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A Note on Interpolating Scaling Functions

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Abstract

In this paper, we are concerned with constructing interpolating scaling functions. The presented construction can be interpreted as a natural generalization of a well–known univariate approach and applies to scaling matrices A satisfying $|\det A| = 2$. The resulting scaling functions automatically satisfy certain Strang–Fix–conditions.

Key Words: Interpolating scaling functions, Strang–Fix–conditions, expanding scaling matrices.

AMS Subject classification: 41A05, 41A30, 41A63

1 Introduction

The problem of constructing interpolating scaling functions has attracted increasing intererst over the last few years, several construction principles for such scaling functions habe been published recently, see, e.g., [2, 3, 4, 5, 6, 7, 13]. Interpolating basis functions of this type are particularly needed for applications in CAGD or for collocation methods for operator equations.

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In this note, we extend the construction principle of Lemarie and Meyer for univariate scaling functions to the construction of multivariate interpolating scaling functions. In general, a function $\phi \in L_2(\mathbf{R}^d)$ is called a *scaling* function or a *refinable* function if it satisfies a *two–scale–relation*

$$\phi(x) = \sum_{k\in\mathbf{Z}^d} a_k\phi(Ax-k), \qquad \mathbf{a} = \{a_k\}_{k\in\mathbf{Z}^d} \in \ell_2(\mathbf{Z}^d), \tag{1.1}$$

where A is an *expanding* integer scaling matrix, i.e., all its eigenvalues have modulus larger than one. For the construction of interpolating scaling functions one requires additionally that ϕ is at least continuous and satisfies

$$\phi(k) = \delta_{0,k} \quad , \qquad k \in \mathbf{Z}^d. \tag{1.2}$$

Furthermore, functions ϕ which are sufficiently smooth and well–located are preferable.

The starting point for the present paper is the natural question, to which extend univariate construction principles carry over to the multi-dimensional case. A survey on the major univariate construction principles and their potential for multidimensional generalizations is contained in [2]. Most of these construction principles have been generalized already. In this paper, we investigate an approach on how to generalise the univariate construction principle of Lemarie and Meyer [10, 12].

This generalized approach yields compactly supported scaling functions which automatically satisfy Strang–Fix–conditions of a certain order. To satisfy Strang–Fix–conditions of high order is desirable for several reasons. First of all they serve as an indicator for a certain smoothness, moreover, they readily imply a certain order of approximation in appropriate function spaces.

The presented approach applies to scaling matrices A satisfying $|\det A| = 2$ in arbitrary spatial dimensions.

This paper is organized as follows. In Section 2, we briefly recall the setting of interpolating scaling functions. In Section 3, we present the main construction and, finally, in Section 4 we discuss some examples in order to demonstrate the applicability of our approach.

For later use, let us fix some notation. Let $q = |\det A| = 2$, furthermore, let $R = \{\rho_0, \rho_1\}, R^T = \{\tilde{\rho}_0, \tilde{\rho}_1\}$ denote complete sets of representatives of $\mathbf{Z}^d/A\mathbf{Z}^d$ and $\mathbf{Z}^d/B\mathbf{Z}^d$, $B = A^T$, respectively. Without loss of generality, we shall always assume that $\rho_0 = \tilde{\rho}_0 = 0$.

2 The Setting

In the sequel, we shall only consider compactly supported scaling functions. Moreover, we shall always assume that supp $\mathbf{a} := \{k \in \mathbf{Z}^d \mid a_k \neq 0\}$ is finite. Computing the Fourier transform of both sides of (1.1) yields

$$\hat{\phi}(\omega) = \sum_{k\in\mathbf{Z}^d} \frac{1}{q} a_k e^{-2\pi i\langle k, B^{-1}\omega\rangle} \hat{\phi}(B^{-1}\omega). \tag{2.1}$$

By iterating (2.1) we obtain

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} m(B^{-j}\omega), \tag{2.2}$$

where the *symbol* $m(\omega)$ is defined by

$$m(\omega) := \frac{1}{q} \sum_{k\in\mathbf{Z}^d} a_k e^{-2\pi i\langle k,\omega\rangle}. \tag{2.3}$$

Equation (2.2) shows that instead of trying to construct a refinable function directly we may also start with a symbol $m(\omega)$. In the following we collect some well-known conditions on the symbol $m(\omega)$ which guarantee that $\hat{\phi}$ according to (2.2) is well–defined in $L_2(\mathbf{R}^d)$ and has some additional desirable properties such as sufficient smoothness. Moreover, for our purposes, we have to clarify how the interpolating property (1.2) can be guaranteed. The following two conditions are necessary:

(C1) $m(0) = 1$;

(C2) $\sum_{\tilde{\rho}\in R^T} m(\omega + B^{-1}\tilde{\rho}) = 1$.

