Universal Constructions for Hopf Algebras

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Abstract

The category of Hopf monoids over an arbitrary symmetric monoidal category as well as its subcategories of commutative and cocommutative objects respectively are studied, where attention is paid in particular to the following questions: (a) When are the canonical forgetful functors of these categories into the categories of monoids and comonoids respectively part of an adjunction? (b) When are the various subcategory-embeddings arsing naturally in this context reflexive or coreflexive? (c) When does a category of Hopf monoids have all limits or colimits? These problems are also shown to be intimately related. Particular emphasis is given to the case of Hopf algebras, i.e., when the chosen symmetric monoidal category is the category of modules over a commutative unital ring.

Key words: Hopf algebras, Hopf monoids, free and cofree constructions, reflections and coreflections, locally presentable category.
2000 MSC: Primary 16W30, Secondary 18A40

1 Introduction

In his seminal book "Hopf Algebras" [12] Sweedler states, after discussing the existence of free bialgebras over coalgebras and cofree bialgebras over algebras, the following concerning analoguous problems for Hopf algebras:

1. Given a coalgebra D, there is
   a.1 a free Hopf algebra on D and

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a.2 a free commutative Hopf algebra on D.
a.3 Both of these are cocommutative if D is cocommutative.

2. Given an algebra A there is
   b.1 a cofree Hopf algebra on A and
   b.2 a cofree cocommutative Hopf algebra on A.
   b.3 Both of these are commutative if A is.

He does not give any proof of these statements, and it took a couple of years until Takeuchi proved — with two completely different constructions — a.1 and a.2 [14]. Concerning Sweedler's further claims the author has not seen any proof in print.

It should be noted at this stage that in the more recent literature there seems to be a tendency to use the term "cofree Hopf algebra" differently from Sweedler's appropriate use, by referring to a Hopf algebra whose underlying coalgebra is cofree over some vector space (see, e.g., [7]).

There is another obviously interesting question concerning (co)universal constructions in the realm of Hopf algebras: is it possible to adjoin (freely or cofreely) an antipode to a bialgebra, in other words: is the category of Hopf algebras a reflexive or coreflexive subcategory of that of bialgebras? (see e.g. [13]). The trivial connection between the existence of (co)frees and (co)reflections clearly is that the latter imply the former due to composition of adjoints (the category of bialgebras has free objects and cofree objects, respectively).

That this implication can be reversed is one of the results of this note, which tries to shed some light on the existence and relationship of such (co)universal constructions by purely categorical methods.

A further non-trivial problem is that of existence of limits and colimits respectively in categories of Hopf algebras due to the algebraic *and* coalgebraic nature of their structure. We will show that the existence of arbitrary free or cofree Hopf algebras implies completeness—even local presentability—of the respective category.

In view of the increasing interest in Hopf algebras not only over a field k but rather over an arbitrary commutative unital ring R (see e.g. [4]) one is led to use as a base category for such studies the category $\mathbf{Mod}_R$ of R–modules. In fact we take a further step of abstraction and use, for developing a framework to deal with these universal constructions, an arbitrary symmetric monoidal category $\mathbb{C}$. This enables us to extend our study to categories of group objects in a suitable category with finite products; this way we have an additional tool to deal with a category like ${}_{\mathbf{coc}}\mathbf{Hopf}_R$, the category of cocommutative Hopf algebras over R, which is known to be the category of group objects in ${}_{\mathbf{coc}}\mathbf{Coalg}_R$, the (cartesian closed — see [3]) category of cocommutative coalgebras.

Our results then include in particular, referring to Sweedler's claims above, purely categorical arguments for

- a.1 $\Rightarrow$ a.2 (Corollary 4.3)
- a.3 for the case of the free commutative Hopf algebra (Corollary 4.3). Concerning the remaining case we only prove the somewhat weaker statement that free cocommutative Hopf algebras over cocommutative coalgebras exist (Theorem 4.5).
- b.1 $\Rightarrow$ b.2 (Corollary 4.4)
- b.3 for the case of the cofree cocommutative Hopf algebra (Corollary 4.4), and again, for the remaining case the weaker statement that cofree commutative Hopf algebras over commutative algebras exist (Theorem 4.8).

We add, as an appendix, a review of Takeuchi's proof mentioned above from a more categorical point of view in order to indicate which role is played by his assumption that the underlying ring is even a field.

