Introduction to pattern formation

Assignment-set 3

Strict deadline: 19 April 2010, 11:15h

1. Let $P_n(y) = \sum_{j=1}^{n} a_j y^j$ be a polynomial of degree n and consider the linear operator on the real line defined by$P_n(\partial_x)$. Show that for odd n the operator is not sectorial, but for even n with $\text{sign}(a_n) = -(-1)^{n/2}$ (using the convention that stable spectrum has positive real part) it satisfies the spectral condition of a sectorial operator.

2. Show that $f(u) = u^r$, $r > 1$ does not map $L^p([0,1])$ to itself for any p.

   (Hint: Do correctly what Jens attempted in the lecture.)

3. Consider the one-dimensional Korteweg de Vries (KdV) equation

$$u_t + u_{xxx} + \left(\frac{1}{2}u^2\right)_x = 0, \ x \in \mathbb{R}.$$

   Show that adding a constant to a solution is the same as going to a co-moving frame with that constant as a velocity. (Such a symmetry is called Galilei-invariance.)

   For each velocity, prove that there exist infinitely many wave trains and solitons. (Wave trains are travelling waves whose profile is periodic, and solitons those whose profile is homoclinic.) Prove that for any fixed value of the asymptotic state there exist a one parameter family of solitons with all possible velocities. What is the qualitative relation between velocity and amplitude among these solitons with the same asymptotic state?

   (Hint: Integrate the travelling wave ODE once.)

4. [Henry excercise 1 page 48:] Consider

$$\begin{aligned} u_t &= u_{xx} + f(x,u), \quad 0 < x < \pi, u = 0 \text{ at } x = 0, \pi \\ f(x,y) &= \sin(x) g(y_1, y_2) + \sin(2x) h(y_1, y_2) \\ y_j &= \frac{2}{\pi} \int_0^{\pi} y(x) \sin(jx) dx, \ j = 1, 2 \end{aligned}$$

   (a) Show that it has solutions of the form $u(x,t) = a(t)\sin(x) + b(t)\sin(2x)$ if

$$a_t = -a + g(a,b), \quad b_t = -4b + h(a,b)$$

   (b) Find g_m, h_m, $m = 1, 2$ such that there is an equilibrium u so that there exist solutions which converge to it exponentially in time and for each x monotone for $m = 1$ and oscillatory for $m = 2$.

5. The aim of this excercise is to construct solutions for the linear wave equation

$$u_{tt} = u_{xx}, \ x \in \mathbb{R}$$

   that blowup at $x = 0$ in finite time, as $t \uparrow T$ for some $T > 0$.

   (a) Give three different transformations of x, t and u that leave the equation invariant.

(b) Using (a), define two algebraic invariants of the transformations, w and y, for the equation. The w should be defined in such a way that we can look for solutions u that blow up as $t \uparrow T$ (with still a freedom in the rate at which the blowup occurs). Give the ODE for w.

(c) Choose the rate constant in w such that the resulting equation for w can be written as

$$(F(w, y))_{yy} = 0.$$

Solve this equation for w. Use this to find an expression for a solution u that blows up as $t \uparrow T$.

6. Consider the Frank-Kamenetskii equation,

$$u_t = u_{xx} + e^u, 0 < x < \pi, t > 0$$

With initial and boundary conditions

$$u(x, 0) = u_0(x) > 0 \text{ for } 0 < x < \pi$$

$$u(0, t) = u(\pi, t) = 0.$$

(a) Show that $u(x, t) > 0$ for all $t > 0$.

(b) Determine a condition on $u_0(x)$ such that there exists a time $T > 0$ for which $|u| \to \infty$ as $t \uparrow T$. Give an upper bound for T.