**Complex Shearlet-Based Detection of Flame Fronts**

Johannes Kiefer¹, Emily J. King², Rafael Reisenhofer²

(1) FG Technische Thermodynamik, Universität Bremen, (2) AG Computational Data Analysis, Universität Bremen

First Applied Mathematics Symposium Münster, Universität Münster, 30 September 2015
OUTLINE

Flame Fronts

Complex Shearlet-Based Edge/Ridge Detection

Results
Planar Laser-Induced Fluorescence (PLIF) recording of a combustion process.

Video from Johannes Kiefer
Flame Fronts

PLIF image of excited OH

PLIF image of excited CH

Images from Johannes Kiefer
Flame Fronts

- Both edges (OH excitation) and ridges (CH excitation) need to be detected

- Local tangent orientations and curvatures are of interest

- Edges can be characterized by both smooth and sharp transitions

- Flame fronts are primarily defined by the visible structure, not by a large contrast

- Ridges can have varying widths
Here, we try to extract information from the signal (edges, ridges, tangent orientations, ...) by analyzing the decomposition of a signal. We don’t care about the reconstruction!
A perfect 1D edge can be modeled as a simple step function

A perfect 1D ridge is something close to a delta distribution
Consider systems of small waves with a strong decay that can be differently scaled (i.e. squeezed and stretched) and translated.

For a generating wavelet $\psi \in L^2(\mathbb{R})$ and indices $\Gamma \subset \mathbb{R}^+ \times \mathbb{R}$, a wavelet system is given by

$$\Psi = \{ \psi_{a,x} = a^{1/2} \psi(a(\cdot - x)) : (a, x) \in \Gamma \}$$

The corresponding wavelet transform for $f \in L^2(\mathbb{R})$ is given by

$$(\mathcal{W}_\psi f)(a, x) = \langle f, \psi_{a,x} \rangle_{L^2}$$
**Complex Wavelets**

- Take a real-valued even-symmetric wavelet generator $\psi^e \in L^2(\mathbb{R})$
- Define complex-valued wavelet $\psi^c$ via Hilbert transform

$$\psi^c = \psi^e + i\mathcal{H}\psi^e$$

- The Hilbert transform is a Fourier multiplier (with factor $-i\text{sgn} (\omega)$) that interchanges sine and cosine

---

Let $f \in L^2(\mathbb{R})$. To determine the presence of a step at a location $x \in \mathbb{R}$, we take

- an even-symmetric real-valued wavelet $\psi^e$
- and an odd-symmetric real-valued wavelet $\psi^o = \mathcal{H}\psi^e$

and analyze the behavior of the coefficients

- $\langle f, \psi^e(a(\cdot - x)) \rangle_{L^2} = \text{Re}(\langle f, \psi^c_{a,x} \rangle_{L^2})$
- $\langle f, \psi^o(a(\cdot - x)) \rangle_{L^2} = \text{Im}(\langle f, \psi^c_{a,x} \rangle_{L^2})$

for varying $a > 0$. 
Step Detection With Complex Wavelets

With $\text{Re}(\psi)$ and $\text{Im}(\psi)$ being $L^1$-normalized
**Definition**

For a signal $f \in L^2(\mathbb{R})$ and a location $x \in \mathbb{R}$, a step measure is given by

$$S_{\psi}(f, x) = \left| \sum_{a \in A} \text{Im}(\langle f, \psi_{a,x}^c \rangle_{L^2}) \right| - \sum_{a \in A} |\text{Re}(\langle f, \psi_{a,x}^c \rangle_{L^2})| + \epsilon,$$

where $A \subset \mathbb{R}^+$ is a set of scaling parameters, $\psi$ is a real-valued symmetric wavelet and $\epsilon$ prevents division by zero.
RIDGE DETECTION WITH COMPLEX WAVELETS

Symmetric Mexican hat wavelets aligned with a ridge

Coefficients of the symmetric Mexican hat wavelet

Odd-symmetric Mexican hat wavelets aligned with a ridge

Coefficients of the odd-symmetric Mexican hat wavelet

With $\text{Re}(\psi)$ and $\text{Im}(\psi)$ being $L^1$-normalized
RIDGE DETECTION WITH COMPLEX WAVELETS

DEFINITION
For a signal $f \in L^2(\mathbb{R})$ and a location $x \in \mathbb{R}$, a ridge measure is given by

$$R_\psi(f, x) = \left| \sum_{a \in A} \text{Re}(\langle f, \psi_{a,x}^c \rangle_{L^2}) \right| - \sum_{a \in A} |\text{Im}(\langle f, \psi_{a,x}^c \rangle_{L^2})|$$

where $A \subset \mathbb{R}^+$ is a set of scaling parameters, $\psi$ is a real-valued symmetric wavelet and $\epsilon$ prevents division by zero and the function $C$ depends only on the real-valued coefficients of the complex wavelet transform.