2 A review of bimonoids

2.1 The definition of bimonoids. Given a symmetric monoidal category $\mathbb{C} = (\mathbf{C}, -\otimes -, I)$, the categories $\mathbf{Mon}\mathbb{C}$ of monoids in $\mathbb{C}$ and the category $\mathbf{Comon}\mathbb{C}$ of comonoids in $\mathbb{C}$ inherit, in a canonical way, the monoidal structure from $\mathbb{C}$ making them symmetric monoidal categories again. Restricting these structures to their subcategories $_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$ of cocommutative comonoids and $_{\mathbf{c}}\mathbf{Mon}\mathbb{C}$ of commutative monoids respectively, here yields the cartesian respectively cocartesian structure. One then has

- $\mathbf{MonComon}\mathbb{C} = \mathbf{ComonMon}\mathbb{C}$, and this category is called the category $\mathbf{Bimon}\mathbb{C}$ of *bimonoids* in $\mathbb{C}$, which has subcategories
- $_{\mathbf{c}}\mathbf{Bimon}\mathbb{C} = \mathbf{Comon}_{\mathbf{c}}\mathbf{Mon}\mathbb{C}$, the category of commutative bimonoids,
- $_{\mathbf{coc}}\mathbf{Bimon}\mathbb{C} = \mathbf{Mon}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$, the category of cocommutative bimonoids.

Somewhat more explicitely and with some redundance, a bimonoid thus is a quintuple $\mathcal{C} = (C, m, e, \mu, \epsilon)$ where $(C, m, e,)$ is a monoid such that m and e are even comonoid homomorphisms, and (C, μ, ϵ) is a comonoid such that μ and ϵ are monoid homomorphisms. The morphims in $\mathbf{Bimon}\mathbb{C}$ are monoid- and comonoid homomorphisms simultanously. For details see [10]. We denote a bimonoid by $\mathcal{C} = (\mathcal{C}^c, \mathcal{C}^m)$ where $\mathcal{C}^c := (C, \mu, \epsilon)$ is the underlying comonoid and $\mathcal{C}^m := (C, m, e)$ the underlying monoid. Note that, assigning to any bimonoid $\mathcal{C}$, the bimonoid $\mathcal{C}^{\mathsf{op}} = \Big((\mathcal{C}^c)^{\mathsf{op}}, (\mathcal{C}^m)^{\mathsf{op}}\Big)$ made up of the opposites of the monoid and comonoid part of $\mathcal{C}$, defines a functorial isomorphism $(-)^{\mathsf{op}}$ on $\mathbf{Bimon}\mathbb{C}$.

2.2 Convolution monoids. For any pair of bimonoids $(\mathcal{C}, \mathcal{D})$ the hom-set $\mathbf{C}(C, D)$ carries the structure of a monoid (in **Set**) given by the multiplication (for $f, g \in \mathbf{C}(C, D)$)

$$f * g\colon = m \circ (f \otimes g) \circ \mu$$

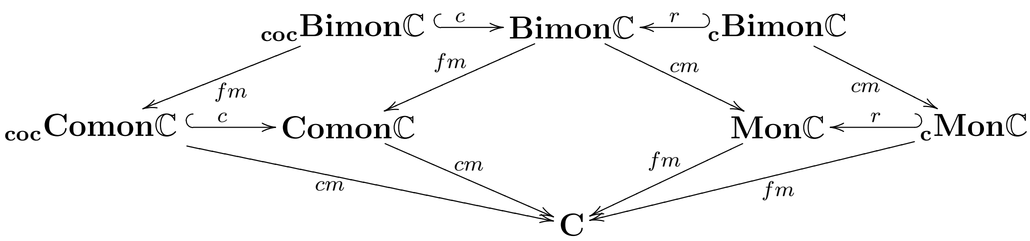
with μ the comultiplication of $\mathcal{C}$ and m the multiplication of $\mathcal{D}$, and the unit

$$C \xrightarrow{\epsilon} 1 \xrightarrow{e} K$$

where ϵ is the counit of $\mathcal{C}$ and e the unit of $\mathcal{D}$. We denote this monoid, called *convolution monoid*, by $Conv(\mathcal{C}, \mathcal{D})$.

2.3 In case the category $\mathbf{C}$ is locally presentable and the functor $- \otimes -$ preserves directed colimits or, for each C in $\mathbf{C}$, the functor $C \otimes -$ preserves at most countable coproducts it holds that (see [10]), in the following diagram,

- all categories are locally presentable,
- all arrows labelled fm are finitary monadic functors,
- all arrows labelled cm are comonadic functors,
- all hooked arrows are accessible embeddings with r denoting a reflective and c a coreflective one.