Very often, also the convenient condition

(C3) $m(\omega) \geq 0$

is required. This condition allows a simplified regularity estimate in Chapter 4.

Usually, conditions (C1)–(C2) are the starting point for the construction of an interpolatory scaling function. Unfortunately, they are not sufficient. Concerning this task, we refer to the following theorem which goes back to Lawton, Lee, and Shen [11].

Theorem 2.1 *Let $m(\omega)$ be a trigonometric polynomial which satisfies condition* (C1). *A necessary and sufficient condition for a continuous refinable function to be interpolatory is that the sequence $\delta = \{\delta_0 = 1,\ \delta_k = 0$ for $k \in \mathbb{Z}^d\backslash\{0\}\ \}$ is the unique eigenvector of the operator*

$$(\mathcal{H}b)_k = \sum_{l\in\mathbf{Z}^d} q a_{Ak-l} b_l, \qquad \{b_k\}_{k\in\mathbf{Z}^d} \in \ell_2(\mathbf{Z}^d) \tag{2.4}$$

corresponding to a simple eigenvalue 1.

In general, one wants to find scaling functions, which have a certain smoothness. To this end, one often requires that the *Strang–Fix–conditions* of order L are satisfied, i.e.,

$$(C4) \qquad \left(\frac{\partial}{\partial\omega}\right)^l m(B^{-1}\tilde{\rho}) = 0 \qquad \text{for all} \quad |l| \leq L \quad \text{and all} \quad \tilde{\rho} \in R^T\backslash\{0\}.$$

In the univariate case, there exist five major approaches to find symbols $m(\omega)$ satisfying (C1)–C(3), see, e.g., [2] for a detailed discussion. There also exist several approaches to generalize some of these concepts to the multivariate case [2, 4]. In this note, we investigate a natural generalization of the following ansatz which is due to Lemarié and Meyer [10, 12]: Define $m(\omega)$ according to

$$m(\omega) := 1 - c_1 \int_0^\omega m_1(t)dt \tag{2.5}$$

$$m_1(t) = \sin^{2K-1}(2\pi t) \tag{2.6}$$

and choose c_1 such that $m(1/2) = 0$, i.e. $c_1 = \left(\int_0^{1/2} m_1(t)dt\right)^{-1}$. Observing that in the univariate case $R = R^T = \{0,1\}$, $B^{-1}\tilde{\rho} = 1/2$, we see that conditions (C1-C3) are satisfied and that the symbol obeys Strang-Fix conditions (C4) of order $L = 2K - 1$.

In order to prepare this construction for an extension to higher dimensions let us observe that conditions (C1) and (C2) are satisfied whenever the defining function m_1 is of periodicity 1 and obeys

$$\int_0^{1/2} m_1(t)\ dt \neq 0 \quad , \quad m_1(t+1/2) = -m_1(t) \ .$$

Hence we can prepare the generralization of the Lemaire-Meyer approach by:

Lemma 2.1 *Let m_1 and m_2 denote integrable functions of periodicity 1, which satisfy*

$$m_1(t+1/2) = -m_1(t) \quad , \quad m_2(t+1/2) = m_2(t) \quad , \int_0^{1/2} m_2(t)\ dt = 0 \quad ,$$

$$\textit{and} \quad \int_0^{1/2} m_1(t)m_2(t)\ dt \neq 0 \quad \textit{or resp.} \quad \int_0^{1/2} m_1(t)\ dt \neq 0.$$

