Wentzell boundary conditions in the context of Dirichlet forms

Hendrik Vogt* and Jürgen Voigt*

Fachrichtung Mathematik, Technische Universität Dresden, D-01062 Dresden, Germany

dedicated to the memory of Klaus Floret (1941 – 2002) whose untimely death we deeply mourn

Abstract

Let X be a locally compact space, m a Radon measure on X, $\mathfrak{h}$ a regular Dirichlet form in $L_2(X,m)$. For a Radon measure μ we interpret $\mathfrak{h}$ as a regular Dirichlet form τ in $L_2(m+\mu)$. We show that μ decomposes as $\mu_r + \mu_s$ where μ_r is coupled to $\mathfrak{h}$ and μ_s decouples from $\mathfrak{h}$. Additionally to this 'space perturbation', a second perturbation is introduced by a measure ν describing absorption.

The main object of the paper is to apply this setting to a study of the Wentzell boundary condition

$$-\alpha Au + n \cdot a\nabla u + \gamma u = 0 \quad \text{on } \partial\Omega$$

for an elliptic operator $A = -\nabla \cdot (a\nabla)$, where $\Omega \subseteq \mathbb{R}^d$ is open, n the outward normal, and α, γ are suitable functions. It turns out that the previous setting can be applied with $\mu = \alpha dS$, $\nu = \gamma dS$, under suitable conditions. Besides the description of the d-dimensional case we give a more detailed analysis of the one-dimensional case.

As a further topic in the general setting we study the question whether mass conservation carries over from the unperturbed form $\mathfrak{h}$ to the space perturbed form τ. In an appendix we extend a known closability criterion from the minimal to the maximal form.

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*partly supported by the DFG

vogt@math.tu-dresden.de, voigt@math.tu-dresden.de

1 Introduction

The object of this paper is to present the treatment of Wentzell boundary conditions in the context of Dirichlet forms.

Let $\Omega \subseteq \mathbb{R}^d$ be open, and $A = -\nabla \cdot (a\nabla)$ an elliptic operator in divergence form. (For precise conditions we refer to Section 3.) Then the Wentzell boundary condition is given by

$$-\alpha Au + n \cdot a\nabla u + \gamma u = 0$$

on (part of) the boundary of Ω, where α, γ are non-negative functions, and n is the outward normal. (See, e.g., [23; formula (3)]; there even more general boundary conditions including diffusion on the boundary and jumps are considered.) In [9], the operator A subject to the Wentzell boundary condition was studied in the L_p-context, where the Lebesgue measure on Ω is augmented by a suitable measure on the boundary. It is the purpose of the present paper to continue and extend this investigation in various directions. The essential new ingredient in our study is putting the problem into the context of Dirichlet forms. This makes it possible to obtain results under rather general hypotheses.

In most of the previous papers, Wentzell boundary conditions have been studied in spaces of continuous functions and with operators that are not in divergence form; see, e.g., [10], [23], [6], [5]. It is only in [9] that the measure space $(\Omega \cup \Gamma, m + \mu)$—where m is the Lebesgue measure on Ω, Γ is part of the boundary of Ω, and μ is a suitable measure on Γ—is proposed as the adequate measure space for studying Wentzell boundary conditions in the L_p-framework. In the one-dimensional case, a space with an additional measure on the boundary was already used in [Feller; Sections 18 and 22] for the L_1-treatment. In [2], more special boundary conditions have been studied thoroughly in the L_p-framework.

The time evolution generated by the operator A presented above describes particles which diffuse in Ω and which may stick (for infinitesimally short times) to the boundary. This behaviour is in contrast to the totally absorbing Dirichlet boundary condition, the totally reflecting Neumann boundary condition, or the partially absorbing, partially reflecting Robin boundary condition. One may think of the Wentzell boundary condition as describing a situation like the Robin boundary condition, with the additional feature that the boundary has the capacity for storing (but not conducting) heat.

Probabilistically speaking, the L_1-treatment means that one wants to describe the time development of measures which are given as densities with respect to a reference measure. As a consequence, if one wants to describe particles which may temporarily concentrate on the boundary, the reference measure should offer this possibility—meaning that the boundary should not be a null set. This aspect is not apparent as long as one studies the problem in spaces of continuous functions.

The paper is organised as follows. In Section 2 we present our abstract version of the Wentzell boundary condition in the context of Dirichlet forms. We start with a Dirichlet form $\mathfrak{h}$ which corresponds to the basic diffusion of particles in a locally compact space X, endowed with a Radon measure m. As a first perturbation we introduce a part of X (described by a measure μ) where additionally particles should stick. The state space X itself is not changed, but the reference measure m is augmented by μ. The interpretation of this perturbation is that the diffusion is slowed down on the support of μ (see, e.g., [11; Sec. 6.2]). A second perturbation is then added as an absorption by a measure ν.

In Section 3 we apply the setting of Section 2 to the situation sketched at the beginning of the introduction. The conclusion of this application is formulated in Theorem 3.4—the main result of the paper. In the first part of the section we give a motivation for our interpretation of the Wentzell boundary condition in the context of Dirichlet forms. As the state space we take $X = \Omega \cup \Sigma$, with Lebesgue measure m, where Σ is the part of $\partial\Omega$ on which we want to pose the Wentzell boundary condition. The form $\mathfrak{h}$ is associated with the differential expression $A = -\nabla \cdot (a\nabla)$. The measures μ and ν are defined as $\mu = \alpha dS$ and $\nu = \gamma dS$, respectively, where dS denotes the surface measure on Σ. The main part of Section 3 is devoted to the investigation of conditions on the coefficients guaranteeing that the results of Section 2 can be applied to the problem under consideration.

In Section 4 we present a more detailed analysis of the one-dimensional case $\Omega = (0,1)$, $\Sigma = \{0,1\}$. We give an explicit description of the L_2-operator associated with the form, and we discuss the various possibilities subsumed in our general approach.

In Section 5 we return to the general context of Section 2 and discuss the question of mass conservation, assuming that the unperturbed situation produces a mass conserving semigroup. Under the additional assumption that m and μ are finite measures it is easy to show that then the semigroup that is space perturbed by μ also is mass conserving. The conjecture that this statement should be true without the additional assumption is answered in the positive only in the case that μ is absolutely continuous with respect to m; see Theorem 5.5. (Unfortunately, the latter result is not applicable in the context of Section 3.) As an auxiliary result we present a partial description of the L_p-generator of the semigroup associated with a Dirichlet form (Proposition 5.2).

In the appendix we show that the maximal form associated with $-\nabla \cdot (a\nabla)$ is closable if $a \in W^1_{2,loc}(\Omega)$. This result (which is well-known for the minimal form) is of independent interest and serves to provide information in the context of Section 3.

2 Space perturbation of Dirichlet forms

In this section, let X be a locally compact space. Let m and μ be two Radon measures on X (i.e., locally finite, inner regular Borel measures), and assume that

$\operatorname{supp} m = X$. Moreover, let $\mathfrak{h}$ be a regular symmetric Dirichlet form in $L_2(X,m)$; cf. [11], [14]. We recall that 'regular' means that $C_c(X) \cap D(\mathfrak{h})$ is dense in $D(\mathfrak{h})$ with respect to the $\mathfrak{h}$-norm and dense in $C_0(X)$ with respect to the sup-norm. In fact, this implies that $C_c(X) \cap D(\mathfrak{h})$ also is dense in $C_c(X)$ with respect to the inductive limit topology of $C_c(X)$, i.e., for each $\varphi \in C_c(X)$ there exists a sequence $(\varphi_n) \subseteq C_c(X) \cap D(\mathfrak{h})$ such that $\bigcup_{n\in\mathbb{N}} \operatorname{supp} \varphi_n$ is relatively compact, and $\varphi_n \to \varphi$ in the sup-norm.

We do not assume countable compactness, metrisability, or separability of X since the properties we are going to use do not depend on these assumptions. In the applications in the two subsequent sections, however, X will be a subset of $\mathbb{R}^d$.

Our aim is to interpret the form $\mathfrak{h}$ as a regular Dirichlet form in the space $L_2(m+\mu)$. As a first step we define a Dirichlet form $\hat{\tau}$ in $L_2(m+\mu)$ by

$$D(\hat{\tau}) := \bigl\{u \in L_2(m+\mu);\, u \in D(\mathfrak{h})\bigr\}, \qquad \hat{\tau}(u,v) := \mathfrak{h}(u,v) \quad \bigl(u,v \in D(\hat{\tau})\bigr).$$

(Saying that $u \in L_2(m+\mu)$ belongs to $D(\mathfrak{h})$ means that u should also be considered as a function in $L_2(m)$. The observation that the mapping $u \mapsto u\colon L_2(m+\mu) \to L_2(m)$ is continuous immediately implies that $\hat{\tau}$ is a closed symmetric form in $L_2(m+\mu)$. Also, $\hat{\tau}$ is densely defined since $C_c(X) \cap D(\mathfrak{h})$ is dense in $C_c(X)$ with respect to the inductive limit topology, and $C_c(X)$ is dense in $L_2(m+\mu)$.)

We now couple μ to the form $\mathfrak{h}$ by restricting the domain of $\hat{\tau}$. Namely, we define

$$\tau := \overline{\hat{\tau}\restriction_{C_c(X)\cap D(\mathfrak{h})}}.$$

Then, by its very definition, τ is a regular symmetric Dirichlet form in $L_2(m+\mu)$.

