Zentrum für Technomathematik

Fachbereich 3 – Mathematik und Informatik

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Report 02–06

Berichte aus der Technomathematik

Report 02–06 **August 2002**

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August 6, 2002

Abstract

This paper is concerned with the construction of generalized uncertainty relations and minimizing states. Starting from a two operator parabola ansatz we derive a new set of uncertainties by extending the parabola ansatz to quadratic forms. This setting can be applied to the computation of uncertainties consisting of more than two operators. The resulting minimizing wavelets are solutions of connected eigenvalue problems. For affine Weyl-Heisenberg groups we derive some examples to confirm the applicability of the presented generalization. These sort of wavelets feature special localization properties. Hence, they are of importance in the context of signal- and image processing.

Key Words: Uncertainty Principles, Infinitesimal Operators, Group Representations, Affine Weyl-Heisenberg Groups, Anisotropic Sobolev Spaces, Wavelets

AMS Subject classification: 42C40, 43A85, 43A99, 46E35, 81R30, 81Q99

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1 Introduction

Wavelet analysis was originally proposed as an alternative to windowed Fourier analysis in a signal processing context. Originally, wavelets were functions generated from a single one by dilation and translation. The so-called mother wavelet was a function of vanishing integral. However, since some years the term of wavelet has to be understood in a more general sense, i.e., a wavelet is a function satisfying a generalized admissibility condition and inducing by inner products a decomposition of the function to be analyzed. From a signal processing point of view it is often very useful to decompose signals by some atomic functions with nice localization properties. It is well-known that localization is related to uncertainty principles. Seen from this angle the present paper is devoted to establish more flexibility in the known uncertainty framework. Using the fact that group theory is the common thread between Gabor- and wavelet analysis, the main goal of this paper is to provide useful representations of underlying groups and to establish related generalized uncertainty principles. Finally, we obtain nice analyzing wavelets by minimizing the new uncertainties.

To be more precisely, we are interested in wavelets with good localization properties related to the representation of the underlying group. In case of the Weyl-Heisenberg group the common representation leads to the windowed Fourier transform. It is known that the localization properties in time and phase space depend on the underlying window function. This fact can be described by Heisenberg's uncertainty relation. In this setting a optimal analyzing wavelet is a window function minimizing Heisenberg's uncertainty. To establish a wider range of useful decompositions we aim at groups containing both the Gabor- and wavelet case. During the last decade, efforts have been made to construct such time-frequency representations in L_2. In [Tor91, Tor92, KT93] it was proposed to consider a bigger group containing both the affine and the Weyl-Heisenberg group. In principle, there are two cases of mixing both groups – the direct and the semi-direct group product. For the semi-direct group product it was shown that the representations of such a bigger group are not square integrable. Hence, one may act on quotient spaces. On the basis of pioneering work of Antoine et.al. and Torresani et.al. on the group theoretical background of Gabor and wavelet analysis we increase the capability of the whole framework by introducing generalized wavelet transforms in anisotropic Sobolev spaces. To obtain the right analyzing wavelets we search for minimizer of related uncertainty relations, see e.g. [DM95] for the affine group. However, it is a known drawback that the structure of uncertainties become more complex if the number of group parameters increases. The classical Heisenberg uncertainty is just a single relation. But in more complicated cases we obtain families of uncertainties and searching for simultaneously minimizing states fails. This problem rests on the classical theory for *two* infinitesimal operators. A r–operator construction is far from being understood. In this paper we present a parabola interpretation of uncertainties. Then, by generalizing this ansatz we obtain matrix-valued commutator relations. This leads to the fact that in our new setting every uncertainty principle can be expressed as a determinant. For two operators the determinant coincides with the classical uncertainty setting.

In the present paper we proceed as follows:

1. In the second section, we review the basic theory on groups, group representations and uncertainty relations. Moreover, we present a new generalized uncertainty setting which can be interpreted as a multidimensional parabola ansatz.

2. In the third section, we introduce the affine Weyl-Heisenberg groups. Furthermore, after establishing anisotropic Sobolev spaces, we derive the related admissibility conditions.

3. In the fourth section, we show by some comprehensive examples the applicability of the generalized uncertainty setting.

4. The appendix contains some basic proofs.

2 Basic Setting and Uncertainty Principles

In this section, we introduce the group theoretical background and the uncertainty framework.

2.1 Group Representations

Let G be a locally compact and topological group with left or right invariant Haar measure $d\mu$. A **representation** π of G in a Hilbert space $\mathcal{H}$ is a homomorphism between G and the group of bounded linear mappings $\mathcal{L}(\mathcal{H})$. We restrict ourselves to the setting of continuous unitary irreducible representations. Such a representation is called **square integrable** if

π is irreducible and there exists a non-trivial vector $\psi \in \mathcal{H}$ such that

$$0 < \int\limits_G |\langle \pi(g)\psi, \psi \rangle_{\mathcal{H}}|^2 d\mu(g) < \infty. \tag{2.1}$$

A vector ψ satisfying (2.1) is called admissible. In case where (2.1) does not exists one may restrict the integration to an adequate subset X. Often one uses $X = G/H$, where H is a closed subgroup of G. Because π acts on G and not on G/H it becomes necessary to embed G/H in G. This will be realized by the canonical fiber bundle structure of G, see [Mac76],

$$\Pi : G \to G/H.$$

Let σ be a section of such a fiber bundle. A representation acting on G/H is called a σ–**modified representation** and is defined by

$$\pi_\sigma = \pi \circ \sigma. \tag{2.2}$$

This leads to a slightly modified definition of admissibility, see [KalTor]. A section σ is called **admissible** if there exists a bounded positive and invertible operator $\mathcal{A}$ with a bounded inverse and a vector ψ such that for all $f \in \mathcal{H}$

$$\int\limits_{G/H} |\langle \pi_\sigma(x)\psi, f \rangle_{\mathcal{H}}|^2 d\mu(x) = \langle f, \mathcal{A}f \rangle_{\mathcal{H}} \tag{2.3}$$

holds, where $d\mu$ is a quasi invariant measure on $L_2(G/H)$. A stronger formulation is given by the following definition. A section σ is called **strictly admissible** if there exists a positive constant K and a vector ψ such that for all $f \in \mathcal{H}$

$$\int\limits_{G/H} |\langle \pi_\sigma(x)\psi, f \rangle_{\mathcal{H}}| d\mu(x) = K\|f\|^2_{\mathcal{H}} \tag{2.4}$$

holds. The general structure of σ was analyzed in [AAG91a, AAG91b]. By π and the definitions of admissibility one has well-defined integral transformations on $\mathcal{H}$. In dependence on the underlying Hilbert-space $\mathcal{H}$ and the special choice of G one has to state explicitly the conditions to specify the set of analyzing function.

2.2 Classical Uncertainty Principles

The starting point to construct uncertainty relations is the underlying group G and its representation π in some Hilbert space $\mathcal{H}$. Let $g = (g_1, \ldots, g_n)$ be an element of G.

Furthermore, let f be a vector belonging to $\mathcal{H}$. With respect to the representation π we define so-called **infinitesimal operators** by

$$[A(g_i)f](x) := \frac{\partial}{\partial g_i}[\pi(g)f](x)|_{g=e} \ . \tag{2.5}$$

Let $A = A(g_i)$ be some infinitesimal operator with $A : \mathcal{D}(A) \to \mathcal{H}$, where $\mathcal{D}(A) \subset \mathcal{H}$ denotes the domain of A. The **expectation** of A with respect to a state $\psi \in \mathcal{D}(A)$ is defined by

$$\mu_\psi(A) := \frac{\langle A\psi, \psi\rangle}{\|\psi\|} \ . \tag{2.6}$$

The **variance** of A with respect to $\psi \in \mathcal{D}(A)$ is defined by

$$\Delta_\psi^2(A) := \mu_\psi((A - \mu_\psi(A))^2) = \mu_\psi(A^2) - \mu_\psi(A)^2 \ . \tag{2.7}$$

If the operator A is self-adjoint and non-commuting, then the following theorem holds.

Theorem 2.1 *Assume that A and B are non-commuting and self-adjoint operators and let the commutator be given by $[A, B] = iC$. Then for all $\psi \in \mathcal{D}([A, B])$ the following uncertainty relation*

$$\mu_\psi(C)^2 \leq 4\mu_\psi(A^2)\mu_\psi(B^2) \tag{2.8}$$

holds. One has equality in (2.8) if and only if there exists a parameter $t \in \mathbb{R}$ with

$$(A - itB)\psi = 0 \quad \textit{or equivalently} \quad (A^2 + t^2B^2)\psi = -tC\psi \ . \tag{2.9}$$

Proof At first, we compute $(A - itB)^*(A - itB) = A^2 + tC + t^2B^2$. This holds for all $t \in \mathbb{R}$. Hence, for all $\psi \in \mathcal{D}([A, B])$, with $\|\psi\| = 1$, we have

$$0 \leq \|(A - itB)\psi\|_{\mathcal{H}}^2 = \mu_\psi(A^2) + t\mu_\psi(C) + t^2\mu_\psi(B^2) \ , \tag{2.10}$$

which is a real and nonnegative parabola in t. Consequently, the condition

$$D = \left(\frac{\mu_\psi(C)}{2\mu_\psi(B^2)}\right)^2 - \frac{\mu_\psi(A^2)}{\mu_\psi(B^2)} \leq 0 \tag{2.11}$$

is fulfilled. This proves inequality (2.8). One has equality in (2.8) if there exists a $t \in \mathbb{R}$ which is a root of second order (this means $D = 0$). This is equivalent to eigenvalue problem $(A - itB)\psi = 0$ or to $(A - itB)^*(A - itB)\psi = 0$ ■

Remark 2.1 *For infinitesimal operators A and B the uncertainty relation can be written as*

$$\langle [A,B]\psi,\psi\rangle^2 \leq 4\Delta_\psi^2(A)\Delta_\psi^2(B).$$

If the commutator (lower bound) vanishes we obtain a trivial uncertainty. Minimizing this inequality leads to the case where one of the variance terms has to be zero. Such a situation might be given if ψ is an eigenvector of A and B respectively. We want to exclude such cases of trivial uncertainties.

2.3 Generalized Uncertainty Principles

In this section, we present a new way to generalize the two-operator setting. The essential idea is to think of vectors of observation variables. We start with r infinitesimal operators and consider suitable linear combinations of the operators. In accordance with Theorem 2.1 we create new commutator conditions to generate non-trivial uncertainties. The new uncertainty principles may be understood in terms of several parabolas.

The **covariance** of A and B with respect to $\psi \in \mathcal{D}(A) \cap \mathcal{D}(B)$ is defined by

$$\Delta_\psi(A,B) := \langle [A-\mu_\psi(A)]\psi, [B-\mu_\psi(B)]\psi\rangle \ . \tag{2.12}$$

Based on Theorem 2.1 the next theorem specifies the commutator conditions which avoid trivial uncertainties.