3 Hopf monoids

3.1 Antipodes. Hopf monoids are defined to be special bimonoids as follows.

Definition A bimonoid $\mathcal{C} = (C, m, e, \mu, \epsilon)$ is called *Hopf monoid*, provided there exists a $\mathbf{C}$-morphism $S\colon C \to C$ satisfying the following equations:

$$m \circ (S \otimes 1_C) \circ \mu = e \circ \epsilon = m \circ (1_C \otimes S) \otimes \mu$$

S is called the *antipode* of $\mathcal{C}$.

We collect some general properties of antipodes as follows.

Facts Let $\mathcal{C}$ be a bimonoid with antipode S; then

1. the antipode is unique (since it is the inverse of the 1_C in the convolution monoid $Conv(\mathcal{C}, \mathcal{C})$.
2. $S \colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathsf{op}}$ is a bimonoid morphism.
3. Every bimonoid homomorphism $f \colon \mathcal{C} \to \mathcal{C}'$, where $\mathcal{C}'$ also is a Hopf monoid with antipode S', f respects the antipodes, i.e., one has

$$S' \circ f = f \circ S.$$

(since $S' \circ f$ and $f \circ S$ both are inverses of f in the convolution monoid $Conv(\mathcal{C}, \mathcal{C}')$).

3.2 The category Hopf$\mathbb{C}$. Since, by the above, bimonoid homomorphisms respect antipodes we define

Definition The full subcategory of **Bimon**$\mathbb{C}$ spanned by all bimonoids with an antipode is called the category **Hopf**$\mathbb{C}$ of Hopf monoids. $_{\mathbf{c}}$**Hopf**$\mathbb{C}$ and $_{\mathbf{coc}}$**Hopf**$\mathbb{C}$ denote the categories of commutative and cocommutative Hopf monoids, respectively.

The following two cases are of particular interest

Car The *cartesian case*, i.e., when the monoidal structure on **C** is just given by cartesian product (and terminal object 1);

Mod The *module case*, i.e., when $\mathbb{C}$ is $\mathbf{Mod}_R$, for some commutative ring R with unit, equipped with the usual tensor product.

3.3 Groups in a category. In the cartesian case the category of comonoids in $\mathbb{C}$ is equivalent to $\mathbb{C}$ since every **C**-object C carries only the trivial comonoid structure $(C, \Delta, !)$ with Δ the diagonal and ! the unique morphism into the terminal object; moreover, the monoidal structure on **Mon**$\mathbb{C}$ inherited from $\mathbb{C}$ is again the cartesian one. Thus, in this case **Bimon**$\mathbb{C}$ = **Mon**$\mathbb{C}$. Moreover, the defining equations for an antipode here agree with the defining equations for group-inversion. Hence, in the cartesian case, the category **Hopf**$\mathbb{C}$ is nothing but the category **Grp(C)** of groups in **C**.

Remark It follows from 2.1 that, in particular,

1. $_{\mathbf{coc}}\mathbf{Hopf}\mathbb{C} = \mathbf{Grp}(_{\mathbf{coc}}\mathbf{Comon}\mathbb{C})$
2. $_{\mathbf{c}}\mathbf{Hopf}\mathbb{C} = \left[\mathbf{Grp}(_{\mathbf{c}}\mathbf{Mon}\mathbb{C})^{\mathsf{op}}\right]^{\mathsf{op}}$.

3.4 Hopf algebras. In the module case one writes $\mathbf{Hopf}_R \colon = \mathbf{HopfMod}_R$ and calls the Hopf monoids in $\mathbf{Mod}_R$ *Hopf algebras.*

4 Universal constructions for $\mathbf{Hopf}\mathbb{C}$

4.A The general case

For the whole of this section we will assume that the category $\mathbf{C}$ is locally presentable and the functor $- \otimes -$ preserves directed colimits or, for each C in $\mathbf{C}$, the functor $C \otimes -$ preserves at most countable coproducts, as in 2.3.

