# Introduction to pattern formation

Assignment-set 1

**Strict deadline**: 1 March 2010, 13:45h

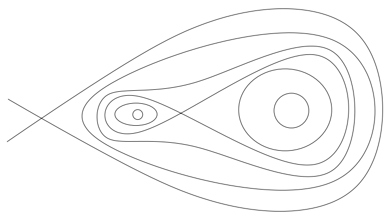
1. Prove that for $0 < a < 1$ and $H(u) = \frac{1}{4}u^4 - \frac{1+a}{3}u^3 + \frac{a}{2}u^2$ we have

$$\sqrt{2}\lim_{\delta \to 0} \int_a^{a+\delta} \frac{du}{\sqrt{H(u) - H(a+\delta)}} = \frac{\pi}{\sqrt{(a-a^2)}}.$$

   (See book pages 23/24 and compare pages 18-20.)

2. (a) Recall the function $H(u, u_x)$ which is constant along trajectories from the example in the lecture (see also book §1.4, page 18). Find the corresponding function for general $u_{xx} + f(u) = 0$ where $u(x,t) \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ continuous, and show that it is conserved, i.e. constant, along trajectories.
   *Bonus*: What is the problem when $u$ is not scalar?

   (b) Give an explicit function $f : \mathbb{R} \to \mathbb{R}$ such that part of the phase portrait of $u_{xx} + f(u) = 0$ qualitatively looks as follows:



   Explain your answer carefully.

3. Let $u_*(x)$, $x \in \mathbb{R}$, be a solution to the equation $u_{xx} + u(1-u) = 0$. Assume that $u_*$ satisfies homogeneous Dirichlet boundary conditions on an interval with length $L > \pi$ and is non-negative on this interval. (See §1.4 pages 17-19, equations (1.4.2), (1.4.3)).

   (a) Show that the above solution $u_*$ is a spatially periodic solution (i.e. a periodic solution in $x$) to the equation $u_t = u_{xx} + u(1-u)$.
   *Hint*: Find the trajectories that intersect $u = 0$ in two points by using a phase plane analysis.

   (b) Show that (a) can fail without the non-negativity assumption on $u_*$.

   (c) Give all values of the domain length $L$ for which $u_*$ gives solutions with homogeneous Dirichlet boundary conditions. Explain.

4. (a) Do excercise 1.8 from the book.

   (b) Replace $\varepsilon$ by $\varepsilon^\alpha$ in the expansion of $u$ and explain why the exponent $\alpha$ has to be chosen as $\alpha = 1$. In other words, follow the arguments sketched in §1.4, pages 21-22 and give an explicit analysis (including all the details) to conclude that $\alpha$ must be taken 1.