Introduction to pattern formation

Assignment-set 2

Strict deadline: 29 March 2010, 11:15h

1. Consider the stationary bistable equation $0 = u_t = u_{xx} + u(u-1)(a-u)$ for $0 < a < \frac{1}{2}$ and $x \in (0, L)$ with inhomogeneous Dirichlet boundary conditions: $u(x) = b$ at $x = 0, L$, where $0 < b$ and b is smaller than the intersection point of the homoclinic orbit with the u-axis.

   Recall (from class) that for $b = a$ a bifurcation occurs at $L = L_* = \pi/\sqrt{a - a^2}$, involving two solutions u_1, u_2 with $u_1(x) > a, u_2(x) < a$ for $0 < x < L$.

   Now set $b \neq a$. Use phase plane analysis and plots (geometrically checking intersections with the line in (u, u_x)-space determined by the boundary conditions) to show:

   (i) For all $L > 0$ there exists a solution with $u(x) \neq a$, $x \in [0, L]$, and this solution is unique for sufficiently small L.

   (ii) Show that there is range of values of L for which the following holds: For each of these values of L there exist at least four solutions $\tilde{u}_j(x)$, $j = 1, \ldots, 4$ and for two of these $\tilde{u}_j(x) - a$ changes sign. (You do not need to provide formulas for this range or any of the solutions. Hint: the solutions emerge in a way related to that for $b = a$.)

2. Consider A_{Neu} and A_{Dir} defined by $A_{\text{Neu/Dir}} u = u_{xx} + cu_x + au$ for homogeneous Neumann/Dirichlet boundary conditions on $x \in [-L, L]$. Recall that both operators have point spectrum only, and for periodic boundary conditions ($u(L) = u(-L)$) it is $\Sigma_L^{\text{Per}} = \{-\kappa^2 + ic\kappa + a | \kappa = k/2L, k \in \mathbb{Z}\}$.

   (i) Give formulas for the eigenvalues of A_{Neu}, A_{Dir} and denote the spectra by Σ_L^{Neu}, Σ_L^{Dir}. In one figure representing the complex plane, give an illustrative plot of Σ_L^{Neu}, Σ_L^{Dir}, Σ_L^{Per}.

   (ii) In each case determine the eigenvalues with largest real part and the corresponding eigenfunction, and mark the location in the plot of (i). Determine the limits of these eigenvalues as $L \to \infty$ and determine whether the eigenfunctions are bounded for $x \in \mathbb{R}$.

   (iii) Determine the values of a and c such that $\max\{\text{Re}(\Sigma_L^{\text{Dir}})\} < 0$ and $\max\{\text{Re}(\Sigma_L^{\text{Per}})\} > 0$.

   (iv) Determine the set of accumulation points of $\Sigma_L^{\text{Neu/Dir}}$ as $L \to \infty$. Accumulation points are $\lambda \in \mathbb{C}$ such that $\forall \varepsilon > 0 \, \exists L_\varepsilon > 0, \lambda_\varepsilon \in \Sigma_{L_\varepsilon}^{\text{Neu/Dir}} : \lambda \neq \lambda_\varepsilon, |\lambda - \lambda_\varepsilon| < \varepsilon$. Recall that for Σ_L^{Per} this is $\{-\kappa^2 + ic\kappa + a | \kappa \in \mathbb{R}\}$. For which c are all these sets of accumulation points the same?

   Remark: Periodic boundary conditions are sometimes used to model large domains. The excercise shows that this often does not work.

3. Do excercise 1.6 from the book, and in addition answer the following questions along the way:

   (i) Show how the system in the travelling wave coordinate is obtained. Give the corresponding boundary conditions.

   (ii) Show how the single equation for $u(z)$ follows. Give the corresponding boundary conditions for this equation.