Form Methods for Evolution Equations

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Preface

Evolution equations are differential equations governing the time evolution of functions describing the state of a system. Many equations from Mathematical Physics belong to this class; a prototype is the heat equation with various boundary conditions whose solutions describe how the temperature distribution in the system evolves in time.

Hilbert spaces are particularly well suited for formulating and treating evolution equations. A physical reason is that the energy is naturally given by an L_2-norm, but much more important is a mathematical reason: the Fréchet–Riesz theorem on the representation of linear functionals by scalar products gives weak solutions to many equations in a straightforward way. A more elaborate version is the Lax–Milgram lemma which is formulated for more general sesquilinear forms. It even turns out that such forms defined on a Hilbert space (under suitable conditions as coercivity) already yield a well-posedness result for the associated evolution equation. More technically speaking, to each (coercive) form defined on a Hilbert space is associated the generator of a holomorphic semigroup which governs the evolution in time.

The study of forms, their associated generators and semigroups, is the subject of the present book. We concentrate on form methods, which are also sometimes called variational methods, and study semigroups mainly on Hilbert spaces. There are many books on semigroups, among the most popular ones, the book by Engel and Nagel [EnNa00]. The book we present here is complementary in many aspects. The book of Engel and Nagel does not contain form methods, whereas the general theory of semigroups is only a very small part of the material presented here. The generation of holomorphic semigroups is treated by using Euler's exponential formula for semigroups, and no contour integrals are needed.

The theory of forms is very rich; important subjects are well-posedness, perturbation, convergence of semigroups and forms, the Trotter product formula and many others. They all are particularly strong versions for semigroups coming from a form. The Trotter product formula for the sum of two forms is a very nice example, and our treatment (of results of T. Kato and B. Simon, Corollary 15.10) is completely new in book form as well as many other things. We mention the recent 'j-method' (see Section 5.3) which leads to an elegant treatment of the Dirichlet-to-Neumann operator. A big role in our book is also played by the Beurling–Deny criteria characterising positivity of the semigroup or more generally the invariance of convex sets.

It is not only the efficiency that makes forms attractive, they also occur naturally in many applications. In the book we present a thorough treatment of parabolic equations with boundary conditions, the Stokes equation is the subject of one chapter, the Dirichlet-to-Neumann operator is analysed, and finally non-autonomous forms are considered at the end of the book. Maximal regularity is a highly topical subject. We give a criterion in the non-autonomous case and show how it can be applied to nonlinear problems in the last chapter.

So far we have given a brief description of the topics of this book. What we do not treat is the interesting relation to stochastics (for which we refer to the book by Ma and Röckner [MaRo92]), or to numerical mathematics. The book by Ouhabaz [Ouh05] shares with ours the description of invariance, but the overlap is small.

Concerning the organisation and style of the book it may be appropriate to mention that it grew out and is an augmented version of the Internet Seminar "Form Methods for Evolution Equations, and Applications" in the academic year 2014/15. The material is presented in 19 chapters, each of which is mainly devoted to one special subject, specified in the title of the chapter. It is one of our principles to provide complete information on subjects from other areas of analysis that are used in the development of the material and in applications, such as elements of operator theory or vector-valued holomorphic functions. To achieve this aim we insert 'interludes' in the chapters where we give full proofs of results exceeding a certain level in analysis and functional analysis.

However, we also provide proofs of various other tools that could not reasonably be included as interludes. These are added as appendices, containing for instance the Stone–Weierstrass theorem, a treatment of Hausdorff measure and an inequality due to Maz'ya related to the isoperimetric inequality, the spectral theorem for self-adjoint operators, a treatment of singular integrals, and the Brouwer and Schauder fixed point theorems.

In detailed Notes at the end of each chapter we explain more general results and give further information on the subject of the chapter.

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Chapter 1

C_0-semigroups

C_0-semigroups serve to describe the time evolution of autonomous linear systems. The objective of the present chapter is to introduce the notion of C_0-semigroups and their generators, and to derive some basic properties. In an "interlude" we provide some fundamental facts concerning operators as well as integration and differentiation of Banach space valued functions.

1.1 Motivation

Let X be a Banach space, and let $\mathcal{L}(X)$ denote the space of bounded linear operators on X. A C_0-semigroup is a function $T\colon [0,\infty) \to \mathcal{L}(X)$ associated with the solutions of the initial value problem for a linear autonomous differential equation on $[0,\infty)$,

$$u' = Au, \quad u(0) = x.$$

Here, A should be a suitable (usually unbounded) linear operator in X, defined on its domain $\mathrm{dom}(A) \subseteq X$. If $x \in \mathrm{dom}(A)$, then the function $u\colon [0,\infty) \to X$ given by $u(t) := T(t)x$ should be the unique solution of the initial value problem given above. These properties will be studied in more detail in the present chapter.

If the operator A is bounded, then the problem can be treated by the usual methods of ordinary differential equations. The aim of the theory of C_0-semigroups is to describe the solution theory for the case when A is a suitable unbounded operator. Typically, in applications the Banach space will be a space of functions, defined on an open subset of $\mathbb{R}^n$, and the operator A will be a partial differential operator. A prototypical example is the heat equation

$$\partial_t u = \Delta u$$

which will be treated in the context of C_0-semigroups.

1.2 Definition and some basic properties

Let X be a (real or complex) Banach space. A **one-parameter semigroup** on X is a function $T\colon [0,\infty) \to \mathcal{L}(X)$ satisfying

(i) $T(t+s) = T(t)T(s)$, for all $t, s \geqslant 0$.

If additionally

(ii) $\lim_{t\to 0+} T(t)x = x$ for all $x \in X$,

then T is called a $\boldsymbol{C_0}$**-semigroup** on X (also called a **strongly continuous semigroup** by many authors).

If T is defined on $\mathbb{R}$ instead of $[0,\infty)$, and property (i) holds for all $t, s \in \mathbb{R}$, then T is called a **one-parameter group**, and if additionally (ii) holds, then T is called a $\boldsymbol{C_0}$**-group**.

1.1 Remarks. (a) Property (i) implies that for $t, s \geqslant 0$ the operators $T(t)$, $T(s)$ commute; also, if $t_1, t_2, \ldots, t_n \geqslant 0$, then $T\big(\sum_{j=1}^n t_j\big) = \prod_{j=1}^n T(t_j)$.

(b) Property (i) implies that $T(0) = T(0)^2$ is a projection.

(c) If T is a C_0-semigroup, then $T(0)x = \lim_{t\to 0+} T(t)T(0)x = \lim_{t\to 0+} T(t)x = x$ for all $x \in X$, i.e. $T(0) = I$, the identity operator in X. △

In property (i) one immediately recognises the functional equation for the exponential function, and in fact this will be our first example for a C_0-group.

1.2 Example. Let $A \in \mathcal{L}(X)$. Then

$$T(t) := \mathrm{e}^{tA} = \sum_{j=0}^{\infty} \frac{1}{j!}(tA)^j$$

converges in $\mathcal{L}(X)$ for all $t \in \mathbb{R}$ and defines a C_0-group T. In fact, $\mathbb{R} \ni t \mapsto T(t) \in \mathcal{L}(X)$ is even continuous with respect to the operator norm.

We leave this as an exercise (see Exercise 1.1). △

1.3 Lemma. *Let T be a one-parameter semigroup on X, and assume that there exists $\delta > 0$ such that $M := \sup_{0\leqslant t<\delta} \|T(t)\| < \infty$. Then there exists $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leqslant M\mathrm{e}^{\omega t} \qquad (t \geqslant 0).$$

Proof. There exists $\omega \geqslant 0$ such that $\|T(\delta)\| \leqslant \mathrm{e}^{\omega\delta}$. For $t \geqslant 0$ there exists $n \in \mathbb{N}_0$ such that $n\delta \leqslant t < (n+1)\delta$. The semigroup property (i) implies $T(t) = T(t-n\delta)T(\delta)^n$, and therefore

$$\|T(t)\| \leqslant \|T(t-n\delta)\|\|T(\delta)\|^n \leqslant M\mathrm{e}^{\omega(n\delta)} \leqslant M\mathrm{e}^{\omega t}.$$ □

1.4 Proposition. *Let T be a C_0-semigroup on X.*

(a) *Then there exist $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leqslant M\mathrm{e}^{\omega t} \qquad (t \geqslant 0).$$

(b) *For all $x \in X$ the function $[0,\infty) \ni t \mapsto T(t)x \in X$ is continuous.*

(c) *If T is a C_0-group on X, then there exist $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leqslant M\mathrm{e}^{\omega|t|} \qquad (t \in \mathbb{R}).$$

For all $x \in X$ the function $\mathbb{R} \ni t \mapsto T(t)x \in X$ is continuous.

Part (b) of this proposition states that the function T is 'strongly continuous'; for the definition of this notion and more information we refer to Subsection 2.1.2.)

Proof of Proposition 1.4. (a) In view of Lemma 1.3 it is sufficient to show that there exists $\delta > 0$ such that $\sup_{0\leqslant t<\delta} \|T(t)\| < \infty$. Assuming that there is no such δ, we can find a null sequence (t_n) such that $\|T(t_n)\| \to \infty$ as $n \to \infty$. However, for all $x \in X$ the sequence $(T(t_n)x)$ is convergent (to x), by property (ii) of C_0-semigroups. Therefore the uniform boundedness theorem (see e.g. [Yos68; II,1 Corollary 1] or [Bre11; Theorem 2.2]) implies that $\sup_{n\in\mathbb{N}} \|T(t_n)\| < \infty$; a contradiction.

(b) Let $x \in X$, $t > 0$. Then $T(t+h)x - T(t)x = T(t)(T(h)x - x) \to 0$ as $h \to 0+$, which proves the right-sided continuity of $T(\cdot)x$. In order to prove the left-sided continuity we let $-t \leqslant h < 0$ and write $T(t+h)x - T(t)x = T(t+h)(x - T(-h)x)$. Then we obtain

$$\|T(t+h)x - T(t)x\| \leqslant \Big(\sup_{0\leqslant s\leqslant t} \|T(s)\|\Big)\|x - T(-h)x\| \to 0$$

as $h \to 0-$.

(c) First we show that, given $x \in X$, the orbit $T(\cdot)x$ is continuous. As the restriction of T to $[0,\infty)$ is a C_0-semigroup it follows from (b) that $T(\cdot)x$ is continuous on $[0,\infty)$. If $t \leqslant 0$, then $T(t+h)x - T(t)x = T(t-1)(T(1+h)x - T(1)x) \to 0$ as $h \to 0$. Hence, $T(\cdot)x$ is continuous on $\mathbb{R}$.

As a consequence, the function $[0,\infty) \ni t \mapsto T(-t) \in \mathcal{L}(X)$ is a C_0-semigroup, and therefore satisfies an estimate as in (a). Putting the estimates for the C_0-semigroups $t \mapsto T(t)$ and $t \mapsto T(-t)$ together one obtains the asserted estimate. □

In examples it is sometimes not immediately clear how to prove the strong continuity property (ii) of a one-parameter semigroup, whereas the boundedness condition of Lemma 1.3 is easy to verify. The following condition is useful for such examples.

1.5 Lemma. *Let T be a one-parameter semigroup on X. Assume that $\sup_{0\leqslant t\leqslant 1} \|T(t)\| < \infty$ and that there exists a dense subset D of X such that $\lim_{t\to 0+} T(t)x = x$ for all $x \in D$. Then T is a C_0-semigroup.*

This lemma is an immediate consequence of the next proposition, a fundamental fact of operator theory.

Let X, Y be Banach spaces. By $\mathcal{L}(X,Y)$ we denote the space of bounded linear operators from X to Y. (Writing $\mathcal{L}(X,Y)$ we will always tacitly assume that the two Banach spaces are over the same scalar field $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$.) A sequence (B_n) in $\mathcal{L}(X,Y)$ is called **strongly convergent** to $B \in \mathcal{L}(X,Y)$, abbreviated $B = \text{s-}\lim_{n\to\infty} B_n$, if $Bx = \lim_{n\to\infty} B_n x$ for all $x \in X$. The uniform boundedness theorem implies that then the sequence (B_n) is bounded.

1.6 Proposition. *Let X, Y be Banach spaces, and let $(B_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{L}(X,Y)$. Assume that the sequence $(B_n x)_{n\in\mathbb{N}}$ is convergent for all x in a dense subset of X.*

Then $Bx := \lim_{n\to\infty} B_n x$ exists for all $x \in X$, and $B\colon X \to Y$ thus defined is an operator $B \in \mathcal{L}(X,Y)$. In other words, $B_n \to B$ strongly for some $B \in \mathcal{L}(X,Y)$.

Proof. A standard $\varepsilon/3$-argument shows that $(B_n x)$ is a Cauchy sequence in Y, for all $x \in X$, and therefore convergent. The linearity and boundedness of B are then easy to show. □

We note that in applications of this proposition, the limiting operator is often already known, but the pointwise convergence is only known on a dense subset. (In the application to Lemma 1.5, for instance, one has $B = I$.)

If Ω is a set and $A \subseteq \Omega$, then $\mathbf{1}_A$ will denote the **indicator function** (also known as the characteristic function) of A,

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in \Omega \setminus A. \end{cases}$$

1.7 Examples (Right translation on $L_p(\mathbb{R})$, $L_p(-\infty, 0)$, $L_p(0, \infty)$ and $L_p(0, 1)$, for $1 \leqslant p < \infty$). (a) On $L_p(\mathbb{R})$: For $t \in \mathbb{R}$ we define $T(t) \in \mathcal{L}(L_p(\mathbb{R}))$ by

$$T(t)f(x) := f(x - t) \qquad \big(x \in \mathbb{R},\ f \in L_p(\mathbb{R})\big).$$

It is clear that $T(t)$ is an isometric isomorphism for all $t \in \mathbb{R}$. Also, it is easy to show that T is a one-parameter group. Let $a, b \in \mathbb{R}$, $a < b$, and put $f := \mathbf{1}_{[a,b]}$. Then it is easy to see that $T(t)f \to f$ as $t \to 0$. This carries over to all linear combinations of such indicator functions. Now the linear span D of all these indicator functions is dense in $L_p(\mathbb{R})$. Therefore Lemma 1.5 implies that T is a C_0-group.

(b) On $L_p(-\infty, 0)$: The operator $T(t)$, for $t \geqslant 0$, is defined by

$$T(t)f(x) := f(x - t) \qquad \big(x \in (-\infty, 0),\ f \in L_p(-\infty, 0)\big).$$

In this case the operators $T(t)$ are not isometric for $t > 0$, but they satisfy $\|T(t)\| = 1$. Again it is easy to see that T is a one-parameter semigroup (not a group), and as in (a) one shows that T is a C_0-semigroup.

(c) On $L_p(0, \infty)$: Denote by S the C_0-semigroup of right translations on $L_p(\mathbb{R})$, defined in (a), but with time parameter t restricted to $[0, \infty)$. Consider $L_p(0, \infty)$ as the subspace $\big\{f \in L_p(\mathbb{R});\ f|_{(-\infty,0)} = 0\big\}$ of $L_p(\mathbb{R})$. Clearly the semigroup operators $S(t)$ leave this subspace invariant, and therefore the restriction T of S to this subspace is a C_0-semigroup. The operator $T(t)$ is isometric for all $t \geqslant 0$. However, for $t > 0$ the operator $T(t)$ is not surjective.

(d) On $L_p(0, 1)$: Analogously to part (b) we can define the C_0-semigroup of right translations S on $L_p(-\infty, 1)$. Similarly to (c) one shows that $S(t)$ leaves the subspace $L_p(0, 1)$ of $L_p(-\infty, 1)$ invariant, and therefore the restriction T of S to $L_p(0, 1)$ is a C_0-semigroup. This semigroup has the property that $\|T(t)\| = 1$ for $0 \leqslant t < 1$, and that $T(t) = 0$ for $t \geqslant 1$. Because of the latter property it is called the *nilpotent* right translation semigroup on $L_p(0, 1)$. △

1.3 Interlude: operators, integration and differentiation

1.3.1 Operators

We have already indicated in Section 1.1 that, besides bounded linear operators between normed spaces, one also needs linear operators defined on subspaces. One can (and should) think of an operator A from X to Y as a linear mapping $A\colon \mathrm{dom}(A) \to Y$, where $\mathrm{dom}(A)$ is a subspace of X. For some purposes, however, it is convenient to go back to the original meaning of a mapping as a relation in $X \times Y$ possessing certain additional properties.

Let X, Y be two vector spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For a **linear relation** A in $X \times Y$, i.e. a subspace of $X \times Y$, we define the **domain** of A,

$$\mathrm{dom}(A) := \{x \in X\,;\ \text{there exists } y \in Y \text{ such that } (x, y) \in A\},$$

the **range** of A,

$$\mathrm{ran}(A) := \{y \in Y\,;\ \text{there exists } x \in X \text{ such that } (x, y) \in A\},$$

and the **kernel** (or **null space**) of A,

$$\ker(A) := \{x \in X\,;\ (x, 0) \in A\}.$$

The linear relation

$$A^{-1} := \{(y, x)\,;\ (x, y) \in A\}$$

in $Y \times X$ is the **inverse (relation)** of A. If B is another linear relation in $X \times Y$, satisfying $A \subseteq B$, then B is called an **extension** of A, and A a **restriction** of B. If the spaces X and Y coincide and A is a linear relation in $X \times X$, then we will also call A a linear relation in X.