Define the univariate symbol $m(\omega)$ by

$$m(\omega) \ = \ 1 - c_{12} \int_0^\omega m_1(t)m_2(t)\ dt \quad , \quad c_{12} = \left(\int_0^{1/2} m_1(t)m_2(t)\ dt \right)^{-1} ,$$

or resp. by

$$m(\omega) \ = \ 1 - c_{12} \int_0^\omega m_1(t)\ dt \int_0^\omega m_2(s)\ ds \ - \ c_1 \int_0^\omega m_1(t)dt \ - \ c_2 \int_0^\omega m_2(s)ds \ ,$$

$$c_1 = \left(\int_0^{1/2} m_1(t)\ dt \right)^{-1} \quad , \quad c_2 = \frac{-c_{12}}{2c_1} \quad .$$

Then $m(\omega)$ satisfies (C1) and (C2).

Proof: Condition (C1) is obvious. (C2) is satisfied if

$$m(\omega) + m(\omega + 1/2) = 1 \quad .$$

In the first case we exploit the definition of c_{12} and the symmetry properties of m_1 and m_2:

$$m(\omega + 1/2) = 1 - c_{12} \int_0^{1/2} m_1(t)m_2(t)dt \; - \; c_{12} \int_{1/2}^{1/2+\omega} m_1(t)m_2(t)dt = c_{12} \int_0^{\omega} m_1(t)m_2(t)dt \quad ,$$

hence we obtain $m(\omega) + m(\omega + 1/2) = 1$ in this case.

The computations in the second case proceed in a similar fashion. Expanding the double integral in the expression for $m(\omega + 1/2)$ and applying the symmetry conditions for m_1 and m_2 lead to:

$$\int_0^{1/2+\omega} m_1(t)\; dt \int_0^{1/2+\omega} m_2(s)\; ds = -\int_0^{\omega} m_1(t)\; dt \int_0^{\omega} m_2(s)\; ds \; + \int_0^{1/2} m_1(t)dt \int_0^{\omega} m_2(s)ds \quad .$$

Hence by the definition of c_2 it follows that

$$m(\omega) + m(\omega + 1/2) = 1 - (2c_2 + c_{12}/c_1) \int_0^{\omega} m_2(s)ds = 1 \; .$$

$\square$

3 The Construction

We want to find multivariate versions of (2.5) for $q = |\det A| = 2$. In a first step, we confine the presentation to the 2D–case. Generalizations to higher–dimensional cases will be discussed later. For notational convenience, we shall always use the abbreviation $\tilde{\rho}_1 = \tilde{\rho}$. (Recall that we always choose $\rho_0 = \tilde{\rho}_0 = 0$).

Starting from the univariate case one might aling the coefficients of the univariate symbol $m(\omega)$ (2.5) along a coordinate axis by

$$m(\omega_1, \omega_2) = 1 - c_K \int_0^{\omega_1} m_1(t)dt \tag{3.1}$$

$$m_1(t) \;\; = \;\; \sin^{2K-1}(\pi (B^{-1}\tilde{\rho})_1^{-1} t) \quad . \tag{3.2}$$

Here $(B^{-1}\tilde{\rho})_1$ denotes the first coefficient of the vector $B^{-1}\tilde{\rho}$. Using the property

$$\sin(\pi(t+1)) = -\sin(\pi t),$$

it is easily checked that such an approach may work in principle. However, it has the disadvantage that it always leads to some kind of 'separable' symbol. We would clearly prefer a 'non–separable', i.e., truly multivariate symbol. To this end we use the results of the previous Lemma 2.1 as the starting point for our generalization. First we outline the general approach, examples using this construction as well as regularity estimates are contained in Section 4.