4.1 Accessability. Concerning categorical properties of the categories of Hopf monoids we then have

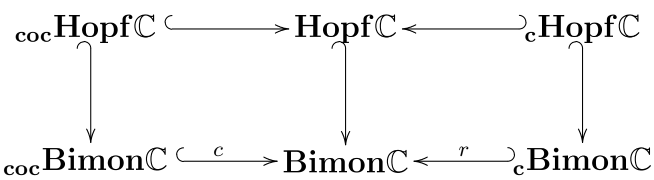
Proposition *Each of the categories* $\mathbf{Hopf}\mathbb{C}$, ${}_{\mathbf{c}}\mathbf{Hopf}\mathbb{C}$, *and* ${}_{\mathbf{coc}}\mathbf{Hopf}\mathbb{C}$ *is accessible.*

Proof: Since Hopf monoids are bimonoids $\mathcal{C}$ equipped with a bimonoid homomorphism $S\colon \mathcal{C} \to \mathcal{C}^{\mathsf{op}}$ they can be considered as $(-)^{\mathsf{op}}$-algebras in the sense of [1]) for the functorial isomorphism $\mathbf{Bimon}\mathbb{C} \xrightarrow{(-)^{\mathsf{op}}} \mathbf{Bimon}\mathbb{C}$ (see 2.1). This functor clearly is accessible and therefore the forgetful functor $\mathbf{Alg}(-)^{\mathsf{op}} \to \mathbf{Bimon}\mathbb{C}$ is accessible (see [1,2]) and so is its composition U with $\mathbf{Bimon}\mathbb{C} \to \mathbf{C}$ by 2.3. We define natural transformations $\varphi^1, \varphi^2, \psi\colon U \longrightarrow U$ as follows (denoting an $\mathbf{Alg}(-)^{\mathsf{op}}$–object as a pair $(\mathcal{C}, S\colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathsf{op}})$ with $\mathcal{C} = (C, m, e, \mu, \epsilon)$)

$$
\begin{aligned}
\varphi^1_{(\mathcal{C},S)} &: = C \xrightarrow{\mu} C \otimes C \xrightarrow{S \otimes 1} C \otimes C \xrightarrow{m} C \\
\varphi^2_{(\mathcal{C},S)} &: = C \xrightarrow{\mu} C \otimes C \xrightarrow{S \otimes 1} C \otimes C \xrightarrow{m} C \\
\psi_{(\mathcal{C},S)} &: = C \xrightarrow{\epsilon} I \xrightarrow{e} C.
\end{aligned}
$$

Then, obviously, $\mathbf{Hopf}\mathbb{C}$ is the equifier of (φ^1, ψ) and (φ^2, ψ) in the sense of [2] and thus an accessible category by [2, 2.76].
The same argument applies if instead of $\mathbf{Bimon}\mathbb{C}$ the categories ${}_{\mathbf{c}}\mathbf{Bimon}\mathbb{C}$ and ${}_{\mathbf{coc}}\mathbf{Bimon}\mathbb{C}$ are considered. □

4.2 Closure properties. It clearly would be interesting to know more about closure properties of the subcategories depicted in the following diagram. By 2.3 we already know that the embeddings in the bottom row are reflexive (labelled r) and coreflexive (c) respectively; they are also closed under directed and absolute colimits (see [10]).



We also have

Proposition **Hopf**$\mathbb{C}$ *is closed in* **Bimon**$\mathbb{C}$ *w.r.t. directed colimits and absolute limits and colimits and so are the other subcategories depicted in the above diagram.*

Proof: Given a directed colimit $\mathcal{C}_i \xrightarrow{\lambda_i} \mathcal{C}$ $(i \in I)$ in **Bimon**$\mathbb{C}$ where each $\mathcal{C}_i$ has an antipode S_i, the colimit property yields a morphism $\mathcal{C} \xrightarrow{S} \mathcal{C}^{\mathsf{op}}$. That S satisfies the relevant equations follows from a simple diagram chase which can be carried out in **C** since the forgetful functor **Bimon**$\mathbb{C} \longrightarrow$ **C** preserves directed colimits. The proof for absolute (co)limits is essentially the same. The same argument works for the commutative and the cocommutative case. □

Corollary *Also the subcategories in the top row of the diagram above are closed in* **Hopf**$\mathbb{C}$ *w.r.t. directed and absolute colimits and absolute limits.*

For additional information in the case of Hopf algebras see Section 5.

4.3 Universal constructions. The following interesting questions now arise naturally:

- Is **Hopf**$\mathbb{C}$ reflexive or coreflexiv in **Bimon**$\mathbb{C}$?
- Is **Hopf**$\mathbb{C}$ (co)monadic over **Mon**$\mathbb{C}$ or **Comon**$\mathbb{C}$?
- Is **Hopf**$\mathbb{C}$ a locally presentable category?

Clearly coreflexivity or comonadicity cannot be expected in general, as the simpliest case $\mathbb{C} =$ **Set** shows. These questions are related as follows.