In this setting, a **linear operator from X to Y** is a linear relation in $X \times Y$ satisfying additionally

$$A \cap (\{0\} \times Y) = \{(0, 0)\}.$$

Then, for all $x \in \mathrm{dom}(A)$, there exists a unique $y \in Y$, such that $(x, y) \in A$, and we will write $Ax = y$. In this sense, A is also a mapping $A\colon \mathrm{dom}(A) \to Y$. As we will consider only linear operators we will mostly drop 'linear' and simply speak of 'operators'. If the spaces X and Y coincide, then we call A an **operator in X**.

Next, let X and Y be Banach spaces. We define a norm on $X \times Y$ by

$$\|(x, y)\|_{X\times Y} := \|x\|_X + \|y\|_Y \qquad ((x, y) \in X \times Y),$$

which makes $X \times Y$ a Banach space. In this context an operator A from X to Y (or a linear relation $A \subseteq X \times Y$) is called **closed** if A is a closed subset of $X \times Y$. A linear operator A from X to Y is **closable** if its **closure** $\overline{A}$ in $X \times Y$ is an operator; we refer to Exercises 1.2 and 1.3 for examples in which $\overline{A}$ is not an operator.

For a subspace $D \subseteq \mathrm{dom}(A)$, the **restriction** of A to D is the operator $A|_D := A \cap (D \times Y)$. The set D is called a **core for** A if A is a restriction of the closure of $A|_D$, i.e. $A \subseteq \overline{A|_D}$.

Finally, if A and B are operators from X to Y, then the **sum** of A and B is the operator defined by

$$\operatorname{dom}(A+B) := \operatorname{dom}(A) \cap \operatorname{dom}(B), \qquad (A+B)x := Ax + Bx \quad (x \in \operatorname{dom}(A+B)),$$

or, expressed differently,

$$A + B = \big\{(x, Ax + Bx)\,;\, x \in \operatorname{dom}(A) \cap \operatorname{dom}(B)\big\} \subseteq X \times Y.$$

We will only use the sum of two operators in the case when one of the operators belongs to $\mathcal{L}(X,Y)$ and hence is defined everywhere.

1.3.2 Integration of continuous functions

Let $a, b \in \mathbb{R}$, $a < b$, and let X be a Banach space. We define the space of **step functions** from $[a,b]$ to X,

$$T([a,b];X) := \operatorname{lin}\{\mathbf{1}_{[s,t]}(\cdot)x\,;\, a \leqslant s \leqslant t \leqslant b,\ x \in X\},$$

where $\mathbf{1}_{[s,t]}(\cdot)x$ denotes the function $r \mapsto \mathbf{1}_{[s,t]}(r)x$ and 'lin' denotes the linear span. (The letter 'T' stands for the German 'Treppenfunktion'.) For $f \in T([a,b];X)$, $f = \sum_{j=1}^{n} \mathbf{1}_{[s_j,t_j]}(\cdot)x_j$, we define the integral

$$\int_a^b f(t)\,\mathrm{d}t := \sum_{j=1}^{n} (t_j - s_j)x_j\,.$$

It is standard to show that this integral is well-defined, that the mapping $f \mapsto \int_a^b f(t)\,\mathrm{d}t$ is linear, and that

$$\left\| \int_a^b f(t)\,\mathrm{d}t \right\| \leqslant (b-a) \sup_{a \leqslant t \leqslant b} \|f(t)\| \qquad \big(f \in T([a,b];X)\big). \tag{1.1}$$

In short, the mapping $T([a,b];X) \ni f \mapsto \int_a^b f(t)\,\mathrm{d}t \in X$ is a bounded linear operator, where $T([a,b];X)$ is provided with the supremum norm. This implies that there is a unique continuous extension of the integral to functions in the closure of $T([a,b];X)$ in the Banach space

$$\ell_\infty([a,b];X) := \{f\colon [a,b] \to X\,;\, f \text{ bounded}\},$$

provided with the supremum norm. We denote this closure by $R([a,b];X)$ (the space of 'regulated functions') and observe that $R([a,b];X)$ contains the subspace of continuous functions $C([a,b];X)$. Moreover, the extension of the integral is linear, and the inequality (1.1) carries over to all $f \in R([a,b];X)$. For $X = \mathbb{R}$, all the functions in $R([a,b];X)$ are Riemann integrable, and the integral defined above is the Riemann integral.

Next we describe how operators act on integrals.

1.8 Theorem. *Let X, Y be Banach spaces, and let $a, b \in \mathbb{R}$, $a < b$.*
(a) *Let $f\colon [a,b] \to X$ be continuous, and let $A \in \mathcal{L}(X,Y)$. Then*

$$A \int_a^b f(t)\,\mathrm{d}t = \int_a^b Af(t)\,\mathrm{d}t.$$

(b) *Let A be a closed operator from X to Y. Let $f\colon [a,b] \to X$ be continuous, $f(t) \in \operatorname{dom}(A)$ for all $t \in [a,b]$, and $t \mapsto Af(t) \in Y$ continuous. Then $\int_a^b f(t)\,\mathrm{d}t \in \operatorname{dom}(A)$, and*

$$A \int_a^b f(t)\,\mathrm{d}t = \int_a^b Af(t)\,\mathrm{d}t.$$

Proof. (a) The equality is clear for step functions and carries over to continuous functions (in fact, even to regulated functions) by continuous extension.

(b) The hypotheses are just a complicated way to express that one is given a continuous function $t \mapsto (f(t), g(t)) \in A \subseteq X \times Y$ (where $g(t) = Af(t)$). Because A is a closed subspace of $X \times Y$, and therefore a Banach space, it follows that $\int_a^b (f(t), g(t))\,\mathrm{d}t \in A$. As the canonical projections from $X \times Y$ to X and Y are bounded linear operators, one concludes from part (a) that

$$\Big(\int_a^b f(t)\,\mathrm{d}t, \int_a^b g(t)\,\mathrm{d}t\Big) = \int_a^b (f(t), g(t))\,\mathrm{d}t \in A,$$

and this shows the assertions. □

The last issue in this interlude is the connection between differentiation and integration, i.e., the fundamental theorem of differential and integral calculus for Banach space valued functions.

1.9 Theorem. *Let $a, b \in \mathbb{R}$, $a < b$.*
(a) *Let $f\colon [a,b] \to X$ be continuous,*

$$F(t) := \int_a^t f(s)\,\mathrm{d}s \qquad (a \leqslant t \leqslant b).$$

Then F is continuously differentiable, and $F' = f$.
(b) *Let $g\colon [a,b] \to X$ be continuously differentiable. Then*

$$\int_a^b g'(t)\,\mathrm{d}t = g(b) - g(a). \tag{1.2}$$

Proof. (a) is proved in the same way as for scalar-valued functions; cf. Exercise 1.7(a).

(b) For $\eta \in X'$ ($= \mathcal{L}(X, \mathbb{K})$, the dual space of X) the function $\eta \circ g$ is continuously differentiable, and one has $(\eta \circ g)' = \eta \circ g'$. The fundamental theorem of differential and integral calculus for $\mathbb{K}$-valued functions then implies that

$$\begin{aligned}\eta\Big(\int_a^b g'(t)\,\mathrm{d}t\Big) &= \int_a^b \eta(g'(t))\,\mathrm{d}t = \int_a^b (\eta \circ g)'(t)\,\mathrm{d}t \\ &= (\eta \circ g)(b) - (\eta \circ g)(a) = \eta\big(g(b) - g(a)\big).\end{aligned}$$

As this holds for all $\eta \in X'$, equation (1.2) follows from an application of the theorem of Hahn–Banach (see e.g. [Bre11; Section 1.1] or [Voi20; Appendix A]). □

1.4 The generator of a C_0-semigroup

Let X be a Banach space. For a C_0-semigroup T on X we define the **generator** (also called the **infinitesimal generator**) A, an operator in X, by

$$A := \big\{(x,y) \in X \times X\,;\, \tfrac{1}{h}(T(h)x - x) \to y \ (h \to 0+)\big\}.$$

In other words, since $T(0)x = x$, the domain of A consists of those $x \in X$ for which the orbit $t \mapsto T(t)x$ is (right-sided) differentiable at $t = 0$, and the image of x under A is the derivative of this orbit at $t = 0$.

1.10 Example. If $A \in \mathcal{L}(X)$, then A is the generator of the C_0-group $(\mathrm{e}^{tA})_{t\in\mathbb{R}}$; see Example 1.2 and Exercise 1.1. △

Similarly to the property that for the ordinary exponential function the derivative at 0 determines the function, we will see that the generator determines the C_0-semigroup.

First we derive some fundamental properties of the generator.

1.11 Theorem. *Let T be a C_0-semigroup on X, with generator A. Then:*

(a) *For $x \in \mathrm{dom}(A)$ one has $T(t)x \in \mathrm{dom}(A)$ for all $t \geqslant 0$, the function $t \mapsto T(t)x$ is continuously differentiable on $[0,\infty)$, and*

$$\frac{\mathrm{d}}{\mathrm{d}t}T(t)x = AT(t)x = T(t)Ax \qquad (t \geqslant 0).$$

(b) *For all $x \in X$, $t > 0$ one has $\int_0^t T(s)x\,\mathrm{d}s \in \mathrm{dom}(A)$,*

$$A\int_0^t T(s)x\,\mathrm{d}s = T(t)x - x.$$

(c) $\mathrm{dom}(A)$ *is dense in X, and A is a closed operator.*

Before the proof we insert a small fact on strong convergence of operators.

1.12 Lemma. *Let X, Y be Banach spaces, and let (B_n) be a sequence in $\mathcal{L}(X,Y)$, $B_n \to B \in \mathcal{L}(X,Y)$ strongly as $n \to \infty$. Let (x_n) in X, $x_n \to x \in X$ as $n \to \infty$.*

Then $B_n x_n \to Bx$ as $n \to \infty$.

Proof. The uniform boundedness theorem implies that $M := \sup_{n\in\mathbb{N}} \|B_n\| < \infty$. Therefore

$$\begin{aligned}\|Bx - B_n x_n\| &\leqslant \|Bx - B_n x\| + \|B_n(x - x_n)\| \\ &\leqslant \|Bx - B_n x\| + M\|x - x_n\| \to 0 \qquad (n \to \infty).\end{aligned}$$ □

Proof of Theorem 1.11. (a) For $t \geqslant 0$, $h > 0$ one has

$$\frac{1}{h}\big(T(t+h)x - T(t)x\big) = \frac{1}{h}(T(h) - I)T(t)x = T(t)\frac{1}{h}(T(h)x - x).$$

As $h \to 0$, the third of these expressions converges to $T(t)Ax$. Looking at the second term, one obtains $T(t)x \in \mathrm{dom}(A)$, and looking at the first term one concludes that $t \mapsto T(t)x$ is right-sided differentiable, with right-sided derivative

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)_{\mathrm{r}} T(t)x = AT(t)x = T(t)Ax.$$

On the other hand, let $t > 0$, $h \in (0,t)$. Then

$$\frac{1}{-h}\big(T(t-h)x - T(t)x\big) = T(t-h)\frac{1}{h}\big(T(h)x - x\big),$$

and this converges to $T(t)Ax$ as $h \to 0$, by Lemma 1.12.

So we have shown that the continuous function $t \mapsto AT(t)x = T(t)Ax$ is the derivative of $t \mapsto T(t)x$.

(b) Let $0 < h < t$. Then Theorem 1.8(a) implies

$$\begin{aligned}
&\frac{1}{h}(T(h) - I)\int_0^t T(s)x\,\mathrm{d}s \\
&\quad = \frac{1}{h}\bigg(\int_h^{t+h} - \int_0^t\bigg)T(s)x\,\mathrm{d}s = \frac{1}{h}\bigg(\int_t^{t+h} - \int_0^h\bigg)T(s)x\,\mathrm{d}s \\
&\quad = (T(t) - I)\,\frac{1}{h}\int_0^h T(s)x\,\mathrm{d}s \to (T(t) - I)x \qquad (h \to 0+),
\end{aligned}$$

where the last convergence follows from Theorem 1.9(a). This shows $\int_0^t T(s)x\,\mathrm{d}s \in \mathrm{dom}(A)$ as well as the asserted equality. (In the above computation we have used basic rules of the Riemann integral that are easily seen to be valid for Banach space valued functions as well.)

(c) Let $x \in X$. Then $h^{-1}\int_0^h T(s)x\,\mathrm{d}s \in \mathrm{dom}(A)$ for all $h > 0$, by part (b). Theorem 1.9(a) implies that $h^{-1}\int_0^h T(s)x\,\mathrm{d}s \to x$ $(h \to 0+)$. This shows that $\mathrm{dom}(A)$ is dense in X.

Let $\big((x_n, y_n)\big)_{n\in\mathbb{N}}$ be a sequence in A, $(x_n, y_n) \to (x, y)$ in $X \times X$. From (a) and Theorem 1.9(b) we obtain

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n\,\mathrm{d}s$$

for all $t > 0$, $n \in \mathbb{N}$. For $n \to \infty$ one concludes that

$$T(t)x - x = \int_0^t T(s)y\,\mathrm{d}s,$$

and then

$$\frac{1}{t}(T(t)x - x) = \frac{1}{t}\int_0^t T(s)y\,\mathrm{d}s \to y \qquad (t \to 0+).$$

This shows that $(x, y) \in A$. □

Theorem 1.11(a) implies in particular that the function $t \mapsto T(t)x$ solves the initial value problem $u' = Au,\ u(0) = x$. We now show that this solution is unique.

1.13 Theorem. *Let T be a C_0-semigroup on X, with generator A.*

(a) *Let $t_0 > 0$, and let $u\colon [0,t_0) \to X$ be continuous, $u(t) \in \operatorname{dom}(A)$ for all $t \in (0,t_0)$, u continuously differentiable on $(0,t_0)$, and $u'(t) = Au(t)$ for all $t \in (0,t_0)$. Then $u(t) = T(t)u(0)$ for all $t \in [0,t_0)$.*

(b) *Let S be a C_0-semigroup on X, with generator $B \supseteq A$. Then $S = T$, $B = A$.*

Proof. (a) Let $0 < t < t_0$. Lemma 1.12 implies that the function $[0,t] \ni s \mapsto T(t-s)u(s) \in X$ is continuous, and it is not difficult to see that this function is differentiable on $(0,t]$, with derivative

$$\frac{\mathrm{d}}{\mathrm{d}s} T(t-s)u(s) = -T(t-s)Au(s) + T(t-s)u'(s) = 0$$

(a kind of product rule). Therefore Theorem 1.9(b) implies

$$T(t)u(0) = \lim_{s\to 0+} T(t-s)u(s) = T(t-t)u(t) = u(t).$$

(b) Let $x \in \operatorname{dom}(A)$, and put $u(t) := T(t)x$ $(t \geqslant 0)$. Then u satisfies the equation $u'(t) = Au(t) = Bu(t)$ $(t \geqslant 0)$. From part (a) it follows that then $u(t) = S(t)u(0) = S(t)x$, for all $t \geqslant 0$. This shows that $S(t) = T(t)$ on $\operatorname{dom}(A)$. As $\operatorname{dom}(A)$ is dense in X one obtains $S(t) = T(t)$ for all $t \geqslant 0$. The equality of the semigroups implies equality of the generators. □

If A is the generator of a C_0-semigroup T, then sometimes one uses the notation $\mathrm{e}^{tA} := T(t)$ $(t \geqslant 0)$, analogously to the case when $A \in \mathcal{L}(X)$.

1.14 Remark. In Theorems 1.11 and 1.13 we have established important properties of C_0-semigroups and their generators which we summarise as follows, putting them into the context of 'well-posedness'. Let A be the generator of a C_0-semigroup T. Then the initial value problem (also called Cauchy problem)

$$u'(t) = Au(t) \quad (t \geqslant 0), \qquad u(0) = x \in \operatorname{dom}(A)$$

is **well-posed**, i.e., (i) it possesses a unique solution (given by $u(t) = T(t)x$), and (ii) the solution depends continuously on the initial value, which in the present (linear) case is expressed by $\sup_{0\leqslant t\leqslant t_0} \|u(t)\| \leqslant \sup_{0\leqslant t\leqslant t_0} \|T(t)\|\|x\|$ for all $t_0 \in (0,\infty)$.

These properties also serve as a motivation to characterise operators which are generators; see Chapter 2. △

For determining the generator of the C_0-semigroups described in Examples 1.7 the following result will be useful.

1.15 Proposition (Nelson's lemma). *Let T be a C_0-semigroup on X, and let A be its generator. Let D be a subspace of* $\operatorname{dom}(A)$ *that is dense in X, and assume that D is invariant under T (i.e. $T(t)(D) \subseteq D$ for all $t \geqslant 0$).*

Then D is a core for A.

