Introduction to pattern formation

Assignment-set 4

Strict deadline: 25 May 2010, 11:15h

1. Exercise 2.4 from the book.
2. A model for enzyme kinetics is given by

$$\begin{aligned} \frac{dy}{dt} &= -y + (y+c)z \\ \varepsilon\frac{dz}{dt} &= y - z(y+1) \end{aligned}$$

   Here, $0 < \varepsilon \ll 1$ and $0 < c < 1$ are constants. Consider the system for general initial conditions $(y(0), z(0))$. For this we will need to use both t and the fast time scale $\tau = \frac{t}{\varepsilon}$ to obtain a matched asymptotic expansion representing the solutions up to $\mathcal{O}(\varepsilon)$.

   (a) For both y and z substitute an asymptotic expansion; set $y = y_0 + \varepsilon y_1 + \cdots$ and $z = z_0 + \varepsilon z_1 + \cdots$. Give the resulting equations up to $\mathcal{O}(\varepsilon)$, in other words the equations for y_0, z_0, y_1 and z_1.

   (b) Solve the equations for y_0 and z_0. Does this give a solution to the posed problem?

   (c) Introduce the fast time variable $\tau = \frac{t}{\varepsilon}$ and write $y = Y(\tau)$ and $z = Z(\tau)$. Again, expand bot Y and Z in asymptotic expansions and give the resulting equations up to $\mathcal{O}(\varepsilon)$.

   (d) Solve the equations for Y_0 and Z_0.

   (e) Match the the solutions Y_0 and Z_0 to the solutions y_0 and z_0.

   (f) Solve the equations at the $\mathcal{O}(\varepsilon)$-level; for both y_1 and z_1 and for Y_1 and Z_1.

   (g) Match the solutions at the $\mathcal{O}(\varepsilon)$-level.

   (b) Can you obtain a higher order expansion?

3. For the following two cases, use the center manifold approach to compute the reduced vector field for the instability of $u = 0$ at $L = \pi$.

   (a) $u_t = u_{xx} + L^2 \sin(u),\ x \in (0,1),\ u(0) = u(1) = 0.$

   (b) $u_t = u_{xx} + L^2(u + u^2),\ x \in (0,1),\ u(0) = u(1) = 0.$

4. Excercise 2.7(a)-(c) from the book. In addition:

   (d) Give the formula for the critical eigenmode, and describe what kind of solutions you expect to see just beyond instability.

   (e) Consider the same problem, but now posed on the real line $x = \mathbb{R}$ and

      (i) Determine the conditions on D, d, a, b for a Turing instability: an instability of the solution, but stability against homogeneous perturbations.

      (ii) Compute the critical wave number k and sketch the spectrum at onset of instability in the (wave number, real part of spectrum)-plane.

5. Consider the scalar Swift-Hohenberg type equation

$$u_t = -u_{xxxx} - \alpha u_{xx} + \beta u - u^3, \; x \in \mathbb{R}.$$

(Since function spaces are unspecified the following are, strictly speaking, formal calculations.)

(a) Find the linearisation in the homogeneous steady state $u \equiv 0$ and use Fourier transform to compute the spectrum.

(b) Find all values of $\alpha, \beta \in \mathbb{R}$ for

i. stable spectrum,

ii. onset of instability against homogeneous perturbations,

iii. onset of Turing instabililty, i.e., unstable against non-homogeneous perturbations and stable against homogeneous perturbations. Sketch the shape of the spectrum in the (real part of spectrum, Fourier wave number)-plane. Describe what kind of solutions you expect to see just beyond instability.

Plot the resulting stability boundaries in the same (α, β)-plane.

(c) Show that adding a term cu_x to the right-hand side only changes the imaginary part of the spectrum, i.e., it does not affect stability. Find the form of the critical mode (for $c = 0$ it is of the form $\exp(k_* x)$.) Sketch the shape of the spectrum in the complex plane for $c > 0$.

6. (This excercise may be omitted if in the lecture the topic cannot be covered timewise.)

Consider $u_{xx} + L^2(u + u^2) = 0$, $u(0) = u(1) = 0$. (Formally, as in the lecture example) solve this by Lyapunov-Schmidt reduction for $L \approx \pi$: Write $u = u_1 \sin(\pi x) + v$, and first project onto the complement of the kernel $\{a \sin(\pi x) | a \in \mathbb{R}\}$, solve this by the implicit function theorem, substitute into the projection onto the kernel and solve the resulting algebraic equation. Compare with excercise 3(b).