is then possible to derive explicit solutions for the finite dimensional problems ( $CP_l^1$ ). In this case, (3) reduces to

$$\Phi(x_{app}(t_i; \hat{\mathbf{u}}), \hat{\mathbf{u}}, h) = A(t_i)x_{app}(t_i; \hat{\mathbf{u}}) + B(t_i)u_i.$$

The recursive evaluation in (3) for Euler's method yields

$$x_{app}(t_{N_t}; \hat{\mathbf{u}}) = \left( \prod_{i=0}^{N_t-1} Q_i \right) x_0 + h \sum_{k=0}^{N_t-1} \left( \prod_{i=k+1}^{N_t-1} Q_i \right) B_k u_k \quad (4)$$

with  $Q_i := I + hA(t_i)$ ,  $B_k := B(t_k)$  and the  $n \times n$ -identity matrix  $I$ . The matrix product  $\prod$  is defined as

$$\prod_{i=k}^j Q_i := Q_j \cdot Q_{j-1} \cdots Q_k.$$

Introducing this expression for  $x_{app}(t_f; \hat{\mathbf{u}})$  in  $(LP_l^1)$  yields the linear program

$$(\mathbf{LP}_l^2) \quad \begin{cases} \text{Maximize} & l^\top \left( \sum_{k=0}^{N_t-1} \left( \prod_{i=k+1}^{N_t-1} Q_i \right) B_k u_k \right) \\ \text{subject to} & u_i \in U, \quad i = 0, 1, \dots, N_t. \end{cases}$$

Note that this linear program has the same solution  $\hat{\mathbf{u}}$  as  $(LP_l^1)$ , whereas the optimal objective function values are different, since we neglected constant terms.

To compute the objective function in  $(LP_l^2)$  very efficiently we introduce additional artificial variables

$$\begin{aligned} \lambda_{N_t}^\top &:= l^\top, \\ \lambda_i^\top &:= \lambda_{i+1}^\top Q_i = \lambda_{i+1}^\top + h\lambda_{i+1}^\top A_i. \end{aligned}$$

These artificial variables are calculated backward in time and correspond to the discretized adjoint variable of the optimal control problem  $(OCP_l)$ .

Then,  $(LP_l^2)$  can be replaced by

$$(\mathbf{LP}_l^3) \quad \begin{cases} \text{Maximize} & \sum_{k=0}^{N_t-1} \lambda_{k+1}^\top B_k u_k \\ \text{subject to} & u_k \in U, \quad k = 0, 1, \dots, N_t. \end{cases}$$

Lemma 2.5 gives us

$$\sum_{k=0}^{N_t-1} \delta^*(\lambda_{k+1}, B_k U) = \sum_{k=0}^{N_t-1} \delta^*(B_k^\top \lambda_{k+1}, U)$$

as optimal value of  $(LP_l^3)$  and hence,  $(u_1, u_2, \dots, u_{N_t-1})$  with the supporting points  $u_k \in Y(B_k^\top \lambda_{k+1}, U)$  as one solution.

In the special of box constraints, that is  $U = \{u \in \mathbb{R}^m \mid \underline{u} \leq u \leq \bar{u}\}$ , we define  $S_k^\top := (S_k^1, \dots, S_k^m) := \lambda_{k+1}^\top B_k \in \mathbb{R}^m$ . Since the objective function

$$\sum_{k=0}^{N_t-1} S_k u_k = \sum_{k=0}^{N_t-1} \sum_{j=1}^m S_k^j \cdot u_k^j$$

is maximized, if each term  $S_k^j \cdot u_k^j$  is maximized, the solution of  $(LP_l^3)$  is given by

$$u_k^j = \begin{cases} \underline{u}^j, & \text{if } S_k^j < 0, \\ \bar{u}^j, & \text{if } S_k^j > 0, \\ \text{arbitrary,} & \text{else.} \end{cases}$$

for  $j = 1, \dots, m$ ,  $k = 0, \dots, N_t - 1$ .

### 3.3 Discrete reachable sets

Discrete reachable sets are the reachable sets of the discretized equations and could be defined as endpoints of discrete solutions of the following problem.

**Problem 3.1** *Given the data in Problem 2.1, the discretized problem depends on the choice of the set  $\mathcal{U}_{app}$  of all discretized control functions and on the Runge-Kutta scheme.*

*For a time discretization (2) with step-size  $h = \frac{t_f - t_0}{N_t}$  and a given discretized control function  $u_{app}(\cdot, \hat{\mathbf{u}})$  we are looking for a solution  $x_{app}(\cdot, \hat{\mathbf{u}})$  at the grid-points  $t_i$ ,  $i = 0, 1, \dots, N_t$ , with*

$$x_{app}(t_{i+1}; \hat{\mathbf{u}}) = x_{app}(t_i; \hat{\mathbf{u}}) + h\Phi(x_{app}(t_i; \hat{\mathbf{u}}), \hat{\mathbf{u}}, h) \quad (5a)$$

$$\text{for } i = 0, 1, \dots, N_t - 1,$$

$$x_{app}(t_0; \hat{\mathbf{u}}) = x_0, \quad (5b)$$

$$u_i \in U, \quad i = 0, 1, \dots, N_t, \quad (5c)$$

$$u_{app}(\cdot, \hat{\mathbf{u}}) \in \mathcal{U}_{app}.$$

**Definition 3.2** *Consider Problem 3.1 with a time discretization (2) and let  $i \in \{0, 1, \dots, N_t\}$ . Then,*

$$\begin{aligned} \mathcal{R}_N(t_i, t_0, x_0) := \{y \in \mathbb{R}^n \mid & \exists u_{app}(\cdot; \hat{\mathbf{u}}) \text{ discretized control function and} \\ & \exists x_{app}(\cdot; \hat{\mathbf{u}}) \text{ corresponding solution of Problem 3.1} \\ & \text{with } x_{app}(t_i; \hat{\mathbf{u}}) = y\} \end{aligned}$$

*is called the discrete reachable set of the corresponding discretized control problem for the time  $t_i$ .*

The definition above shows that each optimizer of problem  $(CP_l^1)$  (resp. the reformulation  $(LP_l^3)$ ) is a supporting point of the discrete reachable set  $\mathcal{R}_N(t_f, t_0, x_0)$  in direction  $l$ . The optimal value of problem  $(CP_l^1)$  coincides with the support function  $\delta^*(l, \mathcal{R}_N(t_f, t_0, x_0))$ . Proposition 2.6 shows that

$$\begin{aligned} \mathcal{R}_N(t_f, t_0, x_0) &= \bigcap_{\|l\|_2=1} \{x \in \mathbb{R}^n \mid l^\top x \leq l^\top x_{app}(t_f; \hat{\mathbf{u}}^*)\}, \\ \mathcal{R}_N(t_f, t_0, x_0) &= \text{co} \left( \bigcup_{\|l\|_2=1} \{x_{app}(t_f; \hat{\mathbf{u}}^*)\} \right). \end{aligned}$$

In practice, only a finite number of different normed directions  $l^i$ ,  $i = 1, \dots, M$ , are chosen.