Proof. Let $x \in \mathrm{dom}(A)$. We have to show that $(x, Ax) \in \overline{A|_D}$.

Let (x_n) be a sequence in D, $x_n \to x$ $(n \to \infty)$ in X. Let $n \in \mathbb{N}$, $t > 0$. The function $[0,t] \ni s \mapsto \big(T(s)x_n, AT(s)x_n\big) \in A|_D$ is continuous; therefore (recall Theorem 1.11(a) and Theorem 1.9(b) for the first equality)

$$\begin{aligned}\Big(\int_0^t T(s)x_n \,\mathrm{d}s, T(t)x_n - x_n\Big) &= \Big(\int_0^t T(s)x_n \,\mathrm{d}s, \int_0^t AT(s)x_n \,\mathrm{d}s\Big) \\ &= \int_0^t \big(T(s)x_n, AT(s)x_n\big) \,\mathrm{d}s \in \overline{A|_D}\,.\end{aligned}$$

Taking $n \to \infty$, we conclude that

$$\Big(\int_0^t T(s)x \,\mathrm{d}s, T(t)x - x\Big) \in \overline{A|_D}\,.$$

Dividing by t and taking $t \to 0+$ one obtains $(x, Ax) \in \overline{A|_D}$. □

1.16 Example. Let T be the C_0-group of right translations on $L_p(\mathbb{R})$ (introduced in Example 1.7(a)). It is not difficult to show that $D := C_c^1(\mathbb{R})$ (the continuously differentiable functions with compact support) is a subspace of $\mathrm{dom}(A)$, and $Af = -f'$ for all $f \in C_c^1(\mathbb{R})$. Also $C_c^1(\mathbb{R})$ is obviously invariant under T, and is a dense subspace of $L_p(\mathbb{R})$. Thus D is a core for A, by Proposition 1.15. For the reader familiar with Sobolev spaces: this implies that $\mathrm{dom}(A) = W_p^1(\mathbb{R})$. For $p = 2$ we refer to Example 4.17(a).

We refer to Exercise 1.4 for the case of translation semigroups on continuous functions. △

Notes

The classical treatise on one-parameter semigroups is the monograph by Hille and Phillips [HiPh57]. The notation 'C_0-semigroup' goes back to Phillips [Phi55]; in [HiPh57; Section 10.6] it is put into context with other properties of one-parameter semigroups. In those days other continuity conditions were also considered, for instance

$$\frac{1}{t}\int_0^t T(s)x \,\mathrm{d}s \to x \qquad (t \to 0)$$

for all $x \in X$, which leads to a 'C_1-semigroup'.

Many authors use the terminology 'strongly continuous semigroup' instead of C_0-semigroup. However, in Chapter 13 we will encounter semigroups T which are strongly continuous at the origin, while $T(0)$ is not necessarily the identity. For this reason we prefer the terminology 'C_0-semigroup'. Here is a – by far incomplete – list of books on C_0-semigroups: [Dav80], [Paz83], [Gol85], [Nag86], [EnNa00].

The idea of using the semigroup invariance to establish a core property, as in Proposition 1.15, is due to Nelson: in [Nel59; proof of Lemma 5.1] he implemented this idea for unitary C_0-groups on Hilbert spaces.

A core can also be seen as a 'space of uniqueness'. In fact, given a subspace D of the domain $\mathrm{dom}(A)$ of the generator A of a C_0-semigroup the following holds. If D is a core for A, then Theorem 1.13(b) implies that A is the only extension of $A|_D$ generating a C_0-semigroup, whereas there are infinitely many such extensions if D is not a core; see [Nag86; Part A-II, Theorem 1.33].

The concept of well-posedness that we have introduced in Remark 1.14 is generally attributed to Jacques Hadamard. Indeed, in [Had02] Hadamard discussed problems for the Laplace equation and the wave equation, and he investigated whether they are "parfaitement bien posé" (perfectly well posed), by which he meant that they are "possible" and "déterminé" (i.e. the solution exists and is unique). Only much later, in [Had23; §19, p. 35], he discussed continuous dependence on the initial data, for the wave equation.

To our knowledge, the three properties of existence, uniqueness and continuous dependence were summed up to one notion for the first time in the book [CoHi37; III, §7] by Courant and Hilbert. For a problem from mathematical physics to be "properly posed" they set "the following basic requirements:

(1) The solution must exist.

(2) The solution should be uniquely determined.

(3) The solution should depend continuously on the data (requirement of stability)."

(Our quotation is from the English edition [CoHi62; III, §6].) In many cases the converse of (3) is obvious, namely that the data depend continuously on the solution. Then, for linear problems, continuous dependence of the solution on the data follows automatically from existence and uniqueness, by the bounded inverse theorem (see e.g. [Yos68; Chapter II, Section 5] or [Bre11; Corollary 2.7]).

We refer to [Goc16; Section 2.3.3] for more information on well-posedness. In Chapter 19 we will investigate some nonlinear problems that are *not* well-posed.

Equations that involve time are frequently called *evolution equations*; the idea is that the solution evolves in time from a given initial value. In this context, an autonomous driving mechanism and well-posedness together with linearity lead naturally to the concept of a C_0-semigroup. In this book we will see that many physical evolutionary problems are well-posed and yield a semigroup. So, Hille's famous sentence from the foreword of his monograph [Hil48] from 1948 will be confirmed: "I hail a semi-group when I see one and I seem to see them everywhere!"

Exercises

1.1 Let X be a Banach space, and let $A, B \in \mathcal{L}(X)$ satisfy $AB = BA$.

(a) Show that $\mathrm{e}^A := \sum_{j=0}^{\infty} \frac{1}{j!} A^j$ is absolutely convergent in $\mathcal{L}(X)$, $\|\mathrm{e}^A\| \leqslant \mathrm{e}^{\|A\|}$, and that $\mathrm{e}^{A+B} = \mathrm{e}^A \mathrm{e}^B = \mathrm{e}^B \mathrm{e}^A$.

(b) Show that $t \mapsto \mathrm{e}^{tA}$ is a one-parameter group, that the function is continuously differentiable as an $\mathcal{L}(X)$-valued function, and that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA} = A\mathrm{e}^{tA}$ $(t \in \mathbb{R})$. (In particular, A is the generator of $(\mathrm{e}^{tA})_{t\in\mathbb{R}}$.)

(c) Show that $\mathrm{e}^{zI} = \mathrm{e}^z I$ for all $z \in \mathbb{K}$.

1.2 Let $x_0 \in [0,1]$, and define the operator A from $L_2(0,1)$ to $\mathbb{K}$ by $\mathrm{dom}(A) := C[0,1]$,

$$Af := f(x_0) \qquad (f \in C[0,1]).$$

Show that $\overline{A} = L_2(0,1) \times \mathbb{K}$. (Thus A is not closable.)

1.3 Let X and Y be Banach spaces.

(a) Let A be an operator from X to Y. Show that A is closable if and only if for every null sequence (x_n) in $\mathrm{dom}(A)$ for which (Ax_n) is convergent one has $\lim_{n\to\infty} Ax_n = 0$.

(b) Let (x_n) be a linearly independent null sequence in X, and let (y_n) be a sequence in Y converging to some element $y \neq 0$ in Y. Define the operator from X to Y by extending the assignments $Ax_n := y_n$ for all $n \in \mathbb{N}$ by linearity to $\mathrm{dom}(A) := \mathrm{lin}\{x_1, x_2, \dots\}$. Show that the operator A is not closable.

1.4 For a locally compact subset G of $\mathbb{R}^n$ we define $C_c(G) := \{f \in C(G)\,;\, \mathrm{spt}\, f \text{ compact}\}$. (Here $C(G)$ denotes the space of continuous $\mathbb{K}$-valued functions, and the **support** of f is defined by $\mathrm{spt}\, f := \overline{[f \neq 0]}^G$, where $[f \neq 0] := \{x \in G\,;\, f(x) \neq 0\}$. We recall that a topological space is called locally compact if every point possesses a compact neighbourhood. Local compactness is not needed for solving this exercise; it is required because the notation C_c and C_0 is only common for locally compact spaces.)

The space $C_0(G)$ is defined as the closure of $C_c(G)$ in $C_b(G)$ (the space of bounded continuous functions, provided with the supremum norm).

(a) Show that every function in $C_0(G)$ is uniformly continuous.

(b) Show that

$$C_0(G) = \{f \in C(G)\,;\, \forall \varepsilon > 0 \colon [|f| \geqslant \varepsilon] \text{ compact}\}$$

(where $[|f| \geqslant \varepsilon] := \{x \in G\,;\, |f(x)| \geqslant \varepsilon\}$). (Hint: For the inclusion $\supseteq$ show first that it is sufficient to treat the case of real-valued functions. Approximate real-valued functions f by $\max\{f - \varepsilon, 0\} + \min\{f + \varepsilon, 0\}$.)

(c) Show that

$$C_0(-\infty, 0] = \big\{f \in C(-\infty, 0]\,;\, \lim_{x\to-\infty} f(x) = 0\big\}$$

and that $C_0(0,1] \cong \{f \in C[0,1]\,;\, f(0) = 0\}$.

1.5 (a) Convince yourself that analogously to Example 1.7 one can define the one-parameter semigroup T of right translations on each of the spaces $C_0(\mathbb{R})$, $C_0(-\infty, 0]$, $C_0(0,\infty)$, $C_0(0,1]$ (defined in Exercise 1.4), and show that T is a C_0-semigroup.

(b) Show that the generator A of T on $C_0(\mathbb{R})$ is given by

$$\mathrm{dom}(A) = \big\{f \in C^1(\mathbb{R})\,;\, f, f' \in C_0(\mathbb{R})\big\}, \qquad Af = -f'.$$

(c) Show that the generator A of T on $C_0(0,1]$ is given by

$$\mathrm{dom}(A) = \big\{f \in C^1(0,1]\,;\, f, f' \in C_0(0,1]\big\}, \qquad Af = -f'.$$

1.6 Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $1 \leqslant p < \infty$. Let $a\colon \Omega \to \mathbb{K}$ be a measurable function.

(a) The **maximal multiplication operator** M_a **induced** by the function a is defined by

$$M_a = \big\{(f,g) \in L_p(\mu) \times L_p(\mu)\,;\, g = af\big\}.$$

Show that M_a is closed and densely defined.

(b) Assume additionally that $\operatorname{Re} a(x) \leqslant 0$ $(x \in \Omega)$. For $t \geqslant 0$, define $T(t) \in \mathcal{L}(L_p(\mu))$ by

$$T(t)f := \mathrm{e}^{ta} f \qquad (f \in L_p(\mu)).$$

Show that T is a C_0-semigroup on $L_p(\mu)$ and that M_a is the generator of T.

1.7 Let X be a Banach space.

(a) Let $f\colon [0,1] \to X$ be continuous. Show that $\lim_{h\to 0+} h^{-1}\int_0^h f(t)\,\mathrm{d}t = f(0)$. (This is the main step for proving Theorem 1.9(a).)

(b) Let T be a one-parameter semigroup on X that is continuous with respect to the operator norm. Show that then the generator A of T is an operator $A \in \mathcal{L}(X)$. (Hint: Show that, for small t, the operator $\int_0^t T(s)\,\mathrm{d}s$ is invertible in $\mathcal{L}(X)$; this is a consequence of the Neumann series, which we will recall in Remark 2.3(a). Note that Theorem 1.11(b) implies that the operator $x \mapsto A\int_0^t T(s)x\,\mathrm{d}s$ belongs to $\mathcal{L}(X)$.)

Chapter 2

Characterisation of generators of C_0-semigroups

Generators of C_0-semigroups have special spectral properties. They will be studied in Section 2.2. The main result of this chapter is the Hille–Yosida theorem which characterises generators of C_0-semigroups. The exponential formula for C_0-semigroups presented in Section 2.3 will be important for applications. We start with an interlude on spectral theory of operators as well as on more integration.

2.1 Interlude: the resolvent of operators, and some more integration

2.1.1 Resolvent set, spectrum and resolvent

Let X be a Banach space over $\mathbb{K}$, and let A be an operator in X.

We define the **resolvent set** of A,

$$\rho(A) := \big\{\lambda \in \mathbb{K};\ \lambda I - A\colon \operatorname{dom}(A) \to X \text{ bijective},\ (\lambda I - A)^{-1} \in \mathcal{L}(X)\big\}.$$

The mapping

$$R(\cdot, A)\colon \rho(A) \to \mathcal{L}(X), \quad \lambda \mapsto R(\lambda, A) := (\lambda I - A)^{-1}$$

is called the **resolvent** of A. The **spectrum** of A is the set

$$\sigma(A) := \mathbb{K} \setminus \rho(A).$$

2.1 Remarks. (a) If $\rho(A) \neq \varnothing$ and $\lambda \in \rho(A)$, then $(\lambda I - A)^{-1} \in \mathcal{L}(X)$ is closed – note that every operator belonging to $\mathcal{L}(X)$ is closed. Hence $\lambda I - A$ is closed, and therefore A is closed, by the reasoning presented subsequently in part (b).

(b) If A is a closed operator and $B \in \mathcal{L}(X)$, then the sum $A + B$ is a closed operator. Indeed, if $\big((x_n, y_n)\big)$ is a sequence in $A + B$, $(x_n, y_n) \to (x, y)$ in $X \times X$ $(n \to \infty)$, then $Bx_n \to Bx$, and therefore $Ax_n = (A + B)x_n - Bx_n \to y - Bx$, and the hypothesis that A is closed implies that $(x, y - Bx) \in A$, i.e., $x \in \operatorname{dom}(A)$ and $y = Ax + Bx$.

(c) Let A be a closed operator. Assume that $\lambda \in \mathbb{K}$ is such that $\lambda I - A\colon \mathrm{dom}(A) \to X$ is bijective. Then the inverse $(\lambda I - A)^{-1}$ is a closed operator which is defined on all of X. Therefore the closed graph theorem (see e.g. [Yos68; II.6, Theorem 1] or [Bre11; Theorem 2.9]) implies that $(\lambda I - A)^{-1} \in \mathcal{L}(X)$. It follows that

$$\rho(A) = \{\lambda \in \mathbb{K};\, \lambda I - A\colon \mathrm{dom}(A) \to X \text{ bijective}\}.$$

(d) Usually, in treatments of operator theory, the above notions are only defined for the case of complex Banach spaces, because many important results of spectral theory depend on complex analysis of one variable. For our purpose it is – for the moment – possible and convenient to include the case of real scalars. △

Before proceeding we include a piece of notation that will be used in different contexts. If (M, d) is a metric space, $x \in M$ and $r \in (0, \infty]$, then

$$B(x, r) := \{y \in M\,;\, d(y, x) < r\} \quad \text{and} \quad B[x, r] := \{y \in M\,;\, d(y, x) \leqslant r\}$$

are the **open ball** and **closed ball** with centre x and radius r, respectively. If necessary, we might also write $B_M(x, r)$ and $B_M[x, r]$ for making it clear in which metric space we are.

The following theorem contains the basic results concerning the resolvent.

2.2 Theorem. *Let A be a closed operator in X.*
(a) *If $\lambda \in \rho(A)$, $x \in \mathrm{dom}(A)$, then $AR(\lambda, A)x = R(\lambda, A)Ax$.*
(b) *For all $\lambda, \mu \in \rho(A)$ one has the* ***resolvent equation***

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\mu, A)R(\lambda, A); \tag{2.1}$$

in particular, the resolvents $R(\mu, A)$, $R(\lambda, A)$ commute.
(c) *For $\lambda \in \rho(A)$ one has $B\big(\lambda, \frac{1}{\|R(\lambda,A)\|}\big) \subseteq \rho(A)$, and for $\mu \in B\big(\lambda, \frac{1}{\|R(\lambda,A)\|}\big)$ one has*

$$R(\mu, A) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda, A)^{n+1}. \tag{2.2}$$

As a consequence, $\rho(A)$ is an open subset of $\mathbb{K}$, and $R(\cdot, A)\colon \rho(A) \to \mathcal{L}(X)$ is analytic.

2.3 Remarks. (a) For the proof of Theorem 2.2 we recall the **Neumann series**: if $B \in \mathcal{L}(X)$ satisfies $\|B\| < 1$, then $I - B$ is invertible in $\mathcal{L}(X)$, and the inverse is given by $(I - B)^{-1} = \sum_{n=0}^{\infty} B^n$, with absolute convergence of the series.

(b) Let $A \in \mathcal{L}(X)$. Then part (a) implies that $\lambda I - A = \lambda\big(I - \frac{1}{\lambda}A\big)$ is invertible in $\mathcal{L}(X)$ for all $\lambda \in \mathbb{K}$ with $|\lambda| > \|A\|$, with inverse

$$(\lambda I - A)^{-1} = \sum_{j=0}^{\infty} \frac{1}{\lambda^{j+1}} A^j.$$

As a consequence, $\{\lambda \in \mathbb{K};\, |\lambda| > \|A\|\} \subseteq \rho(A)$. △

Proof of Theorem 2.2. (a) $AR(\lambda,A)x - \lambda R(\lambda,A)x = -x = R(\lambda,A)Ax - R(\lambda,A)\lambda x$.