Theorem 2.2 *Let a system of infinitesimal and self-adjoint operators $\{A_l\}_{l=1,\dots,r}$ be given. By*

$$A = (A_1,\dots,A_r)\ ,\ K = (-i[A_k,A_l])_{k,l=1,\dots,r} \ \textit{and}\ \ \Sigma = (\Delta(A_k,A_l))_{k,l=1,\dots,r} \tag{2.13}$$

we define a vector of operators, a related commutator matrix and a related matrix of covariances. Assume that we have two linear combinations of the form $B = \gamma \cdot A$ and $C = \alpha \cdot A$, where $\gamma, \alpha \in \mathbb{R}^r$, such that $\gamma' K \alpha \neq 0$. Then, for all $\psi \in \mathcal{D}([B,C])$ the following uncertainty principle

$$\mu_\psi(\gamma' K\alpha)^2 \leq 4\gamma'\Sigma\gamma\alpha'\Sigma\alpha \quad . \tag{2.14}$$

holds. One has equality in (2.14) *if and only if there exists a parameter $t \in \mathbb{R}$ with*

$$(B - itC)\psi = 0 \quad \textit{or equivalently} \quad (C^2 + t^2B^2)\psi = -t\gamma' K\alpha\psi \ . \tag{2.15}$$

Proof Just as in the proof of Theorem 2.1 we compute $(B-itC)^*(B-itC)$. Thus, we have to compute the $\left[\sum_j \gamma_j A_j, \sum_l \alpha_l A_l\right]$. By $[A_j, A_l] = -[A_l, A_j]$ it follows that

$$\begin{aligned}
\left[\sum_j \gamma_j A_j, \sum_l \alpha_l A_l\right] &= \sum_j \sum_l \gamma_j \alpha_l [A_j, A_l] = \sum_{j,l;j\neq l} \gamma_j \alpha_l [A_j, A_l] \\
&= (\gamma_1\alpha_2 - \gamma_2\alpha_1)[A_1, A_2] + (\gamma_1\alpha_3 - \gamma_3\alpha_1)[A_1, A_3] + \dots \\
&\quad (\gamma_2\alpha_3 - \gamma_3\alpha_2)[A_2, A_3] + \dots \\
&= (\gamma_1, \dots, \gamma_r) \begin{pmatrix} 0 & [A_1, A_2] & \dots & [A_1, A_r] \\ -[A_1, A_2] & 0 & & \vdots \\ \vdots & & \ddots & [A_{r-1}, A_r] \\ -[A_1, A_r] & \cdots & -[A_{r-1}, A_r] & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} \\
&= i\gamma' K \alpha \, .
\end{aligned}$$

Hence, we have $(B-itC)^*(B-itC) = B^2 + t\gamma' K\alpha + t^2 C^2$. Consequently, for all $\psi \in \mathcal{D}([B, C])$ and $\|\psi\|_{\mathcal{H}} = 1$ the equation

$$\|(B-itC)\psi\|^2_{\mathcal{H}} = \mu_\psi(B^2) + t\mu_\psi(\gamma' K\alpha) + t^2 \mu_\psi(C^2) \tag{2.16}$$

holds. Non-negativity of (2.16) implies

$$\mu_\psi(\gamma' K\alpha)^2 \leq 4\mu_\psi(B^2)\mu_\psi(C^2) \, . \tag{2.17}$$

One has equality in (2.17) if and only if there exists a parameter $t \in \mathbb{R}$ with $(B-itC)\psi = 0$ or equivalently $(B^2 + t^2C^2)\psi = -t\gamma' K\alpha\psi$. To rewrite (2.14) we use that $B = \gamma A$. Then, it follows immediately that

$$\begin{aligned}
\Delta_f^2(\gamma A) &= \Delta_f(\gamma A, \gamma A) = \langle \gamma A f, \gamma A f \rangle \\
&= \sum_{j=1}^r \sum_{l=1}^r \gamma_j \gamma_l \langle A_j f, A_l f \rangle = \gamma' \Sigma \gamma
\end{aligned}$$

And similarly we have $\Delta_f^2(\alpha A) = \alpha' \Sigma \alpha$. ■

We have to be careful about the choice of ψ. It is not obvious that the domain of the generalized commutator is dense in $\mathcal{H}$. In general, the overall domain is an intersection of all possible domains of all commutators.

Corollary 2.1 *The uncertainty structure may be easily decomposed in*

$$\frac{1}{4}\mu_\psi(\gamma' K\alpha)^2 \leq \gamma' V\gamma\alpha' V\alpha + \gamma' M\alpha \ ,$$

with a variance term

$$V = \mathrm{diag}\left(\Delta_f^2(A_1), \ldots, \Delta_f^2(A_r)\right)$$

and a symmetric term of mixed products

$$M = (\Sigma - V)\gamma'\alpha V + V\gamma'\alpha(\Sigma - V) \ .$$

Every nontrivial uncertainty relation has now the following expression

$$0 < C \leq T_{vv} + T_{vc} + T_{cc} \ , \tag{2.18}$$

where

$$\begin{aligned}
C &= \frac{1}{4}\mu_\psi(\gamma K\alpha')^2 \ , \\
T_{vv} &= \sum_{j,l=1}^{r} \alpha_j^2\gamma_l^2\Delta_f^2(A_j)\Delta_f^2(A_l) \ , \\
T_{vc} &= \sum_{j,l=1;j\neq l}^{r}\sum_{n=1}^{r}\alpha_n^2\gamma_j\gamma_l\Delta_f^2(A_n)\Delta_f(A_j, A_l) \\
&\quad + \sum_{n,m=1;n\neq m}^{r}\sum_{j=1}^{r}\gamma_j^2\alpha_j\alpha_l\Delta_f^2(A_j)\Delta_f(A_n, A_m) \ , \\
T_{cc} &= \left[\sum_{n,m=1;n\neq m}^{r}\alpha_n\alpha_m\Delta_f(A_n, A_m)\right]\left[\sum_{n,m=1;n\neq m}^{r}\gamma_n\gamma_m\Delta_f(A_n, A_m)\right] \ .
\end{aligned}$$

Remark 2.2 *The condition $\gamma' K\alpha \neq 0$ is equivalent to the demand that for all $\psi \in \mathcal{D}([B, C])$ we have to choose vectors γ and α such that $\gamma'\Sigma\gamma \neq 0$ and $\alpha'\Sigma\alpha \neq 0$. To satisfy this condition it is necessary that $\gamma, \alpha \notin \mathcal{N}(\Sigma)$.*

Theorem 2.2 extends Theorem 2.1 by involving all possible commutator relations. But to gather the new objects it seems to be suggestive to formulate the new uncertainties in terms of several parabolas. To this end, we consider the partial differential equation

$$(A - itB)\psi = 0 \ .$$

By substituting $B = \gamma' \tilde{B}$, where $\tilde{B} = (B_1, \ldots, B_r)'$ and $\gamma = (\gamma_1, \ldots, \gamma_r)'$, the resulting partial differential equation has the form

$$(A - it\gamma_1 B_1 - it\gamma_2 B_2 - \ldots - it\gamma_r B_r)\psi = 0$$

and with $\alpha_i = t\gamma_i$ follows

$$(A - i\alpha_1 B_1 - i\alpha_2 B_2 - \ldots - i\alpha_r B_r)\psi = 0 \ .$$

This procedure may be repeated also for $A = \gamma' \tilde{A}$. Finally, this leads to a reformulation of the established r – operator uncertainties in Theorem 2.2.

Theorem 2.3 *Let for $l = 1, \ldots, r$ the infinitesimal and self-adjoint operators A_l be given. Furthermore, we define a matrix F by*

$$F := \begin{pmatrix} A^2 & i/2[A, B_1] & & \cdots & i/2[A, B_r] \\ i/2[A, B_1] & B_1^2 & & & \\ & \ddots & (B_i B_j + B_j B_i)/2 & & \\ \vdots & (B_i B_j + B_j B_i)/2 & \ddots & & \vdots \\ i/2[A, B_r] & \cdots & & & B_r^2 \end{pmatrix} .$$

Let the matrix $\mu_\psi(F)$ be the component-wise expectation of F. If $\psi \in \mathcal{D}([A, \alpha\tilde{B}])$ it follows that the quadratic form $\mu_\psi(F)$ is positive semi-definite, i.e., for all eigenvalues λ_l holds $\lambda_l \geq 0$. Finally, for all $\psi \in \mathcal{D}([A, \alpha\tilde{B}])$ the uncertainty relation with respect to A and $\tilde{B}$ can be expressed in the form

$$\det\left(\mu_\psi(F)\right) \geq 0 \ .$$

Proof At first, we note that we have for all $\alpha_i \in \mathbb{R}$:

$$(A - i\alpha_1 B_1 - \ldots - i\alpha_r B_r)^*(A - i\alpha_1 B_1 - \ldots - i\alpha_r B_r) = (1, \alpha_1, \ldots, \alpha_r) F (1, \alpha_1, \ldots, \alpha_r)' \ .$$

Hence, for all $\psi \in \mathcal{D}([A, \alpha\tilde{B}])$ with $\|\psi\| = 1$ one has

$$0 \leq \|(A - i\alpha\tilde{B})\psi\|^2 = (1, \alpha)\mu_\psi(F) \begin{pmatrix} 1 \\ \alpha \end{pmatrix} ,$$

where $\mu_\psi(F) =$

$$\begin{pmatrix} \mu_\psi(A^2) & \mu_\psi(i/2[A,B_1]) & \cdots & \mu_\psi(i/2[A,B_r]) \\ \mu_\psi(i/2[A,B_1]) & \mu_\psi(B_1^2) & & \\ & \ddots & \mu_\psi((B_iB_j+B_jB_i)/2) & \\ \vdots & \mu_\psi((B_iB_j+B_jB_i)/2) & \ddots & \vdots \\ \mu_\psi(i/2[A,B_r]) & \cdots & & \mu_\psi(B_r^2) \end{pmatrix}.$$

Thus, the form $\mu_\psi(F)$ is positive semi-definite. Consequently, the uncertainty can be written as the determinant. For the two operator case this is just the discriminant (2.11). ■

The condition $\gamma' K\alpha \neq 0$ can by $\tilde{A} = (A, 0, \ldots, 0)$ and $\gamma = (1, 0, \ldots, 0)'$ transformed into

$$\sum_{l=1}^{r} \alpha_l [A, B_l] \neq 0 \ .$$

The mixed terms $\mu_\psi((B_iB_j + B_jB_i))$ in the quadratic form $\mu_\psi(F)$ represent the covariances.