Before analysing τ in more detail we finish the definition of the form we present as the generalisation of the Wentzell boundary condition. For this purpose let ν be a (positive) Borel measure on X which is absolutely continuous with respect to τ-capacity. Then $\tau + \nu$, defined by

$$D(\tau+\nu) := \Bigl\{u \in D(\tau);\, \int |\tilde{u}|^2\, d\nu < \infty\Bigr\},$$
$$(\tau+\nu)(u,v) := \tau(u,v) + \int \tilde{u}\overline{\tilde{v}}\, d\nu \quad \bigl(u,v \in D(\tau+\nu)\bigr)$$

(where $\tilde{u}$ denotes a τ-quasi-continuous version of u) is a symmetric (not necessarily regular) Dirichlet form in $\overline{D(\tau+\nu)}^{L_2(m+\mu)}$. We refer to [21] for this statement and for further information.

Let $\mathfrak{B}_\sigma$ be the σ-ring on X generated by the compact sets. Because of the inner regularity of m, the values of m on $\mathfrak{B}_\sigma$ determine m, and the same is true for μ.

We recall from [21; Prop. 1.1(a)] that μ can be decomposed as

$$\mu = \mu_r + \mu_s,$$

where μ_r—the 'regular' part—is a measure which is absolutely continuous with respect to $\mathfrak{h}$-capacity, and μ_s—the 'singular' part—has the property that for each compact set $K \subseteq X$ there exists a Borel set $N \subseteq K$ of zero $\mathfrak{h}$-capacity such that $\mu_s(K \setminus N) = 0$.

Lemma 2.1. *There exists a local Borel set M such that $\mathfrak{h}\text{-cap}(M \cap K) = 0$ for all compact $K \subseteq X$, and*

$$m + \mu_r = \mathbb{1}_{X \setminus M}(m + \mu), \quad \mu_s = \mathbb{1}_M(m + \mu) \quad \text{on } \mathfrak{B}_\sigma.$$

(In case X is σ-compact, the local Borel set M is automatically a Borel set, and M has zero $\mathfrak{h}$-capacity.)

As consequences, M is locally $(m + \mu_r)$-negligible,

$$\mu_r = \mathbb{1}_{X \setminus M}\mu, \quad \mu_s = \mathbb{1}_M \mu \quad \text{on } \mathfrak{B}_\sigma,$$

and the decomposition of $m+\mu$ given above induces a decomposition of $L_2(m+\mu)$ as an orthogonal sum of $L_2(m + \mu_r)$ and $L_2(\mu_s)$.

Proof. There exists a locally countable disjoint family $(K_\iota)_{\iota \in I}$ of compact subsets of X such that $X \setminus \bigcup_{\iota \in I} K_\iota$ is a local $(m+\mu)$-null set and $\operatorname{supp}\bigl(\mathbb{1}_{K_\iota}(m+\mu)\bigr) = K_\iota$ for all $\iota \in I$ (cf. [4; § 1, n° 4, Prop. 4]; this property expresses the *strict localisability* of the measure $m+\mu$). Defining N_ι corresponding to K_ι as in the paragraph before the lemma, we obtain that the set $M := \bigcup_{\iota \in I} N_\iota$ has the asserted properties. □

The main result of this section is the following description of τ.

Theorem 2.2. *With the above notation,*

$$D(\tau) = \bigl\{u \in L_2(m + \mu) \cap D(\mathfrak{h});\, u = \tilde{u}\ \mu_r\text{-a.e.}\bigr\}$$

(where $\tilde{u}$ denotes an $\mathfrak{h}$-quasi-continuous version of u). The form τ decomposes as $\tau = \tau_r \oplus \tau_s$, where τ_r is the form in $L_2(m + \mu_r)$ defined by $\tau_r(u, v) = \mathfrak{h}(u, v)$ on

$$D(\tau_r) = \bigl\{u \in L_2(m + \mu_r) \cap D(\mathfrak{h});\, u = \tilde{u}\ \mu_r\text{-a.e.}\bigr\},$$

and $\tau_s = 0$ in $L_2(\mu_s)$.

Remark 2.3. (a) The condition '$u = \tilde{u}$ μ_r-a.e.' is the coupling between $\mathfrak{h}$ and μ announced above. It may be considered as a kind of (interior) boundary condition.

(b) Theorem 2.2 states that the form τ decomposes into the interesting part τ_r and the trivial part τ_s. In the application in Sections 3 and 4, it will be one of the issues to indicate conditions excluding the trivial part τ_s.

(c) It is a well-established procedure to associate with a given Dirichlet form $\mathfrak{h}$ in $L_2(m)$ a process on a space $L_2(\tilde{m})$ with a new measure $\tilde{m}$. We refer to [11; Sec. 6.2] for the description of the 'time change' of the process associated with the additive linear functionals generated my $\tilde{m}$. In this case, the space X is assumed to be metric and separable, and $\tilde{m}$ is absolutely continuous with respect to capacity. In our setting we have $\tilde{m} = m + \mu \geqslant m$; this is not assumed in [11].

Assume that X satisfies the additional properties mentioned in the previous paragraph. Having established the decomposition of Theorem 2.2 it is clear that the results of [11] apply to the measure $\tilde{m} = m + \mu_r$.

In the proofs of the next lemmas we will make use of the following fact. Let $u \in D(\mathfrak{h})$, $(u_k) \subseteq D(\mathfrak{h})$ a $\|\cdot\|_\infty$-bounded sequence with $u_k \to u$ in $D(\mathfrak{h})$, and $\psi \in C_c(X) \cap D(\mathfrak{h})$. Then there exists a sequence (v_k) of convex combinations of $\{u_j; j \geqslant k\}$ such that $\psi v_k \to \psi u$ in $D(\mathfrak{h})$ and in $L_2(\mu_r)$. Indeed, $\mathfrak{h}(\psi u_k)^{1/2} \leqslant \|\psi\|_\infty \mathfrak{h}(u_k)^{1/2} + \|u_k\|_\infty \mathfrak{h}(\psi)^{1/2}$ for all $k \in \mathbb{N}$; cf. [11; Thm. 1.4.2(ii)]. Thus, $\psi u_k \to \psi u$ weakly in $D(\mathfrak{h})$. Also, without restriction $\psi u_k \to \psi u$ $\mathfrak{h}$-q.e., hence μ_r-a.e. This implies the assertion.

Lemma 2.4. *Let $u \in L_2(m+\mu) \cap D(\mathfrak{h})$, $u = \tilde{u}$ μ_r-a.e. Then there exists $(\varphi_k) \subseteq C_c(X) \cap D(\mathfrak{h})$ such that $\varphi_k \to u$ in $D(\mathfrak{h})$ and in $L_2(m + \mu_r)$.*

Proof. Without restriction $u \geqslant 0$ and u bounded. By the regularity of $\mathfrak{h}$ there exists $(u_k) \in C_c(X) \cap D(\mathfrak{h})$ with $u_k \to u$ in $D(\mathfrak{h})$; without restriction $u_k \geqslant 0$ $(k \in \mathbb{N})$ and $u_k \to u$ $\mathfrak{h}$-q.e. This implies $u_k \wedge u \to u$ in $D(\mathfrak{h})$ and in $L_2(m + \mu_r)$. Therefore it suffices to approximate each $u_k \wedge u$ or, expressed differently, we may assume $\operatorname{supp} u$ compact to start with.

As before, there exists a $\|\cdot\|_\infty$-bounded sequence $(u_k) \in C_c(X) \cap D(\mathfrak{h})$ with $u_k \to u$ in $D(\mathfrak{h})$. There exists $\psi \in C_c(X) \cap D(\mathfrak{h})$ with $0 \leqslant \psi \leqslant 1$, $\psi = 1$ on $\operatorname{supp} u$. By the remark before the lemma we may assume without restriction that $\varphi_k := \psi u_k \to \psi u = u$ in $D(\mathfrak{h})$ and in $L_2(\mu_r)$, which concludes the proof. $\square$

Lemma 2.5. *Let $u \in L_2(\mu_s)$, $\varepsilon > 0$. Then there exists $\varphi \in C_c(X) \cap D(\mathfrak{h})$ with $\|u - \varphi\|_{L_2(\mu_s)} \leqslant \varepsilon$, $\mathfrak{h}(\varphi) + \|\varphi\|^2_{L_2(m+\mu_r)} \leqslant \varepsilon$.*

Proof. Since $C_c(X) \cap D(\mathfrak{h})$ is dense in $L_2(m + \mu)$, we obtain $\psi \in C_c(X) \cap D(\mathfrak{h})$ such that $\|u - \psi\|_{L_2(\mu_s)} \leqslant \frac{\varepsilon}{2}$. There exists a Borel set $N \subseteq K := \operatorname{supp} \psi$ with $\mathfrak{h}\text{-cap}(N) = 0$, $\mu_s(K \setminus N) = 0$. Since μ_s is inner regular, there exists a compact set $K_0 \subseteq N$ such that $\|\mathbb{1}_{N \setminus K_0} \psi\|_{L_2(\mu_s)} \leqslant \frac{\varepsilon}{2}$. For the compact set K_0 we obtain

$$0 = \mathfrak{h}\text{-cap}(K_0) = \inf\bigl\{\mathfrak{h}(\varphi) + \|\varphi\|^2_{L_2(m)};\, \varphi \in C_c(X) \cap D(\mathfrak{h}),\ \varphi \geqslant 1 \text{ on } K_0\bigr\};$$

cf. [11; Lemma 2.2.7, Problem 1.4.1]. Thus, there exists a sequence $(\varphi_k) \subseteq C_c(X) \cap D(\mathfrak{h})$ with $0 \leqslant \varphi_k \leqslant 1$, $\varphi_k = 1$ on K_0, $\varphi_k \to 0$ in $D(\mathfrak{h})$. As above we may assume without restriction that $\varphi_k \psi \to 0$ in $D(\mathfrak{h})$ and in $L_2(\mu_r)$. Also,

$$\|u - \varphi_k \psi\|_{L_2(\mu_s)} \leqslant \|u - \psi\| + \|\psi - \varphi_k \psi\| \leqslant \frac{\varepsilon}{2} + \|\mathbb{1}_{N \setminus K_0} \psi\| \leqslant \varepsilon \quad (k \in \mathbb{N}).$$

Thus, taking $\varphi := \varphi_k \psi$ for suitably large k, we obtain φ as asserted. $\square$

Proof of Theorem 2.2. First we show $D(\tau) \subseteq \big\{u \in L_2(m+\mu) \cap D(\mathfrak{h});\ u = \tilde{u}\ \mu_r\text{-a.e.}\big\}$. Let $u \in D(\tau)$. By definition there exists $(\varphi_n) \subseteq C_c(X) \cap D(\mathfrak{h})$ with $\varphi_n \to u$ in the $\mathfrak{h}$-norm and in $L_2(m+\mu)$. Without restriction $\varphi_n \to \tilde{u}$ $\mathfrak{h}$-q.e. This implies $\varphi_n \to \tilde{u}$ μ_r-a.e., so $u = \tilde{u}$ μ_r-a.e.