(b) Multiplying the identity

$$(\mu I - A) - (\lambda I - A) = (\mu - \lambda) I|_{\mathrm{dom}(A)}$$

from the right by $R(\lambda,A)$ and from the left by $R(\mu,A)$, one obtains the resolvent equation.

(c) Let $\lambda \in \rho(A)$ and $\mu \in B\big(\lambda, \frac{1}{\|R(\lambda,A)\|}\big)$. Then the operator $I - (\lambda-\mu)R(\lambda,A)$ is invertible in $\mathcal{L}(X)$ since $|\lambda-\mu|\|R(\lambda,A)\| < 1$ (Neumann series). Therefore the identity

$$\mu I - A = (\lambda I - A) - (\lambda - \mu) I = \big(I - (\lambda-\mu)R(\lambda,A)\big)(\lambda I - A)$$

shows that the mapping $\mu I - A \colon \mathrm{dom}(A) \to X$ is bijective, and hence $\mu \in \rho(A)$. Moreover

$$R(\mu,A) = R(\lambda,A)\big(I - (\lambda-\mu)R(\lambda,A)\big)^{-1},$$

and the formula (2.2) for the resolvent is then a consequence of the Neumann series. Now it follows that $\rho(A)$ is an open subset of $\mathbb{K}$, and the analyticity of $R(\cdot,A)$ (meaning that $R(\cdot,A)$ can be written as a power series about every point of $\rho(A)$) is a consequence of (2.2) as well. □

2.4 Remarks. (a) Theorem 2.2(c) shows that $\|R(\lambda,A)\| \geqslant \mathrm{dist}(\lambda,\sigma(A))^{-1}$ for all $\lambda \in \rho(A)$. Therefore the norm of the resolvent has to blow up if λ approaches $\sigma(A)$.

(b) As in first year analysis, the analyticity of $R(\cdot,A)$ implies that $R(\cdot,A)$ is infinitely differentiable, and from the power series (2.2) one can read off the derivatives,

$$\Big(\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big)^n R(\lambda,A) = (-1)^n\, n!\, R(\lambda,A)^{n+1} \qquad (\lambda \in \rho(A),\ n \in \mathbb{N}_0). \qquad \triangle$$

2.1.2 Integration of operator-valued functions, and improper integrals

2.5 Proposition. *Let X, Y be Banach spaces, $a, b \in \mathbb{R}$, $a < b$. Let $F \colon [a,b] \to \mathcal{L}(X,Y)$ be strongly continuous. Then the mapping*

$$X \ni x \mapsto \int_a^b F(t)x \,\mathrm{d}t \in Y$$

belongs to $\mathcal{L}(X,Y)$ and has norm less or equal $\int_a^b \|F(t)\| \,\mathrm{d}t$.

Some comments: **Strongly continuous** means that $t \mapsto F(t)x$ is continuous for all $x \in X$. (In other words, it means that F is continuous with respect to the **strong operator topology** on $\mathcal{L}(X,Y)$, which is defined as the initial topology with respect to the family of mappings $\big(\mathcal{L}(X,Y) \ni A \mapsto Ax \in Y\big)_{x\in X}$.) It will be part of the proof that the function $\|F(\cdot)\|$ is bounded and measurable.

Proof of Proposition 2.5. (i) First we observe that, in view of the uniform boundedness theorem, the boundedness of $\{\|F(t)x\|\,;\ t \in [a,b]\}$ for all $x \in X$ implies that $\{\|F(t)\|\,;\ t \in [a,b]\}$ is bounded.

Next we note that it is easy to see that, for all $\alpha \geqslant 0$, the set $\{t \in [0,1]\,;\, \|F(t)\| \leqslant \alpha\}$ is closed, and this shows that $\|F(\cdot)\|$ is measurable. (For completeness, we mention that the previous property is called **lower semi-continuity** of $\|F(\cdot)\|$.)

(ii) The linearity of the mapping $x \mapsto \int_a^b F(t)x\,\mathrm{d}t$ is obvious. For $x \in X$ we estimate

$$\left\| \int_a^b F(t)x\,\mathrm{d}t \right\| \leqslant \int_a^b \|F(t)x\|\,\mathrm{d}t \leqslant \int_a^b \|F(t)\|\,\mathrm{d}t\,\|x\|,$$

and this shows the norm estimate for the mapping. □

Abbreviating, we will write $\int_a^b F(t)\,\mathrm{d}t$ for the mapping defined in Proposition 2.5. This integral is called the **strong integral**; one has to keep in mind that, in general, it is not the integral of an $\mathcal{L}(X,Y)$-valued function as treated in Subsection 1.3.2.

We will also need 'improper integrals' of continuous Banach space valued functions. For simplicity we restrict our attention to integrals over $[0,\infty)$ (because this is what will be needed next).

2.6 Proposition. *Let X be a Banach space, let $f\colon [0,\infty) \to X$ be continuous, and assume that the function $[0,\infty) \ni t \mapsto \|f(t)\|$ is integrable. Then*

$$\int_0^\infty f(t)\,\mathrm{d}t := \lim_{a\to\infty} \int_0^a f(t)\,\mathrm{d}t$$

exists.

We omit the (easy) proof of this proposition and mention that Proposition 2.5 has its analogue for these improper integrals.

2.2 Characterisation of generators of C_0-semigroups

In this section let X be a Banach space.

2.7 Theorem. *Let T be a C_0-semigroup on X, and let A be its generator. Let $M \geqslant 1$, $\omega \in \mathbb{R}$ be such that*

$$\|T(t)\| \leqslant M\mathrm{e}^{\omega t} \qquad (t \geqslant 0)$$

(cf. Proposition 1.4).

Then $\{\lambda \in \mathbb{K};\, \mathrm{Re}\,\lambda > \omega\} \subseteq \rho(A)$, and for all $\lambda \in \mathbb{K}$ with $\mathrm{Re}\,\lambda > \omega$ one has

$$R(\lambda,A) = \int_0^\infty \mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t \quad (\textit{strong improper integral; see Subsection 2.1.2}),$$

$$\|R(\lambda,A)^n\| \leqslant \frac{M}{(\mathrm{Re}\,\lambda - \omega)^n} \qquad (n \in \mathbb{N}). \tag{2.3}$$

The formula for the resolvent in this theorem means that $\lambda \mapsto R(\lambda,A)$ is the Laplace transform of the semigroup; see [ABHN11; Chapter 2].

In the proof we will use the concept of rescaling. If T is a C_0-semigroup on X with generator A, $\lambda \in \mathbb{K}$, and one defines

$$T_\lambda(t) := \mathrm{e}^{-\lambda t} T(t) \qquad (t \geqslant 0),$$

then T_λ is also a C_0-semigroup, called a **rescaled semigroup**, and the generator of T_λ is given by $A - \lambda I$; see Exercise 2.2.

Proof of Theorem 2.7. Let $\lambda \in \mathbb{K}$, $\operatorname{Re}\lambda > \omega$. Observe that the rescaled semigroup T_λ satisfies the estimate

$$\|T_\lambda(t)\| \leqslant M \mathrm{e}^{(\omega - \operatorname{Re}\lambda)t} \qquad (t \geqslant 0)$$

and that the resolvent of A at λ corresponds to the resolvent of $A - \lambda I$ at 0. This means that it is sufficient to prove the existence and the formula of the resolvent for the case $\omega < 0$ and $\lambda = 0$.

The estimate $\|T(t)\| \leqslant M\mathrm{e}^{\omega t}$ $(t \geqslant 0)$ implies that the strong improper integral

$$R := \int_0^\infty T(t)\,\mathrm{d}t$$

defines an operator $R \in \mathcal{L}(X)$. On the one hand, for all $x \in \operatorname{dom}(A)$ we have

$$RAx = \int_0^\infty T(t)Ax\,\mathrm{d}t = \lim_{a\to\infty} \int_0^a \frac{\mathrm{d}}{\mathrm{d}t} T(t)x\,\mathrm{d}t = \lim_{a\to\infty} (T(a)x - x) = -x.$$

On the other hand, if $x \in X$, then by Theorem 1.11(b) we obtain $A\int_0^a T(t)x\,\mathrm{d}t = T(a)x - x \to -x$ as $a \to \infty$. Since $\int_0^a T(t)x\,\mathrm{d}t \to Rx$ as $a \to \infty$ and A is closed, it follows that $Rx \in \operatorname{dom}(A)$, $ARx = -x$. Thus we have shown the two equalities $RA = -I|_{\operatorname{dom}(A)}$ and $AR = -I$, which imply that $0 \in \rho(A)$ and $R = (-A)^{-1}$.

For the powers of $R(\lambda, A)$ we now obtain (recall Remark 2.4(b))

$$\begin{aligned} R(\lambda, A)^n &= (-1)^{n-1} \frac{1}{(n-1)!} \Big(\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big)^{n-1} \int_0^\infty \mathrm{e}^{-\lambda t} T(t)\,\mathrm{d}t \\ &= \frac{1}{(n-1)!} \int_0^\infty t^{n-1} \mathrm{e}^{-\lambda t} T(t)\,\mathrm{d}t. \end{aligned} \tag{2.4}$$

(The last equality is obtained by differentiation under the integral; we delegate the details to Exercise 2.1. See also the subsequent Remark 2.8.) By Proposition 2.5 this yields the estimate

$$\begin{aligned} \|R(\lambda, A)^n\| &\leqslant \frac{1}{(n-1)!}\, M \int_0^\infty t^{n-1} \mathrm{e}^{(\omega - \operatorname{Re}\lambda)t}\,\mathrm{d}t \\ &= \frac{1}{(n-1)!}\, M \Big(\frac{\mathrm{d}}{\mathrm{d}\omega}\Big)^{n-1} \int_0^\infty \mathrm{e}^{(\omega - \operatorname{Re}\lambda)t}\,\mathrm{d}t \\ &= \frac{1}{(n-1)!}\, M \Big(\frac{\mathrm{d}}{\mathrm{d}\omega}\Big)^{n-1} \frac{1}{\operatorname{Re}\lambda - \omega} = \frac{M}{(\operatorname{Re}\lambda - \omega)^n}\,. \end{aligned}$$

□

2.8 Remark. We will mainly be interested in C_0-semigroups satisfying the estimate of Proposition 1.4(a) with $M = 1$, in which case the semigroup is called **quasi-contractive**. For such C_0-semigroups it is sufficient to prove the resolvent estimate (2.3) in Theorem 2.7 for $n = 1$ (because then taking powers one obtains the estimate for all $n \in \mathbb{N}$). Note that for $n = 1$ the second equality in (2.4) is trivial.

The resolvent estimate (2.3) can also be shown by a reduction to the case of contractive C_0-semigroups; cf. Exercise 2.3. △

Next we show that the necessary conditions for the generator we have derived so far are also sufficient. We restrict ourselves to the quasi-contractive case and delegate the general case to Exercise 2.4.

2.9 Theorem (Hille–Yosida, quasi-contractive case). *Let A be a closed, densely defined operator in X. Assume that there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A)$ and*

$$\|R(\lambda, A)\| \leqslant \frac{1}{\lambda - \omega} \qquad (\lambda \in (\omega, \infty)).$$

Then A is the generator of a C_0-semigroup T satisfying the estimate

$$\|T(t)\| \leqslant \mathrm{e}^{\omega t} \qquad (t \geqslant 0).$$

As a preliminary remark we note that it is sufficient to treat the case $\omega = 0$. Indeed, $\tilde{A} := A - \omega I$ satisfies the conditions of Theorem 2.9 with $\omega = 0$. Having obtained the contractive C_0-semigroup $\widetilde{T}$ with generator $\tilde{A}$ one obtains the C_0-semigroup generated by $A = \tilde{A} + \omega I$ as the rescaled semigroup $\widetilde{T}_{-\omega}$.

We now define the **Yosida approximations**

$$A_n := A\Big(I - \frac{1}{n}A\Big)^{-1} = nAR(n, A) = n^2 R(n, A) - nI \in \mathcal{L}(X) \qquad (n \in \mathbb{N})$$

of A. The proof of Theorem 2.9 will consist of three steps:

In the first step we show that the semigroups generated by A_n are contractive.

In the second step we show that these semigroups converge strongly to a C_0-semigroup.

In the third step we show that A is the generator of the limiting semigroup.

Before proceeding with the proof we explain why the operators A_n can be considered as approximations of A; note that the operator occurring in part (b) of the following lemma equals A_n for $\lambda = n \in \mathbb{N}$.

2.10 Lemma. *Let A be a closed, densely defined operator in X. Assume that there exists $\lambda_0 \geqslant 0$ such that $(\lambda_0, \infty) \subseteq \rho(A)$ and $M := \sup_{\lambda > \lambda_0} \|\lambda R(\lambda, A)\| < \infty$. Then:*

(a) *$\lambda R(\lambda, A)x \to x$ $(\lambda \to \infty)$ for all $x \in X$.*

(b) *$A\big(\lambda R(\lambda, A)\big)x \to Ax$ $(\lambda \to \infty)$ for all $x \in \mathrm{dom}(A)$.*

Proof. (a) If $x \in \mathrm{dom}(A)$, then

$$\lambda R(\lambda, A)x = (\lambda I - A + A)R(\lambda, A)x = x + R(\lambda, A)Ax \to x \qquad (\lambda \to \infty).$$

As $\|\lambda R(\lambda, A)\| \leqslant M$ $(\lambda > \lambda_0)$ and $\mathrm{dom}(A)$ is dense, the convergence carries over to all $x \in X$, by Proposition 1.6.

(b) For $x \in \mathrm{dom}(A)$ the convergence proved in part (a) implies

$$A\big(\lambda R(\lambda, A)\big)x = \lambda R(\lambda, A)Ax \to Ax \qquad (\lambda \to \infty).$$ □

If X is a reflexive Banach space, then the denseness hypothesis for $\mathrm{dom}(A)$ in Lemma 2.10 can be omitted because it follows from the remaining hypotheses; see Exercise 2.6.

Proof of Theorem 2.9. Recall that, without loss of generality, we only treat the case $\omega = 0$.

(i) For $n \in \mathbb{N}$, $t \geqslant 0$ we obtain the estimate

$$\big\|\mathrm{e}^{tA_n}\big\| = \big\|\mathrm{e}^{tn^2R(n,A)}\mathrm{e}^{-tnI}\big\| = \mathrm{e}^{-tn}\left\|\sum_{k=0}^{\infty}\frac{(tn^2R(n,A))^k}{k!}\right\| \leqslant \mathrm{e}^{-tn}\sum_{k=0}^{\infty}\frac{(tn)^k}{k!} = 1.$$

For the first of the equalities recall Exercise 1.1(a), and for the estimate note that $\|R(n,A)\| \leqslant \frac{1}{n}$, because $\omega = 0$.

(ii) For $x \in X$, $t > 0$ and $m, n \in \mathbb{N}$ we compute

$$\begin{aligned}\mathrm{e}^{tA_m}x - \mathrm{e}^{tA_n}x &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{e}^{(t-s)A_n}\mathrm{e}^{sA_m}x\big)\,\mathrm{d}s = \int_0^t \mathrm{e}^{(t-s)A_n}(A_m - A_n)\mathrm{e}^{sA_m}x\,\mathrm{d}s\\ &= \int_0^t \mathrm{e}^{(t-s)A_n}\mathrm{e}^{sA_m}(A_m - A_n)x\,\mathrm{d}s,\end{aligned}$$

where in the last equality we have used the fact that A_m, A_n as well as the generated semigroups commute. Recalling step (i) we obtain the estimate

$$\big\|\mathrm{e}^{tA_m}x - \mathrm{e}^{tA_n}x\big\| \leqslant t\big\|(A_m - A_n)x\big\|. \tag{2.5}$$

Let $a > 0$. For $n \in \mathbb{N}$ we define the operator $\mathcal{T}_n^a \colon X \to C([0,a];X)$ by

$$\mathcal{T}_n^a x := \big[t \mapsto \mathrm{e}^{tA_n}x\big] \qquad (x \in X)$$

(where $C([0,a];X)$ denotes the Banach space of continuous X-valued functions, provided with the supremum norm). Then step (i) shows that $\mathcal{T}_n^a$ is a contraction, and inequality (2.5) shows that

$$\big\|\mathcal{T}_m^a x - \mathcal{T}_n^a x\big\| \leqslant a\|A_m x - A_n x\| \qquad (x \in X,\ m,n \in \mathbb{N}).$$

For $x \in \mathrm{dom}(A)$, this implies that $(\mathcal{T}_n^a x)_{n\in\mathbb{N}}$ is a Cauchy sequence, because $(A_n x)_{n\in\mathbb{N}}$ is convergent (to Ax), by Lemma 2.10(b). Applying Proposition 1.6 we conclude that there exists $\mathcal{T}^a \in \mathcal{L}\big(X, C([0,a];X)\big)$ such that $\mathcal{T}_n^a \to \mathcal{T}^a$ $(n \to \infty)$ strongly.