2.4 Tensor Product Hilbert Spaces

Later on we intend to use anisotropic Hilbert spaces. To this end, we need some basic facts about tensor product Hilbert spaces. Up to equivalence there are three different possibilities to construct a tensor product of two Hilbert spaces, see [DF93]. We use a method presented in [Hoc98].

Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two given Hilbert spaces and $\mathcal{H}_1'$ and $\mathcal{H}_2'$ are its duals. Let $\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')$ be the space of bilinear mappings. Then, the tensor product $x \otimes y$ is defined as the element in $\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')$ with

$$(x \otimes y)(x', y') := x'(x) y'(y) \ , \quad (x', y') \in \mathcal{H}_1' \times \mathcal{H}_2' \ ,$$

where $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$. Consequently, $\mathcal{H}_1 \otimes \mathcal{H}_2$ is generated by tensor products $x \otimes y$, where $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$. Hence, this space is defined as a subspace of $\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')$. Furthermore, we need to establish a "inner product" in this space. Let $f, g \in \mathcal{H}_1 \otimes \mathcal{H}_2$, where $f = \sum_{j=1}^{m_j} c_j u_j \otimes v_j$ and $g = \sum_{k=1}^{m_k} d_k w_k \otimes z_k$ and $u_j, w_k \in \mathcal{H}_1$, $v_j, z_k \in \mathcal{H}_2$. By

$$l(f, g) := \sum_{j=1}^{m_j} \sum_{k=1}^{m_k} c_j d_k \left\langle u_j, w_k \right\rangle_{\mathcal{H}_1} \left\langle v_j, z_k \right\rangle_{\mathcal{H}_2}$$

we define a sesqui-linear functional. Equipped with this sesqui-linear functional, $\mathcal{H}_1 \otimes \mathcal{H}_2$ becomes a pre-Hilbert space. The **tensor product**

$$\mathcal{H}_1 \otimes_l \mathcal{H}_2$$

is then defined as the closure of $\mathcal{H}_1 \otimes \mathcal{H}_2$ with respect to $l(\cdot, \cdot)$.

3 Affine Weyl-Heisenberg Groups and Admissibility

In this section, we specify the groups under consideration, appropriate function spaces, and we determine the related admissibility conditions.

3.1 Weyl-Heisenberg and Affine Group

The **affine group $\mathbf{G}_{aff}$** is given by

$$\mathbf{G}_{aff} = \Big(\{(q, a, R) : \; q \in \mathbb{R}^n, \; a \in \mathbb{R}_+, \; R \in SO(n)\}, \circ \Big),$$

with the group law

$$(q, a, R) \circ (q', a', R') = (q + aRq', aa', RR').$$

Proposition 3.1 *The right and left invariant Haar measures on $\mathbf{G}_{aff}$ are given by*

$$d\mu_R(q, a, R) = a^{-1} dadqdm(R) \quad \textit{and} \quad d\mu_L(q, a, R) = a^{-(n+1)} dadqdm(R),$$

where $dm(R)$ is an invariant measure on $SO(n)$.

By $SO(n)/SO(n-1) \cong S^{n-1}$ we can rewrite every rotation $R_n \in SO(n)$ as a product KR_{n-1}, $K \in S^{n-1}$ and $R_{n-1} \in SO(n-1)$. Very often one is interested in such axial-symmetric functions. We shall use this fact later on for computing admissibility conditions.

The **reduced Weyl-Heisenberg group** is given by

$$\mathbf{G}_{WH} = \left(\{(q, p, \varphi) : \; q \in \mathbb{R}^n, \; p \in \mathbb{R}^n, \; \varphi \in S^1, \circ\} \right),$$

with the group law

$$(q, p, \varphi) \circ (q', p', \varphi') = (q + q', p + p', \varphi + \varphi' + p \cdot q' \; mod \; 2\pi).$$

Proposition 3.2 *The group $\mathbf{G}_{WH}$ is unimodular with the invariant Haar measure*

$$d\mu(g) = dqdpd\varphi \ .$$

This can be found in [SD80].

3.2 Mixed Groups

Beside the groups introduced above we focus mainly on a mixture of the affine and Weyl-Heisenberg group. For that reason we have to put both groups into a uniform setting. This can be done by direct or semi-direct group products.

The **direct affine Weyl-Heisenberg group $\mathbf{G}^d_{aWH}$** is defined as the direct product of $\mathbf{G}_{WH}$ and $\mathbf{G}_{aff}$ an is equipped with the component-wise group law

$$(q, p, \varphi; b, a, R) \circ (q', p', \varphi'; b', a', R')$$

$$= (q + q', p + p', \varphi + \varphi' + pq' \ mod \ 2\pi; b + aRb', aa', RR') \ .$$

There are different ways to generate a related group representation. We could define a matrix-valued mapping (direct sum)

$$\pi_1 + \pi_2 : G \to \mathcal{L}(\mathcal{H}_1 + \mathcal{H}_2), \quad g \mapsto \begin{pmatrix} \pi_1(g) & 0 \\ 0 & \pi_2(g) \end{pmatrix} .$$

However, this representation obviously leads to a component-wise behavior. For our purpose we prefer direct products of representations, see [VK91].

Proposition 3.3 *Assume that π_1 and π_2 are given representations of G and H respectively. Then, $\pi(g, h) = \pi_1(g) \otimes \pi_2(h)$, $g \in G$, $h \in H$ is a representation of $G \times H$.*

A proof of the proposition and basic results for computing the related Haar measures can be found in [VK91, SD80].

Proposition 3.4 *Assume that G, H are given locally compact topological groups with left (right) invariant Haar measures μ_1 and μ_2 respectively. Then, the product $\mu_1 \otimes \mu_2$ is a left (right) Haar measure on the product $G \times H$.*

Consequently, in this setting we can use the known Haar measures of $\mathbf{G}_{WH}$ and $\mathbf{G}_{aff}$. Hence, for the direct product case we have all ingredients. Setting $G = \mathbf{G}_{WH}$ and

$H = \mathbf{G}_{aff}$, we are done.

The second alternative is the construction of a semi-direct product of $\mathbf{G}_{aff}$ and $\mathbf{G}_{WH}$. Starting point is the group generated by translation, modulation, dilation and rotation in $\mathbb{R}^n$. This group was explicitly discussed in [Tor92, Tor94, AM92, KT93, Tor91, TAGM95]. In our elaboration we follow [KT93].

The **semi-direct affine Weyl-Heisenberg group** $\mathbf{G}_{aWH}$ is defined as the semi-direct product of $\mathbf{G}_{WH}$ and $\mathbf{G}_{aff}$ and is equipped with the group law

$$(q, p, a, R, \varphi) \circ (q', p', a', R', \varphi')$$

$$= (q + aRq', p + a^{-1}Rp', aa', RR', \varphi + \varphi' + p(aRq')) \ ,$$

The inverse element of $g \in \mathbf{G}_{aWH}$ is given by

$$(q, p, a, R, \varphi)_R^{-1} = (-a^{-1}R^{-1}q, -aR^{-1}p, a^{-1}, -\varphi + pq).$$

The next proposition states one special representation for $\mathbf{G}_{aWH}$, see [KT93] for more details.

Proposition 3.5 *The representation of* $\mathbf{G}_{aWH}$ *given by*

$$\pi(q, p, a, \varphi)f(x) = \sqrt{a}e^{i(\lambda^* \log(a) + t^*(\varphi + xq))}f(a(x + p)). \tag{3.1}$$

is irreducible and is called Stone-von-Neumann representation.

Proposition 3.6 *The group* $\mathbf{G}_{aWH}$ *is unimodular. The invariant Haar measure is given by*

$$d\mu(q, p, a, R, \varphi) = dqdp\frac{da}{a}dm(R)d\varphi \ ,$$

where $dm(R)$ *denotes again the invariant measure* $SO(n)$*.*

At this point we have appropriated adequate representations of our mixed groups and invariant measures on them.

3.3 Anisotropic Sobolev Spaces and Related Admissibility

The next step is to introduce adequate Bessel potential spaces and to check irreducibility and the existence of at least one non-trivial admissible vector. This includes to state the admissibility conditions.

In many papers [AAG91a, AAG91b, AM92, TAGM95, Tor91, Tor92, Tor94, KT93] of Gabor and wavelet analysis the space under consideration is just the $L_2(\mathbb{R}^n)$. We extend the framework to a wider range of Hilbert spaces. Extending the L_2 – wavelet transform to H^s - one obtains then images in certain fiber spaces, cp. [LMR98]. However, the integral transform is again induced by inner products in $L_2(\mathbb{R}^n)$. A more general way is to define the wavelet transform by inner products in anisotropic Bessel potential spaces. To this end, we start by defining the **anisotropic Sobelev space** by

$$H^{s_1,s_2}_{mix}(\mathbb{R}^m \times \mathbb{R}^n) := \{u \in L_2(\mathbb{R}^n \times \mathbb{R}^m) \ : \quad \|u\|_{s_1,s_2} < \infty\} \quad , \tag{3.2}$$

which is equipped with the norm

$$\|u\|^2_{s_1,s_2} := \int\limits_{\mathbb{R}^m \times \mathbb{R}^n} |(1+\|k_1\|^2)^{s_1/2}(1+\|k_2\|^2)^{s_2/2}\mathcal{F}u(k_1,k_2)|^2 dk_1 dk_2 \ . \tag{3.3}$$

The inner product is given by

$$\langle u,v\rangle_{H^{s_1,s_2}} := \int\limits_{\mathbb{R}^m \times \mathbb{R}^n} (1+\|k_1\|^2)^{s_1}(1+\|k_2\|^2)^{s_2}\mathcal{F}u(k_1,k_2)\overline{\mathcal{F}v}(k_1,k_2) dk_1 dk_2 \ . \tag{3.4}$$

Applying Subsection 2.4 by setting $\mathcal{H}_1 = H^{s_1}(\mathbb{R}^m)$ and $\mathcal{H}_2 = H^{s_2}(\mathbb{R}^n)$ and definition (3.2) one can prove the following theorem, cp. [Hoc98].

Theorem 3.1 *Let $s_1, s_2 \geq 0$ and $n, m \in \mathbb{N}$. Then, we have*

$$H^{s_1}(\mathbb{R}^m) \otimes_l H^{s_2}(\mathbb{R}^n) = H^{s_1,s_2}_{mix}(\mathbb{R}^m \times \mathbb{R}^n) \ .$$

To range $H^{s_1,s_2}_{mix}(\mathbb{R}^m \times \mathbb{R}^n)$ in the scale of other function spaces we introduce two further well-known function spaces, cp. [Tri78, ST87].

