Free Internal Groups

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Abstract

In the first part of this note an elementary proof is given of the fact that algebraic functors, that is, functors induced by morphisms of Lawvere theories, have left adjoints provided that the category $\mathcal{K}$ in which the models of these theories take their values is locally presentable.

The main focus however lies on the special cases of the underlying functor of the category $\mathsf{Grp}(\mathcal{K})$ of internal groups in $\mathcal{K}$ and the embedding of $\mathsf{Grp}(\mathcal{K})$ into $\mathsf{Mon}(\mathcal{K})$, the category of monoids in $\mathcal{K}$: Here a unifying construction of the respective left adjoints is provided which not only works in case $\mathcal{K}$ is a locally presentable category but also when $\mathcal{K}$ is, for example, a particular category of topological spaces such as the category of Hausdorff or Tychonoff spaces or a cartesian closed topological category.

MSC 2000: Primary 18A40, Secondary 18C35, 18C10

Keywords: Algebraic functors; locally presentable categories; group objects; free constructions; reflections.

Introduction

The concept of *internal group* in a category $\mathcal{K}$ with finite products is of wide spread importance. Besides the most familiar examples of ordinary groups (= groups in the category of sets) and (Hausdorff) topological groups (= groups in the category of (Hausdorff) topological spaces) we only mention crossed modules (groups in the category of small categories) and cocommutative Hopf algebras (= groups in the category of cocommutative coalgebras). These examples already show that the categories $\mathcal{K}$ in which one might consider its category $\mathsf{Grp}(\mathcal{K})$ of internal groups are very different in nature: they might be cartesian closed or not, locally presentable or not, (mono)-topological or not. Nevertheless some natural questions have attracted attention in all these examples as, for example, the construction of

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free internal groups over $\mathcal{K}$-objects, in other words, if and how a left adjoint for the forgetful functor $\mathsf{Grp}(\mathcal{K}) \to \mathcal{K}$ can be constructed. Existence of such a left adjoint clearly is related to the problem whether or not $\mathsf{Grp}(\mathcal{K})$ is reflective in the category $\mathsf{Mon}(\mathcal{K})$ of internal monoids in $\mathcal{K}$.

The constructions available for these left adjoints are—not surprisingly in view of the different categorical properties of the various categories $\mathcal{K}$ of interest—of a very different nature, varying from explicit constructions to applications of Freyd's General Adjoint Functor Theorem.

We consider it reasonable to ask to what extent a common approach to these problems is possible. And that is what will be done in the second part of this note: A construction of free internal groups is provided which works at the same time for locally presentable base categories and, for example, for certain categories of topological spaces, thus giving in particular a new categorical proof of the existence of free Hausdorff topological groups (see [14] for other such proofs). The crucial step of this construction is a uniform construction of a reflection of $\mathsf{Mon}(\mathcal{K})$ into $\mathsf{Grp}(\mathcal{K})$ (Theorem 11 below), which can be seen as a generalization of the standard free group construction.

The technical tool employed in our construction is that of factorization structures of sources as in [1], which are shown to exist—though for quite different reasons—over the base categories considered.

The functors of which we thus want to construct a left adjoint are special instances of *algebraic functors*, that is of functors $\Phi^* \colon \mathsf{Alg}(\mathbb{T}, \mathcal{K}) \to \mathsf{Alg}(\mathbb{S}, \mathcal{K})$ induced by a morphism $\Phi \colon \mathbb{S} \to \mathbb{T}$ of Lawvere theories between their respective categories of models in $\mathcal{K}$ (see e.g. [11] or [4]). By Lawvere's classical result in [11] algebraic functors have a left adjoint in the case $\mathcal{K} = \mathbf{Set}$. Various attempts have been made to generalize Lawvere's result. For example, Φ^* also has a left adjoint, if $\mathcal{K}$ is a cartesian closed category or, more generally a so-called π-category (see [8] and [6]). In [21] factorization structures are used to find sufficient conditions for the existence of a left adjoint of an algebraic functor.

There are interesting cases however, where the category $\mathcal{K}$ fails to satisfy any of these conditions, but is locally presentable (for example the category of coalgebras over a commutative unital ring). In an introductory section we therefore give an elementary proof for the existence of left adjoints of algebraic functors over a locally presentable category $\mathcal{K}$, though the result itself (as the referee to this note kindly pointed out to me) also is contained in Lack and Rosický's forthcoming [9], where the problem of finding a left adjoint of an algebraic functor has very elegantly been reduced to general facts on left Kan extensions in enriched category theory.

We here prove moreover that in this case $\mathsf{Alg}(\mathbb{S}, \mathcal{K})$ is monadic over $\mathcal{K}$.