Clearly, if $0 < a < b$, then $\mathcal{T}^b x|_{[0,a]} = \mathcal{T}^a x$ for all $x \in X$, and therefore we can define $T \colon [0,\infty) \to \mathcal{L}(X)$ by

$$T(t)x := \mathcal{T}^a x(t) \qquad (0 \leqslant t \leqslant a,\ x \in X).$$

From $T(\cdot)x|_{[0,a]} = \mathcal{T}^a x$ $(a > 0,\ x \in X)$ we infer that T is strongly continuous. Since $T(t) = \text{s-lim}_{n\to\infty}\mathrm{e}^{tA_n}$ $(t \geqslant 0)$, Lemma 1.12 implies that the semigroup property carries over from the semigroups $(\mathrm{e}^{tA_n})_{t\geqslant 0}$ to T. As a result, T is a C_0-semigroup of contractions.

(iii) Let B be the generator of T. Let $x \in \mathrm{dom}(A)$. Using the notation of step (ii), with $a := 1$, we see that $\mathcal{T}_n^1 x \to \mathcal{T}^1 x$ and $(\mathcal{T}_n^1 x)' = \mathcal{T}_n^1 A_n x \to \mathcal{T}^1 Ax$ $(n \to \infty)$ in $C([0,1];X)$ (recall Lemma 1.12). This implies that $T(\cdot)x|_{[0,1]} = \mathcal{T}^1 x$ is differentiable with continuous derivative $\mathcal{T}^1 Ax$, and therefore $x \in \mathrm{dom}(B)$ and $Bx = Ax$.

So far we have shown that $A \subseteq B$. We also know that $(0,\infty) \subseteq \rho(B)$, by Theorem 2.7, and that $(0,\infty) \subseteq \rho(A)$, by hypothesis. Now from $I - A \subseteq I - B$, the injectivity of $I - B$ and $\mathrm{ran}(I - A) = X$ we obtain $I - A = I - B$, and hence $A = B$. □

2.11 Remarks. (a) The proof of the Hille–Yosida theorem for the general case can be given along the same lines; see Exercise 2.4.

(b) As an interesting feature in the proof of Theorem 2.9 we point out that the approximating semigroups are norm continuous, whereas in general the resulting semigroup is only strongly continuous. The norm continuity is lost because the approximating semigroups only converge strongly (and generally not with respect to the operator norm). △

2.3 Euler's exponential formula

Given $a \in \mathbb{K}$, besides the exponential series there are two well-known ways to approximate e^{ta}, namely

$$\mathrm{e}^{ta} = \lim_{n\to\infty} \Big(1 + \frac{t}{n} a\Big)^n \qquad \text{and} \qquad \mathrm{e}^{ta} = \lim_{n\to\infty} \Big(1 - \frac{t}{n} a\Big)^{-n}.$$

Trying to replace a by an unbounded generator in the first formula leads to hopeless problems with the domains of the powers of the operators involved, whereas the second formula looks more promising because the occurring inverses are just those whose existence is guaranteed by Theorem 2.7. (In fact, the resulting formulas are those known in numerical analysis as 'backward Euler method'.)

We now show that this approximation idea works for arbitrary C_0-semigroups. The proof is independent of the Hille–Yosida theorem (which was shown by a different kind of approximation).

2.12 Theorem (Euler's exponential formula). *Let T be a C_0-semigroup on a Banach space X, with generator A. Then*

$$T(t)x = \lim_{n\to\infty} \Big(I - \frac{t}{n} A\Big)^{-n} x$$

for all $x \in X$, with uniform convergence for t in compact subsets of $[0,\infty)$.

2.13 Remarks. (a) Let $M \geqslant 1$ and $\omega \in \mathbb{R}$ be such that $\|T(t)\| \leqslant M\mathrm{e}^{\omega t}$ $(t \geqslant 0)$; then $(\omega,\infty) \subseteq \rho(A)$, by Theorem 2.7. Let $a > 0$. If $0 < t \leqslant a$, then $\frac{n}{t} \geqslant \frac{n}{a}$, therefore $\frac{n}{t} \in \rho(A)$ if $n > a\omega$, and then

$$\Big(I - \frac{t}{n} A\Big)^{-1} = \frac{n}{t}\Big(\frac{n}{t} I - A\Big)^{-1} \in \mathcal{L}(X).$$

Thus the operator $(I - \frac{t}{n}A)^{-n}$ is defined for all $t \in (0,a]$ if $n > a\omega$. (In fact, if $\omega \leqslant 0$, then this operator is defined for all $t > 0$ and $n \in \mathbb{N}$.)

Applying Theorem 2.7 we conclude that

$$\Big\|\Big(I - \frac{t}{n} A\Big)^{-n}\Big\| \leqslant \Big(\frac{n}{t}\Big)^n M\Big(\frac{n}{t} - \omega\Big)^{-n} = M\Big(1 - \frac{t}{n}\omega\Big)^{-n}$$

for all $t \in (0,a]$, $n > a\omega$.

(b) We note that the expressions $(I - rA)^{-1} = \frac{1}{r}\big(\frac{1}{r} I - A\big)^{-1}$ for small $r > 0$ correspond to expressions $\lambda(\lambda I - A)^{-1}$ for large $\lambda > 0$. This implies that the behaviour of $(I - rA)^{-1}$ for $r \to 0$ is the same as that of $\lambda(\lambda I - A)^{-1}$ for $\lambda \to \infty$. △

Proof of Theorem 2.12. Let $M \geqslant 1$ and $\omega > 0$ be such that $\|T(t)\| \leqslant M\mathrm{e}^{\omega t}$ $(t \geqslant 0)$. Fix $a > 0$ and let $n > a\omega$, as in Remark 2.13(a). For $0 < t \leqslant a$ we compute, using Remark 2.4(b),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(I - \frac{t}{n}A\Big)^{-1} &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{n}{t}\Big(\frac{n}{t}I - A\Big)^{-1}\bigg) = -\frac{n}{t^2}\Big(\frac{n}{t}I - A\Big)^{-1} + \frac{n}{t}\cdot\frac{n}{t^2}\Big(\frac{n}{t}I - A\Big)^{-2} \\ &= \Big(-\frac{n^2}{t^3}I + \frac{n}{t^2}A + \frac{n^2}{t^3}I\Big)\Big(\frac{n}{t}I - A\Big)^{-2} = \frac{1}{n}A\Big(I - \frac{t}{n}A\Big)^{-2},\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(I - \frac{t}{n}A\Big)^{-n} = n\bigg(\Big(I - \frac{t}{n}A\Big)^{-1}\bigg)^{n-1}\frac{1}{n}A\Big(I - \frac{t}{n}A\Big)^{-2} = A\Big(I - \frac{t}{n}A\Big)^{-n-1}.$$

(In the last computation the product rule for differentiation is used. The result of this computation should not be too surprising; if A is a number, then it is a consequence of the chain rule.)

Let $x \in \mathrm{dom}(A)$. Then the function $[0,t] \ni s \mapsto T(t-s)\big(I - \frac{s}{n}A\big)^{-n}x$ is continuously differentiable, with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\bigg(T(t-s)\Big(I - \frac{s}{n}A\Big)^{-n}x\bigg) &= T(t-s)\bigg(-A + A\Big(I - \frac{s}{n}A\Big)^{-1}\bigg)\Big(I - \frac{s}{n}A\Big)^{-n}x \\ &= T(t-s)\Big(I - \frac{s}{n}A\Big)^{-n}\bigg(\Big(I - \frac{s}{n}A\Big)^{-1} - I\bigg)Ax.\end{aligned}$$

By the fundamental theorem of calculus (see Theorem 1.9) it follows that

$$\begin{aligned}&\big\|(I - \tfrac{t}{n}A)^{-n}x - T(t)x\big\| \\ &\quad= \Big\|\int_0^t T(t-s)(I - \tfrac{s}{n}A)^{-n}\big((I - \tfrac{s}{n}A)^{-1} - I\big)Ax\,\mathrm{d}s\Big\| \\ &\quad\leqslant a\sup_{0<s<t\leqslant a}\big\|T(t-s)(I - \tfrac{s}{n}A)^{-n}\big\|\sup_{0<s<a}\big\|\big((I - \tfrac{s}{n}A)^{-1} - I\big)Ax\big\|.\end{aligned} \tag{2.6}$$

Using Remark 2.13(a) one estimates

$$\begin{aligned}\big\|T(t-s)(I - \tfrac{s}{n}A)^{-n}\big\| &\leqslant M^2\mathrm{e}^{\omega(t-s)}(1 - \tfrac{s}{n}\omega)^{-n} \\ &\leqslant M_0 := M^2\mathrm{e}^{\omega a}\sup_{n>a\omega}(1 - \tfrac{a}{n}\omega)^{-n} < \infty,\end{aligned}$$

where the latter supremum is finite because the sequence $\big((1 - \frac{a}{n}\omega)^{-n}\big)_{n>a\omega}$ is convergent (to $\mathrm{e}^{a\omega}$). Hence (2.6) yields

$$\sup_{0\leqslant t\leqslant a}\big\|(I - \tfrac{t}{n}A)^{-n}x - T(t)x\big\| \leqslant aM_0\sup_{0<s<a}\big\|\big((I - \tfrac{s}{n}A)^{-1} - I\big)Ax\big\|,$$

which tends to 0 as $n \to \infty$, by Lemma 2.10(a) (recall Remark 2.13(b)).

Now the proof is completed pretty much as in step (ii) of the proof of Theorem 2.9. For $n > a\omega$ we define $\mathcal{T}_n^a \colon X \to C([0,a];X)$,

$$\mathcal{T}_n^a x := [t \mapsto (I - \tfrac{t}{n}A)^{-n}x] \qquad (x \in X).$$

Then $\big\|\mathcal{T}_n^a\big\| \leqslant M_0$ for all $n > a\omega$, and $\mathcal{T}_n^a x \to T(\cdot)x|_{[0,a]}$ $(n \to \infty)$ for all $x \in \mathrm{dom}(A)$, as shown above. Using the fact that $\mathrm{dom}(A)$ is dense and applying Proposition 1.6 we obtain $\mathcal{T}_n^a x \to T(\cdot)x|_{[0,a]}$ $(n \to \infty)$ for all $x \in X$. □

2.14 Remark. Euler's exponential formula will be used repeatedly later in the book, for instance in the proof of the invariance of a closed set $C \subseteq X$ under a C_0-semigroup, as follows. If C is invariant under $(I - rA)^{-1}$ for small $r > 0$, then Theorem 2.12 implies that C is invariant under T; see also Proposition 9.1. $\triangle$

Notes

The Hille–Yosida theorem is basic for C_0-semigroups and can be found (with varying proofs) in any treatment of C_0-semigroups. Our proof is Yosida's original proof in [Yos48]. A different proof, based on the exponential formula, can be found in §§ 1.2 and 1.4 of the beautiful Chapter IX in [Kat80], which is a concise introduction to semigroups.

The discovery of the Hille–Yosida theorem in 1948 was a major event in functional analysis and operator theory. Known was Stone's theorem from 1930 [Sto30]: an operator A in a complex Hilbert space generates a one-parameter group of unitary operators if and only if iA is self-adjoint. Stone was motivated by quantum mechanics, for which John von Neumann developed the mathematical foundations in the years between 1927 and 1932. Von Neumann's work on this subject culminated in his monograph "Mathematische Grundlagen der Quantenmechanik" [Neu32a], which was the first rigorous mathematical treatment of quantum mechanics focusing on unbounded self-adjoint operators. The theory von Neumann developed is still used today whith great success.

Unitary groups describe the dynamics in quantum mechanics. More generally, one-parameter groups describe evolutionary processes that are reversible and in which the states exist for all times. It is natural to consider time evolution for positive times only; this leads to semigroups and the problem of describing their generators. E. Hille and K. Yosida found the solution independently, for the case of contractive semigroups: Yosida published his proof in the first volume of the Journal of the Mathematical Society of Japan [Yos48], where he introduced the Yosida approximations A_n of the unbounded operator A and constructed the semigroup as a limit of e^{tA_n}, as we did in this chapter. Hille, in his book [Hil48; Sections 12.2 and 12.3], employed the inverse Laplace transform and the exponential formula (see Theorem 2.12) to prove the theorem for a slightly more general than the contractive case.

The general case of the Hille–Yosida theorem was proved independently by Miyadera [Miy52; Theorem 2], Phillips [Phi53; Section 2] and Feller [Fel53; Theorem 3]. (The general version of the theorem is sometimes called "Hille–Yosida–Phillips theorem", maybe because Phillips' paper was submitted first.) Hille's book mentioned above has been extended in collaboration with Phillips to [HiPh57]. In this monograph the authors offer two proofs of the general Hille–Yosida theorem: the first is an adaptation of Yosida's proof to the general case, the second relies on the exponential formula; see [HiPh57; Section 12.3]. It is worth mentioning that the exponential formula can also be derived from the Chernoff product formula, for which we refer to Section 13.3; in particular see Example 13.16(b).

The proof via the exponential formula can be considered as an application of the backward Euler scheme to the numerical solution of the Cauchy problem

$$u'(t) = Au(t) \quad (t \geqslant 0), \qquad u(0) = x.$$

Indeed, fixing $n \in \mathbb{N}$ and subdividing the interval $[0,t]$ into n equal parts, one can discretise the differential equation $u' = Au$ to derive the difference equation

$$\frac{u_k - u_{k-1}}{t/n} = Au_k \qquad (k = 1, \dots, n),$$

where u_k plays the role of $u(\frac{k}{n}t)$. (The scheme is called *backward* because of the term Au_k on the right-hand side, and not Au_{k-1} as in the forward scheme.) Starting from the initial value $u_0 = x$, one easily shows that for large n the solution is given by $u_k = (I + \frac{t}{n}A)^{-k}x$ $(k \in \{0, \dots, n\})$; in particular, for $k = n$ one obtains the approximation $(I + \frac{t}{n}A)^{-n}x$ for $u(t)$ from the exponential formula. We mention that the backward Euler method even works in the theory of nonlinear semigroups, where the central generation theorem is due to Crandall and Liggett; see [CHA&87; Theorem 2.3].

Our entire book demonstrates the power of the Hille–Yosida theorem. We have restricted our proof to the special case of quasi-contractive semigroups; this will be sufficient for our application of the theorem to operators in Hilbert spaces associated with forms, in Section 5.3.

Exercises

2.1 Show the second equality in (2.4).

2.2 Let T be a C_0-semigroup on the Banach space X, with generator A. Let $\lambda \in \mathbb{K}$. Show that

$$T_\lambda(t) := \mathrm{e}^{-\lambda t}T(t) \qquad (t \geqslant 0)$$

defines a C_0-semigroup (the rescaled semigroup) and that the generator of T_λ is given by $A - \lambda I$.

2.3 Let T be a bounded C_0-semigroup on the Banach space X, $M := \sup_{t\geqslant 0} \|T(t)\|$, and let A be its generator.

(a) Show that

$$|||x||| := \sup_{t\geqslant 0} \|T(t)x\| \qquad (x \in X)$$

defines a norm $|||\cdot|||$ on X which is equivalent to $\|\cdot\|$, and that T is a C_0-semigroup of contractions on $(X, |||\cdot|||)$.

(b) For any $\alpha_1, \dots, \alpha_n > 0$, show that

$$\big\|(I - \alpha_1 A)^{-1} \cdots (I - \alpha_n A)^{-1}\big\| \leqslant M.$$

2.4 Show the Hille–Yosida theorem for the general case:

Let A be a closed, densely defined operator in the Banach space X. Assume that there exist $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A)$ and

$$\|R(\lambda, A)^n\| \leqslant \frac{M}{(\lambda - \omega)^n} \qquad (\lambda \in (\omega, \infty),\ n \in \mathbb{N}).$$

Then A is the generator of a C_0-semigroup T satisfying the estimate

$$\|T(t)\| \leqslant M\mathrm{e}^{\omega t} \qquad (t \geqslant 0).$$

(Hint: Proceed as in the proof of Theorem 2.9, with adapted estimates.)

2.5 Let T be a C_0-semigroup on the Banach space X. For $h > 0$ put $A_h := h^{-1}(T(h) - I)$. Show that $\mathrm{e}^{tA_h}x \to T(t)x$ for all $x \in X$ as $h \to 0$, uniformly for t in compact subsets of $[0, \infty)$. (Hint: Use a procedure similar to the proof of the exponential formula, Theorem 2.12.)

2.6 Let X be a reflexive Banach space, let A be a closed operator in X, and assume that there exists $\lambda_0 \geqslant 0$ such that $(\lambda_0, \infty) \subseteq \rho(A)$, $M := \sup_{\lambda > \lambda_0} \|\lambda(\lambda - A)^{-1}\| < \infty$.

Show that $\mathrm{dom}(A)$ is dense in X.