Conversely, let $u \in L_2(m+\mu) \cap D(\mathfrak{h})$, $u = \tilde{u}$ μ_r-a.e. Let $(\varphi_n) \in C_c(X) \cap D(\mathfrak{h})$ be the sequence obtained in Lemma 2.4. By Lemma 2.5 there exists $(\psi_n) \subseteq C_c(X) \cap D(\mathfrak{h})$ with

$$\|(u-\varphi_n) - \psi_n\|_{L_2(\mu_s)} \leqslant \frac{1}{n}, \quad \mathfrak{h}(\psi_n) + \|\psi_n\|^2_{L_2(m+\mu_r)} \leqslant \frac{1}{n} \qquad (n \in \mathbb{N}).$$

This implies that the sequence $(\varphi_n + \psi_n) \subseteq C_c(X) \cap D(\tau)$ converges to u in $D(\tau)$:

$$\begin{aligned}\|u - (\varphi_n + \psi_n)\|_\tau^2 &\leqslant \big(\mathfrak{h}(u-\varphi_n)^{\frac12} + \mathfrak{h}(\psi_n)^{\frac12}\big)^2 \\ &\quad + \big(\|u-\varphi_n\|_{L_2(m+\mu_r)} + \|\psi_n\|_{L_2(m+\mu_r)}\big)^2 + \|u - \varphi_n - \psi_n\|^2_{L_2(\mu_s)} \to 0.\end{aligned}$$

It remains to show that τ decomposes as indicated. For $u \in D(\tau)$ we have $u = \tilde{u}$ μ_r-a.e., thus $u \in D(\tau_r)$ and

$$\tau_r(u,v) = \mathfrak{h}(u,v) = \tau(u,v) \quad \big(u, v \in D(\tau)\big).$$

Conversely, let $u_r \in D(\tau_r)$, $u_s \in L_2(\mu_s)$. Then there exists $u \in L_2(m+\mu)$ with $u_r = u$ $(m+\mu_r)$-a.e., $u_s = u$ μ_s-a.e. From $\tilde{u} = \tilde{u}_r$ $\mathfrak{h}$-q.e. we obtain $u \in D(\tau)$. $\square$

In order to show that the measure ν introduced above decomposes analogously to τ we need the following fact on capacities.

Lemma 2.6. *Let $N \in \mathfrak{B}_\sigma$. Then τ-$\operatorname{cap}(N) = 0$ if and only if $\mathfrak{h}$-$\operatorname{cap}(N) = 0$ and $\mu_s(N) = 0$.*

Proof. Necessity is obvious. For sufficiency, we first note that we may assume that N is contained in a compact set. Let $\varepsilon > 0$. There exists $\psi \in C_c(X) \cap D(\mathfrak{h})$ with $0 \leqslant \psi \leqslant 1$, $\psi = 1$ in a neighbourhood of N, $\int \psi^2\, d\mu_s \leqslant \varepsilon$. Moreover, there exists a sequence $(\varphi_k) \subseteq D(\mathfrak{h})$ with $0 \leqslant \varphi_k \leqslant 1$, $\varphi_k = 1$ in a neighbourhood of N $(k \in \mathbb{N})$, $\varphi_k \to 0$ in $D(\mathfrak{h})$ as $k \to \infty$. As above we may assume that $\varphi_k\psi \to 0$ in $D(\mathfrak{h})$ and in $L_2(m+\mu_r)$ as $k \to \infty$. Since $\varphi_k\psi = 1$ in a neighbourhood of N $(k \in \mathbb{N})$, we obtain

$$\tau\text{-}\operatorname{cap}(N) \leqslant \liminf_{k\to\infty} \Big(\mathfrak{h}(\varphi_k\psi) + \int (\varphi_k\psi)^2\, d(m+\mu)\Big) \leqslant \varepsilon.$$

This proves τ-$\operatorname{cap}(N) = 0$. $\square$

As before, let ν be a Borel measure on X which is absolutely continuous with respect to τ-capacity. We note that only the restriction of ν to $\mathfrak{B}_\sigma$ is relevant for the form $\tau + \nu$. This is due to the fact that, for $u \in D(\tau)$, there exists a quasi-continuous version $\tilde{u}$ such that $[\tilde{u} \neq 0]$ belongs to $\mathfrak{B}_\sigma$. Indeed, there exists a sequence $(u_k) \subseteq C_c(X) \cap D(\tau)$ such that $u_k \to \tilde{u}$ τ-q.e., and we may assume $[\tilde{u} \neq 0] \subseteq \bigcup_{k\in\mathbb{N}} \operatorname{supp} u_k$.

Proposition 2.7. *There exists a decomposition $\nu = \nu_r + \nu_s$ on $\mathfrak{B}_\sigma$ such that ν_r is absolutely continuous with respect to $\mathfrak{h}$-capacity and ν_s is μ_s-absolutely continuous. As a consequence, $\tau + \nu$ decomposes as $\tau + \nu = (\tau_r + \nu_r) \oplus \nu_s$ in $L_2(m + \mu) = L_2(m + \mu_r) \oplus L_2(\mu_s)$.*

Proof. We use the local Borel set M obtained in Lemma 2.1 and claim that $\nu_r := \mathbb{1}_{X\setminus M}\nu$, $\nu_s := \mathbb{1}_M \nu$ constitutes the asserted decomposition.

Let $N \in \mathfrak{B}_\sigma$ be a set of $\mathfrak{h}$-capacity zero. Then $\mu(N \setminus M) = \mu_r(N) = 0$. Lemma 2.6 implies that τ-cap$(N \setminus M) = 0$. Since ν is absolutely continuous with respect to τ-capacity, we obtain $\nu_r(N) = \nu(N \setminus M) = 0$.

On the other hand, let $N \in \mathfrak{B}_\sigma$ be a μ_s-null set. Then $\mu(N \cap M) = \mu_s(N) = 0$, and we have $\mathfrak{h}$-cap$(N \cap M) = 0$ by Lemma 2.1, so Lemma 2.6 implies τ-cap$(M \cap N) = 0$. Thus $\nu_s(N) = \nu(M \cap N) = 0$.

The decomposition of $\tau + \nu$ now follows from the decomposition of τ obtained in Theorem 2.2. □

In part (a) of the following remark we recall some of the far-reaching consequences of the results obtained so far.

Remarks 2.8. (a) By [21; Thm. 4.1(a)] there exists a locally measurable set $X_0 \subseteq X$ such that $L_2(X_0, m + \mu) = \overline{D(\tau + \nu)}^{L_2(m+\mu)}$. Therefore, the form $\tau + \nu$ is associated with a selfadjoint operator A in $L_2(X_0, m + \mu)$. Additionally, the form $\tau + \nu$ is a (symmetric) Dirichlet form which means that it satisfies the Beurling-Deny criteria; we refer to [7; Sec. 1.3], [14; Ch. I, Sec. 4], [21; Sec. 4] for these notions and their consequences. In particular, the C_0-semigroup $(e^{-tA}; t \geqslant 0)$ extends to a positive contractive C_0-semigroup T_p on $L_p(X_0, m + \mu)$, for all $p \in [1, \infty)$. As a side remark we mention that, for $p \in (1, \infty)$, the semigroup T_p is analytic of angle $\arccos |1 - \frac{2}{p}|$, by [13; Cor. 3.2] (see also [17]).

(b) If μ has a non-trivial singular part μ_s then the form $\tau + \nu$ decomposes as described in Proposition 2.7. The singular part ν_s is μ_s-absolutely continuous, thus $\nu_s = \rho_s \mu_s$ for a suitable locally measurable function ρ_s (Indeed, this follows from the usual Radon-Nikodym theorem together with the strict localisability of (X, μ_s) described in the proof of Lemma 2.1). Since τ vanishes on $L_2(\mu_s)$, the part of the form $\tau + \nu$ in $L_2(\mu_s)$ corresponds to the maximal operator of multiplication by ρ_s. Thus, the semigroup T_p acts on $L_p(X_0, \mu_s) = L_p(X_0 \cap M, m + \mu)$ as multiplication by $e^{-t\rho_s}$.

