COMPLEX SHEARLET TRANSFORMS AND APPLICATIONS TO EDGE AND LINE DETECTION

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Universität Bremen
OUTLINE

EDGE AND LINE DETECTION
Real World Problems
Standard Approach to Edge Detection

COMPLEX SHEARLET-BASED EDGE/LINE DETECTION
The 1D Case: Complex Wavelet-Based Step Detection
The 2D Case: Complex Shearlet-Based Edge Detection

RESULTS
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Edge Detection

- Edges can be sharp or smooth transitions
- Edges are not characterized by a high contrast but by a clearly defined structure
- We would like our edge measure to
  - be contrast invariant
  - be stable in the presence of noise
  - provide information about the geometry of an object
LINE DETECTION
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RESULTS
**REAL WORLD DATA: FLAME FRONTS**

Planar Laser-Induced Fluorescence (PLIF) image of excited **OH**

Planar Laser-Induced Fluorescence (PLIF) image of excited **CH**

Images from Johannes Kiefer
REAL WORLD DATA: TIDAL FLATS

Synthetic Aperture Radar (SAR) image of the Wadden flats

Heygster, Dannenberg & Notholt 2010; Li 2014
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RESULTS
Standard Approach to Edge Detection

1. Smooth image
2. Approximate gradients
3. Find local maxima and threshold

- The gradient barely recognizes the geometry of an object
- Gradient-based methods will always pick up two edges instead of one line
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Basic Idea of Applied Harmonic Analysis
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RESULTS
- A perfect 1D edge can be modeled as a simple step function
- A 1D line is something close to a delta distribution
Wavelets

- Quite popular, quite old! (Date back to Haar, 1909)
- Consider systems of small waves with a strong decay that can be differently scaled (i.e. squeezed and stretched) and translated

\[ \Psi = \{ \psi_{a,t} = a^{1/2} \psi(a \cdot -t) : (a, t) \in \Gamma \} \]

- For a generating wavelet \( \psi \in L^2(\mathbb{R}) \) and indices \( \Gamma \subset \mathbb{R}^+ \times \mathbb{R} \), a wavelet system is given by

- The corresponding wavelet transform for \( f \in L^2(\mathbb{R}) \) is given by

\[ (\mathcal{W}_\psi f)(a, t) = \langle f, \psi_{a,t} \rangle_{L^2} \]
**Why Wavelets?**

Wavelets are well suited for sparsely approximating piecewise smooth functions in 1D
Complex Wavelet Transforms

▶ Consider the structure of the complex exponential
\[ e^{ix} = \cos(x) + i \sin(x) \]

▶ Cosine is symmetric
▶ Sine is odd-symmetric
▶ Cosine and sine differ by a $90^\circ$ phase shift
▶ Magnitude response of Fourier transform is perfectly shift invariant
\[ |\langle f, e^{i \cdot} \rangle_{L^2}| = |\langle f(\cdot - t), e^{i \cdot} \rangle_{L^2}| \]

We want to define a complex-valued transform which is well suited for representing 1D piecewise smooth functions and retains some of the nice properties of the Fourier transform

HILBERT TRANSFORM

DEFINITION (HILBERT TRANSFORM)
Let $f \in L^2(\mathbb{R})$, then the Hilbert transform of $f$ is given by

$$(\mathcal{H}f)(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{t-x} dt$$

or equivalently in the Fourier domain

$$\mathcal{F}(\mathcal{H}f)(\xi) = -i\text{sgn}(\xi)\hat{f}(\xi),$$

where $i$ denotes the imaginary unit and $\text{sgn}$ the sign function.

- The Hilbert transform interchanges sine and cosine

$$\mathcal{F}^{-1}\mathcal{H}f)(x) = \frac{1}{\pi} \int_{0}^{\infty} \text{Re}(\hat{f}(\xi))\cos(\xi x) - \text{Im}(\hat{f}(\xi))\sin(\xi x) d\xi$$

$$\mathcal{F}^{-1}\mathcal{H}f)(x) = \frac{1}{\pi} \int_{0}^{\infty} \text{Re}(\hat{f}(\xi))\sin(\xi x) + \text{Im}(\hat{f}(\xi))\cos(\xi x) d\xi$$
**COMPLEX WAVELETS**

- Take a real-valued symmetric wavelet generator $\psi \in L^2(\mathbb{R})$
- Define complex-valued wavelet $\psi^c$ via Hilbert transform

$$\psi^c = \psi + i\mathcal{H}\psi$$
STEP DETECTION WITH COMPLEX WAVELETS

With \( \text{Re}(\psi) \) and \( \text{Im}(\psi) \) being \( L^1 \)-normalized
DEFINITION
For a signal $f \in L^2(\mathbb{R})$ and a location $x \in \mathbb{R}$, a step measure is given by

$$E_\psi(f, x) = \frac{\left| \sum_{a \in A} \text{Im}(\langle f, \psi^c_{a,x}\rangle_{L^2}) \right| - \sum_{a \in A} \left| \text{Re}(\langle f, \psi^c_{a,x}\rangle_{L^2}) \right|}{|A| \max_{a \in A} \left| \text{Im}(\langle f, \psi^c_{a,x}\rangle_{L^2}) \right| + \epsilon},$$

where $A \subset \mathbb{R}^+$ is a set of scaling parameters, $\psi$ is a real-valued symmetric wavelet and $\epsilon$ prevents division by zero.