**Proposition 3.3** *Consider Problem 3.1 with a time discretization (2) and let  $i \in \{0, 1, \dots, N_t\}$ . Then, the corresponding discrete reachable set is convex, compact and nonempty.*

**Proof:** For a chosen discretized control function  $u_{\text{app}}(\cdot, \hat{\mathbf{u}})$ , the discrete solution is defined by (4). The discrete reachable set coincides with the union of all such discrete solutions for all feasible discretized control functions. In the case of Euler and linear approximation of the controls, this corresponds to the union over all vectors  $\hat{\mathbf{u}} \in \mathbb{R}^{m(N+1)}$ . Definition 2.7 shows that the discrete reachable set

$$\mathcal{R}_N(t_f, t_0, x_0) = \left( \prod_{i=0}^{N_t-1} Q_i \right) x_0 + h \sum_{k=0}^{N_t-1} \left( \prod_{i=k+1}^{N_t-1} Q_i \right) B_k U$$

is a scaled Minkowski sum of linearly transformed convex sets  $U$ . Lemma 2.8 proves the wanted properties of the discrete reachable set.  $\blacksquare$

### 3.4 Implementation

In the sequel, we briefly discuss some numerical methods, which are suitable for solving the discretized optimal control problem ( $CP_t^1$ ). Of course, the choice of an appropriate method depends on the explicit representation of the control region  $U$ . Hence, we restrict the discussion to control regions  $U$  defined by

$$U = \{u \in X \mid g_i(u) \leq 0, i = 1, \dots, r\}, \quad (6)$$

where  $X := \{u \in \mathbb{R}^n \mid Au = b, u \geq 0\}$  and the functions  $g_i(\cdot)$ ,  $i = 1, \dots, r$ , could be either linear or nonlinear.

**Remark 3.4** *In the case, that the support function or the supporting points of the convex control set  $U$  are known, general control regions  $U$  can be approximated in another way. Proposition 2.6 suggests to use the approximation*

$$U \approx \bigcap_{i=1, \dots, M} \{x \in \mathbb{R}^n \mid l^i \top x \leq \delta^*(l^i, U)\}$$

resp.

$$U \approx \text{co} \left( \bigcup_{i=1, \dots, M} \{y(l^i, U)\} \right) \text{ with arbitrary } y(l^i, U) \in Y(l^i, U).$$

Herein, the  $M$  different normed directions  $l^i \in \mathbb{R}^n$  should be chosen in an appropriate way in order to approximate the unit sphere.

If the functions  $g_i$  in (6) are affine linear, then problem  $(CP_l^1)$  is a linear optimization problem and can be solved by the well-known simplex method or some interior point method, cf. [42], suitable for linear programs. In the special case of an Euler approximation and  $U$  defined by box constraints only, a very efficient method is described in Section 3.2.

If the functions  $g_i$  are convex and smooth, i.e. at least continuously differentiable, then the resulting problem  $(CP_l^1)$  is a convex but nonlinear programming problem and the sequential quadratic programming (SQP) method is appropriate provided the functions  $g_i$  are defined for infeasible points, cf. [34], [35], [18]. Alternatively, the method of feasible directions is applicable, especially, if the functions  $g_i$  are only defined for admissible points, cf. [43].

If the functions  $g_i$  are convex but nonsmooth, the bundle method respectively the bundle trust region method (BT-method) is suitable, cf. [28], [31], [21], [22], [36]. In addition, Kelly's cutting plane method is also applicable, cf. [20]. Notice, that the BT-method and the cutting plane method are closely related, cf. [21], [36].

## 4 Examples

In the sequel we refer to the optimal control problem  $(OCP_1)$ , the differential equation (1a)-(1b), the control constraint (1c), and the control approximations discussed in (i)-(iii) in Section 3.2.

The following Runge-Kutta methods are used for the numerical computation of reachable sets:

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

Euler's method

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

Heun's method

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1 \end{array}$$

Modified Euler's method

From Definition 2.9 of the Hausdorff distance, it is clear that the approximation of the reachable set corresponds to a uniform convergence of the optimal value functions, whereas the approximation of trajectories corresponds to the uniform convergence of the maximizers and the Demyanov distance. Table I shows the approximated values

$$\max_{i=1,\dots,M} |\delta^*(l_i, \mathcal{R}(1, 0, x_0)) - \delta^*(l_i, \widehat{\mathcal{R}}_{\text{ref}}(0, x_0, ))| \quad (7)$$

resp.

$$\max_{i=1,\dots,M} \|Y(l_i, \mathcal{R}(1, 0, x_0)) - Y(l_i, \widehat{\mathcal{R}}_{\text{ref}}(0, x_0, ))\|_2 \quad (8)$$

at the chosen directions  $l_i$ ,  $i = 1, \dots, M$ , for the two distances

$$d_H(\mathcal{R}(1, 0, x_0), \mathcal{R}_N(1, 0, x_0)) \quad \text{resp.} \quad d_D(\mathcal{R}(1, 0, x_0), \mathcal{R}_N(1, 0, x_0)).$$

**Example 4.1** (see [39, Example in section 4]) *Let us consider the following example with  $n = 2$ ,  $m = 1$ ,  $x_0 = (0, 0)^\top$ ,  $I = [0, 1]$ ,  $U = [0, 1]$ , and*

$$A(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

*In Figure 1 approximations to the reachable set  $\mathcal{R}(1, 0, x_0)$  are shown, in the left picture approximations with Euler's method with piecewise constant selections are shown (first order of convergence), in the right one the corresponding ones for Heun's method with continuous and piecewise linear control approximation (second order of convergence) are depicted. In both cases the set with the solid line shows the reference set (calculated with the corresponding method for  $N = 1280$ ). The dashed lines show the approximations for  $N = 10, 20, 40$  for Euler's method on the left picture (please remark the halvening of the distance of the upper right corner of the sets when the number of subintervals is doubled). At the right one, the dashed lines show the approximations for  $N = 1, 2, 4$  for Heun's method (one could see the more rapid convergence even for these small number of subintervals).*