Proposition *The following are equivalent*

(a) **Hopf**$\mathbb{C} \longrightarrow$ **Comon**$\mathbb{C}$ *has a left adjoint.*
(b) **Hopf**$\mathbb{C}$ *is finitary monadic over* **Comon**$\mathbb{C}$*.*
(c) **Hopf**$\mathbb{C}$ *is closed in* **Bimon**$\mathbb{C}$ *w.r.t. limits.*
(d) **Hopf**$\mathbb{C}$ *is reflexive in* **Bimon**$\mathbb{C}$*.*

Any of these conditions implies that **Hopf**$\mathbb{C}$ *is locally presentable.*

Proof: (a) implies (b) by the Beck–Paré Theorem since **Hopf**$\mathbb{C}$ is closed in **Bimon**$\mathbb{C}$ under absolute coequalizers (4.2) and **Bimon**$\mathbb{C} \to$ **Comon**$\mathbb{C}$ creates those (see 2.3). (b) implies (c) since the forgetful functors of both categories create limits. (c) implies (d) by the reflection theorem for locally presentable categories (see [2, 2.48]). (d) implies (a)statement by composition of adjoints. Given any one of these properties **Hopf**$\mathbb{C}$ is not only accessible (4.1) but also complete. □

For examples of when the above conditions might be satisfied see the following subsections.

Corollary *Assume that the equivalent conditions of the previous proposition are satisfied. Then the following hold:*

1. ${}_{\mathbf{c}}\mathbf{Hopf}\mathbb{C}$ *is reflexive in* $\mathbf{Hopf}\mathbb{C}$.
2. *The forgetful functor* ${}_{\mathbf{c}}\mathbf{Hopf}\mathbb{C} \to \mathbf{Comon}\mathbb{C}$ *has a left adjoint.*
3. ${}_{\mathbf{c}}\mathbf{Hopf}\mathbb{C}$ *is reflexive in* ${}_{\mathbf{c}}\mathbf{Bimon}\mathbb{C}$.
4. *The left adjoint to* ${}_{\mathbf{c}}\mathbf{Hopf}\mathbb{C} \to \mathbf{Comon}\mathbb{C}$ *maps cocommutative comonoids to cocommutative (and commutative) Hopf monoids.*

Proof: Ad 1. We only need to show that ${}_{\mathbf{c}}\mathbf{Hopf}\mathbb{C}$ is closed in $\mathbf{Hopf}\mathbb{C}$ w.r.t. limits, but this is obvious since $\mathbf{Hopf}\mathbb{C}$ and ${}_{\mathbf{c}}\mathbf{Bimon}\mathbb{C}$ are reflexive in $\mathbf{Bimon}\mathbb{C}$. 2. is trivial by composition of adjoints. 3. follows similarly to 1.

Concerning 4. consider the following diagram where $(-)^{\mathsf{op}}$ refers to taking the opposite comonoid structure. Since this is an isomorphism on both categories, the adjoint to $|-|$ will send an object fixed by $(-)^{\mathsf{op}}$ at the bottom row to one fixed by $(-)^{\mathsf{op}}$ at the top row.

$$\begin{array}{ccc} {}_{\mathbf{c}}\mathbf{Hopf}\mathbb{C} & \xrightarrow{(-)^{\mathsf{op}}} & {}_{\mathbf{c}}\mathbf{Hopf}\mathbb{C} \\ {\scriptstyle |-|'}\downarrow & & \downarrow{\scriptstyle |-|} \\ \mathbf{Comon}\mathbb{C} & \xrightarrow[(-)^{\mathsf{op}}]{} & \mathbf{Comon}\mathbb{C} \end{array}$$

□

4.4 Couniversal constructions. There is a result seemingly—but not formally—dual to Proposition 4.3 as follows:

Proposition *The following are equivalent*

(a) $\mathbf{Hopf}\mathbb{C} \longrightarrow \mathbf{Mon}\mathbb{C}$ *has a right adjoint.*
(b) $\mathbf{Hopf}\mathbb{C}$ *is comonadic over* $\mathbf{Mon}\mathbb{C}$.
(c) $\mathbf{Hopf}\mathbb{C}$ *is closed in* $\mathbf{Bimon}\mathbb{C}$ *w.r.t. colimits.*
(d) $\mathbf{Hopf}\mathbb{C}$ *is coreflexive in* $\mathbf{Bimon}\mathbb{C}$.