1 Algebras in a locally presentable category

The results presented below complement existing knowledge about universal algebra in a category different from **Set**: Our base category $\mathcal{K}$ doesn't need to have a closed structure as in [8] or [6] in order to make $\mathsf{Alg}(\mathbb{T},\mathcal{K})$ complete and cocomplete and algebraic functors right adjoint (instead we require local presentability; for an even more general approach see [9]). We include the first statement of the theorem below — though well known and also being a consequence of the second statement — because the proof of it provided within the chain of arguments for the second statement, for which it plays a crucial role, differs from other proofs we know.

1 Theorem *For any Lawvere theory $\mathbb{T}$ and any locally λ-presentable category $\mathcal{K}$*

1. *the category $\mathsf{Alg}(\mathbb{T},\mathcal{K})$ of $\mathbb{T}$-models in $\mathcal{K}$ is locally λ-presentable, and*
2. *its underlying functor into $\mathcal{K}$ is monadic and preserves λ-directed colimits.*

Proof: Let $\mathbb{T}$ be a Lawvere theory and $H_{\mathbb{T}}\colon \mathcal{K} \to \mathcal{K}$ the polynomial functor corresponding to $\mathbb{T}$ on the category $\mathcal{K}$, that is

$$H_{\mathbb{T}}(A) = \sum_{n\in\mathbb{N}}\Big(\sum_{\omega\in\mathbb{T}(n,1)} A^n\Big)$$

$\mathsf{Alg}(\mathbb{T},\mathcal{K})$ now is a full concrete subcategory of the category $\mathsf{Alg}(H_{\mathbb{T}})$ of $H_{\mathbb{T}}$ (functor) algebras. In fact, the assignment $F \mapsto (F1,\alpha)$, with $\alpha\colon H_{\mathbb{T}}(F1) \to F(1)$ acting on the summand $F(n) = F(1)^n$ corresponding to $\omega \in \mathbb{T}(n,1)$ as $F(\omega)$, defines a (faithful) functor from $\mathsf{Alg}(\mathbb{T},\mathcal{K})$ to $\mathsf{Alg}(H_{\mathbb{T}})$ commuting with the canonical underlying functors; and this functor is a full embedding since $\mathbb{T}$-models F are determined by their values $F(\omega)$ on morphisms $\omega\colon n \to 1$, and a family $\lambda = (\lambda_n\colon F(n) \to G(n))_{n\in\mathbb{N}}$ is a morphism between the $\mathbb{T}$-models F and G provided that, for each of theses ω, the diagram

$$\begin{array}{ccc} F(n) & \xrightarrow{\lambda_n} & G(n) \\ {\scriptstyle F(\omega)}\downarrow & & \downarrow{\scriptstyle G(\omega)} \\ F(1) & \xrightarrow[\lambda_1]{} & G(1) \end{array}$$

commutes.

The polynomial functor $H_{\mathbb{T}}$ preserves λ-directed colimits provided that these commute with finite products in $\mathcal{K}$, which certainly is the case if $\mathcal{K}$ is locally λ-presentable. Thus, the forgetful functor $\mathsf{Alg}(H_{\mathbb{T}}) \to \mathcal{K}$ not only creates limits, and absolute colimits (and has a left adjoint) but it also creates λ-directed colimits (see [2]). Moreover, $\mathsf{Alg}(H_{\mathbb{T}})$ is a locally λ-presentable category by [3, 2.75].

Since limits and colimits in the category $\mathcal{K}^{\mathbb{T}}$ of all functors from $\mathbb{T}$ to $\mathcal{K}$ are formed pointwise the category $\mathsf{Alg}(\mathbb{T},\mathcal{K})$ is closed in $\mathcal{K}^{\mathbb{T}}$ under limits and those colimits which commute with finite products in $\mathcal{K}$; in particular the forgetful functor $\mathsf{Alg}(\mathbb{T},\mathcal{K}) \to \mathcal{K}$ preserves limits and λ-directed colimits, and so then does the embedding of $\mathsf{Alg}(\mathbb{T},\mathcal{K})$ into $\mathsf{Alg}(H_{\mathbb{T}})$.

It now follows from [3, 2.48] that $\mathsf{Alg}(\mathbb{T},\mathcal{K})$ is a locally λ-presentable category and that its underlying functor has a left adjoint. This functor then clearly preserves λ-directed colimits and is monadic (in the sense of [5]) by the Beck-Paré Theorem (see e.g. [5, 4.4.4]). □

As a corollary we get in a direct way the following particular instance of a more general result of [9].