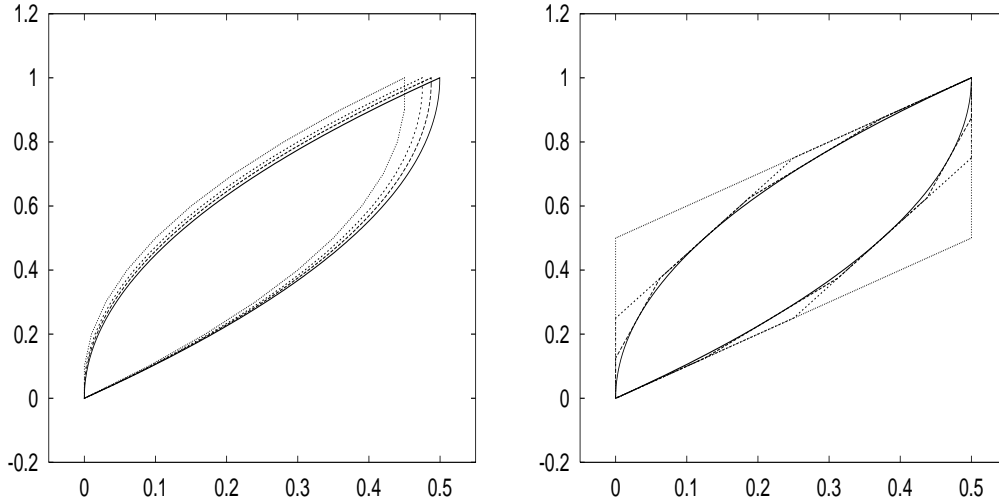


Figure 1: First order contra second order approximations to the reachable set (left: Euler's method with error  $\mathcal{O}(h)$ , right: Heun's method with error  $\mathcal{O}(h^2)$ )

*As Veliov explains in [39], the convergence of the trajectory could not be better than  $\mathcal{O}(h)$  in this example. In Figure 2 the first order approximations to the control and to the state components (coordinates  $x_1$  and  $x_2$ ) are shown for Heun's method with continuous, piecewise linear selections. Again, the reference is computed by the method itself with  $N = 1280$  (solid line) and in dashed lines the*

approximations for  $N = 10, 20, 40$ . As it is clearly seen, the order of convergence is only 1.

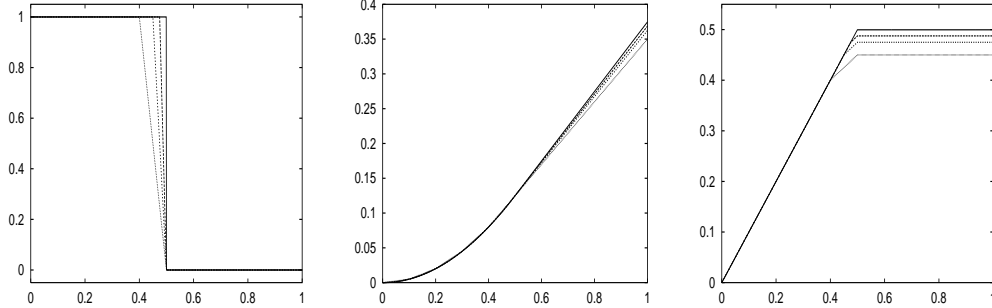


Figure 2: First order approximations to the control (left) and the state components (middle, right) by Heun's method

Here, the combination method of set-valued iterated trapezoidal rule together with Heun's method introduced in [3, 4] with  $N = 1000000$  serves as the reference set  $\widehat{\mathcal{R}}_{\text{ref}}(0, x_0, \cdot)$ . By comparing the different values based on the optimal value function resp. the maximizers, the order of convergence is estimated. The angle  $\varphi$  for the direction  $l \in \mathbb{R}^2$ , in which the maximum in (7) resp. (8) is attained, is shown in the most right column.

$N$	Hausdorff distance	estim. order	angle	$N$	Demyanov distance	estim. order	angle
10	0.05000000	NaN	0.00500	10	0.13702925	NaN	5.55500
20	0.02500000	1.00000	0.00500	20	0.06806368	1.00953	5.55500
40	0.01250000	1.00000	0.00500	40	0.03392323	1.00461	5.51500
80	0.00625000	1.00000	0.00500	80	0.01731662	0.97012	5.53500
160	0.00312500	1.00000	0.00500	160	0.00861479	1.00727	5.53500
320	0.00156250	1.00000	0.00500	320	0.00426388	1.01465	5.53500
640	0.00078125	1.00000	0.00500	640	0.00209303	1.02657	5.62500

Table I: order of convergence for Euler's method (left table: approximation of the reachable set, right table: approximation of the trajectories).

Table I shows the expected order of convergence 1 for reachable set and the trajectories. As remarked above the Hausdorff distance is attained at the upper right corner.

For Heun's method with continuous, piecewise linear control approximation, Table II shows order of convergence 2 for the reachable set and only order 1 for the trajectories.



$N$	Hausdorff distance	estim. order	angle	$N$	Demyanov distance	estim. order	angle
10	0.00124700	NaN	3.09500	10	0.06636590	NaN	5.55500
20	0.00031111	2.00295	3.12000	20	0.03273184	1.01975	5.55500
40	0.00007788	1.99805	6.27500	40	0.01668369	0.97226	2.40000
80	0.00001947	1.99990	3.14000	80	0.00848003	0.97630	5.53500
160	0.00000488	1.99688	6.26000	160	0.00419649	1.01488	5.53500
320	0.00000122	1.99929	3.14500	320	0.00205473	1.03024	5.53500
640	0.00000030	2.00266	6.22500	640	0.00099208	1.05042	5.62500

Table II: order of convergence for Heun's method (left table: approximation of the reachable set, right table: approximation of the trajectories)

**Example 4.2** (see [4, Example 4.4]) *Let us consider the following example with  $n = 2$ ,  $m = 2$ ,  $x_0 = (0, 0)^\top$ ,  $I = [0, 2]$ ,  $U = \{x \in \mathbb{R}^2 \mid \|x\|_2 \leq 1\}$ , and*

$$A(t) = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

*This example introduces the nonlinear constraint*

$$u_1^2 + u_2^2 \leq 1$$

*for the control variable  $u = (u_1, u_2)^\top$ .*

*The second order approximations to the reachable set  $\mathcal{R}(2, 0, x_0)$  calculated by Heun's method with piecewise constant controls resp. with independent control selection are shown in Figure 3.*