Hints: (i) Show that $\lambda R(\lambda, A)x \to x$ $(\lambda \to \infty)$ for all $x \in \mathrm{dom}(A)$; cf. Lemma 2.10. (ii) Now let $x \in X$. Show that there exists a sequence (λ_n) in (λ_0, ∞), $\lambda_n \to \infty$ $(n \to \infty)$ such that $(\lambda_n R(\lambda_n, A)x)$ converges weakly to some $y \in X$. (See e.g. [Yos68; Section V.2, Theorem 1] or [Lax02; Section 10.2, Theorem 7] for the fact that each bounded sequence in a reflexive Banach space contains a weakly convergent subsequence.) (iii) Choose $\mu \in \rho(A)$. Show that $\lambda_n R(\lambda_n, A)R(\mu, A)x \to R(\mu, A)x$, but also $R(\mu, A)\lambda_n R(\lambda_n, A)x \to R(\mu, A)y$ weakly.

Chapter 3

Holomorphic semigroups

The objective of this chapter is to introduce semigroups for which the 'time parameter' t can also be chosen in a complex neighbourhood of the positive reals. The foundations of these 'holomorphic semigroups' will be given in Section 3.2. The generation of such semigroups will be studied in Section 3.3. Finally, in Section 3.4 the special case of holomorphic semigroups on Hilbert spaces will be treated. We start with an interlude on Banach space valued holomorphy.

3.1 Interlude: vector-valued holomorphic functions

In this section let X, Y be complex Banach spaces. The first issue is to show that for Banach space valued functions several notions of holomorphy coincide.

Let $\Omega \subseteq \mathbb{C}$ be open, $f\colon \Omega \to X$. The function f is called **holomorphic** if f is (complex) differentiable (at each point of Ω), and f is **weakly holomorphic** if $x' \circ f$ is holomorphic for all $x' \in X'$ ($= \mathcal{L}(X, \mathbb{C})$, the dual space of X). Moreover, f is called **analytic** if f can be represented as a power series in a neighbourhood of each point of Ω.

3.1 Remarks. (a) It is evident that holomorphy of a function implies weak holomorphy.

(b) Using part (a) we can show the identity theorem for a holomorphic function $f\colon \Omega \to X$, where $\Omega \subseteq \mathbb{C}$ is open and connected: if $[f = 0]$ has a cluster point in Ω, then $f = 0$. (We use the notation $[f = 0] := \{z \in \Omega;\ f(z) = 0\}$; sets like $[f \geqslant 0]$, $[f > 0]$ etc., occuring later in the book, are defined analogously.)

Indeed, for each $x' \in X'$ the zeros of the function $x' \circ f$ have a cluster point in Ω, and therefore $x' \circ f = 0$ follows from the identity theorem for $\mathbb{C}$-valued holomorphic functions. From $x' \circ f = 0$ for all $x' \in X'$ one obtains $f = 0$.

The reasoning presented here is an example of how properties of complex-valued holomorphic functions may be transferred to vector-valued functions. An alternative approach is to assure oneself that the proofs in classical complex analysis also work out in the vector-valued case; cf. part (c).

(c) Let $\Omega \subseteq \mathbb{C}$ be open, and let $f\colon \Omega \to X$ be holomorphic. The following facts can be shown in the same way as in the case of $\mathbb{C}$-valued functions.

If Ω is convex, and γ is a piecewise continuously differentiable closed path in Ω, then $\int_\gamma f(z)\,\mathrm{d}z = 0$ (Cauchy's integral theorem). Note that the path integral is defined by a parametrisation of the path and therefore reduces to integrals over intervals. As a consequence, path integrals fit into the context explained in Subsection 1.3.2.

For $z_0 \in \Omega$, $r > 0$ such that $B[z_0, r] \subseteq \Omega$ the function f satisfies Cauchy's integral formula

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \qquad (z \in B(z_0, r)).$$

The function f is analytic, and one has Cauchy's integral formulas for the derivatives,

$$f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta \qquad (z \in B(z_0, r),\ n \in \mathbb{N}),$$

with z_0 and r as before. △

A set $E \subseteq X'$ is **separating** (**for** X) if for all $0 \neq x \in X$ there exists $x' \in E$ such that $x'(x) \neq 0$. The set E is called **almost norming** (**for** X) if

$$\|x\|_E := \sup\bigl\{|x'(x)| \,;\, x' \in E,\ \|x'\| \leqslant 1\bigr\}$$

defines a norm that is equivalent to the norm on X; it is called **norming** if $\|\cdot\|_E = \|\cdot\|$ on X. It is a consequence of the Hahn–Banach theorem that $E = X'$ is norming for X.

3.2 Theorem. *Let $\Omega \subseteq \mathbb{C}$ be open, $f\colon \Omega \to X$. Then the following properties are equivalent.*

(i) *f is holomorphic.*

(ii) *f is weakly holomorphic.*

(iii) *There exists an almost norming closed subspace $E \subseteq X'$ such that $x' \circ f$ is holomorphic for all $x' \in E$.*

(iv) *f is locally bounded, and there exists an almost norming subset $E \subseteq X'$ such that $x' \circ f$ is holomorphic for all $x' \in E$.*

(v) *f is continuous, and there exists a separating set $E \subseteq X'$ such that $x' \circ f$ is holomorphic for all $x' \in E$.*

Proof. (i) ⇒ (ii) is clear (and was already noted above).

(ii) ⇒ (iii) is clear, with $E = X'$.

(iii) ⇒ (iv). With the canonical embedding $X \hookrightarrow E' = \mathcal{L}(E, \mathbb{C})$ – see Remark 3.3(a) – we can consider f as an $\mathcal{L}(E, \mathbb{C})$-valued function. Since E is a Banach space, the uniform boundedness theorem implies that f is locally bounded with respect to the norm $\|\cdot\|_E$.

(iv) ⇒ (v). Since almost norming subsets are separating we only have to show that f is continuous.

Let $z_0 \in \Omega$, $r > 0$ be such that $B[z_0, r] \subseteq \Omega$; then $M := \sup\bigl\{\|f(\zeta)\| \,;\, |\zeta - z_0| = r\bigr\} < \infty$. For $x' \in E$, $z \in B[z_0, r/2]$ we then obtain, using Cauchy's integral formula for the derivative,

$$\left|\frac{\mathrm{d}}{\mathrm{d}z} x'(f(z))\right| = \left|\frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{x'(f(\zeta))}{(\zeta - z)^2} \, \mathrm{d}\zeta\right| \leqslant \frac{1}{2\pi} 2\pi r \|x'\| \frac{4M}{r^2} = \frac{4M}{r} \|x'\|.$$

For $z', z'' \in B[z_0, r/2]$ this implies

$$|x'(f(z') - f(z''))| \leqslant \frac{4M}{r} |z' - z''| \|x'\| \qquad (x' \in E),$$

and therefore

$$\|f(z') - f(z'')\|_E \leqslant \frac{4M}{r}|z' - z''|.$$

Since the norm $\|\cdot\|_E$ is equivalent to the norm on X, it follows that f is continuous on $B[z_0, r/2]$.

(v) $\Rightarrow$ (i). Let $z_0 \in \Omega$. We show that f can be expanded into a power series about z_0.

Without loss of generality we assume $z_0 = 0$. There exists $r > 0$ such that $B[0, r] \subseteq \Omega$. For $n \in \mathbb{N}_0$ we put

$$a_n := \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{n+1}}\, \mathrm{d}\zeta.$$

With $M := \sup\{\|f(\zeta)\|;\ |\zeta| = r\}$ $(< \infty)$ one obtains $\|a_n\| \leqslant M/r^n$ for all $n \in \mathbb{N}_0$, and therefore the power series $g(z) := \sum_{n=0}^{\infty} z^n a_n$ converges for all $z \in B(0, r)$. For $x' \in E$, $|z| < r$ one has

$$x'(g(z)) = \sum_{n=0}^{\infty} z^n \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{x'(f(\zeta))}{\zeta^{n+1}}\, \mathrm{d}\zeta = \sum_{n=0}^{\infty} \frac{(x' \circ f)^{(n)}(0)}{n!} z^n = x'(f(z)),$$

where the last equality is just the power series expansion of the holomorphic function $x' \circ f$. Since E is separating we therefore obtain $f(z) = g(z)$. □

Note that in the proof of the implication '(v) $\Rightarrow$ (i)' it is also shown that f is analytic.

3.3 Remarks. (a) Let X and Y be normed spaces. Then $X \hookrightarrow Y$ is an abbreviation for saying that X is continuously embedded into Y. In most cases this means that $X \subseteq Y$ in a natural way and that for some constant $c > 0$ one has

$$\|x\|_Y \leqslant c\|x\|_X \qquad (x \in X).$$

Such a constant is called an **embedding constant**.

In the proof of the implication '(iii) $\Rightarrow$ (iv)' of Theorem 3.2 the canonical embedding $X \hookrightarrow E'$ is given by

$$X \ni x \mapsto [E \ni y \mapsto \langle x, y\rangle_{X,X'} \in \mathbb{C}] \in E';$$

it is injective because E is supposed to be almost norming.

(b) If X is a dual Banach space then the predual is a norming closed subspace of its bidual X'. (For instance, c_0 is norming for ℓ_1.) This illustrates a possible application of condition (iii) of Theorem 3.2. △

Next we come to the characterisation of holomorphy for $\mathcal{L}(X, Y)$-valued functions. As $\mathcal{L}(X, Y)$ is a Banach space, all the previous criteria apply. However, weak holomorphy is not a useful concept in this case, because typically one has no explicit description of the dual of $\mathcal{L}(X, Y)$.

3.4 Theorem. *Let $\Omega \subseteq \mathbb{C}$ be open, $F\colon \Omega \to \mathcal{L}(X, Y)$. Let B be a dense subset of X, and let $C \subseteq Y'$ be almost norming for Y. Then the following properties are equivalent.*

(i) *F is holomorphic (as an $\mathcal{L}(X, Y)$-valued function).*

(ii) *F is locally bounded, and $F(\cdot)x$ is holomorphic for all $x \in B$.*

(iii) *F is locally bounded, and $y'(F(\cdot)x)$ is holomorphic for all $x \in B$, $y' \in C$.*

Proof. (i) ⇒ (ii) ⇒ (iii) is clear.

(iii) ⇒ (i). It follows from the hypotheses on B and C that the set

$$E := \big\{A \mapsto y'(Ax);\, x \in B,\ y' \in C\big\} \subseteq \mathcal{L}(X,Y)'$$

is almost norming for $\mathcal{L}(X,Y)$. Therefore Theorem 3.2, (iv) ⇒ (i), implies the assertion. □

Note that the set E in the proof of Theorem 3.4 is not a sub*space* of $\mathcal{L}(X,Y)'$. For this application condition (iv) of Theorem 3.2 is convenient.

The last issue of this section is the convergence of sequences of holomorphic functions.

3.5 Theorem. *Let $\Omega \subseteq \mathbb{C}$ be open, and let (f_n) be a sequence of holomorphic functions $f_n\colon \Omega \to X$. Assume that (f_n) is locally uniformly bounded (i.e., for each $z_0 \in \Omega$ there exists $r > 0$ such that $B(z_0,r) \subseteq \Omega$ and $\sup\big\{\|f_n(z)\|;\, z \in B(z_0,r),\ n \in \mathbb{N}\big\} < \infty$) and that $f(z) := \lim_{n\to\infty} f_n(z)$ exists for all $z \in \Omega$.*

Then (f_n) converges to f locally uniformly, and f is holomorphic.

Proof. In the first step we show that the sequence (f_n) is locally uniformly equicontinuous. Let $z_0 \in \Omega$, $r > 0$ be such that $B[z_0,r] \subseteq \Omega$. Then $M := \sup_{z\in B[z_0,r],\, n\in\mathbb{N}} \|f_n(z)\| < \infty$, by hypothesis. From Cauchy's integral formula for the derivative,

$$f_n'(z) = \frac{1}{2\pi\mathrm{i}} \int_{|\zeta - z_0| = r} \frac{f_n(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta \qquad (z \in B(z_0,r),\ n \in \mathbb{N})$$

we conclude that for all $z \in B[z_0, r/2]$ one has

$$\|f_n'(z)\| \leqslant M\frac{4}{r} \qquad (n \in \mathbb{N});$$

hence, the sequence (f_n) is uniformly equicontinuous on $B[z_0, r/2]$.

The local uniform equicontinuity of the sequence (f_n) together with the pointwise convergence implies that (f_n) converges to f locally uniformly; see Exercise 3.2(a). It follows that $x' \circ f_n \to x' \circ f$ locally uniformly, for all $x' \in X'$; hence, f is weakly holomorphic, therefore holomorphic, by Theorem 3.2. □

3.6 Corollary. *Let $\Omega \subseteq \mathbb{C}$ be open, and let (F_n) be a sequence of holomorphic functions $F_n\colon \Omega \to \mathcal{L}(X,Y)$. Assume that (F_n) is locally uniformly bounded and that $F(z) := \text{s-}\lim_{n\to\infty} F_n(z)$ exists for all $z \in \Omega$.*

Then F is holomorphic.

Proof. The hypotheses in combination with Theorem 3.5 imply that $F(\cdot)x$ is holomorphic for all $x \in X$. Therefore F is holomorphic, by Theorem 3.4. □

3.2 Holomorphic semigroups

Let X be a complex Banach space. For $\theta \in (0, \pi/2]$ we define the (open) sector

$$\Sigma_\theta := \big\{z \in \mathbb{C} \setminus \{0\};\, |\operatorname{Arg} z| < \theta\big\} = \big\{r\mathrm{e}^{\mathrm{i}\alpha};\, r > 0,\ |\alpha| < \theta\big\}.$$

We will also use the notation $\Sigma_{\theta,0} := \Sigma_\theta \cup \{0\}$. A **holomorphic semigroup** on X (**of angle** θ) is a function $T\colon \Sigma_{\theta,0} \to \mathcal{L}(X)$, holomorphic on Σ_θ, satisfying

(i) $T(z_1+z_2)=T(z_1)T(z_2)$ for all $z_1,z_2\in\Sigma_{\theta,0}$.

If additionally

(ii) $\lim_{\Sigma_{\theta'}\ni z\to 0}T(z)x=x$ for all $x\in X$ and all $\theta'\in(0,\theta)$,

then $T(0)=I$, by Remark 1.1(c), and T is called a **holomorphic C_0-semigroup (of angle θ)**.

Saying that T is a holomorphic semigroup we will always mean that T brings along its domain of definition, in particular, *the angle of T* is defined.

3.7 Remarks. (a) It follows from the definition of a holomorphic C_0-semigroup that for all $\theta'\in(0,\theta)$ there exist $M'\geqslant 1$, $\omega'\in\mathbb{R}$ such that

$$\|T(z)\|\leqslant M'\mathrm{e}^{\omega'\operatorname{Re}z}\qquad(z\in\Sigma_{\theta'});$$

see Exercise 3.4.

(b) The reader might wonder why θ is restricted to the interval $(0,\pi/2]$ in the above definition. The reason is that using the same definition with $\theta\in(\pi/2,\pi]$ one automatically obtains a norm continuous semigroup; see Exercise 3.5. In consequence, the semigroup has a holomorphic extension to all of $\mathbb{C}$. △

The following lemma states that it suffices to check the semigroup property for real times.

3.8 Lemma. *Let $\theta\in(0,\pi/2]$, and let $T\colon\Sigma_\theta\to\mathcal{L}(X)$ be holomorphic and such that $T(t+s)=T(t)T(s)$ for all $t,s>0$.*

Then $T(z_1+z_2)=T(z_1)T(z_2)$ for all $z_1,z_2\in\Sigma_\theta$.

Proof. Fixing $z_1\in(0,\infty)$, we know that the functions $\Sigma_\theta\ni z_2\mapsto T(z_1+z_2)$ and $\Sigma_\theta\ni z_2\mapsto T(z_1)T(z_2)$ are holomorphic and coincide on $(0,\infty)$. The identity theorem (see Remark 3.1(b)) implies that they are equal on Σ_θ. Another application of the identity theorem, with fixed $z_2\in\Sigma_\theta$, yields the assertion. □

3.9 Proposition. *Let T be a C_0-semigroup on X, and assume that there exist $\theta\in(0,\pi/2]$ and an extension of T to $\Sigma_{\theta,0}$, also called T, holomorphic on Σ_θ and satisfying*

$$\sup_{z\in\Sigma_\theta,\,|z|<1}\|T(z)\|<\infty.$$

Then $\lim_{\Sigma_\theta\ni z\to 0}T(z)x=x$ for all $x\in X$, and T is a holomorphic C_0-semigroup.

Proof. First note that Lemma 3.8 implies property (i) from above.

Let $x\in D:=\bigcup_{t>0}\operatorname{ran}(T(t))$, i.e., there exist $y\in X$, $t>0$ such that $x=T(t)y$. Then $\lim_{\Sigma_\theta\ni z\to 0}T(z)x=x$, by the continuity of $z\mapsto T(z)x=T(z)T(t)y=T(z+t)y$ at 0. Note that D is dense, because $T(t)\to I$ strongly as $t\to 0$. Therefore the boundedness assumption implies the assertion (recall Proposition 1.6). □

For a holomorphic C_0-semigroup T we now compute its derivative in terms of the generator A of the C_0-semigroup $T|_{[0,\infty)}$, and we discuss C_0-semigroups that are given by the restriction of T to rays $\mathrm{e}^{\mathrm{i}\alpha}[0,\infty)$.