2 Theorem *Let $\Phi\colon \mathbb{S} \to \mathbb{T}$ be a morphism of Lawvere Theories. Then, for every locally presentable category $\mathcal{K}$, the algebraic functor $\Phi^*\colon \mathsf{Alg}(\mathbb{T},\mathcal{K}) \to \mathsf{Alg}(\mathbb{S},\mathcal{K})$ has a left adjoint.*

Proof: Clearly Φ^* commutes with the underlying functors both of which preserve and reflect limits and λ-directed colimits by the above theorem. Hence Φ^* is a limit and λ-directed colimit preserving functor between locally λ-presentable categories and, thus, has a left adjoint by [3, 2.45]. □

3 Remark Our result on left adjoints of algebraic functors, which rather should be seen as a triangle theorem, differs from similar results (see e.g. [21]) in that we don't need to require productivity of a suitable class of morphisms. A suitable such class seems to exist in a locally presentable category $\mathcal{K}$ only in very special cases, e.g., when $\mathcal{K}$ is a variety.

It seems unlikely however, that our conditions on $\mathcal{K}$ are sufficient to guarantee structure-semantics adjointness, as in the cases considered in [8, 6].

2 Internal groups

2.1 Internal monoids and groups

As from now $\mathcal{K}$ always denotes a category with finite products.

4 Definition A *monoid in $\mathcal{K}$* is a triple $(M, m\colon M \times M \to M, e\colon 1 \to M)$ with an object M and morphisms m and e in $\mathcal{K}$ making the following diagrams commute:

$$
\begin{array}{ccc}
M \times M \times M & \xrightarrow{m \times \mathsf{id}_M} & M \times M \\
{\scriptstyle \mathsf{id}_M \times m}\big\downarrow & & \big\downarrow{\scriptstyle m} \\
M \times M & \xrightarrow[m]{} & M
\end{array}
$$

$$
\begin{array}{ccc}
M \times M & \xrightarrow{m} & M \\
& {\scriptstyle \mathsf{id}_M \times e}\searrow & \big\| \\
& & M \times 1
\end{array}
\qquad
\begin{array}{ccc}
M \times M & \xrightarrow{m} & M \\
& {\scriptstyle e \times \mathsf{id}_M}\searrow & \big\| \\
& & 1 \times M
\end{array}
$$

A homomorphism of monoids $(M, m, e) \to (M', m', e')$ is a $\mathcal{K}$-morphism $f\colon M \to M'$ such that $m' \circ (f \times f) = f \circ m$ and $f \circ e = e'$.

The dual monoid of $M = (M, m, e)$, that is, the monoid $(M, m \circ \sigma, e)$ with $\sigma\colon M \times M \to M \times M$ the switch morphism, is denoted by M^{op}. Every monoid homomorphism $f\colon (M, m, e) \to (M', m', e')$ then also is a monoid homomorphism $f\colon (M, m, e)^{\mathsf{op}} \to (M', m', e')^{\mathsf{op}}$ which, for clarity, will be denoted by f^{op}.

By $\mathsf{Mon}(\mathcal{K})$ we denote the category of monoids. Clearly, $\mathsf{Mon}(\mathcal{K})$ is equivalent to the category of models in $\mathcal{K}$ of the Lawvere theory of monoids.

It is well known that $\mathsf{Mon}(\mathcal{K})$ is monadic over $\mathcal{K}$ provided that its forgetful functor into $\mathcal{K}$ has a left adjoint. Concerning the existence of free monoids we will need the following.

5 Lemma *The forgetful functor $U\colon \mathsf{Mon}(\mathcal{K}) \longrightarrow \mathcal{K}$ has a left adjoint provided that $\mathcal{K}$ satisfies one of the following conditions:*

(CC) $\mathcal{K}$ has all finite and countable coproducts and, for each K in $\mathcal{K}$, the functor $K \times -\colon \mathcal{K} \longrightarrow \mathcal{K}$ preserves these.

(DC) $\mathcal{K}$ has λ–directed colimits for some regular cardinal λ and the functor $(-)^2\colon \mathcal{K} \longrightarrow \mathcal{K}$, that is, $K \mapsto K \times K$, preserves these; $\mathcal{K}$ also has binary coproducts.

That condition (CC) implies the existence of free monoids is well known (see [12]). It is satisfied for example by every epireflective subcategory of **Top**, closed under topological sums, thus in particular by the categories $\mathbf{Top}_2$ of Hausdorff spaces and **Tych**, the category of Tychonoff spaces; also every cocomplete cartesian closed category has this property. Concerning condition (DC) see [16]. (DC) is, in particular, satisfied by every locally presentable category; note that in this case the existence of free monoids would also follow from Theorem 1.