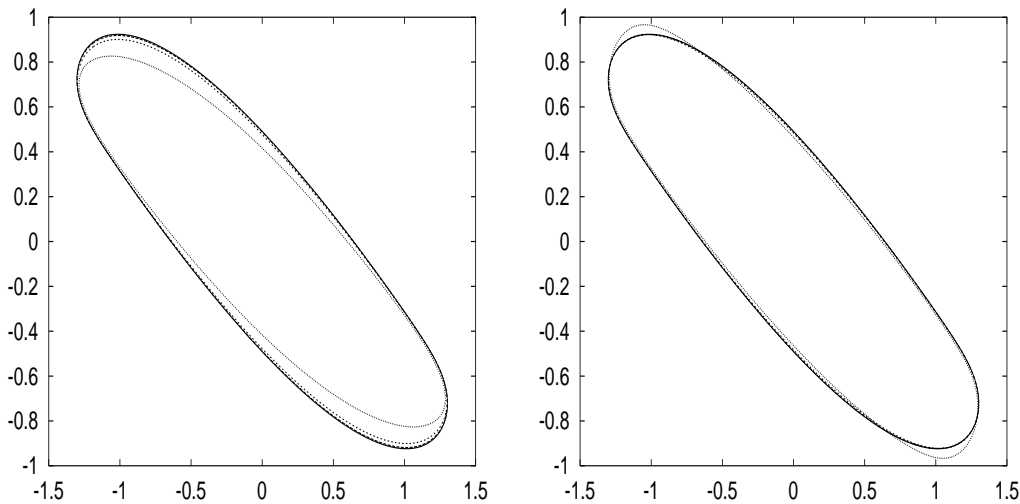


Figure 3: second order approximations to the reachable set for Heun's method with piecewise constant control approximation (left) resp. independent control selection (right).

*The set with the solid line shows the reference set (calculated with the corresponding method for  $N = 160$ ) and the dashed lines represent the approximations for  $N = 5, 10, 20$ . At the left picture the convergence order  $\mathcal{O}(h^2)$  can be seen by studying the boundary of the sets near by  $y = 1$ .*

*Both selection strategies seems to converge with order 2 which is assured by Tables III and IV.*

*Nevertheless, Figure 4 shows that the choice of the selection strategies for the control should depend on the Runge-Kutta method. In Figure 4 the piecewise constant selection strategy is compared with two independent selections for modified Euler's method. The latter selection strategy destroys order of convergence 2 of the Runge-Kutta method. This is verified in the Tables V (order  $\mathcal{O}(h^2)$ ) and VI (only order  $\mathcal{O}(h)$ ) for the convergence to the reachable set and the trajectories.*

$N$	Hausdorff distance	Order	angle	$N$	Demyanov distance	Order trajectory	angle
5	0.10328935	NaN	1.37000	5	0.37223126	NaN	0.90500
10	0.02307167	2.16250	1.53000	10	0.07159599	2.37825	0.88500
20	0.00521186	2.14625	1.57500	20	0.01535558	2.22112	4.02500
40	0.00123195	2.08086	4.73500	40	0.00355544	2.11066	4.02500
80	0.00029922	2.04164	1.60000	80	0.00085565	2.05493	4.02500
160	0.00007372	2.02105	4.74500	160	0.00020992	2.02719	4.02500

Table III: Order of Convergence for Heun's method with piecewise constant control approximation.

$N$	Hausdorff distance	Order	angle	$N$	Demyanov distance	Order trajectory	angle
5	0.04517018	NaN	1.72000	5	0.16781544	NaN	1.18500
10	0.00772443	2.54787	4.23500	10	0.04611042	1.86371	0.87500
20	0.00203009	1.92789	4.30000	20	0.01077148	2.09788	4.01500
40	0.00051385	1.98211	4.33500	40	0.00257389	2.06520	0.87500
80	0.00012897	1.99429	1.21000	80	0.00062808	2.03492	0.87500
160	0.00003229	1.99784	1.22000	160	0.00015506	2.01802	4.01500

Table IV: Order of Convergence for Heun's method with two independent selections.

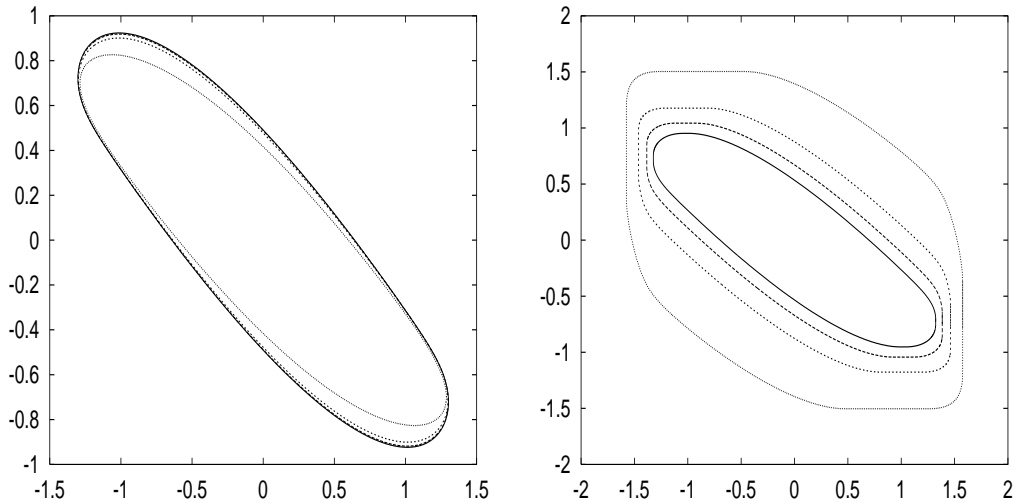


Figure 4: approximations to the reachable set for  $N = 160$  (solid) and  $N = 5, 10, 20$  (dashed) computed by modified Euler's method with piecewise constant (left) resp. independent control selection (right).

$N$	Hausdorff distance	Order	angle	$N$	Demyanov distance	Order trajectory	angle
5	0.10328935	NaN	1.37000	5	0.37223121	NaN	0.90500
10	0.02307167	2.16250	1.53000	10	0.07159599	2.37825	0.88500
20	0.00521186	2.14625	1.57500	20	0.01535559	2.22112	4.02500
40	0.00123195	2.08086	4.73500	40	0.00355571	2.11056	4.02500
80	0.00029922	2.04164	1.60000	80	0.00085566	2.05503	0.88500

Table V: Order of Convergence for the modified Euler's method with piecewise constant control approximation.

$N$	Hausdorff distance	Order	angle	$N$	Demyanov distance	Order trajectory	angle
5	0.83583108	NaN	4.03000	5	1.03202096	NaN	0.73000
10	0.33319435	1.32685	0.85500	10	0.36562913	1.49702	3.85000
20	0.15333206	1.11970	5.34000	20	0.16060144	1.18690	3.76000
40	0.07575471	1.01725	5.36000	40	0.07933801	1.01740	4.72000
80	0.03762644	1.00959	2.22500	80	0.03952243	1.00534	4.72000

Table VI: Order of Convergence for the modified Euler's method with free selection.

## 5 Outline of Further Research

It is known that set valued quadrature methods in [4] could lead to a order of convergence greater than two, if the problem satisfies additional smoothness conditions, cf. [3]. In this case, selection strategies with piecewise constant controls are no longer appropriate. Preliminary computer experiments with the classical Runge-Kutta method show that order of convergence greater than two is attainable. But for these Runge-Kutta methods suitable selection strategies have to be studied in more detail. In this context, additional difficulties arise if state constraints are present, because these constraints should be fulfilled also at the intermediate stages of the Runge-Kutta scheme (as in [8]).