3.10 Theorem. *Let T be a holomorphic C_0-semigroup of angle $\theta \in (0, \pi/2]$, and let A be the generator of the C_0-semigroup $T|_{[0,\infty)}$. Then:*

(a) *For all $z \in \Sigma_\theta$ one has $\operatorname{ran}(T(z)) \subseteq \operatorname{dom}(A)$ and $T'(z) = AT(z)$. For all $x \in \operatorname{dom}(A)$, $z \in \Sigma_\theta$ one has $T'(z)x = T(z)Ax$.*

(b) *For all $x \in \operatorname{dom}(A)$, $\theta' \in (0, \theta)$ one has*

$$\lim_{\Sigma_{\theta'} \ni z \to 0} \frac{1}{z}\big(T(z)x - x\big) = Ax.$$

(c) *For each $\alpha \in (-\theta, \theta)$ the mapping $[0, \infty) \ni t \mapsto T_\alpha(t) := T(\mathrm{e}^{\mathrm{i}\alpha}t)$ is a C_0-semigroup. The generator A_α of T_α equals $\mathrm{e}^{\mathrm{i}\alpha}A$.*

Proof. (a) Let $x \in X$, $z \in \Sigma_\theta$. Then

$$T'(z)x = \lim_{h \to 0+} \frac{1}{h}\big(T(z+h)x - T(z)x\big) = \lim_{h \to 0+} \frac{1}{h}(T(h) - I)T(z)x.$$

This implies that $T(z)x \in \operatorname{dom}(A)$ and $T'(z)x = AT(z)x$.

If $x \in \operatorname{dom}(A)$, then also $T'(z)x = T(z) \lim_{h \to 0+} \frac{1}{h}(T(h)x - x) = T(z)Ax$.

(b) Let $x \in \operatorname{dom}(A)$, $\theta' \in (0, \theta)$. Then the restriction of $T(\cdot)Ax$ to $\Sigma_{\theta',0}$ is continuous. Using part (a) one finds for $z \in \Sigma_{\theta'}$ that

$$\frac{1}{z}\big(T(z)x - x\big) = \frac{1}{z}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} T(sz)x \,\mathrm{d}s = \int_0^1 T(sz)Ax \,\mathrm{d}s,$$

and the latter tends to Ax as $z \to 0$ in $\Sigma_{\theta'}$.

(c) It is clear that T_α is a C_0-semigroup. Let $x \in \operatorname{dom}(A)$. Then by part (b) one obtains

$$\frac{1}{t}\big(T_\alpha(t)x - x\big) = \mathrm{e}^{\mathrm{i}\alpha} \frac{1}{\mathrm{e}^{\mathrm{i}\alpha}t}\big(T(\mathrm{e}^{\mathrm{i}\alpha}t)x - x\big) \to \mathrm{e}^{\mathrm{i}\alpha}Ax$$

as $t \to 0$, and hence $x \in \operatorname{dom}(A_\alpha)$, $A_\alpha x = \mathrm{e}^{\mathrm{i}\alpha}Ax$. This shows that $\mathrm{e}^{\mathrm{i}\alpha}(\lambda - A) \subseteq \mathrm{e}^{\mathrm{i}\alpha}\lambda - A_\alpha$ for all $\lambda > 0$. Now choose λ so large that $\lambda \in \rho(A)$ and $\mathrm{e}^{\mathrm{i}\alpha}\lambda \in \rho(A_\alpha)$. Then the above operator inclusion is in fact an equality because a surjective mapping cannot have a proper injective extension, and it follows that $\mathrm{e}^{\mathrm{i}\alpha}A = A_\alpha$. □

Let T be a holomorphic C_0-semigroup. In the light of Theorem 3.10(b) it is justified to say that the generator of the C_0-semigroup $T|_{[0,\infty)}$ is also the **generator** of the holomorphic C_0-semigroup T. Clearly, the application of Theorem 2.7 yields estimates for the resolvents of the generator of T. We will not pursue this issue in full generality but restrict our attention to contractive holomorphic semigroups.

3.11 Remarks. (a) Note that holomorphic C_0-semigroups are much more regular on $(0, \infty)$ than general C_0-semigroups. They are always norm continuous, even infinitely differentiable with respect to the operator norm.

(b) Theorem 3.10(a) implies that, for all $x \in X$, the function $[0, \infty) \ni t \mapsto T(t)x \in X$ is continuous, continuously differentiable on $(0, \infty)$, and solves the initial value problem

$$u'(t) = Au(t) \quad (t \in (0, \infty)), \qquad u(0) = x.$$

By Theorem 1.13(a) the solution is unique. △

3.3 Generation of contractive holomorphic semigroups

As before, let X be a complex Banach space. We call a holomorphic semigroup of angle $\theta \in (0, \pi/2]$ **contractive** if $\|T(z)\| \leqslant 1$ for all $z \in \Sigma_{\theta,0}$.

The following theorem characterises the generation of contractive holomorphic C_0-semigroups.

3.12 Theorem. *For $\theta \in (0, \pi/2]$ and an operator A in X, the following properties are equivalent.*

(i) *A is the generator of a contractive holomorphic C_0-semigroup of angle θ.*

(ii) *For all $\alpha \in (-\theta, \theta)$, $\mathrm{e}^{\mathrm{i}\alpha}A$ is the generator of a contractive C_0-semigroup.*

(iii) *A is closed and densely defined, $\Sigma_\theta \subseteq \rho(A)$, and*

$$\big\|(\lambda I - A)^{-1}\big\| \leqslant \frac{1}{|\lambda|} \qquad (\lambda \in \Sigma_\theta).$$

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 3.10(c).

(ii) $\Rightarrow$ (iii). 'Closed' and 'densely defined' are clear by Theorem 1.11(c). For $\alpha \in (-\theta, \theta)$, we know from Theorem 2.7 that $(0, \infty) \subseteq \rho(\mathrm{e}^{\mathrm{i}\alpha}A)$ and $\|(\mu I - \mathrm{e}^{\mathrm{i}\alpha}A)^{-1}\| \leqslant 1/\mu$ for all $\mu > 0$. The identity $(\mu I - \mathrm{e}^{\mathrm{i}\alpha}A) = \mathrm{e}^{\mathrm{i}\alpha}(\mathrm{e}^{-\mathrm{i}\alpha}\mu I - A)$ then shows that $\{\mathrm{e}^{-\mathrm{i}\alpha}\mu;\, \mu > 0\} \subseteq \rho(A)$ and $\|(\mathrm{e}^{-\mathrm{i}\alpha}\mu I - A)^{-1}\| \leqslant 1/\mu$ for all $\mu > 0$.

(iii) $\Rightarrow$ (i). We employ the exponential formula, Theorem 2.12. For $n \in \mathbb{N}$ we define the holomorphic function $F_n \colon \Sigma_\theta \to \mathcal{L}(X)$,

$$F_n(z) := (I - \tfrac{z}{n}A)^{-n} = \big(\tfrac{n}{z}(\tfrac{n}{z} - A)^{-1}\big)^n.$$

(For the holomorphy of F_n recall Remark 2.4(b).) Then the hypotheses imply that $\|F_n(z)\| \leqslant 1$ for all $z \in \Sigma_\theta$, $n \in \mathbb{N}$. They further imply that for each $\alpha \in (-\theta, \theta)$ the operator $\mathrm{e}^{\mathrm{i}\alpha}A$ generates a contractive C_0-semigroup T_α, by Theorem 2.9. Let $z \in \Sigma_\theta$, $z = \mathrm{e}^{\mathrm{i}\alpha}t$ with suitable $t > 0$, $\alpha \in (-\theta, \theta)$. Then $F_n(z) = (I - \frac{t}{n}\mathrm{e}^{\mathrm{i}\alpha}A)^{-n} \to T_\alpha(t)$ strongly as $n \to \infty$, by Theorem 2.12, so

$$T(z) := \text{s-lim}_{n\to\infty} F_n(z)$$

exists for all $z \in \Sigma_{\theta,0}$. Then $T|_{[0,\infty)} = T_0$ is the C_0-semigroup generated by A. Combining Corollary 3.6 and Proposition 3.9 we conclude that T is a holomorphic C_0-semigroup of angle θ. □

A further equivalence to the conditions stated in Theorem 3.12 needs some more preparation and will be presented in Theorem 3.20.

We add some comments on the generation of holomorphic semigroups that are not necessarily contractive.

3.13 Remarks. (a) The same proof as given for Theorem 3.12 can be used to show the following equivalence.

Let $\theta \in (0, \pi/2]$, $M \geqslant 1$. Then an operator A is the generator of a **bounded holomorphic C_0-semigroup** of angle θ and with bound M if and only if for each $\alpha \in (-\theta, \theta)$ the

operator $e^{i\alpha}A$ is the generator of a bounded C_0-semigroup with bound M. (Note that we call the holomorphic semigroup T of angle θ bounded if $\sup_{z\in\Sigma_\theta}\|T(z)\|<\infty$. We point out that the terminology 'bounded holomorphic semigroup' is sometimes used differently.)

(b) If A is the generator of a bounded C_0-semigroup with bound $M\geqslant 1$, then $[\mathrm{Re}>0]:=\{\lambda\in\mathbb{C};\,\mathrm{Re}\,\lambda>0\}\subseteq\rho(A)$ and

$$\bigl\|(\lambda I-A)^{-1}\bigr\|\leqslant\frac{M}{\mathrm{Re}\,\lambda}\qquad(\mathrm{Re}\,\lambda>0),$$

by Theorem 2.7. If $\theta'\in(0,\pi/2)$ and $\lambda\in\Sigma_{\theta'}$, then $\frac{\mathrm{Re}\,\lambda}{|\lambda|}\geqslant\cos\theta'$, and this implies

$$\bigl\|(\lambda I-A)^{-1}\bigr\|\leqslant\frac{M}{\mathrm{Re}\,\lambda}\leqslant\frac{M}{\cos\theta'}\frac{1}{|\lambda|}\,.$$

Together with Theorem 3.10(c) the above considerations show: if A is the generator of a bounded holomorphic semigroup of angle $\theta\in(0,\pi/2]$, then

$$\Sigma_{\theta+\pi/2}\subseteq\rho(A)\quad\text{and}\quad\sup_{\lambda\in\Sigma_{\theta'}}\bigl\|\lambda(\lambda I-A)^{-1}\bigr\|<\infty\quad(\theta'\in(0,\theta+\pi/2)).\tag{3.1}$$

This fact has a kind of converse. Here is the complete information: a closed, densely defined operator A in X is the generator of a holomorphic C_0-semigroup of angle $\theta\in(0,\pi/2]$ that is bounded on all sectors $\Sigma_{\theta'}$ with $\theta'\in(0,\theta)$ if and only if (3.1) holds (see [Kat80; IX, § 1.6], [EnNa00; Section II.4.a]). △

3.4 The Lumer–Phillips theorem

Let H be a Hilbert space over $\mathbb{K}$. The scalar product of two elements $x,y\in H$ will be denoted by $(x\,|\,y)$, and it is defined to be linear in the first and antilinear in the second argument.

An operator A in H is called **accretive** if

$$\mathrm{Re}(Ax\,|\,x)\geqslant 0\qquad(x\in\mathrm{dom}(A)).$$

3.14 Lemma. *Let A be an operator in H.*

(a) *Then A is accretive if and only if*

$$\|(\lambda I+A)x\|\geqslant\lambda\|x\|\qquad(x\in\mathrm{dom}(A))\tag{3.2}$$

for all $\lambda>0$.

(b) *For $\lambda>0$ with $\mathrm{ran}(\lambda I+A)=H$, inequality (3.2) holds if and only if $\lambda\in\rho(-A)$ and $\|(\lambda I+A)^{-1}\|\leqslant\lambda^{-1}$.*

Proof. (a) Assume that A is accretive, and let $\lambda>0$, $x\in\mathrm{dom}(A)$. Then

$$\|(\lambda I+A)x\|\|x\|\geqslant|((\lambda I+A)x\,|\,x)|\geqslant\mathrm{Re}((\lambda I+A)x\,|\,x)\geqslant(\lambda x\,|\,x)=\lambda\|x\|^2.$$

This shows the asserted inequality (3.2).

On the other hand, assume that (3.2) holds, and let $x \in \mathrm{dom}(A)$. Then

$$0 \leqslant \|(\lambda I + A)x\|^2 - \lambda^2\|x\|^2 = 2\lambda \operatorname{Re}(Ax \,|\, x) + \|Ax\|^2$$

for all $\lambda > 0$, and this implies $\operatorname{Re}(Ax \,|\, x) \geqslant 0$.

(b) follows from the next remark, which is valid in a more general context. □

3.15 Remark. Let X, Y be Banach spaces, B an operator from X to Y, and $\alpha > 0$. Then

$$\|Bx\|_Y \geqslant \alpha\|x\|_X \qquad (x \in \mathrm{dom}(B))$$

if and only if B is injective and $\|B^{-1}y\| \leqslant \alpha^{-1}\|y\|$ for all $y \in \mathrm{ran}(B)$. If these properties are satisfied and additionally $\mathrm{ran}(B) = Y$, then $B^{-1} \in \mathcal{L}(Y, X)$ and $\|B^{-1}\| \leqslant \alpha^{-1}$. △

An accretive operator A satisfying $\mathrm{ran}(I + A) = H$ is called **m-accretive**. A historical note on the prefix 'm': it should be remindful of the word 'maximal'. In fact, a densely defined accretive operator is m-accretive if and only if it has no proper accretive extensions; see [Phi59; Corollary of Theorem 1.1.1]. However, there exist maximal accretive operators that are not m-accretive (and not densely defined); see [Phi59; footnote (6)].

3.16 Theorem (Lumer–Phillips). *Let A be an operator in H. Then $-A$ is the generator of a contractive C_0-semigroup if and only if A is m-accretive.*

Note the minus sign that makes $-A$ a generator. The background is that one likes the operator A to have the "positivity" property of accretivity; see also Remark 3.21.

An essential part of the proof of Theorem 3.16 is contained in the following lemma.

3.17 Lemma. *Let A be an accretive operator in H, and assume that there exists $\lambda_0 > 0$ such that* $\mathrm{ran}(\lambda_0 I + A) = H$. *Then* $[\mathrm{Re} > 0] = \{\lambda \in \mathbb{K};\ \mathrm{Re}\,\lambda > 0\} \subseteq \rho(-A)$ *(in particular, A is m-accretive),*

$$\big\|(\lambda I + A)^{-1}\big\| \leqslant \frac{1}{\lambda} \qquad (\lambda > 0),$$

A is closed, and $\mathrm{dom}(A)$ *is dense in H.*

Proof. From Lemma 3.14 we obtain $\lambda_0 \in \rho(-A)$. Moreover, if $0 < \lambda \in \rho(-A)$, then $\|(\lambda I + A)^{-1}\| \leqslant \lambda^{-1}$, and Theorem 2.2(c) implies $B(\lambda, \lambda) \subseteq B\big(\lambda, \|(\lambda I + A)^{-1}\|^{-1}\big) \subseteq \rho(-A)$. Combining these statements one shows $[\mathrm{Re} > 0] \subseteq \rho(-A)$. In particular it follows that A is closed, by Remark 2.1(a).

Let $x \in \mathrm{dom}(A)^\perp$. Then $y := (I + A)^{-1}x \in \mathrm{dom}(A)$, and hence

$$0 = \operatorname{Re}(x \,|\, y) = \operatorname{Re}((I + A)y \,|\, y) \geqslant \|y\|^2.$$

This implies that $y = 0$, $x = (I + A)y = 0$; therefore $\mathrm{dom}(A)^\perp = \{0\}$, and it follows that $\mathrm{dom}(A)$ is dense in H. □

3.18 Remark. Lemma 3.17 implies that every accretive operator $A \in \mathcal{L}(H)$ is automatically m-accretive. Indeed, $\{\lambda \in \mathbb{R};\ \lambda > \|A\|\} \subseteq \rho(-A)$, by Remark 2.3(b). △

Proof of Theorem 3.16. The necessity is an immediate consequence of Theorem 2.7 and Lemma 3.14. The sufficiency follows from Lemma 3.17 and Theorem 2.9. □

For the remainder of this section we assume that H is a complex Hilbert space. In order to formulate a conclusion concerning the generation of holomorphic semigroups we need the following notions. For an operator A in H we define the **numerical range** $\operatorname{num}(A) := \big\{(Ax \,|\, x);\, x \in \operatorname{dom}(A),\ \|x\| = 1\big\}$. We call A **sectorial of angle** $\theta \in [0, \pi/2)$ if

$$\operatorname{num}(A) \subseteq \overline{\Sigma_\theta} = \big\{z \in \mathbb{C} \setminus \{0\};\, |\operatorname{Arg} z| \leqslant \theta\big\} \cup \{0\},$$

and we call A **m-sectorial** (**of angle** θ) if additionally $\operatorname{ran}(I + A) = H$. Here we supplement the notation introduced in Section 3.2 by $\Sigma_0 := (0, \infty)$. (Recall that previously Σ_θ was defined only for $0 < \theta \leqslant \pi/2$.)