6 Definition A *group in* $\mathcal{K}$ is a monoid (M, m, e) in $\mathcal{K}$ equipped with a $\mathcal{K}$–morphism $i\colon M \longrightarrow M$, called *inversion*, such that

$$M \xrightarrow{\Delta} M \times M \xrightarrow{i \times \mathsf{id}_M} M \times M \xrightarrow{m} M = M \xrightarrow{!} 1 \xrightarrow{e} M$$

and

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\mathsf{id}_M \times i} M \times M \xrightarrow{m} M = M \xrightarrow{!} 1 \xrightarrow{e} M.$$

We denote the full subcategory of $\mathsf{Mon}(\mathcal{K})$ spanned by all groups in $\mathcal{K}$ by $\mathsf{Grp}(\mathcal{K})$. $\mathsf{Grp}(\mathcal{K})$ is equivalent to the category of algebras of the Lawvere theory of groups.

Note that, if (M, m, e, i) is a group in $\mathcal{K}$, then the inversion i is a monoid homomorphism $(M, m, e, i) \to (M, m, e, i)^{\mathsf{op}}$.

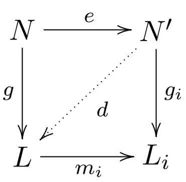
The construction of left adjoints for $\mathsf{Grp}(\mathcal{K})$'s forgetful functor into $\mathcal{K}$ and its embedding into $\mathsf{Mon}(\mathcal{K})$ (which essentially are algebraic functors) we are going to present below requires (in addition to the existence of free monoids) a certain factorization structure on $\mathsf{Mon}(\mathcal{K})$. We therefore recall the following definition from [1].

7 Definition A pair $(E, \mathcal{M})$ consisting of collections E of epimorphisms and $\mathcal{M}$ of sources $(K, (m_i)_{i \in I})$ in $\mathcal{K}$, both closed under composition with isomorphisms, is called a *factorization structure* of sources on $\mathcal{K}$, provided that

1. each source $(M, (M \xrightarrow{f_i} M_i)_{i \in I})$ has an $(E, \mathcal{M})$-factorization, i.e., there exists some $M \xrightarrow{e} L \in E$ and a source $(L, (L \xrightarrow{m_i} M_i)_{i \in I} \in \mathcal{M}$ such that, for each $i \in I$,

$$M \xrightarrow{f_i} M_i = M \xrightarrow{e} L \xrightarrow{m_i} M_i$$

2. E is orthogonal to $\mathcal{M}$, that is, each commutative diagram



with $e \in E$ and $(L, (m_i)_{i \in I}) \in \mathcal{M}$ has a unique diagonal d.

Note that here I might be empty or even a proper class. In case of an empty source condition 1. above simply means: there exists an epimorphism $M \xrightarrow{e} L \in E$ such that the empty cone on L belongs to $\mathcal{M}$; in other words, such that every $N \xrightarrow{g} L$ factorizes through every $N \xrightarrow{r} N' \in E$. The structure *(surjective maps, point-separating families of maps)* on **Set** is a typical example. Here only 1 and $\emptyset$ carry empty point-separating source.

Concerning the existence of such structures, the following results will be useful:

8 Lemma *Let $\mathcal{K}$ have an $(E, \mathcal{M})$–factorization structure for sources. Then* $\mathsf{Mon}(\mathcal{K})$ *has an $(U^{-1}[E], U^{-1}[\mathcal{M}])$–factorization structure, created by U, provided that, for each $q \in E$, also $q \times q \in E$.*

Proof: If $f_i\colon (M, m, e) \to (M_i, m_i, e_i)$ $(i \in I)$ is a non-empty family of monoid homomorphisms and $M \xrightarrow{q} H \xrightarrow{s_i} M_i$ its $(E, \mathcal{M})$–factorization in $\mathcal{K}$ the required multiplication on H is obtained as the "diagonal" in the diagram

$$\begin{array}{ccccc}
M \times M & \xrightarrow{q \times q} & H \times H & \xrightarrow{s_i \times s_i} & M_i \times M_i \\
{\scriptstyle m}\downarrow & & \vdots & & \downarrow{\scriptstyle m_i} \\
M & \xrightarrow[q]{} & H & \xrightarrow[s_i]{} & M_i
\end{array}$$

The neutral element of H is obtained analogously and one now easily checks, using the fact that $q \times q$ is an epimorphism, that the resulting structure on H defines a monoid and also that $(U^{-1}[E], U^{-1}[\mathcal{M}])$ has the orthogonality property of a factoriation structure.