Further research can be conducted towards the study of Runge-Kutta schemes as in [29], [13], [27], where the selection strategy is motivated by multiple control integrals. In the special case of two selections per Runge-Kutta step this leads to alternative selection sets of type  $\left(u_{app}^{(i)}(t_i + \gamma_1 h; \hat{\mathbf{u}}), u_{app}^{(i)}(t_i + \gamma_2 h; \hat{\mathbf{u}})\right) \in \hat{U} \subset U \times U$ , where  $U \times U$  corresponds to case (iii) of independent selections in Section 3.2. This set  $\hat{U}$  can be described by finitely many nonlinear inequalities and equalities, which can be easily imposed as additional constraints in the discretized optimal control problems.

The proposed method itself can be easily adapted to the calculation of convex reachable sets for nonlinear differential inclusions. For the numerical solution of discretized optimal control problems efficient algorithms are available, cf., e.g., [6], [16, 17]. In the more general case of nonconvex reachable sets suitable modifications of our approach have to be studied. Theoretical results in this direction can be found in [12], [41], [40] for Runge-Kutta methods of order one and two. A survey of other methods is given in [11] and [8].

However, those Runge-Kutta methods with appropriate selection strategies, which show higher order of convergence in the linear case, are worth being investigated also in the nonlinear case. In addition, these methods have to be compared with set-valued Runge-Kutta methods based on set arithmetics, cf. [8], which work also on the general nonlinear case. First steps in this direction can be found in [8, Example 5.3.1].

## References

- [1] J.-P. Aubin and A. Cellina. *Differential Inclusions*, volume 264 of *Grundlehren der mathematischen Wissenschaften*. Springer Verlag, Berlin–Heidelberg–New York–Tokyo, 1984.
- [2] R. J. Aumann. Integrals of Set-Valued Functions. *J. Math. Anal. Appl.*, 12(1):1–12, 1965.

- [3] R. Baier. Mengenwertige Integration und die diskrete Approximation erreichbarer Mengen. *Bayreuth. Math. Schr.*, 50:xxii + 248 S., 1995.
- [4] R. Baier and F. Lempio. Computing Aumann's integral. In A. B. Kurzhanski and V. M. Veliov, editors, *Modeling Techniques for Uncertain Systems, Proceedings of a Conference held in Sopron, Hungary, July 6-10, 1992*, volume 18 of *Progress in Systems and Control Theory*, pages 71–92, Basel, 1994. Birkhäuser.
- [5] V. I. Blagodatskikh and A. F. Filippov. Differential inclusions and optimal control. In *Proc. Steklov Inst. Math.*, 4, pages 199–259. North-Holland, Amsterdam, 1986.
- [6] C. Büskens. *Optimierungsmethoden und Sensitivitätsanalyse für optimale Steuerprozesse mit Steuer- und Zustandsbeschränkungen*. PhD thesis, Fachbereich Mathematik, Westfälische Wilhelms-Universität Münster, 1998.
- [7] J. C. Butcher. *The Numerical Analysis of Ordinary Differential Equations—Runge-Kutta and General Linear Methods*. John Wiley and Sons, Chichester–New York–Brisbane–Toronto–Singapore, 1987.
- [8] I. A. Chahma. Set-valued discrete approximation of state-constrained differential inclusions. *Bayreuth. Math. Schr.*, 67:3–162, 2003.
- [9] P. Diamond, P. Kloeden, A. Rubinov, and A. Vladimirov. Comparative Properties of Three Metrics in the Space of Compact Convex Sets. *Set-Valued Anal.*, 5(3):267–289, 1997.
- [10] A. L. Dontchev, W. W. Hager, and V. M. Veliov. Second-Order Runge-Kutta Approximations in Control Constrained Optimal Control. *SIAM Journal on Numerical Analysis*, 38(1):202–226, 2000.
- [11] A. L. Dontchev and F. Lempio. Difference methods for differential inclusions: A survey. *SIAM Rev.*, 34(2):263–294, 1992.
- [12] A. L. Dontchev and E. M. Farkhi. Error Estimates for Discretized Differential Inclusions. *Computing*, 41:349–358, 1989.
- [13] R. Ferretti. High-Order Approximations of Linear Control Systems via Runge-Kutta Schemes. *Computing*, 58(4):351–364, 1997.
- [14] A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Mathematics and Its Applications (Soviet Series). Kluwer Academic Publishers, Dordrecht–Boston–London, 1988.
- [15] J. E. Gayek. Approximating reachable sets for a class of linear control systems. *Internat. J. Control*, 43(2):441–453, 1986.

- [16] M. Gerdts. Numerische Methoden optimaler Steuerprozesse mit differential-algebraischen Gleichungssystemen höheren Indexes und ihre Anwendungen in der Kraftfahrzeugsimulation und Mechanik. volume 61 of *Bayreuther Mathematische Schriften*, Bayreuth, 2001.
- [17] M. Gerdts. Direct Shooting Method for the Numerical Solution of Higher Index DAE Optimal Control Problems. *Journal of Optimization Theory and Applications*, 117(2):267–294, 2003.
- [18] P. E. Gill, W. Murray, M. A. Saunders, and M. H. Wright. User’s guide for NPSOL 5.0: A FORTRAN package for nonlinear programming. *Technical Report NA 98-2, Department of Mathematics, University of California, San Diego, California*, 1998.
- [19] J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms I. Fundamentals*, volume 305 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin–Heidelberg–New York–London–Paris–Tokyo–Hong Kong–Barcelona–Budapest, 1993.
- [20] J. E. Kelley. The cutting-plane method for solving convex programs. *J. Soc. Ind. Appl. Math.*, 8:703–712, 1960.
- [21] K. C. Kiwiel. *Methods of Descent for Nondifferentiable Optimization*, volume 1133 of *Lecture Notes in Math*. Springer, Berlin-Heidelberg-New York.Tokyo, 1985.
- [22] K. C. Kiwiel. A Constraint Linearization Method for Nondifferentiable Convex Minimization. *Numer. Math.*, 51:395–414, 1987.
- [23] A. B. Kurzhanski and I. Vályi. *Ellipsoidal Calculus for Estimation and Control*. Systems & Control: Foundations & Applications. Birkhäuser, Boston–Basel–Berline, 1997.
- [24] A. B. Kurzhanski and P. Varaiya. Ellipsoidal techniques for reachability analysis: internal approximation. *Systems Control Lett.*, 41:201–211, 2000.
- [25] A. B. Kurzhanski and P. Varaiya. On Ellipsoidal Techniques for Reachability Analysis. part I: External Approximations. *Optim. Methods Softw.*, 17(2):177–206, 2002.
- [26] A. B. Kurzhanski and P. Varaiya. On Ellipsoidal Techniques for Reachability Analysis. part II: Internal Approximations Box-valued Constraints. *Optim. Methods Softw.*, 17(2):207–237, 2002.
- [27] P. E. Kloeden L. Grüne. Higher order numerical schemes for affinely controlled nonlinear systems. *Numer. Math.*, 89:669–690, 2001.