3 Second order elliptic operators

Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $\Sigma \subseteq \partial\Omega$ be relatively open in the C^1-part of $\partial\Omega$. Let $a\colon \Omega \to \mathbb{R}^{d\times d}$ be measurable, $a(x)$ symmetric and positive semi-definite for all $x \in \Omega$.

We want to use the setting of Section 2 in order to obtain an operator realisation of $A = -\nabla \cdot (a\nabla)$, with zero Dirichlet boundary condition on $\partial\Omega \setminus \Sigma$ and Wentzell boundary condition

$$-\alpha Au + n \cdot a\nabla u + \gamma u = 0 \quad \text{on } \Sigma. \tag{3.1}$$

Here we assume $\gamma\colon \Sigma \to [0,\infty]$ measurable, $0 \leqslant \alpha \in L_{1,loc}(\Sigma, dS)$, where dS denotes the $(d-1)$-dimensional surface measure on Σ, and n denotes the outward normal on Σ.

In the following we motivate our formulation of the Wentzell boundary condition. For an analogous motivation in terms of resolvents we refer to [9; Sec. 2].

We want to define $A = -\nabla \cdot (a\nabla)$ by the form method, with boundary conditions as above. Taking u satisfying the above boundary conditions and $Au \in L_2(\Omega)$, moreover v vanishing on $\partial\Omega \setminus \Sigma$ in a suitable sense, and applying the divergence theorem purely formally, we obtain

$$\begin{aligned}\int_\Omega Au\,\overline{v}\,dx &= -\int_\Omega \nabla \cdot (a\nabla u)\overline{v}\,dx = \int_\Omega a\nabla u \cdot \nabla\overline{v}\,dx - \int_\Sigma n \cdot a\nabla u\,\overline{v}\,dS \\ &= \int_\Omega a\nabla u \cdot \nabla\overline{v}\,dx - \int_\Sigma Au\,\overline{v}\,\alpha dS + \int_\Sigma u\overline{v}\,\gamma dS.\end{aligned}$$

Looking at the right hand side one realises that the part containing Au should rather appear on the left hand side. The only way how to achieve this is including the measure $\mu := \alpha dS$ in the measure for the L_2-space in which A should act, for then the above formula can be written as

$$(Au|v)_{L_2(m+\mu)} = \int_\Omega a\nabla u \cdot \nabla\overline{v}\,dx + \int_\Sigma u\overline{v}\,\gamma dS. \tag{3.2}$$

The right hand side defines a symmetric form with a domain which will be specified subsequently. So A should be the operator in $L_2(m+\mu)$ associated with this form.

Remark 3.1. (a) On those parts where one does not want to pose zero Dirichlet boundary conditions, the coefficient of $n \cdot a\nabla u$ in (3.1) should be different from zero; otherwise one could not replace this term in the above computation. This assumption also appears in [9; Sec. 2]. We simply choose the corresponding coefficient as 1.

(b) In [9], the special case $\Sigma = \partial\Omega$ and $a = \tilde{a}\,\mathrm{id}$, with a scalar function $\tilde{a}$ is studied. In that paper, the Wentzell boundary condition is formulated as $-Au + \tilde{\beta} n \cdot \nabla u + \tilde{\gamma} u = 0$, with $\tilde{\beta}$ strictly positive. Rewriting this in our notation we obtain $\alpha = \frac{\tilde{a}}{\tilde{\beta}}$, $\gamma = \frac{\tilde{a}\tilde{\gamma}}{\tilde{\beta}}$. Thus, implicitly, $\alpha = \gamma = 0$ on $[a = 0]$ is assumed in [9]. This amounts to posing a Neumann boundary condition on $[a = 0]$. It turns out, however, that even on this set a Wentzell boundary condition—formulated in our way—may be meaningful if a does not vanish too strongly; see Propositions 3.7 and 4.2 below.

We define the symmetric form $\tilde{\mathfrak{h}}$ in $L_2(\Omega)$ by

$$D(\tilde{\mathfrak{h}}) := \big\{u \in L_2(\Omega);\, \nabla u \in L_{1,loc}(\Omega),\ \int_\Omega a(x)\nabla u(x)\cdot\nabla\overline{u}(x)\,dx < \infty\big\},$$
$$\tilde{\mathfrak{h}}(u,v) := \int_\Omega a(x)\nabla u(x)\cdot\nabla\overline{v}(x)\,dx \quad \big(u,v \in D(\tilde{\mathfrak{h}})\big).$$

We assume that

(H1) $\tilde{\mathfrak{h}}$ is densely defined and closable.

Then the closure of $\tilde{\mathfrak{h}}$ is a symmetric Dirichlet form.

Remark 3.2. Obviously, $\tilde{\mathfrak{h}}$ is densely defined if

$$a \in L_{1,loc}(\Omega).$$

A sufficient condition for $\tilde{\mathfrak{h}}$ to be closed is given by

$$a \text{ a.e. invertible, } a^{-1} \in L_{1,loc}(\Omega)$$

(cf. [20; Thm. 3.2]).

In the appendix we show that (H1) also holds if $a \in W^1_{2,loc}(\Omega)$.

Moreover we assume that

(H2) $C_c(\Omega\cup\Sigma)\cap D(\tilde{\mathfrak{h}})$ is dense in $C_0(\Omega\cup\Sigma)$ with respect to the sup-norm.

Remark 3.3. Assumption (H2) is satisfied, e.g., if

$$a \in L_{1,loc}(\Omega\cup\Sigma).$$

Indeed, then $C^1_c(\Omega\cup\Sigma)$ is contained in $D(\tilde{\mathfrak{h}})$, and it is easy to show that $C^1_c(\Omega\cup\Sigma)$ is dense in $C_c(\Omega\cup\Sigma)$ and hence in $C_0(\Omega\cup\Sigma)$.

In the one-dimensional case, however, (H2) is satisfied if only $a \in L_{1,loc}(\Omega)$; see Section 4.

If (H1) and (H2) hold then

$$\mathfrak{h} := \overline{\tilde{\mathfrak{h}}\restriction_{C_c(\Omega\cup\Sigma)\cap D(\tilde{\mathfrak{h}})}}$$

is a regular symmetric Dirichlet form in $L_2(X,m)$, with $X = \Omega\cup\Sigma$ and m the Lebesgue measure on X. The operator associated with this form should be considered as the realisation of $-\nabla\cdot(a\nabla)$ with zero Dirichlet boundary condition on $\partial\Omega\setminus\Sigma$ and Neumann boundary condition on Σ.

The hypothesis $\alpha \in L_{1,loc}(\Sigma, dS)$ implies that $\mu = \alpha dS$ is a Radon measure on X. Starting from the form $\mathfrak{h}$ above, we thus can define the form τ in $L_2(X, m+\mu)$ as in Section 2. As indicated by formula (3.2), our realisation of $A = -\nabla\cdot(a\nabla)$, with boundary condition (3.1), will be the operator associated with the form $\tau+\nu$ in $L_2(m+\mu)$, with $\nu := \gamma dS$. For the form $\tau+\nu$ to be well-defined, we have to assume that

(H3) $\nu = \gamma dS$ is absolutely continuous with respect to τ-capacity.

Theorem 3.4. *Let $X = \Omega \cup \Sigma$, a, α, and γ be as above. Assume that* (H1) *and* (H2) *are satisfied. Let $\mathfrak{h}$ and τ be as above, and assume that* (H3) *is satisfied.*

Then the form $\tau + \nu$ is associated with a selfadjoint operator $A \geqslant 0$ in $L_2(X_0, m+\mu) = \overline{D(\tau+\nu)}^{L_2(m+\mu)}$ (see Remark 2.8(a) for the meaning of X_0). The C_0-semigroup $(e^{-tA}; t \geqslant 0)$ on $L_2(X_0, m+\mu)$ is positivity preserving and contractive in $L_p(X_0, m+\mu)$, for all $1 \leqslant p \leqslant \infty$.

The proof of this theorem is contained in the explanations in Remark 2.8(a).

Remark 3.5. The preceding result summarises our procedure to associate a Dirichlet form with the differential operator $-\nabla\cdot(a\nabla)$ subject to Wentzell boundary conditions (3.1), under rather general assumptions. Having achieved this goal one knows that this Dirichlet form gives rise to a C_0-semigroup on L_2 which in turn acts as a positive contractive C_0-semigroup on L_p, for all $p \in [1, \infty)$. The negative generator A_p of this C_0-semigroup on L_p duly should be considered as the L_p-realisation of the differential operator subject to the boundary condition.

It is then a separate (difficult) task to obtain an explicit description of the operator A_p. In [9], a different approach is chosen: an L_p-operator is defined, and the authors endeavour to show that the closure of this operator generates a contractive C_0-semigroup.

In the remainder of this section we will discuss properties of $\mu = \alpha dS$ and $\nu = \gamma dS$; in particular we will indicate conditions implying condition (H3).

Remarks 3.6. (a) If $\gamma = 0$ dS-a.e. on $[\alpha = 0]$ then ν is μ-absolutely continuous, so ν is absolutely continuous with respect to τ-capacity.

(b) We do not exclude α to be zero on part of Σ. In these points (3.1) becomes

$$n \cdot a\nabla u + \gamma u = 0,$$

i.e., a Robin boundary condition. In this way, the Robin boundary conditions can be realised as a measure perturbation of the Dirichlet form for the Neumann boundary condition (or rather 'intermediate Neumann form' because of our definition of $\mathfrak{h}$); see also [16; § 4.11.6].

Posing the Robin boundary condition is only possible on those parts of Σ where γdS is absolutely continuous with respect to $\mathfrak{h}$-capacity.