Theorem 3.1 *Suppose that $m_1(t_1)$, $m_2(t_2)$ are trigonometric polynomials satisfying*

$$m_1((B^{-1}\tilde{\rho})_1 + t) = -m_1(t), \qquad m_2((B^{-1}\tilde{\rho})_2 + t) = m_2(t), \tag{3.3}$$

$$\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t)dt \neq 0 \quad , \quad \int_0^{(B^{-1}\tilde{\rho})_2} m_2(t)dt = 0, \tag{3.4}$$

and

$$\left(\frac{d}{dt}\right)^k m_i((B^{-1}\tilde{\rho})_i) = 0 \quad \text{for all} \;\; k \leq L-1, \; i = 1, 2. \tag{3.5}$$

Furthermore, let the constant c_1 be defined by

$$c_1 := \left(\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1\right)^{-1} \tag{3.6}$$

and suppose that c_2 and c_{12} are related by

$$c_2 = -\frac{c_{12}}{2c_1}. \tag{3.7}$$

Then the symbol

$$m(\omega_1, \omega_2) = 1 - c_{12}\int_0^{\omega_1}\int_0^{\omega_2} m_1(t_1)m_2(t_2)dt_1dt_2 - c_1\int_0^{\omega_1} m_1(t_1)dt_1 - c_2\int_0^{\omega_2} m_2(t_2)dt_2 \tag{3.8}$$

satisfies (C1), (C2) *and Strang–Fix conditions* (C4) *of order L.*

Proof: Let us start by varifying the Strang–Fix conditions (C4). For $l_1, l_2 > 0$, we obtain by exploiting assumption (3.5)

$$\begin{aligned}\left(\frac{\partial}{\partial\omega}\right)^l (m(B^{-1}\tilde{\rho})) &= -c_{12}\left(\frac{d}{dt_1}\right)^{l_1-1} m_1((B^{-1}\tilde{\rho})_1)\left(\frac{d}{dt_2}\right)^{l_2-1} m_2((B^{-1}\tilde{\rho})_2) \\ &\quad -c_1\left(\frac{d}{dt_1}\right)^{l_1-1} m_1((B^{-1}\tilde{\rho})_1) - c_2\left(\frac{d}{dt_2}\right)^{l_2-1} m_2((B^{-1}\tilde{\rho})_2) = 0.\end{aligned}$$

The cases $l_1 = 0,\; l_2 > 0$ and $l_2 = 0,\; l_1 > 0$ can be treated analogously. It remains to study the case $l_1 = l_2 = 0$. By using (3.4) and (3.6) we get

$$\begin{aligned}m(B^{-1}\tilde{\rho}) &= 1 - c_{12}\int_0^{(B^{-1}\tilde{\rho})_1}\int_0^{(B^{-1}\tilde{\rho})_2} m_1(t_1)m_2(t_2)dt_1dt_2 - c_1\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 \\ &\quad -c_2\int_0^{(B^{-1}\tilde{\rho})_2} m_2(t_2)dt_2 \\ &= 1 - c_1\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 \\ &= 0.\end{aligned}$$

The next step is to check the condition (C2), which is a straightforward but lengthy calculation by applying the symmetry properties of m_1 and m_2: Splitting up the integrals yields

$$
\begin{aligned}
&m(\omega) + m(\omega + B^{-1}\tilde{\rho}) \\
&= 2 - c_{12}\int_0^{\omega_1}\int_0^{\omega_2} m_1(t_1)m_2(t_2)dt_1dt_2 \quad - c_1\int_0^{\omega_1} m_1(t_1)dt_1 - c_2\int_0^{\omega_2} m_2(t_2)dt_2 \\
&\quad -c_{12}\int_0^{\omega_1+(B^{-1}\tilde{\rho})_1}\int_0^{\omega_2+(B^{-1}\tilde{\rho})_2} m_1(t_1)m_2(t_2)dt_1dt_2 \quad - c_1\int_0^{\omega_1+(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 \\
&\quad -c_2\int_0^{\omega_2+(B^{-1}\tilde{\rho})_2} m_2(t_2)dt_2 \\
&= 2 - c_{12}\int_0^{\omega_1}\int_0^{\omega_2} m_1(t_1)m_2(t_2)dt_1dt_2 \quad - c_1\int_0^{\omega_1} m_1(t_1)dt_1 - c_2\int_0^{\omega_2} m_2(t_2)dt_2 \\
&\quad -c_{12}\left(\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 + \int_{(B^{-1}\tilde{\rho})_1}^{(B^{-1}\tilde{\rho})_1+\omega_1} m_1(t_1)dt_1\right)\left(\int_0^{(B^{-1}\tilde{\rho})_2} m_2(t_2)dt_2 + \int_{(B^{-1}\tilde{\rho})_2}^{(B^{-1}\tilde{\rho})_2+\omega_2} m_2(t_2)dt_2\right) \\
&\quad -c_1\left(\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 + \int_{(B^{-1}\tilde{\rho})_1}^{\omega_1+(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1\right) - c_2\left(\int_0^{(B^{-1}\tilde{\rho})_2} m_2(t_2)dt_2 + \int_{(B^{-1}\tilde{\rho})_2}^{\omega_2+(B^{-1}\tilde{\rho})_2} m_2(t_2)dt_2\right).
\end{aligned}
$$