The case of empty families is trivial: the terminal object of $\mathcal{K}$ is terminal in $\mathsf{Mon}(\mathcal{K})$ by means of its unique (trivial) monoid structure. □

9 Example The previous lemma applies, for example, in the following cases:

1. $\mathcal{K}$ is monotopological (see [1]) over **Set**. Here $U^{-1}[E]$ consists of all morphisms with surjective underlying map, while $U^{-1}[\mathcal{M}]$ consists of all initial point-separating families.

2. $\mathcal{K}$ is a quasivariety. Here $U^{-1}[E]$ consists of all surjective homomorphisms, while $U^{-1}[\mathcal{M}]$ consists of all point-separating families of homomorphisms.

A similar result holds when $\mathcal{K}$ is locally presentable. By Theorem 1 then $\mathsf{Mon}(\mathcal{K})$ is locally presentable as well and has a right adjoint forgetful functor into $\mathcal{K}$. Since locally presentable categories, being co-wellpowered and cocomplete, carry an $(Epi, extremal\ monosource)$-factorization structure (see e.g. [1, 15.17]) and right adjoints preserve monosources, we thus have

10 Lemma *Let $\mathcal{K}$ be locally presentable. Then* $\mathsf{Mon}(\mathcal{K})$ *has an $(E, \mathcal{M})$–factorization structure for sources such that, for each $(m_i)_I \in \mathcal{M}$, the family $(Um_i)_I$ is jointly monomorphic in $\mathcal{K}$.*

2.2 Free internal groups

In view of the previous results the following construction is applicable quite universally. It can be seen as a direct generalization of the case $\mathcal{K} = \mathbf{Set}$. Admittedly one would prefer conditions on $\mathcal{K}$ rather then on $\mathsf{Mon}(\mathcal{K})$ in the theorem below; the diversity of reasons however, for which $\mathsf{Mon}(\mathcal{K})$ might satisfy these assumptions, seems to make that an unreasonable wish.

11 Theorem *Let $U\colon \mathsf{Mon}(\mathcal{K}) \to \mathcal{K}$ have a left adjoint F and let $\mathsf{Mon}(\mathcal{K})$ have an $(E, \mathcal{M})$–factorization system with $(Um_i)_I$ jointly monomorphic for each $(m_i)_I \in \mathcal{M}$. For every monoid (M, m, e) which admits a coproduct $M \xrightarrow{\iota_1} M^* \xleftarrow{\iota_2} M^{\mathsf{op}}$ in $\mathsf{Mon}(\mathcal{K})$ there exists an E-quotient $M^* \xrightarrow{q} G_M$ such that $M \xrightarrow{\iota_1} M^* \xrightarrow{q} G_M$ is a reflection of M into $\mathsf{Grp}(\mathcal{K})$.*

In particular, G_{FK} is a free internal group over K, for each $\mathcal{K}$-object K.

Proof: Let $I^*\colon M^* \longrightarrow M^{*\mathsf{op}}$ be the homomorphism with

$$I^* \circ \iota_1 = \iota_2^{\mathsf{op}},\ I^* \circ \iota_2 = \iota_1^{\mathsf{op}}. \tag{1}$$

Denote by $\mathcal{S}$ the class of all homomorphisms $f\colon M^* \to G$ where G is a group in $\mathcal{K}$ with inversion i such that the diagram below commutes.

$$\begin{array}{ccc} M^* & \xrightarrow{\quad f \quad} & G \\ {\scriptstyle I^*}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ M^{*\mathsf{op}} & \xrightarrow[\quad f^{\mathsf{op}} \quad]{} & G^{\mathsf{op}} \end{array}$$

By a slight abuse of notation we will by $\mathcal{S}$ also denote the source $(M^*, \mathcal{S})$.

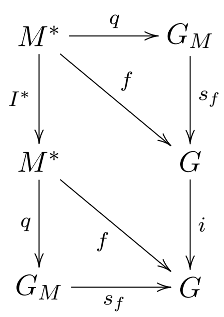
By assumption there is a factorization of $\mathcal{S}$ as

$$f = M^* \xrightarrow{q} G_M \xrightarrow{s_f} G \quad (f \in \mathcal{S})$$

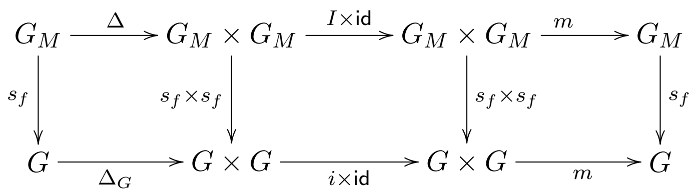
with the family (s_f) being jointly monomorphic and $q \in E$, thus an epimorphism. We are going to show

1. G_M is a group in $\mathcal{K}$,
2. $M \xrightarrow{\iota_1} M^* \xrightarrow{q} G_M$ is a reflection for M into $\mathsf{Grp}(\mathcal{K})$.