(c) Assume that $\mu = \alpha dS$ is not absolutely continuous with respect to $\mathfrak{h}$-capacity, i.e., μ as non-trivial singular part μ_s. From Lemma 2.1 we know that there exists a Borel set $\Sigma_0 \subseteq \Sigma$ such that $\mu_s = \mathbb{1}_{\Sigma_0} \alpha dS$. Then the density of ν_s with respect to μ_s is given by $\frac{\gamma}{\alpha}$, and the semigroup generated by $-A$ acts as $(e^{-t\frac{\gamma}{\alpha}}; t \geqslant 0)$ on $L_2(X_0, \mu_s) = L_2(X_0 \cap \Sigma_0, \alpha dS)$, by Remark 2.8(b).

The following statement supplies a condition guaranteeing that μ and ν are absolutely continuous with respect to $\mathfrak{h}$-capacity. Then μ has no singular part μ_s, and ν is absolutely continuous with respect to τ-capacity.

Proposition 3.7. (a) *In addition to the hypotheses* (H1), (H2) *assume that for all $x \in \Sigma$ there exist an open neighbourhood U and vector fields $Y, Z\colon U \cap X \to \mathbb{R}^d$, Y continuous, satisfying*

$$Y = a^{1/2} Z, \quad \nabla \cdot Y \in L_2(U \cap X), \quad Z \in L_2(U \cap X)^d, \quad n \cdot Y \geqslant 1 \text{ on } U \cap \Sigma.$$

Then the surface measure dS on Σ is absolutely continuous with respect to $\mathfrak{h}$-capacity. As a consequence, (H3) *is satisfied.*

(b) *The additional hypothesis formulated in* (a) *is satisfied if a is a.e. invertible and*

$$a^{-1} \in L_{1,loc}(\Omega \cup \Sigma).$$

For the proof we need the divergence theorem in the following general form.

Lemma 3.8. *Let $w \in C_c(X)^d$ with $\nabla \cdot w \in L_1(\Omega)$ (where $\nabla \cdot w$ is taken in the sense of distributions on Ω). Then*

$$\int_\Omega \nabla \cdot w(x)\, dx = \int_\Sigma w(x) \cdot n(x)\, dS(x),$$

where n is the outward normal on Σ.

Proof. Using a partition of unity one can approximate w by functions having a continuous extension to $\mathbb{R}^d$, with divergence in $L_1(\mathbb{R}^d)$. For such functions, the assertion follows by approximation by smooth functions. □

Proof of Proposition 3.7. (a)(i) In the first step we show that functions in $D(\mathfrak{h})$ have $L_{1,loc}$-trace on Σ.

Let $x \in \Sigma$, and U, Y, Z as in the hypothesis. Let $\varphi \in C_c^1(\mathbb{R}^d)$, $\operatorname{supp} \varphi \subseteq U$, $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ in a neighbourhood of x. For $u \in C_c(X) \cap D(\tilde{\mathfrak{h}})\ (\subseteq W_1^1(\Omega))$, Lemma 3.8 is applicable to the function $\varphi|u|Y$ on X, and we obtain

$$\begin{aligned}
\int_\Sigma \varphi|u|\, dS &\leqslant \int_\Sigma \varphi|u| Y \cdot n\, dS = \int_\Omega \nabla \cdot (\varphi|u|Y)\, dx \\
&= \int_\Omega \nabla\varphi \cdot Y|u|\, dx + \int_\Omega \varphi Y \cdot \nabla|u|\, dx + \int_\Omega \varphi|u|\nabla \cdot Y\, dx \\
&\leqslant \|\nabla\varphi \cdot Y\|_2 \|u\|_2 + \|\varphi Z \cdot a^{1/2}\nabla u\|_1 + \|\varphi \nabla \cdot Y\|_2 \|u\|_2.
\end{aligned}$$

The second term on the right hand side can be estimated by $\|\varphi Z\|_2 \mathfrak{h}(u)^{\frac{1}{2}}$. Thus, the trace mapping $C_c(X) \cap D(\tilde{\mathfrak{h}}) \ni u \mapsto u\restriction_\Sigma \in L_{1,loc}(\Sigma)$ is continuous with respect to the $\mathfrak{h}$-norm on $C_c(X) \cap D(\tilde{\mathfrak{h}})$. Since $C_c(X) \cap D(\tilde{\mathfrak{h}})$ is dense in $D(\mathfrak{h})$ (by definition), we obtain the assertion.

(ii) Let $N \subseteq \Sigma$ be a set of $\mathfrak{h}$-capacity zero, N relatively compact in Σ. Then there exists a sequence $(u_n) \subseteq D(\mathfrak{h})_+$ with $\bigcup_n \operatorname{supp} u_n$ relatively compact in X, $u_n \geqslant 1$ in a neighbourhood of N ($n \in \mathbb{N}$), and $\mathfrak{h}(u_n) \to 0$ as $n \to \infty$. By (i) this implies $\|\mathbb{1}_N\|_{L_1(\Sigma)} \leqslant \|u_n\|_{L_1(\Sigma)} \to 0$ as $n \to \infty$, i.e., N is a dS-null set.

This shows that each set of $\mathfrak{h}$-capacity zero is a dS-null set.

(b) Let $x \in \Sigma$, and let n_x be the outward normal in x. Then, for any $c > 1$, the hypothesis of (a) holds for the constant vector field $Y := cn_x$, $Z := a^{-1/2}Y$, and a suitably small neighbourhood U of x. $\square$

4 The one-dimensional case

In this section we discuss the example $d = 1$, $\Omega = (0,1)$, $\Sigma = \{0,1\}$ in more detail, where the notation is as in Section 3. Then $X = [0,1]$, so $C_c(X) = C[0,1]$. Throughout we assume that $a, a^{-1} \in L_{1,loc}(0,1)$, so (H1) holds (and $\tilde{\mathfrak{h}}$ is closed), by Remark 3.2. A standard hypothesis in papers dealing with the present context is that a is continuous.

In view of the abundant literature for the one-dimensional case, most of the facts presented in this section are likely to be known. The purpose of the following discussion is presenting these facts under the point of view of Sections 2 and 3.

First of all we show that $\tilde{\mathfrak{h}}$ is a regular Dirichlet form; in particular, $\mathfrak{h} = \tilde{\mathfrak{h}}$, and assumption (H2) is automatically fulfilled.

Proposition 4.1. (a) *$D(\tilde{\mathfrak{h}}) \subseteq C(0,1)$.*

(b) *Let*

$$D_0 := \big\{u \in D(\tilde{\mathfrak{h}});\, u\restriction_{[0,\varepsilon]},\, u\restriction_{[1-\varepsilon,1]} \text{ are constant for some } \varepsilon > 0\big\}.$$

Then $D_0 \subseteq C[0,1] \cap D(\tilde{\mathfrak{h}})$, and D_0 is dense in $D(\tilde{\mathfrak{h}})$ as well as in $C[0,1]$. In particular, $\tilde{\mathfrak{h}}$ is a regular Dirichlet form.

Proof. (a) is clear since $D(\tilde{\mathfrak{h}}) \subseteq W^1_{1,loc}(0,1)$.

(b) The first assertion follows from (a). Let $u \in D(\tilde{\mathfrak{h}})$. Then

$$D_0 \ni u_\varepsilon := \mathbb{1}_{(0,\varepsilon)}u(\varepsilon) + \mathbb{1}_{[\varepsilon,1-\varepsilon]}u + \mathbb{1}_{(1-\varepsilon,1)}u(1-\varepsilon) \longrightarrow u$$

in $D(\tilde{\mathfrak{h}})$ as $\varepsilon \to 0$. On the other hand,

$$\big\{u \in C^1[0,1];\, u\restriction_{[0,\varepsilon]},\, u\restriction_{[1-\varepsilon,1]} \text{ are constant for some } \varepsilon > 0\big\} \subseteq D_0$$

is dense in $C[0,1]$. $\square$

As an immediate consequence of $D(\mathfrak{h}) = D(\tilde{\mathfrak{h}}) \subseteq C(0,1)$, for all $x \in (0,1)$ the set $\{x\}$ has positive $\mathfrak{h}$-capacity. In the following we characterise when the endpoints of the interval have positive $\mathfrak{h}$-capacity.

Proposition 4.2. *(cf. Proposition 3.7) The set $\{0\}$ has positive $\mathfrak{h}$-capacity if and only if $a^{-1}\restriction_{(0,\frac{1}{2})} \in L_1$. Analogously, $\mathfrak{h}$-cap$(\{1\}) > 0$ if and only if $a^{-1}\restriction_{(\frac{1}{2},1)} \in L_1$.*

Proof. If $a^{-1}\restriction_{(0,\frac12)} \in L_1$ then $u' = a^{-\frac12}a^{\frac12}u' \in L_{1,loc}[0,1)$ for all $u \in D(\tilde{\mathfrak{h}})$ and hence $D(\tilde{\mathfrak{h}}) \subseteq C[0,1)$. Thus, $\{0\}$ has positive $\mathfrak{h}$-capacity.

Assume now that $a^{-1}\restriction_{(0,\frac12)} \notin L_1$. Define the function $f \in W^1_{1,loc}(0,1)$ by

$$f(x) := 1 + \int_x^{\frac12} \frac{dt}{a(t)} \quad (x \leqslant \tfrac12), \qquad f(x) := 1 \quad (x > \tfrac12).$$

For $u := \ln f$ we then have $u' = \frac{f'}{f} = -\frac{1}{af}$ on $(0,\frac12)$, and hence

$$a|u'|^2 = au' \cdot u' = -\frac{1}{f} \cdot \frac{f'}{f} = \Big(\frac{1}{f}\Big)'.$$

Since $f(x) \to \infty$ as $x \to 0$, this yields $\int_0^1 a|u'|^2 = 1$.