Therefore, by employing the conditions (3.3) and (3.4), we get

$$
\begin{aligned}
&m(\omega) + m(\omega + B^{-1}\tilde{\rho}) \\
&= 2 - c_{12}\int_0^{\omega_1}\int_0^{\omega_2} m_1(t_1)m_2(t_2)dt_1dt_2 \quad - c_1\int_0^{\omega_1} m_1(t_1)dt_1 - c_2\int_0^{\omega_2} m_2(t_2)dt_2 \\
&\quad -c_{12}\left(\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 - \int_0^{\omega_1} m_1(t_1)dt_1\right)\int_0^{\omega_2} m_2(t_2)dt_2 \\
&\quad -c_1\left(\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 - \int_0^{\omega_1} m_1(t_1)dt_1\right) - c_2\int_0^{\omega_2} m_2(t_2)dt_2 \\
&= 2 - c_2\int_0^{\omega_2} m_2(t_2)dt_2 - c_{12}\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1\int_0^{\omega_2} m_2(t_2)dt_2 \\
&\quad -c_1\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 - c_2\int_0^{\omega_2} m_2(t_2)dt_2.
\end{aligned}
$$

By using (3.6), we end up with

$$
m(\omega) + m(\omega + B^{-1}\tilde{\rho}) = 1 + (-2c_2 - c_{12}c_1^{-1})\int_0^{\omega_2} m_2(t_2)dt_2
$$

and (C2) follows from (3.7). It is obvious that the symbol $m(\omega_1, \omega_2)$ satisfies (C1). The theorem is proved. □

Remark 3.1 *The reader should observe that Theorem 3.1 can in fact be used simultaneously for a whole class of matrices satisfying* $|\det A| = 2$. *Assume that a second scaling matrix* M *exists with a representative* $\tilde{\delta}$ *such that* $A^{-T}\tilde{\rho} = M^{-T}\tilde{\delta}$ *holds. Then a symbol* m *constructed according to* (3.8) *for* A *also works for* M. *Nevertheless, from* (2.2) *it is clear that the resulting refinable functions may differ dramatically.*

Theorem 3.1 obviously generalizes to higher dimensional cases, although everything becomes more complicated from the notational point of view. Therefore we only state one possible 3D–version of our approach. Several other variants are possible.

Theorem 3.2 *Suppose that $m_1(t_1), m_2(t_2)$ and $m_3(t_3)$ are trigonometric polynomials satisfying* (3.5)*. Let us furthermore assume that m_2 and m_3 both satisfy* (3.4) *and that*

$$m_1((B^{-1}\tilde{\rho})_1 + t) = -m_1(t), \quad m_2((B^{-1}\tilde{\rho})_2 + t) = m_2(t), \quad m_3((B^{-1}\tilde{\rho})_3 + t) = m_3(t). \tag{3.9}$$

Let c_1 be defined by (3.6) *and suppose that $c_{1,2,3}$ and $c_{2,3}$ are related by*

$$c_{2,3} = -\frac{c_{1,2,3}}{2c_1}. \tag{3.10}$$

Then the symbol

$$\begin{aligned} m(\omega_1, \omega_2, \omega_3) &= 1 - c_{1,2,3} \int_0^{\omega_1} \int_0^{\omega_2} \int_0^{\omega_3} m_1(t_1) m_2(t_2) m_3(t_3) dt_1 dt_2 dt_3 \\ &\quad - c_{2,3} \int_0^{\omega_2} \int_0^{\omega_3} m_2(t_2) m_3(t_3) dt_2 dt_3 - c_1 \int_0^{\omega_1} m_1(t_1) dt_1 \end{aligned} \tag{3.11}$$

satisfies (C1), (C2) *and* (C4).