Assume that $\bar{s} = (s_1, s_2) \in \mathbb{R} \times \mathbb{R}$ and $n, m \in \mathbb{N}$. The spaces

$$H^{\bar{s}}_A(\mathbb{R}^{m+n}) := \{u \in S'(\mathbb{R}^{m+n}) \ : \quad \|u\|_{A,\bar{s}} < \infty\} \tag{3.5}$$

and

$$H^{s_1,s_2}_T(\mathbb{R}^{m+n}) := \{f \in S'(\mathbb{R}^{m+n}) \ : \quad \|u\|_{T,\bar{s}} < \infty\} \tag{3.6}$$

equipped with the norms

$$\|u\|^2_{A,\bar{s}} := \int\limits_{\mathbb{R}^{m+n}} |\left((1+\|k_1\|^2)^{s_1/2} + (1+\|k_2\|^2)^{s_1/2}\right)\mathcal{F}u(k_1,k_2)|^2 dk_1 dk_2 \tag{3.7}$$

and

$$\|u\|^2_{T,\bar{s}} := \int\limits_{\mathbb{R}^{m+n}} |(1+\|(k_1,k_2)\|^2)^{s_1/2}(1+\|k_1\|^2)^{s_2/2}\mathcal{F}u(k_1,k_2)|^2 dk_1 dk_2 \ . \tag{3.8}$$

are called **anisotropic Bessel potential spaces**.

Proposition 3.7 *Assume that* $0 \leq s_1, s_2 \leq \mathbb{R}$ *and* $m, n \in \mathbb{N}$. *Moreover, assume that* $\bar{s} = (s_1, s_2)$ *and* $\tilde{s} = (s_1, \max(s_1, s_2))$. *Then, the following relation holds*

$$H_A^{2\tilde{s}} \subseteq H_T^{\bar{s}} \subseteq H_{mix}^{\bar{s}} \quad .$$

After establishing our Hilbert spaces, we aim at constructing related representations of G. Let us start by the common Sobolev space $H^s(\mathbb{R}^n)$. To establish a suitable group representation in $H^s(\mathbb{R}^n)$ we define the operator Λ_s by its Fourier-transform

$$\Lambda_s \ : \ S'(\mathbb{R}^n) \to S'(\mathbb{R}^n), \quad \text{where} \ \ (\Lambda_s f)^\wedge(\omega) = (1+\|\omega\|^2)^{s/2}\hat{f}(\omega). \tag{3.9}$$

A representation of some group G in $H^s(\mathbb{R}^n)$ can now be defined by

$$\pi_s(g) := \Lambda_{-s}\pi(g)\Lambda_s \ : \ \ H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n) \ , \tag{3.10}$$

where π is a representation of G in $L_2(\mathbb{R}^n)$. To apply definition (3.9) we have to move from π to the equivalent representation $\hat{\pi}$ in phase space via

$$\pi = \mathcal{F}^{-1}\hat{\pi}\mathcal{F} \quad \text{ if and only if } \quad \mathcal{F}\pi = \hat{\pi}\mathcal{F} \ . \tag{3.11}$$

Consequently, by using well-known group representations π in $L_2(\mathbb{R}^n)$ of $\mathbf{G}_{WH}$ and $\mathbf{G}_{aff}$ we can establish representations of mixed groups in anisotropic Sobolev spaces. Let us start by the direct group product $\mathbf{G}_{WH} \times \mathbf{G}_{aff}$. For the related representation in accordance with Proposition 3.3 a natural candidate for the representation space is $H^{s_1}(\mathbb{R}^m) \otimes_l H^{s_2}(\mathbb{R}^n)$. Hence, a suitable product representation of $\mathbf{G}^d_{aWH}$ in phase space can be defined by

$$\begin{aligned}\hat{\pi}_{s_1,s_2}(g,h)(\hat{f})(\omega,\eta) \ &:= \ (1+\|\omega\|^2)^{-s_1/2}(1+\|\eta\|^2)^{-s_2/2}e^{i(\varphi-q'\omega)}a^{n/2}e^{-ib'\eta} \quad \times \\ &\quad\ (1+\|\omega+p\|^2)^{s_1/2}(1+\|aR\eta\|^2)^{s_2/2}\hat{f}(\omega+p, aR\eta) \ .\end{aligned}$$

By defining

$$(\Lambda_{s_1,s_2} f)^{\wedge}(k_1,k_2) := (1+\|k_1\|^2)^{s_1/2}(1+\|k_2\|^2)^{s_2/2}\hat{f}(k_1,k_2)$$

the representation in time space is given by

$$\pi_{s_1,s_2}(g,h)f(x,y) = \Lambda_{-s_1,-s_2}\pi(g,h)\Lambda_{s_1,s_2}f(x,y)\ .$$

The following theorem ensures at least square integrability of our representation π_{s_1,s_2} of $\mathbf{G}_{WH} \times \mathbf{G}_{aff}$.

Theorem 3.2 *The representation π_{s_1,s_2} of $\mathbf{G}^d_{aWH}$ in $H^{s_1}(\mathbb{R}^m) \otimes_l H^{s_2}(\mathbb{R}^n)$ is homomorph, continuous, unitary and square integrable.*

Proof Assume $s_1 = s \geq 0$ and $s_2 = t \geq 0$. Then its obvious that $\pi_{s,t}$ is a homomorphism:

$$\begin{aligned}
\pi_{s,t}((g,h)\circ(g',h')) &= \pi_{s,t}(g\circ g', h\circ h') = \pi_{1,s}(g\circ g')\otimes\pi_{2,t}(h\circ h')\\
&= \pi_{1,s}(g)\pi_{1,s}(g')\otimes\pi_{2,t}(h)\pi_{2,t}(h')\\
&= (\pi_{1,s}(g)\otimes\pi_{2,t}(h))(\pi_{1,s}(g')\otimes\pi_{2,t}(h'))\\
&= \pi_{s,t}(g,h)\pi_{s,t}(g',h')
\end{aligned}$$

We know that $\pi_{1,s}$, the representation of $\mathbf{G}_{WH}$, and that $\pi_{2,t}$, the representation of $\mathbf{G}_{aff}$, are continuous. Hence, we conclude that $\pi_{s,t}$ continuous. To show that $\pi_{s,t}$ is square integrable one has to show that $\pi_{s,t}$ is irreducible and there exists a non-trivial vector. Assume $\pi_{s,t}$ is reducible. This means there exists a closed, non-trivial subspace $V \subseteq H^s(\mathbb{R}^m) \otimes_l H^t(\mathbb{R}^n)$ with $\pi_{s,t}(g)V \subseteq V\ \forall g \in \mathbf{G}^d_{aWH}$. Under this assumption there exist non-trivial functions $h \in V$ and $f \in V^\perp$.
Let $f, h \in H^s(\mathbb{R}^m) \otimes_l H^t(\mathbb{R}^n)$ and

$$(\Lambda_{s,t} f)^{\wedge}(k_1,k_2) = (1+\|k_1\|^2)^{s/2}(1+\|k_2\|^2)^{t/2}\hat{f}(k_1,k_2)\ .$$

Then, we have by $\Lambda_{s,t}f =: F$ and $\Lambda_{s,t}h =: H$

$$\langle \pi_{s,t}(g)f,h\rangle_{H^s(\mathbb{R}^m)\otimes_l H^t} = \langle \pi(g)F,H\rangle_{L_2(\mathbb{R}^{m+n})} = \left\langle \hat{\pi}(g)\hat{F},\hat{H}\right\rangle_{L_2(\mathbb{R}^{m+n})}\ .$$

Because of

$$\hat{\pi}(g)\hat{F}(\omega,\eta) = a^{n/2}e^{i\varphi}e^{-i(\omega,\eta)(q,b)'}\hat{F}(\omega+p, aR'\eta)$$

it follows that

$$\left\langle \hat{\pi}(g)\hat{F},\hat{H}\right\rangle_{L_2(\mathbb{R}^{m+n})} = a^{n/2}e^{i\varphi}(2\pi)^{\frac{m+n}{2}}\mathcal{F}\left(K(\cdot,\cdot,p,a,R)\right)(q,b)\ , \qquad (3.12)$$

where $K(\omega,\eta,p,a,R) = \hat{F}(\omega+p,aR'\eta)\bar{\hat{H}}(\omega,\eta)$. By $\Omega = \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}_+ \times SO(n)$ and some substitutions we obtain

$$\int\limits_{\mathbf{G}^d_{aWH}} |\left\langle \pi(p,q,\varphi,b,a,R)F,H\right\rangle_{L_2}|^2 d\mu d\nu = (2\pi)^{m+n+1}\int\limits_{\mathbb{R}^{m+n}} |\hat{H}(\omega,\eta)|^2 I(\eta)d\omega d\eta\ , \qquad (3.13)$$

where

$$I(\eta) = \int\limits_{\mathbb{R}^m\times\mathbb{R}_+\times SO(n)} |\hat{F}(\xi,aR'\eta)|^2 a^{-1}d\xi da dm(R)\ .$$

Using Euler-angle representation of $R \in SO(n)$ and some standard substitutions $I(\eta)$ can be expressed by

$$I(\eta) = vol(SO(n-1))\int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^n}\frac{|\hat{F}(\xi,\zeta)|^2}{\|\zeta\|^n}d\zeta d\xi\ .$$

Hence, (3.13) is equal to

$$(2\pi)^{m+n+1}\cdot vol(SO(n-1))\cdot\|h\|^2_{s,t}\cdot\int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^n}(1+\|\xi\|^2)^s(1+\|\zeta\|^2)^t\frac{|\hat{f}(\xi,\zeta)|^2}{\|\zeta\|^n}d\zeta d\xi\ .$$

Assume that $\pi_{s,t}$ is reducible. Then, there exist vectors

$$0 \neq h \in V \subset H^s(\mathbb{R}^m)\otimes_l H^t(\mathbb{R}^n) \ \text{ and } \ 0 \neq f \in V^\perp$$

such that

$$0 = \left\langle \pi(p,q,\varphi,b,a,R)F,H\right\rangle_{L_2}\ .$$

But this implies that

$$0 = \|h\|^2_{s,t}\cdot\int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^n}(1+\|\xi\|^2)^s(1+\|\zeta\|^2)^t\frac{|\hat{f}(\xi,\zeta)|^2}{\|\zeta\|^n}d\zeta d\xi$$

is true. But this is contradictory to $0 \neq f, h$. If $\hat{f}$ vanishes in a neighborhood of the origin and setting $f = h \neq 0$ it follows that

$$0 < \|f\|^2_{s,t}\cdot\int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^n}(1+\|\xi\|^2)^s(1+\|\zeta\|^2)^t\frac{|\hat{f}(\xi,\zeta)|^2}{\|\zeta\|^n}d\zeta d\xi < \infty.$$

Such a function f evidently exists. ■

Remark 3.1 *We call the related left transform in $H^s(\mathbb{R}^m)\otimes_l H^t(\mathbb{R}^n)$* **anisotropic Gabor–Wavelet–transform**. *The integral transform maps $H^s(\mathbb{R}^m)\otimes_l H^t(\mathbb{R}^n)$ isometrically onto $L_2(\mathbf{G}^d_{aWH}, d\mu_L)$.*

By the weight functions

$$\begin{aligned}
(\Lambda_{A,\bar{s}}f)^\wedge(k_1,k_1) &:= \left[(1+\|k_1\|^2)^{s_1/2} + (1+\|k_2\|^2)^{s_2/2)}\right]\hat{f}(k_1,k_1) \text{ and} \\
(\Lambda_{T,\bar{s}}f)^\wedge(k_1,k_1) &:= \left[(1+\|k_1\|^2)^{s_1/2}(1+\|(k_1,k_2)\|^2)^{s_2/2)}\right]\hat{f}(k_1,k_1)
\end{aligned}$$

we obtain representations

$$\begin{aligned}
\pi_{A,\bar{s}}(g,h)f(x_1,x_2) &:= \Lambda_{A,-\bar{s}}\pi(g,h)\Lambda_{A,\bar{s}}f(x_1,x_2) \\
\pi_{T,\bar{s}}(g,h)f(x_1,x_2) &:= \Lambda_{T,-\bar{s}}\pi(g,h)\Lambda_{T,\bar{s}}f(x_1,x_2).
\end{aligned}$$

in H^{s_1,s_2}_A and H^{s_1,s_2}_T.