Finally, let $u_n := (\frac{u}{n}) \wedge 1$. Then $u_n \in C[0,1] \cap D(\mathfrak{h})$, $u_n(0) = 1$, $\mathfrak{h}(u_n) \leqslant \frac{1}{n^2}$ for all $n \in \mathbb{N}$, $u_n \to 0$ in $L_2(0,1)$, and therefore $\mathfrak{h}\text{-cap}(\{0\}) = 0$. $\square$

Let α, γ, μ, ν be as in the previous section. Then $\mu = \alpha(0)\delta_0 + \alpha(1)\delta_1$, $\nu = \gamma(0)\delta_0 + \gamma(1)\delta_1$, where δ_j is the Dirac measure in $\{j\}$, for $j = 0, 1$. Assume that, for $j = 0, 1$, $\alpha(j) = \gamma(j) = 0$ if $\mathfrak{h}\text{-cap}(\{j\}) = 0$. Then μ, ν are absolutely continuous with respect to $\mathfrak{h}$-capacity. Moreover, let $\gamma(0), \gamma(1) < \infty$ (see Remark 4.4(b) concerning the case that this is not satisfied). Then the form $\tau + \nu$ is densely defined. Let A be the selfadjoint operator in $L_2(m+\mu)$ and T_p the C_0-semigroup on $L_p(m+\mu)$ associated with $\tau + \nu$.

Proposition 4.3. (a) *The operator A is given by*

$$\begin{aligned} D(A) = D_A := \big\{ u \in D(\tau);\ & au' \textit{ absolutely continuous},\ (au')' \in L_2(0,1), \\ & (-1)^{j+1}(au')(j) + (\gamma u)(j) = 0 \textit{ if } \alpha(j) = 0\ (j = 0,1) \big\} \end{aligned}$$

(where $(\gamma u)(j) = 0$ if $\gamma(j) = \mathfrak{h}\text{-cap}(\{j\}) = 0$; note that u may have no trace at j if $\mathfrak{h}\text{-cap}(\{j\}) = 0$),

$$\begin{aligned} Au &= -(au')' \ m\text{-a.e.}, \\ (Au)(j) &= \tfrac{1}{\alpha(j)}\big((-1)^{j+1}(au')(j) + (\gamma u)(j)\big) \quad \textit{if } \alpha(j) > 0 \quad (j = 0,1). \end{aligned}$$

(b) *If $\frac1a \in L_1(0,1)$ then $T_1(t)\colon L_1(m+\mu) \to C[0,1]$ for all $t > 0$,*

$$\|T_1(t)\colon L_1(m+\mu) \to C[0,1]\| \leqslant ct^{-1} \quad (0 < t \leqslant 1)$$

for some $c > 0$, and $\big(T_1(t)\restriction_{C[0,1]}; t \geqslant 0\big)$ is a C_0-semigroup on $C[0,1]$. In particular, T_1 has the strong Feller property (in the sense of [18; Sec. 2]).

Proof. (a) Observe that $D(\tau+\nu) = D(\tau) = D(\tilde{\mathfrak{h}})$ with equivalence of the norms. Thus, the set D_0 defined in Proposition 4.1(b) is dense in $D(\tau+\nu)$. Let $u \in D(A)$, $f := Au$. Then $u \in D(\tau)$ and $(f|v)_{L_2(m+\mu)} = (\tau+\nu)(u,v)$, i.e.,

$$\int_0^1 fv\,dm + \sum_{j=0}^{1}(\alpha f v)(j) = \int_0^1 au'v'\,dm + \sum_{j=0}^{1}(\gamma uv)(j) \quad (v \in D_0) \tag{4.1}$$

(where $(\alpha f v)(j) = 0$ if $\alpha(j) = 0$; note that $f(j)$ is not defined and v may have no trace at j in the case $\alpha(j) = 0$). For $v \in C_c^\infty(0,1) \subseteq D_0$ the second term on both sides vanishes. We obtain $(au')' = -f\restriction_{(0,1)}$ m-a.e., so au' is absolutely continuous and $Au = -(au')'$ m-a.e. We infer that

$$\int_0^1 fv\,dm = -\int_0^1 (au')'v\,dm = -\lim_{\varepsilon\to 0} au'v|_\varepsilon^{1-\varepsilon} + \int_0^1 au'v'\,dm \quad (v \in D_0) \tag{4.2}$$

and hence, by (4.1),

$$\sum_{j=0}^{1}(\alpha f v)(j) = \lim_{\varepsilon\to 0} au'v|_\varepsilon^{1-\varepsilon} + \sum_{j=0}^{1}(\gamma uv)(j) \quad (v \in D_0). \tag{4.3}$$

This shows $(\alpha Au)(j) = (\alpha f)(j) = (-1)^{j+1}(au')(j) + (\gamma u)(j)$ for $j = 0, 1$. Thus $u \in D_A$, and the representation for Au is valid.

Conversely, let $u \in D_A$. Define the function $f \in L_2(m+\mu)$ by

$$f = -(au')' \ m\text{-a.e.}, \quad f(j) = \tfrac{1}{\alpha(j)}\big((-1)^{j+1}(au')(j) + (\gamma u)(j)\big) \quad \text{if } \alpha(j) > 0$$

for $j = 0, 1$. Then (4.2) and (4.3) are easily verified, so (4.1) holds. Since D_0 is dense in $D(\tau+\nu)$, this shows $u \in D(A)$, $Au = f$.

(b) As in the beginning of the proof of Proposition 4.2 we obtain $D(\tau+\nu) \hookrightarrow C[0,1]$. Since T_2 is an analytic semigroup on $L_2(m+\mu)$, we have $T_2(t)\colon L_2(m+\mu) \to D(\tau+\nu)\,(\subseteq C[0,1])$, thus also $T_1(t)\colon L_1(m+\mu) \to L_2(m+\mu)$, by symmetry and duality. This shows the first assertion. Since $\|u\|_2^4 \leqslant \|u\|_\infty^2\|u\|_1^2 \leqslant c\big((\tau+\nu+1)(u)\big)\|u\|_1^2$ for all $u \in D(\tau+\nu)$, the norm estimate is an easy consequence of [7; Cor. 2.4.7, (ii) $\Longrightarrow$ (i)], applied with $\mu = 2$.

By Proposition 4.1(b), $D(\tau+\nu) \supseteq D_0$ is dense in $C[0,1]$. Moreover, $\big(T_2(t)\restriction_{D(\tau+\nu)}; t \geqslant 0\big)$ is a C_0-semigroup on $D(\tau+\nu)$. This implies the last assertion since T_2 is L_∞-bounded. $\square$

Remarks 4.4. (a) The assumption that $\alpha(j) = \gamma(j) = 0$ if $\mathfrak{h}$-cap$(\{j\}) = 0$ was made for simplicity. It suffices to assume $\gamma(j) = 0$ if $\alpha(j) = \mathfrak{h}$-cap$(\{j\}) = 0$, for $j = 0, 1$. Then ν is still absolutely continuous with respect to τ-capacity. But, as discussed in Remark 2.8(b), T_2 acts trivially on $L_2\big(\{j \in \{0,1\};\ \alpha(j) > 0,\ \mathfrak{h}\text{-cap}(\{j\}) = 0\}, \alpha dS\big)$.

As a consequence, if $\frac{1}{a} \notin L_1(0, \frac{1}{2})$, $\alpha(0) \neq 0$ (or $\frac{1}{a} \notin L_1(\frac{1}{2}, 1)$, $\alpha(1) \neq 0$) then T_2 in general does not act on $C[0,1]$.

(b) We could also allow γ to assume the value ∞. This would lead to Dirichlet boundary conditions on $[\gamma = \infty]$. If $\Sigma_0 := [\gamma = \infty] \cap [\alpha > 0] \neq \varnothing$ then $\tau + \nu$ is not densely defined, and $\overline{D(\tau+\nu)}^{L_2(m+\mu)} = L_2(m + \mu\restriction_{[0,1]\setminus\Sigma_0})$.

(c) It follows from Remark 5.1(b) below that T_1 is stochastic if $\gamma = 0$.

(d) The representation of the operator A given in Proposition 4.3(a) illustrates that the Wentzell boundary condition in L_p actually does not constitute a boundary condition but determines A on $\partial\Omega$. Only in $C[0,1]$ it really is a boundary condition.

5 Conservation of mass

Let $\mathfrak{h}$, μ, τ be as in Section 2, T the C_0-semigroup on $L_2(m)$ associated with $\mathfrak{h}$, and T_μ the C_0-semigroup on $L_2(m+\mu)$ associated with τ. The interpretation of T_μ is that the 'diffusion' described by T is slowed down corresponding to μ. It is thus natural to conjecture that T_μ is stochastic whenever T is stochastic. It is easy to see that this is true if m and μ are finite (see Remark 5.1(b) below).

For general m and μ we cannot answer the question, but in case μ is absolutely continuous with respect to m the answer is positive; see Theorem 5.5. We prove this in the following more general setting.

Let (X, m) be a measure space, $\mathfrak{h}$ a (not necessarily symmetric) Dirichlet form in $L_2(m)$. Let $-A_p$ denote the generator of the associated C_0-semigroup T_p on $L_p(m)$, for $1 \leqslant p < \infty$.