4 Examples

First of all we apply the presented construction to the notorious quincunx case $d = 2$, $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. In this case, $|\det A| = 2$ as required and we may choose $\tilde{\rho} = \binom{1}{0}$ as the second representative. Quite natural choices for $m_1(t_1)$, $m_2(t_2)$ are given by

$$m_1(t_1) = \sin^{2K-1}(2\pi t_1), \qquad m_2(t_2) = \sin^{2K-1}(4\pi t_2). \tag{4.1}$$

The case $K = 1$ is of minor interest, hence let us start with a discussion of the case $K = 2$. Then

$$c_1 = \frac{3\pi}{2}, \qquad c_2 = \frac{-c_{1,2}}{3\pi} \tag{4.2}$$

and (3.8) yields

$$\begin{aligned} & m(\omega_1, \omega_2) \\ &= 1 - \frac{c_{1,2}}{72\pi^2} \Big(-\cos(2\pi\omega_1)(2 + \sin^2(2\pi\omega_1)) + 2\Big) \Big(-\cos(4\pi\omega_2)(2 + \sin^2(4\pi\omega_2)) + 2\Big) \\ &\quad - \frac{1}{4} \Big(-\cos(2\pi\omega_1)(2 + \sin^2(2\pi\omega_1)) + 2\Big) + \frac{c_{1,2}}{36\pi^2} \Big(-\cos(4\pi\omega_2)(2 + \sin^2(4\pi\omega_2)) + 2\Big). \end{aligned} \tag{4.3}$$

The nonvanishing coefficients of the resulting mask can be computed as follows.

$$a_{(0,0)} = \frac{1}{2}; \tag{4.4}$$

$$
\begin{aligned}
a_{(1,2)} &= a_{(1,-2)} = a_{(-1,2)} = a_{(-1,-2)} = -\frac{81c_{1,2}}{4608\pi^2};\\
a_{(1,6)} &= a_{(1,-6)} = a_{(-1,6)} = a_{(-1,-6)} = a_{(3,2)} = a_{(3,-2)} = a_{(-3,2)} = a_{(-3,-2)} = \frac{9c_{1,2}}{4608\pi^2};\\
a_{(-3,-6)} &= a_{(-3,6)} = a_{(3,-6)} = a_{(3,6)} = -\frac{c_{1,2}}{4608\pi^2};\\
a_{(-1,0)} &= a_{(1,0)} = \frac{9c_{1,2}}{288\pi^2} + \frac{9}{32};\\
a_{(3,0)} &= a_{(-3,0)} = -\frac{c_{1,2}}{288\pi^2} - \frac{1}{32}.
\end{aligned}
$$

A typical symbol $m(\omega_1, \omega_2)$ obtained by this procedure is displayed in Figure 1.

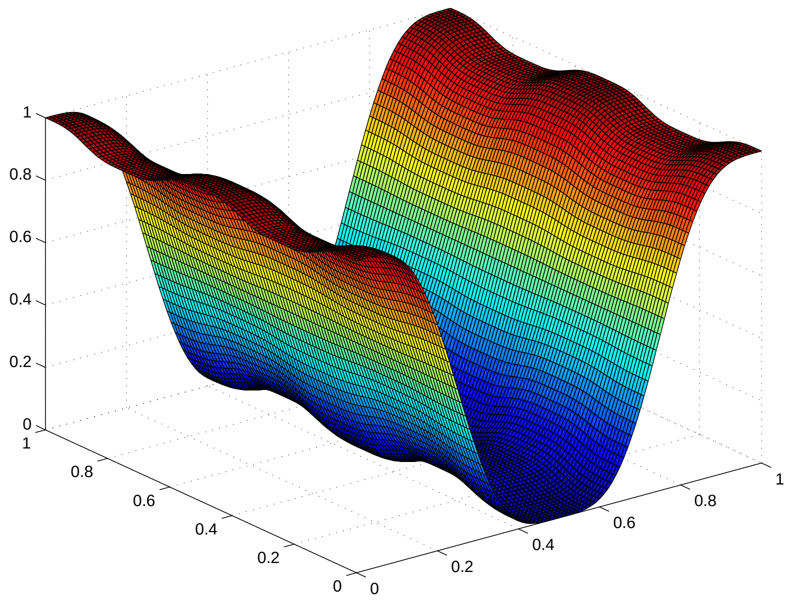

Figure 1: $m(\omega_1, \omega_2)$ for $c_{1,2} = -5$

It remains to estimate the smoothness of the resulting refinable function ϕ, i.e., we want to find