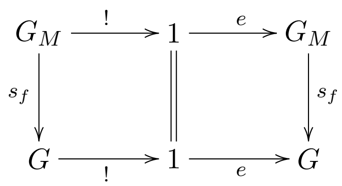
By definition of $\mathcal{S}, q$ and s_f the following diagram commutes for each $f \in \mathcal{S}$



and, thus, admits a unique diagonal $I\colon G_M \longrightarrow G_M$ (omitting the obvious use of $()^{\mathsf{op}}$). In order to show that I is a group inversion on G_M consider the diagrams in $\mathcal{K}$



and



both of which obviously commute (as does the first one with $1 \times I$ instead of $I \times 1$). It follows that, for each $f \in \mathcal{S}$, we have

$$\begin{aligned} s_f \circ (m \circ (I \times \mathsf{id}) \circ \Delta) &= s_f \circ (e \circ\, !) \\ &= s_f \circ (m \circ (\mathsf{id} \times I) \circ \Delta) \end{aligned}$$

since each G is a group.

Now the family $(s_f)_{f \in \mathcal{S}}$ is jointly monomorphic in $\mathcal{K}$, too (right adjoints preserve jointly monomorphic families) and 1. is proved.

For proving condition 2. we note first that every homomorphism $h\colon M \to G$ into a group G admits a unique extension $\bar{h}\colon M^* \to G \in \mathcal{S}$ with $\bar{h} \circ \iota_1 = h$. Simply put

$$\bar{h} \circ \iota_1 = h, \ \bar{h} \circ \iota_2 = i \circ h^{\mathsf{op}}. \tag{2}$$

To prove that $\bar{h} \in \mathcal{S}$ it suffices to show (again omitting $(-)^{\mathsf{op}}$)

α) $(i \circ \bar{h}) \circ \iota_1 = (\bar{h} \circ I^*) \circ \iota_1$

β) $(i \circ \bar{h}) \circ \iota_2 = (\bar{h} \circ I^*) \circ \iota_2$

Indeed, by first employing equations (2) and then equation (1) we get

$$i \circ \bar{h} \circ \iota_1 = i \circ h = \bar{h} \circ \iota_2 = \bar{h} \circ I^* \circ \iota_1$$

while the same equations together, with the fact that the order of a group-inversion i is 2, gives

$$i \circ \bar{h} \circ \iota_2 = i \circ i \circ h = h = \bar{h} \circ \iota_1 = \bar{h} \circ I^* \circ \iota_2$$

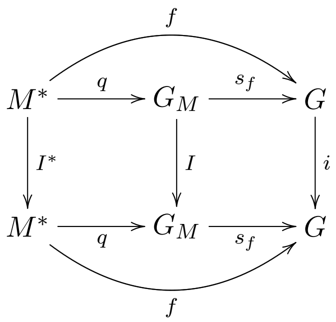
This extension of h indeed is unique since $\bar{h} \circ i_1 = h$ and $\bar{h} \in \mathcal{S}$ imply

$$\begin{aligned}
\bar{h} \circ \iota_2 &= i \circ i \circ \bar{h} \circ \iota_2 & \\
&= i \circ \bar{h} \circ I^* \circ \iota_2 & \text{since } \bar{h} \in \mathcal{S} \\
&= i \circ \bar{h} \circ \iota_1 & \text{since } \bar{h} \circ i_1 = h \\
&= i \circ h. & \text{by Equation (2)}
\end{aligned}$$

Now the given factorization of $\mathcal{S}$ provides an $m_{\bar{h}} \colon G_M \longrightarrow G$ with $\bar{h} = m_{\bar{h}} \circ q$ and we obtain

$$h = \bar{h} \circ \iota_1 = m_{\bar{h}} \circ (q \circ \iota_1).$$

It remains to prove that $m_{\bar{h}}$ is unique for a factorization of h over $q \circ \iota_1$. Assume $h = h^* \circ (q \circ \iota_1)$ for some homomorphism $h^* \colon G_M \longrightarrow G$. If q would be in $\mathcal{S}$ we were done by uniqueness of h's extension to an element of $\mathcal{S}$ and the fact that q is epic. Thus only $q \in \mathcal{S}$ remains to be shown, but this follows from commutativity of the following diagram (for each $f \in \mathcal{S}$) by jointly cancelling the family (s_f).