- Similar ideas have been developed by Kovesi (phase congruency) and du Buf
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The 2D Case
**Shearlets**

- Quite new (Kutyniok, Labate et al. 2005), not so famous (yet!)
- **Anisotropic scaling** is used to better represent 2D features

- Orientation has to be changed
- **Shears** are used instead of rotations to keep the integer grid intact
To construct a shearlet system, one needs
- a shearlet generator $\psi \in L^2(\mathbb{R}^2)$
- anisotropic scaling and shear matrices

$$A_{a} = \begin{pmatrix} a & 0 \\ 0 & a^{\alpha} \end{pmatrix} \quad \text{and} \quad S_{s} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

with parameters $a \in \mathbb{R}^+$ and $s \in \mathbb{R}$
- $\alpha \in [0, 1]$ determines the degree of anisotropy
- For a set of parameters $\Gamma \subset \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$, a shearlet system is given by

$$\Psi_{\psi} = \left\{ \psi_{a,s,t} = a^{(1+\alpha)/2} \psi(S_{s}A_{a}(\cdot - t)) : (a, s, t) \in \Gamma \right\}$$

The corresponding shearlet transform for $f \in L^2(\mathbb{R}^2)$ is given by

$$\left( S_{\psi} f \right)(a, s, t) = \langle f, \psi_{a,s,t} \rangle_{L^2}$$

Guo, Kutyniok & Labate 2005; Grohs, Keiper, Kutyniok & Schäfer 2014
Complex Shearlet Transform

- Hilbert transform can be generalized to 2D
- Given a real-valued (typically symmetric) shearlet generator $\psi \in L^2(\mathbb{R}^2)$, a complex-valued shearlet generator is given by

$$\psi^c = \psi + i\mathcal{H}\psi$$

- The Hilbert transform commutes with scalings, translates and shears
- For a real-valued generator $\psi \in L^2(\mathbb{R}^2)$ and a set of parameters $\Gamma \subset \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$, a complex shearlet-based dictionary is given by

$$\Psi^c_{\psi} = \{\psi^c_{a,s,t} = (\psi^c)(S_sA_a(\cdot - t)) : (a,s,t) \in \Gamma\}$$

Storath 2013; R 2014
Edge Detection With Complex Shearlets

Definition
For an image \( f \in L^2(\mathbb{R}^2) \), a location \( x \in \mathbb{R}^2 \) and a shear parameter \( s \in \mathbb{R} \), an edge measure is given by

\[
E_\psi(f, x, s) = \left| \sum_{a \in A} \text{Im}(\langle f, \psi_{a,s,x}^c \rangle_{L^2}) \right| - \left| \sum_{a \in A} \text{Re}(\langle f, \psi_{a,s,x}^c \rangle_{L^2}) \right|
\]

\[
\frac{|A| \max_{a \in A} \left| \text{Im}(\langle f, \psi_{a,s,x}^c \rangle_{L^2}) \right| + \epsilon}{\left| \sum_{a \in A} \text{Im}(\langle f, \psi_{a,s,x}^c \rangle_{L^2}) \right| - \left| \sum_{a \in A} \text{Re}(\langle f, \psi_{a,s,x}^c \rangle_{L^2}) \right|},
\]

where \( A \subset \mathbb{R}^+ \) is a set of scaling parameters, \( \psi \) is a real-valued symmetric shearlet and \( \epsilon \) prevents division by zero.

- A specific orientation parameter \( s \) can be chosen...
  - beforehand, by searching for the largest odd-symmetric coefficient over all orientations
  - in hindsight, by searching for the largest edgeness over all orientations
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Live Demo
Line Detection With Complex Shearlets

\[ E_\psi(f, x, s) = \frac{\sum_{a \in A} \text{Im}(\langle f, \psi^c_{a,s,x} \rangle_{L^2}) - \sum_{a \in A} \text{Re}(\langle f, \psi^c_{a,s,x} \rangle_{L^2})}{|A| \max_{a \in A} \left| \text{Im}(\langle f, \psi^c_{a,s,x} \rangle_{L^2}) \right| + \epsilon} \]

We just reverse the roles of \( \text{Im}(\langle f, \psi^c_{a,s,x} \rangle_{L^2}) \) and \( \text{Re}(\langle f, \psi^c_{a,s,x} \rangle_{L^2}) \)

\[ L_\psi(f, x, s) = \frac{\sum_{a \in A} \text{Re}(\langle f, \psi^c_{a,s,x} \rangle_{L^2}) - \sum_{a \in A} \text{Im}(\langle f, \psi^c_{a,s,x} \rangle_{L^2})}{|A| \max_{a \in A} \left| \text{Re}(\langle f, \psi^c_{a,s,x} \rangle_{L^2}) \right| + \epsilon} \]
SUMMARY AND OUTLOOK

- Interplay of even-symmetric and odd-symmetric basis functions can be used to
  - detect edges and lines
  - approximate tangential directions
- The method is stable in the presence of noise and by construction contrast invariant
- Can the parameters - especially for the construction of the analyzing functions - be chosen automatically?
- Method can easily be generalized to 3D data
- There are many connections to things in the primary visual cortex, but I didn’t have time to tell you