- [28] C. Lemarechal, J. J. Strodiot, and A. Bihain. On a Bundle Algorithm for Nonsmooth Optimization. In O. L. Mangasarian, R. R. Meyer, and S. M. Robinson, editors, *Nonlinear Programming 4*, pages 245–282, New York, 1981. Academic Press.
- [29] F. Lempio and V. Veliov. Discrete Approximations of Differential Inclusions. *Bayreuth. Math. Schr.*, 54:149–232, 1998.
- [30] K. Malanowski, C. Büskens, and H. Maurer. Necessary Conditions for Optimal Control Problems Involving Nonlinear Differential Algebraic Equations. In Anthony Fiacco, editor, *Mathematical programming with data perturbations*, volume 195, pages 253–284. Dekker. Lect. Notes Pure Appl. Math., 1997.
- [31] R. Mifflin. A modification and an extension of Lemarechal’s algorithm for nonsmooth minimization. *Math. Program. Study*, 17:77–90, 1982.
- [32] R. T. Rockafellar. *Convex Analysis*, volume 28 of *Princeton Mathematical Series*. Princeton University Press, Princeton, New Jersey, 2<sup>nd</sup> edition, 1972.
- [33] P. Saint-Pierre. Approximation of the viability kernel. *Appl. Math. Optim.*, 29:187–209, 1994.
- [34] K. Schittkowski. On the Convergence of a Sequential Quadratic Programming Method with an Augmented Lagrangian Line Search Function. *Math. Operationsforsch. Stat., Ser. Optimization*, 14(2):197–216, 1983.
- [35] K. Schittkowski. NLPQL: A Fortran subroutine for solving constrained nonlinear programming problems. *Ann. Oper. Res.*, 5:484–500, 1985.
- [36] H. Schramm. Eine Kombination von Bundle- und Trust-Region-Verfahren zur Lösung nichtdifferenzierbarer Optimierungsprobleme. *Bayreuth. Math. Schr.*, 30, 1989.
- [37] L. M. Sonneborn and F. S. van Vleck. The bang-bang principle for linear control problems. *SIAM J. Control, Ser. A*, 2(2):151–159, 1965.
- [38] P. Varaiya. Reach set computation using optimal control. In M. K. Inan and R. P. Kurshan, editors, *Verification of digital and hybrid systems. Proceedings of the NATO ASI, Antalya, Turkey, May 26–June 6, 1997*, volume 170 of *NATO ASI Ser, Ser. F, Comput. Syst. Sci.*, pages 323–331. Springer, 2000.
- [39] V. M. Veliov. Approximations to differential inclusions by discrete inclusions. Working Paper WP-89-017, IIASA, Laxenburg, Austria, 1989.



- [40] V. M. Veliov. Second order discrete approximations to strongly convex differential inclusions. *Systems Control Lett.*, 13:263–269, 1989.
- [41] V. M. Veliov. Second Order Discrete Approximation to Linear Differential Inclusions. *SIAM J. Numer. Anal.*, 29(2):439–451, 1992.
- [42] S. E. Wright. *Primal-Dual Interior-Point Methods*. SIAM, Philadelphia, PA, 1997.
- [43] G. Zoutendijk. *Methods of Feasible Directions*. Elsevier Publishing Company, Amsterdam, Netherlands, 1960.



## Reports

Stand: 12. August 2004

- 98–01. Peter Benner, Heike Faßbender:  
*An Implicitly Restarted Symplectic Lanczos Method for the Symplectic Eigenvalue Problem*, Juli 1998.
- 98–02. Heike Faßbender:  
*Sliding Window Schemes for Discrete Least-Squares Approximation by Trigonometric Polynomials*, Juli 1998.
- 98–03. Peter Benner, Maribel Castillo, Enrique S. Quintana-Ortí:  
*Parallel Partial Stabilizing Algorithms for Large Linear Control Systems*, Juli 1998.
- 98–04. Peter Benner:  
*Computational Methods for Linear–Quadratic Optimization*, August 1998.
- 98–05. Peter Benner, Ralph Byers, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:  
*Solving Algebraic Riccati Equations on Parallel Computers Using Newton’s Method with Exact Line Search*, August 1998.
- 98–06. Lars Grüne, Fabian Wirth:  
*On the rate of convergence of infinite horizon discounted optimal value functions*, November 1998.
- 98–07. Peter Benner, Volker Mehrmann, Hongguo Xu:  
*A Note on the Numerical Solution of Complex Hamiltonian and Skew-Hamiltonian Eigenvalue Problems*, November 1998.
- 98–08. Eberhard Bänsch, Burkhard Höhn:  
*Numerical simulation of a silicon floating zone with a free capillary surface*, Dezember 1998.
- 99–01. Heike Faßbender:  
*The Parameterized SR Algorithm for Symplectic (Butterfly) Matrices*, Februar 1999.
- 99–02. Heike Faßbender:  
*Error Analysis of the symplectic Lanczos Method for the symplectic Eigenvalue Problem*, März 1999.
- 99–03. Eberhard Bänsch, Alfred Schmidt:  
*Simulation of dendritic crystal growth with thermal convection*, März 1999.
- 99–04. Eberhard Bänsch:  
*Finite element discretization of the Navier-Stokes equations with a free capillary surface*, März 1999.
- 99–05. Peter Benner:  
*Mathematik in der Berufspraxis*, Juli 1999.
- 99–06. Andrew D.B. Paice, Fabian R. Wirth:  
*Robustness of nonlinear systems and their domains of attraction*, August 1999.