$$\alpha^* := \sup\{\alpha : \ \phi \in C^\alpha\}.$$

It is well–known that $\alpha^* \geq \kappa_{\text{sup}}$, where κ_{sup} is defined by

$$\kappa_{\text{sup}} := \sup\{\kappa : \ \int_{\mathbf{R}^d} (1 + |\omega|)^\kappa |\hat{\phi}(\omega)| d\omega < \infty\}. \tag{4.5}$$

The regularity problem, i.e., the problem of estimating κ_{sup} from below, has attracted substantial research in the last few years, see, e.g., [1, 8, 13, 14]. A typical result in this direction reads as follows.

Theorem 4.1 *For an integer L, let*

$$V_L := \{v \in \ell_0(\mathbf{Z}^d) \ : \ \sum_{k\in\mathbf{Z}^d} p(k)v_k = 0, \quad \textit{for all polynomials} \ \ p \in \Pi_L\},$$

where Π_L denotes the polynomials of total degree L. Assume that A is a dilation matrix with a complete set of orthonormal eigenvectors. If the symbol $m(\omega)$ according to (2.3) *is nonnegative and satisfies Strang–Fix–conditions* (C4) *of order L, then for a suitable choice Ω with supp $\mathbf{a} \subseteq \Omega$, V_L is invariant under the matrix*

$$\mathcal{H} := [q a_{Ak-l}]_{k,l\in\Omega}\,.$$

Let ϱ be the spectral radius of $\mathcal{H}|_{V_L}$. Then the exponent $\kappa_{\sup}$ satisfies

$$\kappa_{\sup} \geq -\frac{\log(\varrho)}{\log(|\lambda_{\max}|)}. \tag{4.6}$$

We used Theorem 4.1 to test several values of $c_{1,2}$. The results are shown in the following table.

$c_{1,2}$	$-\log(\varrho)/\log(\vert\lambda_{\max}\vert)$
-50	0.26569
-10	0.55643
-5	0.60106
-3	0.61971
-1	0.63884
-0.5	0.6437
0	0.6486
0.5	0.65352
1	0.65848
3	0.67864
50	0.7298
100	0.0054245

Remark 4.1 *i) We see that the regularity of the resulting interpolating scaling functions decreases significantly for large values of $|c_{1,2}|$. For very large values of $|c_{1,2}|$, one does not even get an L_2–function.*

ii) We also observe that in order to increase the smoothness of the corresponding scaling function it seems to be a good idea to use positive values of $c_{1,2}$. However, in order to use Theorem 4.1, we have to work with a nonnegative symbol. But it can be easily checked, that this is only the case for $c_{1,2}$ in a certain interval contained in $(-\infty, 0]$. Therefore the results for positive values of $c_{1,2}$ do not relate to regularity estimates of the corresponding scaling functions by Theorem 4.1 directly. Nevertheless, the requirement of a nonnegative symbol in Theorem 4.1 is a sufficient but not necessary condition. Experience from numerical experiments indicates, that the given figures still give lower bounds to the order of regularity even for positive values of c_{12}.

As already stressed in Remark 3.1, the symbol computed according to Theorem 3.1 can also be used for other scaling matrices. In our case, it is easy to check that e.g. for the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\tilde{\delta} = \binom{1}{0}$ the conditions of Remark 3.1 are satisfied. It turns out that for this matrix the resulting refinable functions are in fact much smoother as can be seen from the following table.