Remark 3.2 *Abbreviating the weight functions by Λ_* one has a general admissibility condition for functions f belonging to some Hilbert spaces H^s_**

$$0 < (2\pi)^{m+n+1} vol(SO(n-1))\|f\|^2_* \int_{\mathbb{R}^{m+n}} \frac{|(\Lambda_{*,\bar{s}}f)^\wedge(k_1,k_2)|^2}{\|k_2\|^n} d^m k_1 d^n k_2 < \infty \,. \tag{3.14}$$

Consequently, in all cases the operator $\mathcal{A}$ in condition (2.3) is given by

$$\mathcal{A} = Id \cdot C \int_{\mathbb{R}^{m+n}} \frac{|(\Lambda_*\psi)^\wedge(k_1,k_2)|^2}{\|k_2\|^n} d^m k_1 d^n k_2 \tag{3.15}$$

and hence, by (2.4) we have isometrical mappings.

After establishing representations of $\mathbf{G}^d_{aWH}$ in Bessel potential spaces H^s_* we consider now the semi-direct group product $\mathbf{G}_{aWH}$ and related representations.

Proposition 3.8 *The representation (3.1) of the semi-direct product $\mathbf{G}_{aWH}$ in $L_2(\mathbb{R}^n)$ is not square integrable.*

To overcome this deficiency we proceed as in Section 2.1 and restrict the integration to suitable cosets X. To keep notations at a reasonable level we restrict the computations to the classical Sobolev space. Everything holds for ansisotropic spaces $H^s_*(\mathbb{R}^n)$ too. To construct a restriction of π on a homogeneous space X we may consider for instance the following subgroup of $\mathbf{G}_{aWH}$

$$\Gamma = \{(0, p, 1, 1, \varphi) \in \mathbf{G}_{aWH}\} \ . \tag{3.16}$$

Other cosets may be chosen, see [KT93]. However, at first we have to specify a admissible section σ to embed $X = \Gamma \setminus \mathbf{G}_{aWH}$ in $\mathbf{G}_{aWH}$.

Proposition 3.9 *Let $\psi \in H^s(\mathbb{R}^n)$ and $\sigma(q, a, R) = (q, \beta(a, R), a, R, 0)$, where $\beta : X \to \Gamma$ is a piecewise differentiable mapping. Furthermore, the representation $\pi_{s,\sigma}$ is given by $\Lambda_{-s}(\pi \circ \sigma)\Lambda_s$. Then, the section σ is strictly admissible for a constant co-vector v if*

$$0 < \int_{\mathbb{R}^n} \frac{|(\Lambda_s \psi)^\wedge(k)|^2}{|1 - \langle v, k \rangle\,|} dk < \infty \ . \tag{3.17}$$

The proof of Proposition 3.17 for the simplified $L_2(\mathbb{R}^n)$ case can be found in [KT93] and can be applied as well for $H^s(\mathbb{R}^n)$.

Remark 3.3 $\mathbf{G}_{aWH}$ *acts from right. However,* $\mathbf{G}_{aWH}$ *is unimodular and thus the invariant Haar measure is given by $dqdadm(R)/a$. A possible structure of β is given by*

$$\beta(a, R) = \frac{a^{-1} R v'}{\|v\|^2} + \varrho(S) \ ,$$

where $S \in SO(n-1)$, v is a $SO(n-1)$-invariant co-vector and ϱ is smooth enough on $SO(n-1)$.

4 Uncertainties and Wavelets in Anisotropic Sobolev Spaces

In this section, we compute generalized uncertainties und minimizing wavelets in anisotropic Sobolev spaces. We consider both, the semi-direct product $\mathbf{G}_{aWH}$ and the direct group product $\mathbf{G}^d_{aWH}$.

Before computing some uncertainties we have to check that our modified representations

$$\pi_{*,s} = \Lambda_{*,-s} \pi \Lambda_{*,s} \tag{4.1}$$

induce self-adjoint infinitesimal operators in $H^s_*(\mathbb{R}^n)$. Similar to (2.5) we define by

$$[A_{*,s}(g_i) f](x) := \frac{\partial}{\partial g_i} [\pi_{*,s}(g) f](x)|_{g=e} \tag{4.2}$$

infinitesimal operators on $H^s_*(\mathbb{R}^n)$.

Lemma 4.1 *Assumed that $A(g_i)$, defined by (2.5), is some self-adjoint operator on L_2. Then, $A_{*,s}(g_i)$, defined by (4.2), is some self-adjoint operator on H_*^s.*

Proof At first, we remark that for some $\phi \in H_*^s$ we have

$$\begin{aligned}
[A_{*,s}(g_i)\phi](x) &= \frac{\partial}{\partial g_i}[\pi_{*,s}(g)\phi](x)|_{g=e} \\
&= \Lambda_{*,-s}\frac{\partial}{\partial g_i}[\pi(g)\Lambda_{*,s}\phi](x)|_{g=e} \\
&= \Lambda_{*,-s}[A(g_i)\Lambda_{*,s}\phi](x) \ .
\end{aligned}$$

Finally, we obtain for $\phi, \psi \in H_*^s$ that

$$\begin{aligned}
\langle A_{*,s}(g_i)\phi, \psi\rangle_{H_*^s} &= \langle \Lambda_{*,s}\Lambda_{*,-s}A(g_i)\Lambda_{*,s}\phi, \Lambda_{*,s}\psi\rangle_{L_2} \\
&= \langle \Lambda_{*,s}\phi, A(g_i)\Lambda_{*,s}\psi\rangle_{L_2} \\
&= \langle \phi, \Lambda_{*,-s}A(g_i)\Lambda_{*,s}\psi\rangle_{H_*^s} \\
&= \langle \phi, A_{*,s}(g_i)\psi\rangle_{H_*^s} \ .
\end{aligned}$$

■

Consequently, if we switch from L_2 to the H_*^s framework we can apply Proposition 4.1 to establish uncertainty principles by Theorem 2.1 and 2.2 respectively.

4.1 Uncertainties related to $\mathbf{G}_{aWH}^d$

In this subsection, we establish in accordance with Theorem 2.2 a generalized uncertainty. Let $\mathbf{G}_{aWH}^d$ be the underlaying group and $H_{mix}^{\bar{s}}(\mathbb{R}^2)$ the representation space under consideration. Hence, we have the following infinitesimal operators: $A(q)$, $A(p)$, $A(a)$, $A(b)$ and $A(\varphi)$. Let $A = (A(q), A(p), A(a), A(b))$ be the vector of operators and let γ and α be given by

$$\gamma = (0,1,1,0) \quad \text{and} \quad \alpha = (1,0,0,1) \ .$$

Then, we have

$$K = -i \begin{pmatrix} 0 & [A(q),A(p)] & [A(q),A(a)] & [A(q),A(b)] \\ -[A(q),A(p)] & 0 & [A(p),A(a)] & [A(p),A(b)] \\ -[A(q),A(a)] & -[A(p),A(a)] & 0 & [A(a),A(b)] \\ -[A(q),A(b)] & -[A(p),A(b)] & -[A(a),A(b)] & 0 \end{pmatrix}$$

$$
= \begin{pmatrix} 0 & A(\varphi) & 0 & 0 \\ -A(\varphi) & 0 & 0 & 0 \\ 0 & 0 & 0 & A(b) \\ 0 & 0 & -A(b) & 0 \end{pmatrix}
$$

and thus $\gamma K \alpha' = (0,1,1,0) \cdot (0, -A(\varphi), A(b), 0)' = -A(\varphi) + A(b) \neq 0$. This represents the generalized commutator. To derive the uncertainty structure we have to compute the following terms

$$
\Sigma - V \;=\; \begin{pmatrix} 0 & \Delta_{\bar{s},f}(A(q),A(p)) & \Delta_{\bar{s},f}(A(q),A(a)) & \Delta_{\bar{s},f}(A(q),A(b)) \\ \overline{\Delta_{\bar{s},f}(A(q),A(p))} & 0 & \Delta_{\bar{s},f}(A(p),A(a)) & \Delta_{\bar{s},f}(A(p),A(b)) \\ \overline{\Delta_{\bar{s},f}(A(q),A(a))} & \overline{\Delta_{\bar{s},f}(A(p),A(a))} & 0 & \Delta_{\bar{s},f}(A(a),A(b)) \\ \overline{\Delta_{\bar{s},f}(A(q),A(b))} & \overline{\Delta_{\bar{s},f}(A(p),A(b))} & \overline{\Delta_{\bar{s},f}(A(a),A(b))} & 0 \end{pmatrix} ,
$$

$$
V = \begin{pmatrix} \Delta^2_{\bar{s},f}(A(q)) & 0 & 0 & 0 \\ 0 & \Delta^2_{\bar{s},f}(A(p)) & 0 & 0 \\ 0 & 0 & \Delta^2_{\bar{s},f}(A(a)) & 0 \\ 0 & 0 & 0 & \Delta^2_{\bar{s},f}(A(b)) \end{pmatrix} ,
$$

$$
\gamma V \gamma' = \Delta^2_{\bar{s},f}(A(p)) + \Delta^2_{\bar{s},f}(A(a)) \ ,
$$

$$
\alpha V \alpha' = \Delta^2_{\bar{s},f}(A(q)) + \Delta^2_{\bar{s},f}(A(b)) \ ,
$$

$$
(\Sigma - V)\gamma'\alpha V = \begin{pmatrix} & 0 & 0 & \\ (\Sigma - V)\gamma'\Delta^2_{\bar{s},f}(A(q)) & 0 & 0 & (\Sigma - V)\gamma'\Delta^2_{\bar{s},f}(A(b)) \\ & 0 & 0 & \\ & 0 & 0 & \end{pmatrix} ,
$$

$$
V\gamma'\alpha(\Sigma - V) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \multicolumn{4}{c}{\alpha(\Sigma - V)\Delta^2_{\bar{s},f}(A(p))} \\ \multicolumn{4}{c}{\alpha(\Sigma - V)\Delta^2_{\bar{s},f}(A(a))} \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

and consequently, we have

$$\begin{aligned}\gamma M \alpha' &= \left[\Delta^2_{\bar{s},f}(A(q)) + \Delta^2_{\bar{s},f}(A(b))\right]\left[\Delta_{\bar{s},f}(A(p),A(a)) + \overline{\Delta_{\bar{s},f}(A(p),A(a))}\right] \\ &\quad + \left[\Delta^2_{\bar{s},f}(A(p)) + \Delta^2_{\bar{s},f}(A(a))\right]\left[\Delta_{\bar{s},f}(A(q),A(b)) + \overline{\Delta_{\bar{s},f}(A(q),A(b))}\right] \ .\end{aligned}$$