- 99–07. Peter Benner, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:  
*Balanced Truncation Model Reduction of Large-Scale Dense Systems on Parallel Computers*, September 1999.
- 99–08. Ronald Stöver:  
*Collocation methods for solving linear differential-algebraic boundary value problems*, September 1999.
- 99–09. Huseyin Akcay:  
*Modelling with Orthonormal Basis Functions*, September 1999.
- 99–10. Heike Faßbender, D. Steven Mackey, Niloufer Mackey:  
*Hamilton and Jacobi come full circle: Jacobi algorithms for structured Hamiltonian eigenproblems*, Oktober 1999.
- 99–11. Peter Benner, Vincente Hernández, Antonio Pastor:  
*On the Kleinman Iteration for Nonstabilizable System*, Oktober 1999.
- 99–12. Peter Benner, Heike Faßbender:  
*A Hybrid Method for the Numerical Solution of Discrete-Time Algebraic Riccati Equations*, November 1999.
- 99–13. Peter Benner, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:  
*Numerical Solution of Schur Stable Linear Matrix Equations on Multicomputers*, November 1999.
- 99–14. Eberhard Bänsch, Karol Mikula:  
*Adaptivity in 3D Image Processing*, Dezember 1999.
- 00–01. Peter Benner, Volker Mehrmann, Hongguo Xu:  
*Perturbation Analysis for the Eigenvalue Problem of a Formal Product of Matrices*, Januar 2000.
- 00–02. Ziping Huang:  
*Finite Element Method for Mixed Problems with Penalty*, Januar 2000.
- 00–03. Gianfrancesco Martinico:  
*Recursive mesh refinement in 3D*, Februar 2000.
- 00–04. Eberhard Bänsch, Christoph Egbers, Oliver Meincke, Nicoleta Scurtu:  
*Taylor-Couette System with Asymmetric Boundary Conditions*, Februar 2000.
- 00–05. Peter Benner:  
*Symplectic Balancing of Hamiltonian Matrices*, Februar 2000.
- 00–06. Fabio Camilli, Lars Grüne, Fabian Wirth:  
*A regularization of Zubov's equation for robust domains of attraction*, März 2000.
- 00–07. Michael Wolff, Eberhard Bänsch, Michael Böhm, Dominic Davis:  
*Modellierung der Abkühlung von Stahlbrammen*, März 2000.
- 00–08. Stephan Dahlke, Peter Maaß, Gerd Teschke:  
*Interpolating Scaling Functions with Duals*, April 2000.
- 00–09. Jochen Behrens, Fabian Wirth:  
*A globalization procedure for locally stabilizing controllers*, Mai 2000.

- 00–10. Peter Maaß, Gerd Teschke, Werner Willmann, Günter Wollmann:  
*Detection and Classification of Material Attributes – A Practical Application of Wavelet Analysis*, Mai 2000.
- 00–11. Stefan Boschert, Alfred Schmidt, Kunibert G. Siebert, Eberhard Bänsch, Klaus-Werner Benz, Gerhard Dziuk, Thomas Kaiser:  
*Simulation of Industrial Crystal Growth by the Vertical Bridgman Method*, Mai 2000.
- 00–12. Volker Lehmann, Gerd Teschke:  
*Wavelet Based Methods for Improved Wind Profiler Signal Processing*, Mai 2000.
- 00–13. Stephan Dahlke, Peter Maass:  
*A Note on Interpolating Scaling Functions*, August 2000.
- 00–14. Ronny Ramlau, Rolf Clackdoyle, Frédéric Noo, Girish Bal:  
*Accurate Attenuation Correction in SPECT Imaging using Optimization of Bilinear Functions and Assuming an Unknown Spatially-Varying Attenuation Distribution*, September 2000.
- 00–15. Peter Kunkel, Ronald Stöver:  
*Symmetric collocation methods for linear differential-algebraic boundary value problems*, September 2000.
- 00–16. Fabian Wirth:  
*The generalized spectral radius and extremal norms*, Oktober 2000.
- 00–17. Frank Stenger, Ahmad Reza Naghsh-Nilchi, Jenny Niebsch, Ronny Ramlau:  
*A unified approach to the approximate solution of PDE*, November 2000.
- 00–18. Peter Benner, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:  
*Parallel algorithms for model reduction of discrete-time systems*, Dezember 2000.
- 00–19. Ronny Ramlau:  
*A steepest descent algorithm for the global minimization of Tikhonov–Phillips functional*, Dezember 2000.
- 01–01. Efficient methods in hyperthermia treatment planning:  
*Torsten Köhler, Peter Maass, Peter Wust, Martin Seebass*, Januar 2001.
- 01–02. Parallel Algorithms for LQ Optimal Control of Discrete-Time Periodic Linear Systems:  
*Peter Benner, Ralph Byers, Rafael Mayo, Enrique S. Quintana-Ortí, Vicente Hernández*, Februar 2001.
- 01–03. Peter Benner, Enrique S. Quintana-Ortí, Gregorio Quintana-Ortí:  
*Efficient Numerical Algorithms for Balanced Stochastic Truncation*, März 2001.
- 01–04. Peter Benner, Maribel Castillo, Enrique S. Quintana-Ortí:  
*Partial Stabilization of Large-Scale Discrete-Time Linear Control Systems*, März 2001.
- 01–05. Stephan Dahlke:  
*Besov Regularity for Edge Singularities in Polyhedral Domains*, Mai 2001.
- 01–06. Fabian Wirth:  
*A linearization principle for robustness with respect to time-varying perturbations*, Mai 2001.

- 01–07. Stephan Dahlke, Wolfgang Dahmen, Karsten Urban:  
*Adaptive Wavelet Methods for Saddle Point Problems - Optimal Convergence Rates*, Juli 2001.
- 01–08. Ronny Ramlau:  
*Morozov's Discrepancy Principle for Tikhonov regularization of nonlinear operators*, Juli 2001.
- 01–09. Michael Wolff:  
*Einführung des Drucks für die instationären Stokes–Gleichungen mittels der Methode von Kaplan*, Juli 2001.
- 01–10. Stephan Dahlke, Peter Maaß, Gerd Teschke:  
*Reconstruction of Reflectivity Densities by Wavelet Transforms*, August 2001.
- 01–11. Stephan Dahlke:  
*Besov Regularity for the Neumann Problem*, August 2001.
- 01–12. Bernard Haasdonk, Mario Ohlberger, Martin Rumpf, Alfred Schmidt, Kunibert G. Siebert:  
 *$h$ - $p$ -Multiresolution Visualization of Adaptive Finite Element Simulations*, Oktober 2001.
- 01–13. Stephan Dahlke, Gabriele Steidl, Gerd Teschke:  
*Coorbit Spaces and Banach Frames on Homogeneous Spaces with Applications to Analyzing Functions on Spheres*, August 2001.
- 02–01. Michael Wolff, Michael Böhm:  
*Zur Modellierung der Thermoelasto-Plastizität mit Phasenumwandlungen bei Stählen sowie der Umwandlungsplastizität*, Februar 2002.
- 02–02. Stephan Dahlke, Peter Maaß:  
*An Outline of Adaptive Wavelet Galerkin Methods for Tikhonov Regularization of Inverse Parabolic Problems*, April 2002.
- 02–03. Alfred Schmidt:  
*A Multi-Mesh Finite Element Method for Phase Field Simulations*, April 2002.
- 02–04. Sergey N. Dachkovski, Michael Böhm:  
*A Note on Finite Thermoplasticity with Phase Changes*, Juli 2002.
- 02–05. Michael Wolff, Michael Böhm:  
*Phasenumwandlungen und Umwandlungsplastizität bei Stählen im Konzept der Thermoelasto-Plastizität*, Juli 2002.
- 02–06. Gerd Teschke:  
*Construction of Generalized Uncertainty Principles and Wavelets in Anisotropic Sobolev Spaces*, August 2002.
- 02–07. Ronny Ramlau:  
*TIGRA – an iterative algorithm for regularizing nonlinear ill-posed problems*, August 2002.
- 02–08. Michael Lukaschewitsch, Peter Maaß, Michael Pidcock:  
*Tikhonov regularization for Electrical Impedance Tomography on unbounded domains*, Oktober 2002.