Remark 5.1. (a) It is easy to see that T_1 is stochastic (i.e., T_1 positive and $\|T_1(t)f\|_1 = \|f\|_1$ for all $f \in L_1(m)_+$) if and only if

$$\int A_1 u\, dm = 0 \quad \big(u \in D(A_1)\big).$$

(b) If m is finite then (a) implies that T_1 is stochastic if and only if

$$1 \in D(\mathfrak{h}) \quad \text{and} \quad \mathfrak{h}(u, 1) = 0 \quad \big(u \in D(\mathfrak{h})\big).$$

(For necessity we note that $T_1(t)\mathbb{1} = \mathbb{1}$ since $T_1(t)\mathbb{1} \leqslant \mathbb{1}$ and $\int T_1(t)\mathbb{1}\, dm = \int \mathbb{1}\, dm$; for sufficiency recall that $D(A_1) \cap D(\mathfrak{h})$ is a core for A_1.)

The following proposition is an auxiliary result describing $A_p\restriction_{D(A_p)\cap D(\mathfrak{h})}$ in terms of the form $\mathfrak{h}$.

Proposition 5.2. *Let $p \in [1, \infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $D \subseteq D(\mathfrak{h}) \cap L_{p'}(m)$ a sublattice, D dense in $D(\mathfrak{h})$. Let $u \in L_p(m) \cap D(\mathfrak{h})$, $v \in L_p(m)$. Then the following are equivalent:*

(i) $u \in D(A_p),\ A_p u = v;$
(ii) $\mathfrak{h}(u,\varphi) = \int v\overline{\varphi}\,dm$ *for all* $\varphi \in D(\mathfrak{h}) \cap L_{p'}(m)$*;*
(iii) $\mathfrak{h}(u,\varphi) = \int v\overline{\varphi}\,dm$ *for all* $\varphi \in D$*.*

For the proof of the implication '(iii) $\Longrightarrow$ (ii)' we need the density argument stated in the following lemma. The proof of this result is adapted from [15; Kor. 11.2.5].

Lemma 5.3. *Let* $\eta\colon X \to [0,\infty]$ *be measurable. Let* $1 \leqslant p \leqslant \infty$*,* $D \subseteq D(\mathfrak{h}) \cap L_p(\eta m)$ *a sublattice,* D *dense in* $D(\mathfrak{h})$*.*

(a) *If* $1 \leqslant p < \infty$ *then* D *is dense in* $D(\mathfrak{h}) \cap L_p(\eta m)$*.*

(b) $D \cap B_{L_\infty(\eta m)}$ *is dense in* $D(\mathfrak{h}) \cap B_{L_\infty(\eta m)}$ *with respect to the topology of* $D(\mathfrak{h})$*. (Here,* $B_{L_\infty(\eta m)}$ *denotes the closed unit ball of* $L_\infty(\eta m)$*).*

Proof. (a) Let $u \in D(\mathfrak{h}) \cap L_p(\eta m)$; without restriction $u \geqslant 0$.

In the first step we assume additionally that there exists $\psi \in D$ such that $u \leqslant \psi$. There exists $(\tilde{u}_k) \subseteq D$, $\tilde{u}_k \to u$ in $D(\mathfrak{h})$; without restriction $\tilde{u}_k \to u$ m-a.e. Then $D \ni u_k := (\operatorname{Re}\tilde{u}_k)^+ \wedge \psi \to u$ in $D(\mathfrak{h}) \cap L_p(\eta m)$.

For the general case let $(\tilde{u}_k) \subseteq D$, $\tilde{u}_k \to u$ in $D(\mathfrak{h})$; without restriction $\tilde{u}_k \to u$ m-a.e. Then $u_k := (\operatorname{Re}\tilde{u}_k)^+ \wedge u$ is in the closure of D in $D(\mathfrak{h}) \cap L_p(\eta m)$, by the first step, and $u_k \to u$ in $D(\mathfrak{h}) \cap L_p(\eta m)$.

(b) The proof is analogous to the proof of (a). □

Proof of Proposition 5.2. (i) $\Longrightarrow$ (ii),(iii) Firstly, $\frac{1}{t}(u - T_p(t)u) \to v$ in $L_p(m)$ as $t \to 0$. Secondly, $u \in D(\mathfrak{h})$ implies

$$\big(\tfrac{1}{t}(u - T_p(t)u)\big|\varphi\big) \to \mathfrak{h}(u,\varphi) \quad \big(\varphi \in D(\mathfrak{h})\big)$$

(see, e.g., [19; Lemma 1.1]). This implies the assertion.

(iii) $\Longrightarrow$ (ii) For $1 < p < \infty$ this is immediate from Lemma 5.3(a); for $p = 1$ it follows from Lemma 5.3(b) and the dominated convergence theorem.

(ii) $\Longrightarrow$ (i) First observe that the complex case is an easy consequence of the real case, so we assume $\mathbb{K} = \mathbb{R}$. Let $\check{A}_p := A_p\restriction_{D(A_p)\cap D(A_2)}$. It is easy to see that $D(A_p) \cap D(A_2)$ is a core for A_p, so we obtain that $R(1 + \check{A}_p)$ is dense in $L_p(m)$. Let $\hat{A}_p$ be defined by

$$D(\hat{A}_p) := \big\{u \in D(\mathfrak{h}) \cap L_p(m);\ \exists\, v =: \hat{A}_p u \in L_p(m)\, \forall\, \varphi \in D(\mathfrak{h}) \cap L_{p'}(m) : \\ \mathfrak{h}(u,\varphi) = \textstyle\int v\varphi\,dm\big\}.$$

Then $\check{A}_p \subseteq \hat{A}_p$, by '(i) $\Longrightarrow$ (ii)'. In order to show the assertion it now suffices to show that $\hat{A}_p$ is accretive (i.e., $-\hat{A}_p$ is dissipative). (Then $(1+\check{A}_p)^{-1} \subseteq (1+\hat{A}_p)^{-1}$ have unique continuous extensions to $L_p(m)$ coinciding with $(1 + A_p)^{-1}$; thus $\hat{A}_p \subseteq A_p$.)

Let $\varphi\colon \mathbb{R} \to \mathbb{R}$, $\varphi(t) := |t|^{p-1} \operatorname{sgn} t$ ($= \operatorname{sgn} t$ if $p = 1$). Let $u \in D(\hat{A}_p)$. Then we have to show

$$\int \hat{A}_p u(\varphi \circ u)\, dm \geqslant 0. \tag{5.1}$$

For $k \in \mathbb{N}$ we define $\varphi_k(t) := \bigl(|t|^{p-1} \wedge (k|t|)\bigr) \operatorname{sgn} t$. Then $|\varphi_k(t)| \leqslant |\varphi(t)|$, $\varphi_k(0) = 0$, φ_k monotone increasing and Lipschitz continuous. This implies $\varphi_k \circ u \in D(\mathfrak{h}) \cap L_{p'}(m)$, and by [3; Cor. 9] we obtain

$$\int \hat{A}_p u(\varphi_k \circ u)\, dm = \mathfrak{h}(u, \varphi_k \circ u) \geqslant 0.$$

Finally, $\varphi_k \to \varphi$ pointwise, so (5.1) follows by the dominated convergence theorem. □

Let now $\eta\colon X \to [1, \infty]$ be measurable, and assume that $D(\mathfrak{h}) \cap L_2(\eta m)$ is dense in $D(\mathfrak{h})$ as well as in $L_2(\eta m)$. Similarly to the setup of Section 2, with the measure $\mu := (\eta - 1)m$ (here absolutely continuous with respect to m), we define the Dirichlet form τ in $L_2(\eta m)$ by

$$\tau(u, v) := \mathfrak{h}(u, v) \quad \text{on } D(\tau) := D(\mathfrak{h}) \cap L_2(\eta m).$$

Remarks 5.4. (a) Let D be a sublattice of $D(\mathfrak{h}) \cap L_2(\eta m)$, D dense in $D(\mathfrak{h})$. Then D is dense in $D(\mathfrak{h}) \cap L_2(\eta m) = D(\tau)$, by Lemma 5.3(a). Thus, $\overline{\tau\restriction_D} = \tau$.

(b) For the application to Section 2 we obtain the following. Assume that μ is absolutely continuous with respect to m, $\mu = (\eta - 1)m$. Then $D := C_c(X) \cap D(\mathfrak{h})$ is a sublattice of $D(\mathfrak{h}) \cap L_2(\eta m) = D(\mathfrak{h}) \cap L_2(m + \mu)$ since $m + \mu$ is a Radon measure. Part (a) shows that the form τ defined above coincides with the form τ introduced in Section 2.

A version of part (b) of the following theorem, in a different setting, is contained in [8; Thm. 1].

For $1 \leqslant p < \infty$ we denote by $T_{\eta,p}$ the C_0-semigroup on $L_p(\eta m)$ associated with τ, and by $-A_{\eta,p}$ its generator.

Theorem 5.5. (a) *Let $1 \leqslant p < \infty$, $u \in D(A_{\eta,p}) \cap D(\mathfrak{h})$, $\eta A_{\eta,p} u \in L_p(m)$. Then $u \in D(A_p)$, $\eta A_{\eta,p} u = A_p u$.*

(b) *The operator $A_{\eta,1}$ is given by*

$$D(A_{\eta,1}) = D(A_1) \cap L_1(\eta m), \quad \eta A_{\eta,1} u = A_1 u \quad \bigl(u \in D(A_{\eta,1})\bigr).$$

(c) *Assume that T_1 is stochastic. Then $T_{\eta,1}$ is stochastic.*

Proof. (a) For $\varphi \in D(\tau) \cap L_{p'}(\eta m) =: D$, Proposition 5.2, (i) $\Longrightarrow$ (ii), implies

$$\mathfrak{h}(u, \varphi) = \tau(u, \varphi) = \int A_{\eta,p} u\, \overline{\varphi}\, \eta dm = \int \eta A_{\eta,p} u\, \overline{\varphi}\, dm.$$

Recall that $D(A_{\eta,2}) \cap L_{p'}(\eta m) \subseteq D$ is a core for $A_{\eta,2}$ and hence for τ. Thus, D is dense in $D(\mathfrak{h})$, and D is a sublattice of $D(\mathfrak{h}) \cap L_{p'}(\eta m)$. By Proposition 5.2, (iii) $\Longrightarrow$ (i), we obtain the assertion.