Concluding, we obtain the following uncertainty

$$\begin{aligned}\mu_{\bar{s},f}(-A(\varphi)+A(b))^2 &\leq 4\left[\Delta^2_{\bar{s},f}(A(p)) + \Delta^2_{\bar{s},f}(A(a))\right]\left[\Delta^2_{\bar{s},f}(A(q)) + \Delta^2_{\bar{s},f}(A(b))\right] \\ &\quad + \left[\Delta^2_{\bar{s},f}(A(q)) + \Delta^2_{\bar{s},f}(A(b))\right]\left[\Delta_{\bar{s},f}(A(p),A(a)) + \overline{\Delta_{\bar{s},f}(A(p),A(a))}\right] \\ &\quad + \left[\Delta^2_{\bar{s},f}(A(p)) + \Delta^2_{\bar{s},f}(A(a))\right]\left[\Delta_{\bar{s},f}(A(q),A(b)) + \overline{\Delta_{\bar{s},f}(A(q),A(b))}\right] \ .\end{aligned}$$

The minimizing vector f is given in phase space as a solution of the following partial differential equation

$$(A(a) + A(p) - \mu_1)\,\hat{f} = it\,(A(b) + A(q) - \mu_2)\,\hat{f} \ . \tag{4.3}$$

A function satisfying (4.3) is given by

$$\hat{f}(\omega,\eta) = \eta^{1+s_2} e^{t\omega^2/2 - \mu_2 t\omega - \omega/2 - i\mu_1\omega + \eta t - s_2\omega - \omega}\left(1+\omega^2\right)^{-s_1/2}\left(1+\eta^2\right)^{-s_2/2} \ ,$$

where the parameters must be chosen such that the admissibility condition is satisfied, i.e., $\eta > 0$, $t < 0$, and for $\eta \leq 0$ the function $\hat{f} = 0$, see Figure 1.

4.2 Uncertainties related to $\mathbf{G}_{aWH}$

In this section, we derive uncertainties related to $\mathbf{G}_{aWH}$. We proceed in two steps. At first, we start by a homogeneous space X and compute directly some uncertainty structures. Secondly, we observe that this special structure can be obtained by applying Theorem 2.2.

As in Section 3.3 we restrict $\mathbf{G}_{aWH}$ to the homogeneous space X induced by the subgroup X given by (3.16). To keep notations at a reasonable level we choose $\mathcal{H} = H^s(\mathbb{R}^2)$. By Remark 3.3 we know how to choose β. The next proposition exhibits the infinitesimal operators in dependence on the section σ and the embedding β.

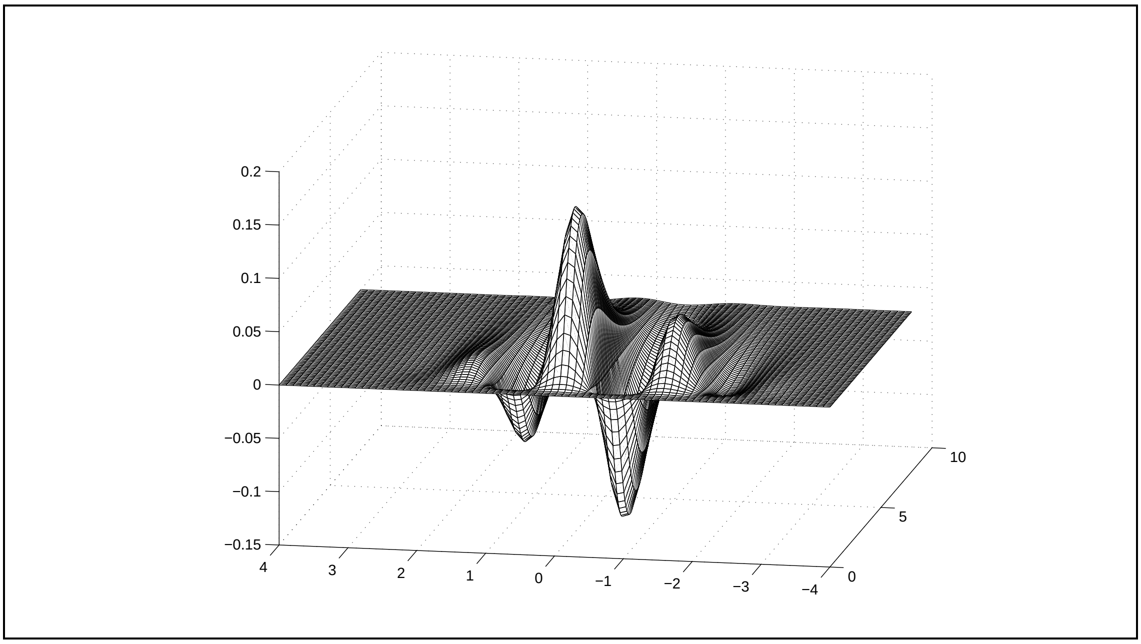

Figure 1: Wavelet function based on $\mathbf{G}^d_{aWH}$ for $s_1 = 1$, $s_2 = 5$, $t = -1$, $\mu_1 = -4$ and $\mu_2 = 6$.

Proposition 4.1 *Let $\mathbf{G}_{aWH}$ be given and let $\pi_{s,\sigma} = \Lambda_{-s}(\pi \circ \sigma)\Lambda_s$ be the square integrable representation of X in $H^s(\mathbb{R}^2)$. If the embedding function is given by*

$$\beta(a, R) = a^{-1} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{4.4}$$

where $a \in \mathbb{R}_+$ and $\theta \in [0, 2\pi)$, then, the infinitesimal operators are given by

$$A^\sigma(a) = A(a) + A(p_2) \ , \quad A^\sigma(q_j) = A(q_j) \text{ and } A^\sigma(\theta) = A(\theta) + A(p_1) \ . \tag{4.5}$$

Moreover, the infinitesimal operators satisfy the following commutator relations

$$[A^\sigma(a), A^\sigma(q_j)] = iA(q_j) + i\delta_{2j}A(\varphi) \ , \quad [A^\sigma(a), A^\sigma(\theta)] = 2iA(p_1) \ ,$$

$$[A^\sigma(\theta), A^\sigma(q_1)] = iA(q_2) + iA(\varphi) \ , \quad [A^\sigma(\theta), A^\sigma(q_2)] = -iA(q_1) \text{ and } [A^\sigma(q_1), A^\sigma(q_2)] = 0 \ . \tag{4.6}$$

Consequently, for the homogeneous space X there exist for $n = 2$ five uncertainty principles.

Proof We choose $n = 2$, $v = (0, 1)$, $S \in SO(n-1)$, $\tau = -1$ and finally $\varphi(S) = (0, \tau)' = (0, -1)'$. Hence, the structure of (4.4) is satisfied

$$\beta(a, R) = a^{-1}(-\sin\theta, \cos\theta)' - (0, 1)' = \frac{a^{-1}Rv'}{\|v\|^2} + \varphi(S) \ .$$

Using this embedding and $\lambda(\omega) = (1 + \|\omega\|^2)^{-s/2}$ we derive the infinitesimal operators

$$\begin{aligned}
\tilde{A}^\sigma(a)\hat{\psi}(\omega) &= \frac{\partial}{\partial a}\hat{\pi}_{s,\sigma}\hat{\psi}(\omega)\Big|_e = \lambda(\omega)(1+\|\omega\|^2)^{s/2}\hat{\psi}(\omega) + \\
&\quad \lambda(\omega)s(\|\omega\|^2+\omega_2)(1+\|\omega\|^2)^{s/2-1}\hat{\psi}(\omega) + \lambda(\omega)(1+\|\omega\|^2)^{s/2}\nabla\hat{\psi}\cdot\begin{pmatrix}\omega_1\\ \omega_2+1\end{pmatrix} \\
&= \hat{\psi}(\omega) + \frac{s\|\omega\|^2}{1+\|\omega\|^2}\hat{\psi}(\omega) + \nabla\hat{\psi}\cdot\omega + \frac{s\omega_2}{1+\|\omega\|^2}\hat{\psi}(\omega) + \frac{\partial}{\partial\omega_2}\hat{\psi}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{A}^\sigma(\theta)\hat{\psi}(\omega) &= \frac{\partial}{\partial \theta}\hat{\pi}_{s,\sigma}\hat{\psi}(\omega)\Big|_e = \lambda(\omega)s\omega_1(1+\|\omega\|^2)^{s/2-1}\hat{\psi}(\omega) + \\
&\quad \lambda(\omega)(1+\|\omega\|^2)^{s/2}\nabla\hat{\psi}\cdot\begin{pmatrix}\omega_2+1\\ -\omega_1\end{pmatrix} \\
&= \frac{s\omega_1}{1+\|\omega\|^2}\hat{\psi}(\omega) + \nabla\hat{\psi}\cdot\begin{pmatrix}\omega_2\\ -\omega_1\end{pmatrix} + \frac{\partial}{\partial\omega_1}\hat{\psi}(\omega)\ .
\end{aligned}$$

To make them self-adjoint we modify them by

$$i\tilde{A}^\sigma(a) =: A^\sigma(a) = A(a) + A(q_1) \quad \text{and} \quad i\tilde{A}^\sigma(\theta) =: A^\sigma(\theta) = A(\theta) + A(p_1)\quad .$$