- Similar ideas have been developed by Kovesi (1999) and du Buf & Deemter (2000)

King, R 2015
Shearlets

- Introduced by Kutyniok, Labate et al. in 2005
- **Anisotropic scaling** is used to better represent 2D features
- Orientation has to be changed
- **Shears** are used instead of rotations to keep the integer grid intact
To construct a shearlet system, one needs

- a shearlet generator $\psi \in L^2(\mathbb{R}^2)$
- anisotropic scaling and shear matrices

$$A_a = \begin{pmatrix} a & 0 \\ 0 & a^\alpha \end{pmatrix} \quad S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

with parameters $a \in \mathbb{R}^+$ and $s \in \mathbb{R}$

- $\alpha \in [0, 1]$ determines the degree of anisotropy

For a set of parameters $\Gamma \subset \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$, a shearlet system is given by

$$\Psi_\psi = \left\{ \psi_{a,s,x} = a^{(1+\alpha)/2} \psi(S_sA_a(\cdot - x)) : (a, s, x) \in \Gamma \right\}$$

The corresponding shearlet transform for $f \in L^2(\mathbb{R}^2)$ is given by

$$(S_\psi f)(a, s, x) = \langle f, \psi_{a,s,x} \rangle_{L^2}$$

Guo, Kutyniok & Labate 2005; Grohs, Keiper, Kutyniok & Schäfer 2014
Complex Shearlet Transform

- Hilbert transform can be generalized to 2D
- Given a real-valued even-symmetric shearlet generator $\psi^e \in L^2(\mathbb{R}^2)$, a complex-valued shearlet generator is given by

$$\psi^c = \psi^e + i \mathcal{H} \psi^e$$

- The Hilbert transform commutes with scalings, translates and shears
- For a real-valued generator $\psi^e \in L^2(\mathbb{R}^2)$ and a set of parameters $\Gamma \subset \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$, a complex shearlet-based dictionary is given by

$$\Psi^c_{\psi} = \{ \psi^c_{a,s,x} = (\psi^c)(S_sA_a(\cdot-x)): (a, s, x) \in \Gamma \}$$

Storath 2013; R 2014
**Definition**

For an image \( f \in L^2(\mathbb{R}^2) \), a location \( x \in \mathbb{R}^2 \) and a shear parameter \( s \in \mathbb{R} \), an edge measure is given by

\[
E_{\psi}(f, x, s) = \frac{\left| \sum_{a \in \mathcal{A}} \operatorname{Im}(\langle f, \psi^c_{a,s,x}\rangle_{L^2}) \right| - \sum_{a \in \mathcal{A}} |\operatorname{Re}(\langle f, \psi^c_{a,s,x}\rangle_{L^2})|}{|\mathcal{A}| \max_{a \in \mathcal{A}} |\operatorname{Im}(\langle f, \psi^c_{a,s,x}\rangle_{L^2})| + \epsilon},
\]

where \( \mathcal{A} \subset \mathbb{R}^+ \) is a set of scaling parameters, \( \psi \) is a real-valued symmetric shearlet and \( \epsilon \) prevents division by zero.

- A specific orientation parameter \( s \) can be chosen...
  - beforehand, by searching for the **largest odd-symmetric coefficient** over all orientations
  - in hindsight, by searching for the **largest edgeness** over all orientations
Edge Detection With Complex Shearlets

Video and Live Demo
SUMMARY AND OUTLOOK

- Interplay of even-symmetric and odd-symmetric basis functions can be used to
  - detect edges and ridges
  - approximate tangential directions

- The method is stable in the presence of noise and by construction contrast invariant

- Method could easily be generalized to 3D data

- Can parameters - especially for the construction of the analyzing functions - be chosen automatically?

- There are many connections to things in the primary visual cortex, but I didn’t have time to tell you

R. Reisenhofer, J. Kiefer and E. J. King; Shearlet-Based Detection of Flame Fronts; Preprint.

E. J. King and R. Reisenhofer; Complex Shearlet-Based Edge and Ridge Detection; In Preperation.

R. Reisenhofer; The Complex Shearlet Transform and Applications to Image Quality Assessment; Master’s Thesis; Technische Universität Berlin, 2014.

You can download the CoShREM Toolbox from www.math.uni-bremen.de/~reisenho (on Friday, I promise)
Thank you!