$c_{1,2}$	$-\log(\varrho)/\log(\lvert\lambda_{\max}\rvert)$
-10	0.96322
-5	1.2694
-1	1.7589
-0.5	1.8665
0	2
0.2	1.9678
1	1.8562
10	1.2073
50	0.045414

We have also studied the case $K = 3$. In this case, eq. (3.8) yields

$$\begin{aligned} & m(\omega_1, \omega_2) \\ = \; & 1 - \frac{c_{1,2}}{\pi^2}\left(-\frac{5}{16}\cos(2\pi\omega_1) + \frac{5}{96}\cos(6\pi\omega_1) - \frac{1}{160}\cos(10\pi\omega_1) + \frac{4}{15}\right) \\ & \quad \cdot \left(-\frac{5}{32}\cos(4\pi\omega_2) + \frac{5}{192}\cos(12\pi\omega_2) - \frac{1}{320}\cos(20\pi\omega_2) + \frac{2}{15}\right) \\ & -\frac{15}{8}\left(-\frac{5}{16}\cos(2\pi\omega_1) + \frac{5}{96}\cos(6\pi\omega_1) - \frac{1}{160}\cos(10\pi\omega_1) + \frac{4}{15}\right) \\ & +\frac{4c_{1,2}}{15\pi^2}\left(-\frac{5}{32}\cos(4\pi\omega_2) + \frac{5}{192}\cos(12\pi\omega_2) - \frac{1}{320}\cos(20\pi\omega_2) + \frac{2}{15}\right). \end{aligned}$$

For the sake of completeness we state the corresponding filter coefficientes explicitely.

$$\begin{aligned} a_{(0,0)} &= \frac{1}{2}; \qquad (4.7) \\ a_{(1,2)} &= a_{(1,-2)} = a_{(-1,2)} = a_{(-1,-2)} = -\frac{25c_{1,2}}{2048\pi^2}; \\ a_{(1,6)} &= a_{(1,-6)} = a_{(-1,6)} = a_{(-1,-6)} = \frac{25c_{1,2}}{12288\pi^2}; \\ a_{(1,10)} &= a_{(1,-10)} = a_{(-1,10)} = a_{(-1,-10)} = -\frac{5c_{1,2}}{20480\pi^2}; \\ a_{(1,0)} &= a_{(-1,0)} = \frac{75}{256} + \frac{c_{1,2}}{48\pi^2}; \\ a_{(3,2)} &= a_{(3,-2)} = a_{(-3,2)} = a_{(-3,-2)} = \frac{45c_{1,2}}{12288\pi^2}; \\ a_{(-3,-6)} &= a_{(-3,6)} = a_{(3,-6)} = a_{(3,6)} = -\frac{45c_{1,2}}{73728\pi^2}; \end{aligned}$$

$$\begin{aligned}
a_{(3,10)} &= a_{(3,-10)} = a_{(-3,10)} = a_{(-3,-10)} = \frac{9c_{1,2}}{122880\pi^2};\\
a_{(3,0)} &= a_{(-3,0)} = -\frac{9c_{1,2}}{1440\pi^2} - \frac{75}{1536};\\
a_{(5,2)} &= a_{(5,-2)} = a_{(-5,2)} = a_{(-5,-2)} = -\frac{5c_{1,2}}{20480\pi^2};\\
a_{(5,6)} &= a_{(5,-6)} = a_{(-5,6)} = a_{(-5,-6)} = \frac{5c_{1,2}}{122880\pi^2};\\
a_{(5,10)} &= a_{(5,-10)} = a_{(-5,10)} = a_{(-5,-10)} = -\frac{c_{1,2}}{204800\pi^2};\\
a_{(5,0)} &= a_{(-5,0)} = \frac{15}{2560} + \frac{c_{1,2}}{2400\pi^2}.
\end{aligned}$$

The regularity of the corresponding scaling functions can again be estimated by using Theorem 4.1.

$c_{1,2}$	$-\log(\varrho)/\log(\lvert\lambda_{\max}\rvert)$
-50	0.42988
-10	0.5938
-3	0.61571
-1	0.62137
-0.5	0.62275
0	0.6241
3	0.63181
10	0.64683
20	0.66002
30	0.625
50	0.4986

Remark 4.2 *A* MATLAB *program to compute the regularity of refinable functions according to Theorem 4.1 can be found on the IGPM–homepage, see* `http://elc2.igpm.rwth-aachen.de/barinka/mattoys/soft.html`.

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