This shows (4.5). Furthermore, we have

$$\tilde{A}^\sigma(q_j)\hat{\psi}(\omega) = \frac{\partial}{\partial q_j}\hat{\pi}_{s,\sigma}\hat{\psi}(\omega)\Big|_e = -i\omega_j\hat{\psi}(\omega\ ) \text{ and } i\tilde{A}^\sigma(q_j) =: A^\sigma(q_j) = A(q_j)\quad .$$

Hence, the relations (4.6) are obvious

$$[A^\sigma_a, A^\sigma(q_j)] = [A(a), A(q_j)] + [A(p_2), A(q_j)] = iA(q_j) + i\delta_{j2}A(\varphi)\ ,$$

$$[A^\sigma(q_1), A^\sigma(q_2)] = [A(q_1), A(q_2)] = 0\ ,$$

$$[A^\sigma_a, A^\sigma(\theta)] = [A(p_2), A(\theta)] + [A(a), A(p_1)] = 2iA(p_1)\ ,$$

$$[A^\sigma(\theta), A^\sigma(q_1)] = [A(\theta), A(q_1)] + [A(p_1), A(q_1)] = iA(q_2) + iA(\varphi) \quad \text{and}$$

$$[A^\sigma(\theta), A^\sigma(q_2)] = [A(\theta), A(q_2)] + [A(p_1), A(q_2)] = -iA(q_1)\ .$$

■

The essential message of Proposition 4.1 is that some of the infinitesimal operators are given by sums of operators. This reflects the influence of the embedding function β. To analyze these cases we consider the uncertainties with respect to $A^\sigma(\theta)$ and $A^\sigma(q_l)$, for $l = 1, 2$.

Proposition 4.2 *Let X be the homogeneous space under consideration and let $\psi \in H^s$ such that $\|\psi\|_s = 1$. In accordance with Theorem 2.2 the uncertainties with respect to $A^\sigma(\theta)$ and $A^\sigma(q_l)$, $l = 1, 2$, are given by*

$$\begin{aligned}\frac{1}{4}C_l &\leq \Delta^2_{s,\psi}(A(\theta))\Delta^2_{s,\psi}(A_{q_l}) + \Delta^2_{s,\psi}(A(p_1))\Delta^2_{s,\psi}(A_{q_l}) + \\ & \quad 2\Delta^2_{s,\psi}(A_{q_l})\left[\Re\mu_{s,\psi}(A(\theta)A(p_1)) - \mu_{s,\psi}(A(\theta))\mu_{s,\psi}(A(p_1))\right] , \end{aligned} \tag{4.7}$$

with lower bounds

$$\begin{aligned} C_1 &= \mu^2_{s,\psi}(A(q_2) + A(\varphi)) \quad \text{and} \\ C_2 &= \mu^2_{s,\psi}(A(q_1)). \end{aligned} \tag{4.8}$$

Proof This result can be proved more generally. Assume that A, B and C are the given self-adjoint operators. Then by

$$\begin{aligned}\Delta^2_{s,\psi}(A+B) &= \Delta^2_{s,\psi}(A) + \Delta^2_{s,\psi}(A) + \langle (AB+BA)\psi, \psi\rangle_s - 2\langle A\psi,\psi\rangle_s \langle B\psi,\psi\rangle_s \\ &= \Delta^2_{s,\psi}(A) + \Delta^2_{s,\psi}(A) + 2\Re\langle B\psi, A\psi\rangle_s - 2\langle A\psi,\psi\rangle_s \langle B\psi,\psi\rangle_s\end{aligned}$$

we deduce (4.7) and (4.8)

$$\begin{aligned}\Delta^2_{s,\psi}(A+B)\Delta^2_{s,\psi}(C) &= \Delta^2_{s,\psi}(A)\Delta^2_{s,\psi}(C) + \Delta^2_{s,\psi}(B)\Delta^2_{s,\psi}(C) \\ &\quad + 2\Delta^2_{s,\psi}(C)\left[\Re\langle B\psi, A\psi\rangle_s - \langle A\psi,\psi\rangle_s \langle B\psi,\psi\rangle_s\right] .\end{aligned}$$

■

Using the definition of the covariance we specify (4.7) in Proposition 4.2

$$\begin{aligned}\frac{1}{4}C_l &\leq \Delta^2_{s,\psi}(A(\theta))\Delta^2_{s,\psi}(A_{q_l}) + \Delta^2_{s,\psi}(A(p_1))\Delta^2_{s,\psi}(A_{q_l}) + \\ &\quad \Delta^2_{s,\psi}(A_{q_l})\Delta_{s,\psi}(A(\theta), A(p_1)) + \Delta^2_{s,\psi}(A_{q_l})\Delta_{s,\psi}(A(p_1), A(\theta)) .\end{aligned} \tag{4.9}$$

To realize how this interacts with Theorem 2.2 we start again by the same set of infinitesimal operators $A(p_1)$, $A(p_2)$, $A(q_1)$, $A(q_2)$, $A(\theta)$, $A(a)$. Assume that

$$A = (A(p_1), A(p_2), A(q_1), A(q_2), A(\theta), A(a))$$

and we have chosen vectors

$$\gamma = (1,0,0,0,1,0)' \quad \text{and} \quad \alpha_1 = (0,0,-1,0,0,0)' , \alpha_2 = (0,0,0,-1,0,0)'.$$

Then, it follows

$$K = \begin{pmatrix} 0 & 0 & -A(\varphi) & 0 & A(p_2) & -A(p_1) \\ 0 & 0 & 0 & -A(\varphi) & A(p_1) & -A(p_1) \\ A(\varphi) & 0 & 0 & 0 & A(q_2) & -A(q_1) \\ 0 & A(\varphi) & 0 & 0 & A(q_1) & -A(q_2) \\ -A(p_2) & -A(p_1) & -A(q_2) & -A(q_1) & 0 & 0 \\ A(p_1) & A(p_1) & A(q_1) & A(q_2) & 0 & 0 \end{pmatrix}$$

and hence, $\gamma' K \alpha_1 = A(\varphi) + A(q_2) \neq 0$ and $\gamma' K \alpha_2 = A(q_1) \neq 0$. Moreover, we have

$$\begin{aligned} \gamma' V \gamma &= \Delta^2_{s,\psi}(A(p_1)) + \Delta^2_{s,\psi}(A(\theta)), \\ \alpha_l' V \alpha_l &= \Delta^2_{s,\psi}(A_{q_l}) \text{ and} \\ \gamma' M \alpha_l &= \Delta^2_{s,\psi}(A_{q_l}) \left[\Delta_{s,\psi}(A(\theta), A(p_1)) + \overline{\Delta_{s,\psi}(A(\theta), A(p_1))} \right] . \end{aligned}$$

Consequently, we obtain the following uncertainty principles

$$\begin{aligned} \frac{1}{4}\mu_{s,\psi}(A(\varphi) + A(q_2))^2 \leq\ & \left[\Delta^2_{s,\psi}(A(p_1)) + \Delta^2_{s,\psi}(A(\theta))\right] \Delta^2_{s,\psi}(A(q_1)) \\ & + \Delta^2_{s,\psi}(A_{q_1}) \left[\Delta_{s,\psi}(A(\theta), A(p_1)) + \overline{\Delta_{s,\psi}(A(\theta), A(p_1))} \right] \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \frac{1}{4}\mu_{s,\psi}(A(q_1))^2 \leq\ & \left[\Delta^2_{s,\psi}(A(p_1)) + \Delta^2_{s,\psi}(A(\theta))\right] \Delta^2_{s,\psi}(A(q_2)) \\ & + \Delta^2_{s,\psi}(A_{q_2}) \left[\Delta_{s,\psi}(A(\theta), A(p_1)) + \overline{\Delta_{s,\psi}(A(\theta), A(p_1))} \right] . \end{aligned} \tag{4.11}$$

Uncertainties (4.10) and (4.11) correspond exactly to (4.9).

Finally, we derive some minimizing elements of (4.10) and (4.11) for the special case $s = 0$. For other cases it is to difficult to find a analytical solution. Using $A^\sigma(\theta) = A(\theta) + A(p_1)$, $A^\sigma_{q_l} = A_{q_l}$ and

$$A(\theta) = i\nabla\hat{\psi} \cdot \begin{pmatrix} \omega_2 \\ -\omega_1 \end{pmatrix} , \quad A(p_1) = i\frac{s\omega_1}{1 + \|\omega\|^2}\hat{\psi} + i\frac{\partial}{\partial\omega_1}\hat{\psi} \text{ and } A_{q_l} = \omega_l\hat{\psi}$$

we have to solve the following partial differential equations

$$i\nabla\hat{\psi} \cdot \begin{pmatrix} \omega_2 \\ -\omega_1 \end{pmatrix} + i\frac{s\omega_1}{1 + \|\omega\|^2}\hat{\psi} + i\frac{\partial}{\partial\omega_1}\hat{\psi} - \mu_1\hat{\psi} = it\left(\omega_l\hat{\psi} - \mu_2\hat{\psi}\right).$$

Lemma 4.2 *Let $\psi \in H^s(\mathbb{R}^2)$ and $\hat{\phi}$ be smooth enough. For a adequate choice of $\hat{\phi}$ and $s = 0$ explicit solutions are given by*

$$i\nabla\hat{\psi} \cdot \begin{pmatrix} \omega_2 \\ -\omega_1 \end{pmatrix} + i\frac{s\omega_1}{1+\|\omega\|^2}\hat{\psi} + i\frac{\partial}{\partial\omega_1}\hat{\psi} - \mu_1\hat{\psi} = it\left(\omega_l\hat{\psi} - \mu_2\hat{\psi}\right), \quad \text{for } l = 1, 2$$

and for $l = 1$ and $1 + \omega_2 \geq 0$ by

$$\hat{\psi}(\omega) = \hat{\phi}\left(\frac{\|\omega\|^2}{2} + \omega_2\right)(1 + i\omega_1 + \omega_2)^{-\mu_1 + it\mu_2}\, e^{(1+\omega_2)t}$$

and for $l = 2$ and $1 + \omega_2 \geq 0$ by

$$\hat{\psi}(\omega) = \hat{\phi}\left(\frac{\|\omega\|^2}{2} + \omega_2\right)(1 + i\omega_1 + \omega_2)^{\mu_1 - it(\mu_2+1)}\, e^{\omega_1 t}\ .$$

A straightforward proof of Lemma 4.2 can be found in [Tes01].