- 02–09. Volker Dicken, Peter Maaß, Ingo Menz, Jenny Niebsch, Ronny Ramlau:  
*Inverse Unwuchtidentifikation an Flugtriebwerken mit Quetschöldämpfern*, Oktober 2002.
- 02–10. Torsten Köhler, Peter Maaß, Jan Kalden:  
*Time-series forecasting for total volume data and charge back data*, November 2002.
- 02–11. Angelika Bunse-Gerstner:  
*A Short Introduction to Iterative Methods for Large Linear Systems*, November 2002.
- 02–12. Peter Kunkel, Volker Mehrmann, Ronald Stöver:  
*Symmetric Collocation for Unstructured Nonlinear Differential-Algebraic Equations of Arbitrary Index*, November 2002.
- 02–13. Michael Wolff:  
*Ringvorlesung: Distortion Engineering 2*  
*Kontinuumsmechanische Modellierung des Materialverhaltens von Stahl unter Berücksichtigung von Phasenumwandlungen*, Dezember 2002.
- 02–14. Michael Böhm, Martin Hunkel, Alfred Schmidt, Michael Wolff:  
*Evaluation of various phase-transition models for 100Cr6 for application in commercial FEM programs*, Dezember 2002.
- 03–01. Michael Wolff, Michael Böhm, Serguei Dachkovski:  
*Volumenanteile versus Massenanteile - der Dilatometerversuch aus der Sicht der Kontinuumsmechanik*, Januar 2003.
- 03–02. Daniel Kessler, Ricardo H. Nochetto, Alfred Schmidt:  
*A posteriori error control for the Allen-Cahn Problem: circumventing Gronwall's inequality*, März 2003.
- 03–03. Michael Böhm, Jörg Kropp, Adrian Muntean:  
*On a Prediction Model for Concrete Carbonation based on Moving Interfaces - Interface concentrated Reactions*, April 2003.
- 03–04. Michael Böhm, Jörg Kropp, Adrian Muntean:  
*A Two-Reaction-Zones Moving-Interface Model for Predicting  $\text{Ca}(\text{OH})_2$  Carbonation in Concrete*, April 2003.
- 03–05. Vladimir L. Kharitonov, Diederich Hinrichsen:  
*Exponential estimates for time delay systems*, May 2003.
- 03–06. Michael Wolff, Michael Böhm, Serguei Dachkovski, Günther Löwisch:  
*Zur makroskopischen Modellierung von spannungsabhängigem Umwandlungsverhalten und Umwandlungsplastizität bei Stählen und ihrer experimentellen Untersuchung in einfachen Versuchen*, Juli 2003.
- 03–07. Serguei Dachkovski, Michael Böhm, Alfred Schmidt, Michael Wolff:  
*Comparison of several kinetic equations for pearlite transformation in 100Cr6 steel*, Juli 2003.
- 03–08. Volker Dicken, Peter Maass, Ingo Menz, Jenny Niebsch, Ronny Ramlau:  
*Nonlinear Inverse Unbalance Reconstruction in Rotor dynamics*, Juli 2003.

- 03–09. Michael Böhm, Serguei Dachkovski, Martin Hunkel, Thomas Lübben, Michael Wolff:  
*Übersicht über einige makroskopische Modelle für Phasenumwandlungen im Stahl*,  
Juli 2003.
- 03–10. Michael Wolff, Friedhelm Frerichs, Bettina Suhr:  
*Vorstudie für einen Bauteilversuch zur Umwandlungsplastizität bei der perlitischen Umwandlung des Stahls 100 Cr6*,  
August 2003.
- 03–11. Michael Wolff, Bettina Suhr:  
*Zum Vergleich von Massen- und Volumenanteilen bei der perlitischen Umwandlung der Stähle 100Cr6 und C80*,  
September 2003.
- 03–12. Rike Grotmaack, Adrian Muntean:  
*Stabilitätsanalyse eines Moving-Boundary-Modells der beschleunigten Karbonatisierung von Portlandzementen*,  
September 2003.
- 03–13. Alfred Schmidt, Michael Wolff, Michael Böhm:  
*Numerische Untersuchungen für ein Modell des Materialverhaltens mit Umwandlungsplastizität und Phasenumwandlungen beim Stahl 100Cr6 (Teil 1)*,  
September 2003.
- 04–01. Liliana Cruz Martin, Gerd Teschke:  
*A new method to reconstruct radar reflectivities and Doppler information*,  
Januar 2004.
- 04–02. Ingrid Daubechies, Gerd Teschke:  
*Wavelet based image decomposition by variational functionals*,  
Januar 2004.
- 04–03. N. Guglielmi, F. Wirth, M. Zennaro:  
*Complex polytope extremality results for families of matrices*,  
März 2004.
- 04–04. I. Daubechies, G. Teschke:  
*Variational image restoration by means of wavelets: simultaneous decomposition, deblurring and denoising*,  
April 2004.
- 04–05. V.L. Kharitonov, E. Plischke:  
*Lyapunov matrices for time-delay systems*,  
April 2004.
- 04–06. Ronny Ramlau:  
*On the use of fixed point iterations for the regularization of nonlinear ill-posed problems*,  
Juni 2004.
- 04–07. Christof Büskens, Matthias Knauer:  
*Higher Order Real-Time Approximations In Optimal Control of Multibody-Systems For Industrial Robots*,  
August 2004.



04–08. Christof Büskens, Roland Griesse:

*Computational Parametric Sensitivity Analysis of Perturbed PDE Optimal Control Problems with State and Control Constraints,*  
August 2004.

04–09. Christof Büskens:

*Higher Order Real-Time Approximations of Perturbed Control Constrained PDE Optimal Control Problems with State and Control Constraints,*  
August 2004.

04–10. Christof Büskens, Matthias Gerdt:

*Differentiability of Consistency Functions,*  
August 2004.

04–11. Robert Baier, Christof Büskens, Ilyes Aïssa Chama, Matthias Gerdt:

*Approximation of Reachable Sets by Direct Solution Methods of Optimal Control Problems,*  
August 2004.