□

If, under the general assumptions of the previous theorem, $M + M^{\mathsf{op}}$ does not exist in general in $\mathsf{Mon}(\mathcal{K})$ and therefore a reflection of $\mathsf{Mon}(\mathcal{K})$ into $\mathsf{Grp}(\mathcal{K})$ cannot be constructed and, consequently, also no left adjoint of the forgetful functor $\mathsf{Grp}(\mathcal{K}) \to \mathcal{K}$ by simple composition of adjoints, one still might be able to obtain at least the latter by the above construction as follows:

Assume that, for some object K, the coproduct $K + K$ in $\mathcal{K}$ exists; then there exists a free group–object over K: simply put $M = FK$, the free monoid over K, and observe that $M + M^{\mathsf{op}} = FK + (FK)^{\mathsf{op}} \cong FK + FK \cong F(K + K)$. Now apply the previous theorem. We thus get, as a corollary,

12 Proposition *Let $U\colon \mathsf{Mon}(\mathcal{K}) \to \mathcal{K}$ satisfy the assumptions of Theorem 11 and let $\mathcal{K}$ have binary coproducts. Then the forgetful functor $\mathsf{Grp}(\mathcal{K}) \to \mathcal{K}$ has a left adjoint.*

2.3 Applications

2.3.1 When K is locally presentable

If $\mathcal{K}$ be a locally λ-presentable category, it follows independently from both, Section 1 and the previous results, that the category $\mathsf{Grp}(\mathcal{K})$ is locally λ-presentable, reflexively embedded in $\mathsf{Mon}(\mathcal{K})$, and λ-ary monadic over $\mathcal{K}$. Moreover, the construction of Theorem 11 applies. Examples of this situation include

a) $\mathsf{Grp}(\mathbf{Cat})$, the category of groups in the category of small categories, which is the same as the category of internal categories in the category of groups, or the category of crossed modules (see [12]).

   Note that reflexivity of $\mathsf{Grp}(\mathbf{Cat})$ in $\mathsf{Mon}(\mathbf{Cat})$ also is a consequence of the facts that the embedding is an an algebraic functor (over $\mathbf{Cat}$) and that $\mathbf{Cat}$ is cartesian closed (see [8, 6]).

b) $\mathsf{Grp}({}_{coc}\mathbf{Coalg}_R)$, the category of groups in the category of cocommutative coalgebras over a commutative ring R. This is the category ${}_{coc}\mathbf{Hopf}_R$ of cocommutative Hopf algebras over R (sometimes also called "formal groups over R" [7]). ${}_{coc}\mathbf{Hopf}_R$ thus, in particular, is reflexive in the category of cocommutative bialgebras over R. For R a field the existence of left adjoint to ${}_{coc}\mathbf{Hopf}_R \longrightarrow {}_{coc}\mathbf{Coalg}_R$ has already been mentioned (without indicating a proof) by Sweedler [19]. For local presentability of ${}_{coc}\mathbf{Coalg}_R$ see [15].

c) For the sake of completeness only we add the trivial examples

   - $\mathsf{Grp}(\mathbf{Grp})$, the category of groups in the category of groups which is $\mathbf{Ab}$, the category of Abelian groups, by the Eckmann-Hilton argument. It coincides with $\mathsf{Mon}(\mathbf{Grp})$.
   - $\mathsf{Grp}(\mathbf{Set}) = \mathbf{Grp}$ is clearly reflexive in the category $\mathbf{Mon}$ of monoids over $\mathbf{Set}$ and has free objects.

2.3.2 Commutative Hopf algebras

The category ${}_{c}\mathbf{Hopf}_R$ of commutative Hopf algebras over a commutative ring R (also called the category of affine groups or affine group schemes [7, 20]) is known to be the dual of the category of groups in the dual of the category ${}_{c}\mathbf{Alg}_R$ of commutative algebras over R, that is,

$${}_{c}\mathbf{Hopf}_R^{\mathsf{op}} = \mathsf{Grp}({}_{c}\mathbf{Alg}_R^{\mathsf{op}}).$$

Further, $\mathsf{Mon}({}_c\mathbf{Alg}_R^{\mathsf{op}}) = (\mathbf{Comon}_c\mathbf{Alg}_R)^{\mathsf{op}} = {}_c\mathbf{Bialg}_R^{\mathsf{op}}$, the dual of the category of commutative bialgebras over R. Since ${}_c\mathbf{Bialg}_R \longrightarrow {}_c\mathbf{Alg}_R$ has a right adjoint (see [16]), $\mathsf{Mon}({}_c\mathbf{Alg}_R^{\mathsf{op}}) \longrightarrow {}_c\mathbf{Alg}_R^{\mathsf{op}}$ has a left adjoint. ${}_c\mathbf{Alg}_R^{\mathsf{op}}$ – being the dual of locally presentable category – is cocomplete and co-wellpowered; in particular it carries a ((strong) epi, monosource)–factorization structure, the epis in ${}_c\mathbf{Alg}_R^{\mathsf{op}}$ being injective maps. Since, in ${}_c\mathbf{Alg}_R^{\mathsf{op}}$ the product $- \times -$ is given by tensor product $- \otimes -$ we can apply Theorem 11, using Lemma 8, provided that we require R to be a von Neumann–regular ring. We thus obtain