Any of these conditions implies that $\mathbf{Hopf}\mathbb{C}$ *is locally presentable.*

Proof: (a) implies (b) by the Beck–Paré Theorem since $\mathbf{Hopf}\mathbb{C}$ is closed in $\mathbf{Bimon}\mathbb{C}$ under absolute equalizers (4.2) and $\mathbf{Bimon}\mathbb{C} \to \mathbf{Comon}\mathbb{C}$ creates those (see 2.3). (b) implies (c) since the forgetful functors of both categories create colimits. (c) implies (d) by the Special Adjoint Functor Theorem (by 3. $\mathbf{Hopf}\mathbb{C}$ is cocomplete, thus locally presentable; now recall that a locally presentable category is cowellpowered and has a generator). (d) implies (a) by composition of adjoints. □

As above one now gets

Corollary *Assume that the equivalent conditions of the previous proposition are satisfied. Then the following hold:*

1. ${}_{\mathbf{coc}}\mathbf{Hopf}\mathbb{C}$ *is coreflexive in* $\mathbf{Hopf}\mathbb{C}$.
2. *The forgetful functor* ${}_{\mathbf{coc}}\mathbf{Hopf}\mathbb{C} \to \mathbf{Mon}\mathbb{C}$ *has a right adjoint.*
3. ${}_{\mathbf{coc}}\mathbf{Hopf}\mathbb{C}$ *is coreflexive in* ${}_{\mathbf{coc}}\mathbf{Bimon}\mathbb{C}$.
4. *The right adjoint to* ${}_{\mathbf{coc}}\mathbf{Hopf}\mathbb{C} \to \mathbf{Mon}\mathbb{C}$ *maps commutative monoids to commutative (and cocommutative) Hopf monoids.*

4.B The cartesian case

It has been shown in [11] that—given the general assumptions of this section—in the cartesian case, i.e., when Hopf monoids in $\mathbb{C}$ are just the group objects in $\mathbf{C}$, $\mathbf{Hopf}\mathbb{C}$ is a reflexive subcategory of $\mathbf{Bimon}\mathbb{C}$. By Remark 3.3 we thus obtain

4.5 Theorem *For every commutative unital ring R the following hold.*

1. ${}_{\mathbf{coc}}\mathbf{Hopf}_R$ *is a reflexive subcategory of* ${}_{\mathbf{coc}}\mathbf{Bialg}_R$.
2. *The forgetful functor* ${}_{\mathbf{coc}}\mathbf{Hopf}_R \to {}_{\mathbf{coc}}\mathbf{Coalg}_R$ *has a left adjoint.*
3. *The category* ${}_{\mathbf{coc}}\mathbf{Hopf}_R$ *is locally presentable.*

As mentioned before we cannot expect an example towards the application of 4.4 in the cartesian case.

4.C The module case

4.6 Free Hopf algebras. Concerning the question as to when the statements of Proposition 4.3 might be true, in the module case the following is known: statement (a) "free Hopf algebras exist" holds for $R = k$ a field by a classical result of Takeuchi's [14] (see Appendix); (d) "free adjunction of an antipode is possible" is shown to hold in [13] in case every element in an R-coalgebra is contained in a subcoalgebra which, as an R-module, is finitely generated and projective (which certainly is the case for R a field).

4.7 Cofree Hopf algebras. It is claimed in [12] that, for R a field, a cofree Hopf algebra can be constructed over every algebra; so this would be a situation, where 4.4 applies.

4.8 Commutative Hopf algebras. In 4.2 we have seen that in general the category ${}_{\mathbf{c}}\mathbf{Hopf}\mathbb{C}$ will be closed in ${}_{\mathbf{c}}\mathbf{Bimon}\mathbb{C}$ only with respect to absolut and directed colimits. In case of $\mathbb{C} = \mathbf{Mod}_R$ we can prove more.

Proposition *${}_{\mathbf{c}}\mathbf{Hopf}_R$ is closed in ${}_{\mathbf{c}}\mathbf{Bialg}_R$ w.r.t. all colimits.*

Proof: By 4.2 we only need to prove closure w.r.t. coequalizers and binary coproducts. For coequalizers the argument from 4.2 can be adopted, since the forgetful functor ${}_{\mathbf{c}}\mathbf{Bialg}_R \longrightarrow {}_{\mathbf{c}}\mathbf{Alg}_R \longrightarrow \mathbf{Mod}_R$ sends a coequalizer to a surjection (${}_{\mathbf{c}}\mathbf{Bialg}_R \longrightarrow {}_{\mathbf{c}}\mathbf{Alg}_R$ preserves coequalizers, and these are surjective in ${}_{\mathbf{c}}\mathbf{Alg}_R$ since this is a variety).