5 Appendix

Proof of Proposition 3.1

To show the machinery for computing Haar measures, we prove this proposition. Let $H \subset \Omega = R^n \times \mathbb{R}_+ \times SO(n)$. Then we have to show that $\mu_L(g \circ H) = \mu_L(H)$ and $\mu_R(H \circ g) = \mu_R(H)$. Let $\Phi_L(H) = g' \circ H$, that is

$$\Phi_L(q, a, R) = (q', a', R') \circ (q, a, R) = (q' + a'R'q, a'a, R'R).$$

By Haar's theorem there exists a non-negative weight function γ with

$$\mu_L(H) = \int_H \gamma(q, a, R)dqdadm(R).$$

Hence, we have

$$\begin{aligned} \mu_L(g' \circ H) &= \int_{g' \circ H} \gamma(q, a, R)dqdadm(R) \\ &= \int_H \gamma(\Phi_L(q, a, R))|\det J_{\Phi_L}(q, a, R)|dqdadm(R)\ . \end{aligned}$$

Using Euler angles, the Jacobian is of the following structure

$$J_{\Phi_L} = \begin{pmatrix} a'R' & 0 & 0 \\ 0 & a' & 0 \\ 0 & 0 & Id_{n(n-1)/2} \end{pmatrix} \quad \text{and thus,} \quad |\det J_{\Phi_L}(q, a, R)| = (a')^{n+1}\ .$$

Up to a constant we deduce that $\gamma(q,a,R) = a^{-(n+1)}$ and consequently, we have

$$\mu_L(q,a,R) = a^{-(n+1)} dq da dm(R) \ .$$

We compute the right measure analogously. By

$$\Phi_L(q,a,R) = (q,a,R) \circ (q',a',R') = (q + aRq', aa', RR')$$

we obtain the measure

$$\mu_R(q,a,R) = a^{-1} dq da dm(R) \ .$$

■

Proof of Proposition 3.5

We proceed as follows. We start by computing the adjoint group action in the corresponding Lie algebra. Therewith we are able to derive the co-adjoint action on the dual Lie algebra. By this mapping the co-adjoint orbit follows instantly. Finally, by Pukanzky's condition we can derive irreducible representations as characters with respect to certain functionals belonging to the dual Lie algebra.

The first goal is to construct the adjoint action of $\mathbf{G}_{aWH}$ on the related Lie algebra $\mathcal{G}_{aWH}$. For that reason we conceive a element of $\mathbf{G}_{aWH}$ as a matrix

$$g = \begin{pmatrix} 1 & (aR'q)' & \varphi \\ 0 & aR & q \\ 0 & 0 & 1 \end{pmatrix} \ ,$$

where group law is preserved by matrix multiplication. The corresponding Lie algebra $\mathcal{G}_{aWH}$ can be conceived as a matrix-valued Lie algebra

$$\mathcal{G}_{aWH} \cong \left\{ \begin{pmatrix} 0 & \zeta' & t \\ 0 & r^{\lambda} & x \\ 0 & 0 & 0 \end{pmatrix} \ , \ \zeta, x \in \mathbb{R}^n, t \in \mathbb{R}, r \in so(n) \right\} .$$

Every element $X \in \mathcal{G}_{aWH}$ can be written as

$$X = x^i \cdot Q_i + \zeta^i \cdot P_i + \lambda K + R_i^j \cdot J_j^i + tT \ ,$$

where $\lambda, t \in \mathbb{R}$, $x, \zeta \in \mathbb{R}^n$, R_i^j are coefficients of $r \in so(n)$. The connection between $\mathbf{G}_{aWH}$ and $\mathcal{G}_{aWH}$ is given by the exp - mapping. Now, we compute the adjoint action of $\mathbf{G}_{aWH}$

in $\mathcal{G}_{aWH}$. Let $g = (q, p, a, \varphi) \in \mathbf{G}_{aWH}$ and (x, ζ, λ, t) are the coordinates of a element $X \in \mathcal{G}_{aWH}$ with respect to the basis $\{Q, P, K, T\}$. By

$$Ad(g)X = gXg^{-1} = (ax - \lambda q, a^{-1}\zeta + \lambda p, \lambda, t + apx - a^{-1}q\zeta - pq\lambda)$$

it follows the adjoint action

$$Ad(g) = \begin{pmatrix} a & 0 & -q & 0 \\ 0 & a^{-1} & p & 0 \\ 0 & 0 & 1 & 0 \\ ap & -a^{-1}q & -pq & 1 \end{pmatrix}.$$

With respect to the basis $\{Q^*, P^*, K^*, T^*\}$ we can compute the co-adjoint action

$$Ad^*(g) = Ad'(g^{-1}) \begin{pmatrix} a^{-1} & 0 & 0 & -p \\ 0 & a & 0 & q \\ a^{-1}q & -ap & 1 & -pq \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A functional $F \in \mathcal{G}^*_{aWH}$ has the expression

$$F = x_0^* Q^* + \zeta_0^* P^* + \lambda_0^* K^* + t_0^* T^*.$$

Consequently, we can deduce the co-adjoint orbits

$$\begin{cases} x_0^* \to x^* = a^{-1}x_0^* - pt_0^* \\ \zeta_0^* \to \zeta^* = a\zeta_0^* + qt_0^* \\ \lambda_0^* \to \lambda^* = a^{-1}qx_0^* - ap\zeta_0^* + \lambda_0^* - pqt_0^* \\ t_0^* \to t^* = t_0^* \end{cases}.$$

We are interested in Stone-von-Neumann type representations. The related orbits are described by setting $t^* \neq 0$. In this case we can find for every initial value x_0^*, ζ_0^* numbers q, p and a such that $x^* = \zeta^* = 0$. Hence, without loss of generality we may choose $x_0^* = \zeta_0^* = 0$ and therewith we have the following orbits

$$\begin{cases} x^* = -pt_0^* \\ \zeta^* = qt_0^* \\ \lambda^* = \lambda_0^* + \frac{x^*\zeta^*}{t^*} \\ t^* = t_0^* \end{cases}.$$

These orbits are characterized by $t^*, \lambda^* \in \mathbb{R}$. Its knows that for the co-adjoint orbit the relation $O_F \cong \mathbf{G}_{aWH}/G_F$ holds, where G_F is the stabilizer of F with respect to $\mathbf{G}_{aWH}$

$$G_F = \{g \in \mathbf{G}_{aWH} : Ad^*(g)F = F\} \quad \text{and} \quad F = t^*T^* + \lambda^*K^* .$$

The co-adjoint mappings $Ad^*(g)$ acts as follows

$$\begin{aligned} Ad^*(g)F &= -pt^*T^* + qt^*T^* + \lambda^*K^* - pqt^*T^* \\ &= (q - p - pq + 1)t^*T^* + \lambda^*K^* \ . \end{aligned}$$

To preserve the invariance property we have to ensure that $p = q = 0$. Consequently, we have

$$G_F = \{(0, 0, a, \varphi) : \quad a \in \mathbb{R}_+, \varphi \in \mathbb{R}\} \ .$$

At this point we use Pukanzky's condition.

Lemma 5.1 *Assume that H is a polarization in $\mathcal{G}_{aWH}$ and that the condition*

$$Ad^*(\exp(H))F = F + H^\perp$$

is satisfied. Then, the characters of $\mathbf{H}$ *with respect to F induce irreducible representations π. The characters are of the following structure $\chi_F(h) = e^{i\langle F,X\rangle}$, where $h = \exp(X) \in$* $\mathbf{H} = \exp(H)$.

A polarization with respect to F is given by

$$H = \mathbb{R} \cdot Q + \mathbb{R} \cdot K + \mathbb{R} \cdot T \ .$$

Let $X = (x, 0, \lambda, t) = xQ + \lambda K + tT \in H \subset \mathcal{G}_{aWH}$. Then, we can identify the related group elements by $\exp(X) = (x(e^\lambda - 1)/\lambda, 0, e^\lambda, t)' \in \mathbf{H}$. For checking Pukanzky's condition we need

$$Ad^*(exp(X)) = \begin{pmatrix} e^{-\lambda} & 0 & 0 & 0 \\ 0 & e^{-\lambda} & 0 & x(e^\lambda - 1)/\lambda \\ e^{-\lambda}x(e^\lambda - 1)/\lambda & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Denoting $F = (0, 0, \lambda^*, t^*)'$ we have that

$$Ad^*(\exp(H))F = \begin{pmatrix} 0 \\ t^*x(e^\lambda - 1)/\lambda \\ \lambda^* \\ t^* \end{pmatrix} = F + \underbrace{\begin{pmatrix} 0 \\ t^*x(e^\lambda - 1)/\lambda \\ 0 \\ 0 \end{pmatrix}}_{\in H^\perp}$$

and F defines the following characters χ_F of $\mathbf{H}$ by $\chi_F(q, 0, a, \varphi) = e^{i(\lambda^* \log(a) + t^*\varphi)}$. Let $(q, 0, a, \varphi) = h \in \mathbf{H}$, $(0, x, 1, 0) = g_h \in \mathbf{H} \setminus \mathbf{G}_{aWH}$ and $(q, p, a, \varphi) = g \in \mathbf{G}_{aWH}$ Then we deduce

$$(0, x, 1, 0) \circ (q, p, a, \varphi) = (q, x + p, a, \varphi + xq) = (q, 0, a, \varphi + xq) \circ (0, 0, a(x + p), 0)$$

and with $f(h \circ g) = \chi_F(h)f(g)$ we finally obtain

$$
\begin{aligned}
\pi(g)f(x) &= \chi_F(q,0,a,\varphi+xq)f(a(x+p)) \\
&= e^{\langle (0,0,\lambda^*,t^*),(\cdot,\cdot,\log(a),\varphi+xq)\rangle} f(a(x+p)) \\
&= e^{i(\lambda^* \log(a)+t^*(\varphi+xq))} f(a(x+p))
\end{aligned}
$$

■

Proof of Proposition 3.6

The strategy is the same as in the proof of Proposition 3.1. The difference is the transformation $\Phi_L : \Omega \to \Omega$ which is defined by

$$
\begin{aligned}
\Phi_L(q,p,a,R,\varphi) &= (q',p',a',R',\varphi') \circ (q,p,a,R,\varphi) \\
&= (q'+a'R'q, p'+a'^{-1}R'p, a'a, R'R, \varphi'+\varphi+p'(a'R'q)).
\end{aligned}
$$

This leads to the condition $\gamma(\Phi_L(q,p,a,R,\varphi))a' \overset{!}{=} \gamma(q,p,a,R,\varphi)$. A possibly weight function is then given by $\gamma(q,p,a,R,\varphi) = a^{-1}$. Hence, we obtain

$$
d\mu_L(q,p,a,R,\varphi) = \gamma(q,p,a,R,\varphi)dq\ dp\ dm(R)\ da\ d\varphi = a^{-1}dq\ dp\ dm(R)\ da\ d\varphi.
$$

For $d\mu_R$ we obtain the same weight function. ■

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Berichte aus der Technomathematik ISSN 1435-7968

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