13 Proposition *Let R be a von–Neumann–regular ring. Then ${}_c\mathbf{Hopf}_R$ is coreflexive in ${}_c\mathbf{Bialg}_R$ and comonadic over ${}_c\mathbf{Alg}_R$.*

2.3.3 Topological groups

On each of the categories **Top** of topological spaces, $\mathbf{Top}_2$ of Hausdorff spaces and **Tych** of Tychonoff spaces each functor $X \times -$ preserves coproducts. Thus, by the starting remarks of Section 2, $\mathsf{Mon}(\mathbf{Top})$, $\mathsf{Mon}(\mathbf{Top}_2)$ and $\mathsf{Mon}(\mathbf{Tych})$ have free monoids. In view of Example 9 (1), the general assumptions of Theorem 11 are therefore satisfied. By means of Corollary 12 we obtain a new construction of free Hausdorff topological groups over any Hausdorff or Tychonoff space (see e.g. [14] for a survey of other constructions).

The construction above can also be applied for every "convenient" (that is cartesian closed) topological category (see e.g. [10]), where the first of the Examples 9 together with condition (CC) applies. For the same reason also the corresponding results of [18] are covered (and generalized) by our approach.

References

[1] Adámek, J., Herrlich H., and Strecker, G.E., *Abstract and Concrete Categories.* John Wiley, New York (1990).

[2] Adámek, J. and H.–E. Porst, On varieties and covarieties in a category. *Math. Struct. in Comp. Science* **13** (2003), 201–232.

[3] Adámek, J. and J. Rosický, *Locally Presentable and Accessible Categories.* Cambridge University Press, Cambridge (1994).

[4] Barr, M., and C. Wells, *Toposes, Triples, and Theories*, Grundlehren der math. Wissenschaften 278, Springer-Verlag, 1985.

[5] Borceux, F., *Handbook of Categorical Algebra 2*, Cambridge University Press, Cambridge (1994).

[6] Borceux, F. and B. Day, Universal algebra in a closed category. *J. Pure Appl. Algebra* **16** (1980), 133-147.

[7] Dăscălescu, S., et al., *Hopf Algebras - An Introduction.* Marcel Dekker, New York – Basel (2001).

[8] Gray, J., *Universal algebra in a cartesian closed category.* E.T.H. Zürich (1975).

[9] Lack, S. and J. Rosický, Notions of Lawvere theory, to appear in *Appl. Categ. Structures.*

[10] Lamartin, W.F., *On the Foundations of k-group Theory.* Dissertationes Math. CXLVI, Warsaw (1977).

[11] Lawvere, F. W., *Functorial Semantics of Algebraic Theories*, PhD Thesis, Columbia University, 1963. (Available with commentary as TAC Reprint 5.)

[12] Mac Lane, S., *Categories for the working mathematician.* 2nd ed., Springer, New York (1998).

[13] Markov, A.A., On Free Topological Groups. *Izv. Akad. Nauk SSSR Ser. Mat.* **21** (1945), 3-64 [*Amer. Math. Soc. Transl.* **30** (1950), 11–88]

[14] Porst, H.–E., On the Existence and Structure of Free Topological Groups. in: H. Herrlich and H.-E. Porst (eds.), *Category Theory at Work*, Heldermann-Verlag, Berlin (1991), 165–176.

[15] Porst, H.–E., On Corings and Comodules. *Arch. Math. (Brno)* **42** (2006), 419–425.

[16] Porst, H.–E., On Categories of Monoids, Comonoids, and Bimonoids. *Quaestiones Math.* **31** (2008), 127–139.

[17] Samuel, P., On Universal Mappings and Free Topological Groups. *Bull. Amer. Math. Soc.* **54** (1948), 591–598.

[18] Seal, G.J., Free Modules over Cartesian Closed Toplogical Categories. *Appl. Categ. Structures* **13** (2005), 181–187

[19] Sweedler, M.E., *Hopf Algebras.* Benjamin, New York (1969).

[20] Street, R., *Quantum Groups.* Cambridge University Press, Cambridge (2007).

[21] Tholen, W., *Relative Bildzerlegungen und algebraische Kategorien*, Dissertation, Universität Münster (1974).