Let now $\mathcal{C} \xrightarrow{i_1} \mathcal{C}_1 + \mathcal{C}_2 \xleftarrow{i_2} \mathcal{C}_2$ be a coproduct in ${}_{\mathbf{c}}\mathbf{Bialg}_R$ with each $\mathcal{C}_i$ having an antipode S_i. Then $\mathcal{C}_1^a \xrightarrow{i_1} \mathcal{C}_1^a \otimes \mathcal{C}_2^a \xleftarrow{i_2} \mathcal{C}_2^a$ is a coproduct in ${}_{\mathbf{c}}\mathbf{Alg}_R$, and $i_1(c_1) = c_1 \otimes e_2,\ i_2(c_2) = e_1 \otimes c_2$ for all $c_1 \in C_1,\ c_2 \in C_2$ (see 2.1). Let $S\colon \mathcal{C}_1 + \mathcal{C}_2 \longrightarrow (\mathcal{C}_1 + \mathcal{C}_2)^{\mathsf{op}}$ be the morphism induced by the coproduct property. Since S_1 and S_2 are antipodes we can conclude (writing ${}^a\mathcal{C}_1 +^a \mathcal{C}_2 = (C_1 \otimes C_2, m, e, \mu, \epsilon)$).

$$m \circ (1 \otimes S) \circ \mu(x) = \epsilon \circ e(x) = m \circ (S \otimes 1) \circ \mu(x) \qquad (*)$$

for all $x \in \{c_1 \otimes e_2 \mid c_1 \in C_1\} \cup \{e_1 \otimes c_2 \mid c_2 \in C_2\}$.

By a well known result (see e.g. [5, 4.3.3]) it follows that (*) also holds for all $x \in \{c_1 \otimes c_2 \mid c_1 \in C_1, c_2 \in C_2\}$. Thus, the linear maps $C_1 \otimes C_2 \longrightarrow C_1 \otimes C_2$ which are to be equal coincide on a generating set. □

As an immediate corollary we get the following result which, despite its complete duality to Theorems 4.5 concerning the results stated, by no means appears as that theorems formal dual.

Theorem *For any commutative unital ring R the following hold:*

1. *${}_{\mathbf{c}}\mathbf{Hopf}_R$ is a locally presentable category.*
2. *${}_{\mathbf{c}}\mathbf{Hopf}_R$ is coreflexive in ${}_{\mathbf{c}}\mathbf{Bialg}_R$.*
3. *${}_{\mathbf{c}}\mathbf{Hopf}_R$ is comonadic over ${}_{\mathbf{c}}\mathbf{Alg}_R$.*

Proof: 1. is a consequence of the fact that ${}_{\mathbf{c}}\mathbf{Hopf}_R$—being accessible by 4.1—is also cocomplete by the proposition above. For 2. and 3. use the Special Adjoint Functor Theorem, whose assumptions are satisfied by 1., and the Beck–Paré Theorem. □

Remark Using different methods, results 2. and 3. of the theorem above have been obtained in [11] for von-Neumann-regular rings only.

5 Reflections and coreflections for Hopf algebras

We summarize our results concerning the existence of reflections and coreflections in the realm of the various categories of Hopf- and bialgebras in the following diagram

$$\begin{array}{ccccc}
{}_{\mathbf{coc}}\mathbf{Hopf}_R & \xhookrightarrow[(4.4)]{c^\star} & \mathbf{Hopf}_R & \xhookleftarrow[(4.3)]{r*} & {}_{\mathbf{c}}\mathbf{Hopf}_R \\
{\scriptstyle (4.5)\ r}\big\downarrow{\scriptstyle c^\star (4.4)} & & {\scriptstyle (4.4)\ c^\star}\big\downarrow{\scriptstyle r^*(4.3)} & & \big\downarrow{\scriptstyle r\ (4.8)} \\
{}_{\mathbf{coc}}\mathbf{Bialg}_R & \xhookrightarrow[(2.3)]{c} & \mathbf{Bialg}_R & \xhookleftarrow[(2.3)]{r} & {}_{\mathbf{c}}\mathbf{Bialg}_R
\end{array}$$

where r and c mark embeddings being unconditionally reflective and coreflective respectively while r^* and $c^\star$ mark embeddings whose (co)reflexivity depends on whether $\mathbf{Hopf}_R$ has free and cofree objects respectively. The labels in brackets show where to find the respective argument in this paper.

6 Appendix: Takeuchi's Free-Hopf-Algebra construction revisited

Takeuchi [14] proves the existence of a free Hopf algebra $H(C)$ over a coalgebra C in a four-step process generalizing the construction of a free group on a set.