(b) Let $u \in D(A_{\eta,1})$. Since $D(A_{\eta,1}) \cap D(A_{\eta,2})$ is a core for $A_{\eta,1}$, there exists a sequence $(u_k) \subseteq D(A_{\eta,1}) \cap D(A_{\eta,2})$ with $u_k \to u$, $A_{\eta,1}u_k \to A_{\eta,1}u$ in $L_1(\eta m)$. Then $\eta A_{\eta,1}u_k \in L_1(m)$, and by part (a) we obtain $A_1u_k = \eta A_{\eta,1}u_k \to \eta A_{\eta,1}u$ in $L_1(m)$. Therefore $u \in D(A_1)$, $A_1u = \eta A_{\eta,1}u$.

So we have shown $A_{\eta,1} \subseteq \frac{1}{\eta}\check{A}_1$, where $\check{A}_1 := A_1{\restriction}_{D(A_1)\cap L_1(\eta m)}$. In the relation

$$1 + A_{\eta,1} \subseteq 1 + \tfrac{1}{\eta}\check{A}_1 = \tfrac{1}{\eta}(\check{A}_1 + \eta) \tag{5.2}$$

we know that $1 + A_{\eta,1}$ is surjective since $A_{\eta,1}$ generates a contractive C_0-semigroup. On the other hand, $A_1 + (\eta - 1)$ is an accretive operator in $L_1(m)$, and thus $\check{A}_1 + \eta$ is injective. Moreover, $\frac{1}{\eta}\check{A}_1$ is an operator in $L_1(\eta m)$. This implies equality in (5.2).

(c) For $u \in D(A_{\eta,1}) = D(A_1) \cap L_1(\eta m)$ we have $\int A_{\eta,1}u\,\eta dm = \int A_1 u\, dm$. Thus, the assertion follows from Remark 5.1(a). $\square$

Appendix

Let $\tilde{\mathfrak{h}}$ be the maximal form defined in Section 3. It is well-know that the restriction of $\tilde{\mathfrak{h}}$ to $C_c^\infty(\Omega)$ is closable if $a \in W^1_{2,loc}(\Omega)$ (see, e.g., [7; Thm. 1.2.5]). We show that even $\tilde{\mathfrak{h}}$ is closable in this case.

Proposition A.1. *Assume that $a \in W^1_{2,loc}(\Omega)$. Then $\tilde{\mathfrak{h}}$ is closable. Thus, assumption* (H1) *of Section 3 holds.*

In [15; Prop. 14.2.7] this result is shown under the additional assumption $a \in L_{\infty,loc}(\Omega)$. In order to remove this restriction, we need the following product rule for weak derivatives. The crucial point in the lemma is that only the scalar product $\nabla u \cdot v$ is assumed to be locally integrable, not the single components $v_j\partial_j u$.

Lemma A.2. *Let $\mathbb{K} = \mathbb{R}$, and $u \in W^1_{1,loc}(\Omega)$, $v \in W^1_{1,loc}(\Omega)^d$ with*

$$uv_j,\ u\partial_j v_k,\ \nabla u \cdot v \in L_{1,loc}(\Omega) \quad (j, k = 1, \ldots, d).$$

Then $uv \in W^1_{1,loc}(\Omega)^d$, $\nabla \cdot (uv) = \nabla u \cdot v + u\nabla \cdot v$.

Proof. If u and v are bounded then the assertion follows by a standard convolution argument. We deduce the general case from the case of bounded functions.

For $n \in \mathbb{N}$ let $u_n := (u \wedge n) \vee (-n)$, $w_n := (1 + \frac{|v|}{n})^{-1}$, and $v_n := w_n v$. Then $u_n \in W^1_{1,loc}(\Omega)$ and $\nabla u_n = \mathbb{1}_{[|u|<n]}\nabla u$. We claim that $v_n \in W^1_{1,loc}(\Omega)^d$ and

$$\nabla \cdot v_n = w_n \nabla \cdot v - w_n \frac{v}{|v|} \cdot \sum_{j=1}^{d} \frac{v_j}{|v| + n}\partial_j v. \tag{A.3}$$

Indeed, let $\varphi(x) := (1+\frac{|x|}{n})^{-1}x$ and $j \in \{1,\dots,d\}$. Then $v_n = \varphi \circ v$, $\varphi_j \in C_b^1(\mathbb{R}^d)$, $\nabla\varphi_j(x) = (1+\frac{|x|}{n})^{-1}e_j - \frac{1}{n}(1+\frac{|x|}{n})^{-2}\frac{x}{|x|}x_j$. Thus, $\varphi_j \circ v \in W^1_{1,loc}(\Omega)$ and

$$\partial_j(\varphi_j \circ v) = (\nabla\varphi_j)\circ v \cdot \partial_j v = w_n \partial_j v_j - (n+|v|)^{-1} w_n \frac{v}{|v|} v_j \cdot \partial_j v.$$

This shows (A.3).

Since u_n, v_n are bounded, we have

$$\nabla \cdot (u_n v_n) = \nabla u_n \cdot v_n + u_n \nabla \cdot v_n.$$

Now the assertion follows since $u_n v_n \to uv$ in $L_{1,loc}(\Omega)^d$ and

$$\nabla u_n \cdot v_n = \mathbb{1}_{[|u|<n]} w_n \nabla u \cdot v \to \nabla u \cdot v,$$

$$u_n \nabla \cdot v_n = w_n u_n \nabla \cdot v - w_n \frac{v}{|v|} \cdot \sum_{j=1}^{d} \frac{v_j}{|v|+n} u_n \partial_j v \to u \nabla \cdot v$$

in $L_{1,loc}(\Omega)$ as $n \to \infty$, by the dominated convergence theorem. □

Proof of Proposition A.1. Since $\tilde{\mathfrak{h}}$ is real (i.e., $\tilde{\mathfrak{h}}(u) = \tilde{\mathfrak{h}}(\operatorname{Re} u) + \tilde{\mathfrak{h}}(\operatorname{Im} u)$ for all $u \in D(\tilde{\mathfrak{h}})$), we can assume $\mathbb{K} = \mathbb{R}$. We have to show that for all $(u_n) \subseteq D(\tilde{\mathfrak{h}})$ with $\tilde{\mathfrak{h}}(u_n - u_m) \to 0$, $\|u_n\|_2 \to 0$ as $n, m \to \infty$ one has $\tilde{\mathfrak{h}}(u_n) \to 0$ as $n \to \infty$. It is easy to see that it suffices to show $\tilde{\mathfrak{h}}(u_n, \varphi) \to 0$ for all $\varphi \in D(\tilde{\mathfrak{h}})$.

Let $W_a := \big\{ f \in L_{1,loc}(\Omega)^d;\ af \cdot f \in L_1(\Omega) \big\}$. Then $\langle f, g\rangle_a := \int af \cdot g\, dx$ defines a semi-scalar product on W_a. It is readily seen that $C_c^\infty(\Omega)^d$ is dense in W_a with respect to this semi-scalar product. Moreover, $D(\tilde{\mathfrak{h}}) = \{u \in L_2(\Omega);\ \nabla u \in W_a\}$. We have to show

$$\tilde{\mathfrak{h}}(u_n, \varphi) = \langle \nabla u_n, \nabla\varphi\rangle_a \to 0 \quad \big(\varphi \in D(\tilde{\mathfrak{h}})\big).$$

The sequence (∇u_n) is bounded in W_a since $\|\nabla u_n - \nabla u_m\|^2_{W_a} = \tilde{\mathfrak{h}}(u_n - u_m) \to 0$ as $n, m \to \infty$, so it suffices to show

$$\langle \nabla u_n, \Phi\rangle_a \to 0 \quad \text{for all } \Phi \in C_c^\infty(\Omega)^d.$$

From $u_n \in L_2(\Omega)$, $a\Phi \in W^1_{2,c}(\Omega)^d$, and $\nabla u_n, \Phi \in W_a$ it follows that

$$u_n a\Phi \in L_1(\Omega)^d, \quad u_n \partial_j (a\Phi)_k \in L_1(\Omega) \quad (j,k = 1,\dots,d), \quad \nabla u_n \cdot a\Phi \in L_1(\Omega).$$

By Lemma A.2 we thus obtain $u_n a\Phi \in W^1_{1,c}(\Omega)^d$ and

$$\nabla \cdot (u_n a\Phi) = \nabla u_n \cdot a\Phi + u_n \nabla \cdot (a\Phi).$$

Since $\int \nabla \cdot f\, dx = 0$ for all $f \in W^1_{1,c}(\Omega)^d$, $u_n \to 0$ in $L_2(\Omega)$, and $\nabla \cdot (a\Phi) \in L_2(\Omega)$, we conclude that

$$\langle \nabla u_n, \Phi\rangle_a = -\int u_n \nabla \cdot (a\Phi)\, dx \to 0 \quad (n \to \infty). \qquad \square$$

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