Step 1 Given C construct $V = C + C^{\mathsf{op}} + C + C^{\mathsf{op}} + \ldots$ and define an obvious shift morphism $: V \longrightarrow V^{\mathsf{op}}$.
Step 2 Apply the "free functor" $T\colon \mathbf{Coalg}_R \longrightarrow \mathbf{Bialg}_R$ to $S\colon V \longrightarrow V^{\mathsf{op}}$ (one clearly might have changed steps one and two).
Step 3 Factor the bialgebra TV suitably such that the map induced by S on the quotient makes this quotient into a (universal) Hopf algebra.

Clearly, the crucial step is the third one. Takeuchi achieves this by

- forming the subset $U \subset TV$:

$$U = \{S * 1_{TV}(x) - e\epsilon(x) \mid x \in V\} \cup \{1_{TV} * S(x) - e\epsilon(x) \mid x \in V\}$$

  and forming the ideal $I = \langle U \rangle$ generated by U in TV,
- showing that I in fact is a coideal, such that $TV/_I$ becomes a bialgebra,
- showing that I is invariant under S, such that it induces a bialgebra map $\hat{S}\colon TV/_I \longrightarrow (TV/_I)^{\mathsf{op}}$.

It is then more or less obvious that $TV/_I$ is a Hopf algebra with antipode $\hat{S}$, such that it remains to prove

Step 4 $(TV/_I, \hat{S})$ is free over C (by means of $C \hookrightarrow V \longrightarrow TV \longrightarrow TV/_I$).

A slightly better categorical understanding of the crucial Steps 3 and 4 might be obtained by the following suggestion of an alternative description of the ideal I occuring in Step 3.

Consider the class $\mathcal{S}$ of all bialgebra homomorphisms $f\colon TV \longrightarrow H$, into a Hopf algebra H which respect S in the sense that

$$f \circ S = S_H \circ f,$$

where S_H is the antipode of H. Let J be the ideal

$$J = \bigcap_{f \in \mathcal{S}} \ker f.$$

We claim $J = I$. $J \subset I$ is obvious since the quotient map $q\colon TV \longrightarrow TV/_I$ clearly belongs to $\mathcal{S}$. It thus suffices to show $U \subset J$, i.e., to prove

$$\forall f \in \mathcal{S} \quad u \in U \Longrightarrow f(u) = 0.$$

But this is trivial since, e.g., for $u = S * 1(x) - e\epsilon(x)$, $x \in V$, we have $f\big(e\epsilon(x)\big) = e_H\epsilon_H(f(x))$ (since f is bialgebra homomorphism), $f\big(\hat{S} * 1(x)\big) = S_H * 1_H\big(f(x)\big)$ (since f, in addition, respects $\hat{S}$), and, thus,

$$f(u) = S_H * 1_H - e_H\epsilon_H\big(f(x)\big) = 0$$

since H is a Hopf algebra.

Observe that, in any case, J is invariant under S: if $x \in \bigcap_{f \in \mathcal{S}} \ker f$ then, for each $f \in \mathcal{S}$,

$$f\big(S(x)\big) = S_H\big(f(x)\big) = S_H(0) = 0.$$

Thus, as soon as $J = \bigcap_{f \in \mathcal{S}} \ker f$ is a coideal, we obtain a bialgebra $T/_J$ together with a linear map $\hat{S}\colon TV/_J \longrightarrow TV/_J$ induced by S. And this makes $TV/_J$ a Hopf algebra: the family of maps

$$m_f\colon TV/_{\bigcap_{f \in \mathcal{S}} \ker f} \longrightarrow H$$

induced by $f \in \mathcal{S}$ is clearly a point–separating family of linear maps and the category of Hopf algebras is closed under such families in the category of bialgebras as a simple diagram chase shows.

That, finally, $C \overset{i}{\hookrightarrow} V \overset{\eta}{\longrightarrow} TV \overset{q}{\longrightarrow} TV/_J$ makes $TV/_J$ the free Hopf algebra over C follows from the observation that any coalgebra map $g\colon C \longrightarrow H$ into a Hopf algebra H extends uniquely to a coalgebra map $V \xrightarrow{\hat{g}} H$ with $\hat{g} \circ i = g$ and the homomorphic extension of $\hat{g}$ belonging to $\mathcal{S}$.

We finally remark that, without resorting to Takeuchi's proof, the above method thus provides a free Hopf algebra construction for any ring R, for which kernels of bialgebra homomorphisms and intersections of those are coideals.

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