We note that our definition of 'sectorial' is slightly more restrictive than the one used in [Kat80; V, § 3.10]. Unhappily, our terminology also conflicts with a notion introduced later in [PrSo93; Section 3] that has become an important concept in the functional calculus for operators.

3.19 Remarks. Let A be an operator in H.

(a) Obviously A is accretive if and only if $\operatorname{num}(A) \subseteq [\operatorname{Re} \geqslant 0]$.

(b) We note that for any angle α one has $\operatorname{num}(\mathrm{e}^{\mathrm{i}\alpha}A) = \mathrm{e}^{\mathrm{i}\alpha} \operatorname{num}(A)$. Let $\theta \in (0, \pi/2]$. Then it follows from (a) that $\mathrm{e}^{\mathrm{i}\alpha}A$ is accretive for all $\alpha \in (-\theta, \theta)$ if and only if $\operatorname{num}(A) \subseteq \overline{\Sigma_{\pi/2-\theta}}$. △

We now draw a conclusion of the Lumer–Phillips theorem for generators of holomorphic semigroups which are contractive on a sector.

3.20 Theorem. *Let A be an operator in the complex Hilbert space H, and let $\theta \in (0, \pi/2]$. Then $-A$ generates a contractive holomorphic C_0-semigroup of angle θ if and only if A is m-sectorial of angle $\pi/2 - \theta$.*

Proof. By Theorems 3.12 and 3.16 it suffices to show the equivalence

$$\mathrm{e}^{\mathrm{i}\alpha}A \text{ is m-accretive for all } \alpha \in (-\theta, \theta) \iff A \text{ is m-sectorial of angle } \pi/2 - \theta.$$

Either property implies that A is m-accretive, whereupon $\Sigma_\theta \subseteq [\operatorname{Re} > 0] \subseteq \rho(-A)$, by Lemma 3.17. Thus the equivalence follows from Remark 3.19(b). □

3.21 Remark. In the context of generators of contractive C_0-semigroups in Banach spaces, one usually considers *dissipative* instead of *accretive* operators. An operator A is called **dissipative** if $-A$ is accretive. One reason we prefer using the notion of accretive operators is that they will arise naturally in the context of forms. △

Notes

The equivalence of (i), (ii), (iii) in Theorem 3.2 is due to Dunford [Dun38; Theorem 76]. Theorem 3.4 is also due to Dunford; see [Hil39; footnote to Theorem 1], [HiPh57; Theorem 3.10.1].

Surprisingly, the continuity assumption in property (v) of Theorem 3.2 can be further relaxed: f is holomorphic if f is locally bounded and there exists a separating set $E \subseteq X'$ such that $x' \circ f$ is holomorphic for all $x' \in E$. This generalisation is due to Grosse-Erdmann [GrE92] (published in [GrE04]); an elegant short proof, based on the theorem of Krein–Šmulian, has been given in [ArNi00; Theorem 3.1], see also [Voi20; Example 12.2]. An even further weakening can be found in [ABK20; Theorem 1.1], where it is shown that instead of local boundedness of f it suffices to assume that there exists a function $g \in L_{1,\mathrm{loc}}(\Omega)$ such that $\|f(z)\| \leqslant g(z)$ for all $z \in \Omega$. An extra assumption of this kind cannot be omitted entirely: in [ArNi00; Theorem 1.5] a function f with a discontinuity is constructed such that $x' \circ f$ is holomorphic for all x' from a norming (non-closed!) subspace of X'; cf. [ArNi00; Remark 1.4 f)].

The contents of Section 3.2 are standard and can be found in most books on C_0-semigroups. The statement of Theorem 3.12 can be considered as standard. Its proof, however, deviates from the standard proofs. Classically, the generation theorem for holomorphic semigroups is treated by defining the semigroup as a contour integral. We refer to the literature for this kind of proof. Our proof follows the approach presented in [AEH97; Section 4] and [ArEl12a; Section 2]. We note that the characterisation stated at the end of Remark 3.13(b) can also be proved without contour integrals; see [AEH97; Theorem 4.3].

The notion of accretivity and the Lumer–Phillips theorem can be generalised to Banach spaces; see [Paz83; Section 1.4], [EnNa00; Section II.3.b]. The Hilbert space case is all we need in this book, and the restriction to this case makes our treatment significantly simpler.

Exercises

3.1 Define the subspace E of $c_0' = \ell_1$ by

$$E := \Big\{ x = (x_n) \in \ell_1 \,;\, \sum_{n=1}^{\infty} x_n = 0 \Big\}.$$

Show that E is almost norming, but not norming for c_0.

3.2 (a) Let M be a compact metric space, X a Banach space, (f_n) an equicontinuous sequence in $C(M;X)$. Show that the sequence (f_n) ist uniformly equicontinuous.

Assume further that $f(s) := \lim_{n\to\infty} f_n(s)$ exists for all $s \in M$. Show that $f \in C(M;X)$ and that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

(b) Let X, Y be Banach spaces, (T_n) a sequence in $\mathcal{L}(X,Y)$, $T_n \to T \in \mathcal{L}(X,Y)$ strongly. Let $M \subseteq X$ be compact. Show that $T_n x \to Tx$ $(n \to \infty)$ uniformly for $x \in M$. (Hint: Recall the uniform boundedness theorem.)

3.3 Let $\Omega \subseteq \mathbb{C}$ be an open set. Let (f_n) be a bounded sequence of bounded holomorphic functions $f_n \colon \Omega \to \mathbb{C}$ (i.e. $\sup_{z\in\Omega,\, n\in\mathbb{N}} |f_n(z)| < \infty$). Define the function $f \colon \Omega \to \ell_\infty$ by $f(z) := (f_n(z))_{n\in\mathbb{N}}$.

(a) Show that f is holomorphic. (Hint: Find a suitable norming set for ℓ_∞, and use Theorem 3.2.)

(b) Assume additionally that $f_\infty(z) := \lim_{n\to\infty} f_n(z)$ exists for all $z \in \Omega$. Show that then $f\colon \Omega \to c$ is holomorphic (where c denotes the subspace of ℓ_∞ consisting of the convergent sequences), and that f_∞ is holomorphic.

(c) Show that the convergence $f_n \to f_\infty$ is locally uniform. (Hint: Use the continuity of $f\colon \Omega \to c$.)

(Comment: The whole setup of this exercise could also start with X-valued functions, thereby finally yielding an alternative proof of Theorem 3.5.)

3.4 Let X be a complex Banach space, and let T be a holomorphic C_0-semigroup of angle $\theta \in (0, \pi/2]$ on X.

(a) Show that, for each $\theta' \in (0, \theta)$, there exists $\delta > 0$ such that

$$\sup_{z\in\Sigma_{\theta'},\, \mathrm{Re}\, z\leqslant\delta} \|T(z)\| < \infty.$$

(b) Show Remark 3.7(a).

(c) Show that the estimate in Remark 3.7(a) can be written equivalently as follows: for each $\theta' \in (0, \theta)$, there exist $M'' \geqslant 1$, $\omega'' \in \mathbb{R}$ such that

$$\|T(z)\| \leqslant M'' \mathrm{e}^{\omega''|z|} \qquad (z \in \Sigma_{\theta'}).$$

3.5 Let X be a complex Banach space, and let $\pi/2 < \theta \leqslant \pi$. Suppose that T is a holomorphic C_0-semigroup of angle θ (with the same definition as in Section 3.2, but now for $\theta \in (\pi/2, \pi]$). Show that the generator of T belongs to $\mathcal{L}(X)$. (Hint: Show that the C_0-group S defined by $S(s) := T(\mathrm{i}s)$ $(s \in \mathbb{R})$ is norm continuous.)

3.6 Let T be a bounded holomorphic C_0-semigroup, with generator A, on a complex Banach space X. Show that there exists $c > 0$ such that $\|AT(t)\| \leqslant \frac{c}{t}$ for all $t \geqslant 0$. (Hint: Use Cauchy's integral formula.)

3.7 Let T be a bounded holomorphic C_0-semigroup of angle $\theta \in (0, \pi/2]$, with generator A.

(a) Show that there exists a strongly continuous extension (also called T) to the closure of $\Sigma_{\theta,0}$. (Hint: Show first that the extension can be defined on $\bigcup_{t>0} \mathrm{ran}(T(t))$.)

(b) Show that $T_{\pm\theta}$, defined by $T_{\pm\theta}(t) := T\big(\mathrm{e}^{\pm\mathrm{i}\theta}t\big)$ $(t \geqslant 0)$, are C_0-semigroups (the **boundary semigroups** of T), with generators $\mathrm{e}^{\pm\mathrm{i}\theta}A$. (Hint concerning the generator property: For $x \in \mathrm{dom}(A)$ and $|\alpha| < \theta$ consider the functions $t \mapsto T(\mathrm{e}^{\mathrm{i}\alpha}t)x$ and their derivatives; then take the limits as $\alpha \to \pm\theta$.)

(c) If $\theta = \pi/2$, then show that $T_{\pi/2}(t) := T(\mathrm{i}t)$ $(t \in \mathbb{R})$ defines a C_0-group $T_{\pi/2}$ (the **boundary group** of T), with generator $\mathrm{i}A$.

3.8 Let A be an operator in a complex Hilbert space.

(a) Let $0 < \theta < \pi/2$ and suppose that $\mathrm{e}^{\pm\mathrm{i}\theta}A$ generate contractive C_0-semigroups. Show that A generates a contractive holomorphic C_0-semigroup of angle θ. (Hint: Use Theorem 3.20.)

(b) Show that the assertion in (a) does not hold for $\theta = \pi/2$.

Chapter 4

The Sobolev space H^1, and applications

In Section 4.1 we present the definition and some basic properties of the Sobolev space H^1. The treatment is prepared by several important tools from analysis. The main objective of this chapter is the Hilbert space treatment of the Laplace operator in Section 4.2. In particular, the Dirichlet Laplacian will be presented as our first (non-trivial) example of a generator of a contractive holomorphic C_0-semigroup.

4.1 The Sobolev space H^1

4.1.1 Convolution

Convolution is an important tool for regularizing locally integrable functions. We recall that the space of **locally integrable functions** on an open subset Ω of $\mathbb{R}^n$ is given by

$$L_{1,\mathrm{loc}}(\Omega) := \big\{ f\colon \Omega \to \mathbb{K};\ \forall x \in \Omega\ \exists r > 0\colon B(x,r) \subseteq \Omega,\ f|_{B(x,r)} \in L_1(B(x,r)) \big\}$$

(where as usual functions are identified if they agree a.e.). Moreover, $C_{\mathrm{c}}^k(\Omega) := C^k(\Omega) \cap C_{\mathrm{c}}(\Omega)$ is the space of k times continuously differentiable functions with compact support, for $k \in \mathbb{N}_0$, and similarly $C_{\mathrm{c}}^\infty(\Omega) := C^\infty(\Omega) \cap C_{\mathrm{c}}(\Omega)$.

4.1 Lemma. *Let $f \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$, $k \in \mathbb{N}_0 \cup \{\infty\}$, $\rho \in C_{\mathrm{c}}^k(\mathbb{R}^n)$. We define the* ***convolution*** *of ρ and f,*

$$\rho * f(x) := \int_{\mathbb{R}^n} \rho(x-y) f(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \rho(y) f(x-y)\,\mathrm{d}y \qquad (x \in \mathbb{R}^n).$$

Then $\rho * f \in C^k(\mathbb{R}^n)$, and for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leqslant k$, one has

$$\partial^\alpha(\rho * f) = (\partial^\alpha \rho) * f.$$

Proof. (i) The integral exists because ρ is bounded and has compact support.

(ii) Continuity of $\rho * f$: There exists $R > 0$ such that $\operatorname{spt}\rho \subseteq B(0,R)$. Let $R' > 0$, $\delta > 0$. For $x, x' \in B(0,R')$, $|x - x'| < \delta$, one obtains

$$\begin{aligned} |\rho * f(x) - \rho * f(x')| &= \left| \int_{B(0,R+R')} \big(\rho(x-y) - \rho(x'-y)\big) f(y)\,\mathrm{d}y \right| \\ &\leqslant \sup\big\{ |\rho(z) - \rho(z')|;\ |z - z'| < \delta \big\} \int_{B(0,R+R')} |f(y)|\,\mathrm{d}y. \end{aligned}$$

The second factor in the last expression is finite because f is locally integrable, and the first factor becomes small for small δ because ρ is uniformly continuous. This completes the proof if $k = 0$. For the remainder of the proof let $k \geqslant 1$.

(iii) Let $1 \leqslant j \leqslant n$. The existence of the partial derivative of $\rho * f$ with respect to the j-th variable and the equality $\partial_j(\rho * f) = (\partial_j \rho) * f$ are a consequence of the differentiability of integrals with respect to a parameter. The function $(\partial_j \rho) * f$ is continuous, by step (ii) above, and therefore $\rho * f$ is continuously differentiable with respect to the j-th variable.

(iv) Induction shows the assertion for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leqslant k$. □

A sequence $(\rho_k)_{k\in\mathbb{N}}$ in $C_c(\mathbb{R}^n)$ is called a **delta sequence** if $\rho_k \geqslant 0$, $\int \rho_k(x)\,\mathrm{d}x = 1$ and $\operatorname{spt}\rho_k \subseteq B[0, 1/k]$ for all $k \in \mathbb{N}$. (The term 'delta sequence' is motivated by the fact that the sequence approximates the 'δ-distribution' $C_c^\infty(\mathbb{R}^n) \ni \varphi \mapsto \varphi(0)$.)

4.2 Remarks. (a) We recall the standard example of a C_c^∞-function φ. The source of this function is the well-known function $\psi \in C^\infty(\mathbb{R})$,

$$\psi(t) := \begin{cases} 0 & \text{if } t \leqslant 0, \\ \mathrm{e}^{-1/t} & \text{if } t > 0. \end{cases}$$

Then $\varphi(x) := \psi(1 - |x|^2)$ $(x \in \mathbb{R}^n)$ defines a function $0 \leqslant \varphi \in C_c^\infty(\mathbb{R}^n)$, with the property that $\varphi(x) \neq 0$ if and only if $|x| < 1$.

(b) If $0 \leqslant \rho \in C_c(\mathbb{R}^n)$, $\int \rho(x)\,\mathrm{d}x = 1$, $\operatorname{spt}\rho \subseteq B[0, 1]$, and we define

$$\rho_k(x) := k^n \rho(kx) \qquad (x \in \mathbb{R}^n,\ k \in \mathbb{N}),$$

then (ρ_k) is a delta sequence. △

4.3 Proposition. *Let (ρ_k) be a delta sequence in $C_c(\mathbb{R}^n)$.*

(a) *Let $f \in C(\mathbb{R}^n)$. Then $\rho_k * f \to f$ uniformly on compact subsets of $\mathbb{R}^n$ as $k \to \infty$.*

(b) *Let $1 \leqslant p \leqslant \infty$, $f \in L_p(\mathbb{R}^n)$. Then $\rho_k * f \in L_p(\mathbb{R}^n)$,*

$$\|\rho_k * f\|_p \leqslant \|f\|_p \qquad (k \in \mathbb{N}).$$

If $1 \leqslant p < \infty$, then

$$\|\rho_k * f - f\|_p \to 0 \qquad (k \to \infty).$$

Proof. (a) The assertion is an easy consequence of the uniform continuity of f on compact subsets of $\mathbb{R}^n$.

(b) (i) For $p = \infty$ the estimate is straightforward.

If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we estimate, using Hölder's inequality in the second step,

$$\begin{aligned} |\rho_k * f(x)| &= \left| \int \rho_k(x-y)^{\frac{1}{q}+\frac{1}{p}} f(y)\,\mathrm{d}y \right| \\ &\leqslant \left(\int \rho_k(x-y)\,\mathrm{d}y \right)^{1/q} \left(\int \rho_k(x-y)|f(y)|^p\,\mathrm{d}y \right)^{1/p} \\ &= \left(\int \rho_k(x-y)|f(y)|^p\,\mathrm{d}y \right)^{1/p}. \end{aligned}$$

This estimate also holds (trivially) for $p = 1$. Then, with Fubini's theorem in the second step,

$$\int |\rho_k * f(x)|^p \,\mathrm{d}x \leqslant \iint \rho_k(x-y)|f(y)|^p \,\mathrm{d}y \,\mathrm{d}x$$
$$= \iint \rho_k(x-y) \,\mathrm{d}x \,|f(y)|^p \,\mathrm{d}y = \|f\|_p^p .$$

(ii) Now let $1 \leqslant p < \infty$. For $k \in \mathbb{N}$ we define $T_k \in \mathcal{L}(L_p(\mathbb{R}^n))$ by $T_k g := \rho_k * g$ for $g \in L_p(\mathbb{R}^n)$; then step (i) shows that $\|T_k\| \leqslant 1$.

If $g \in C_c(\mathbb{R}^n)$, then $T_k g \to g$ in $L_p(\mathbb{R}^n)$ as $k \to \infty$. Indeed, there exists $R > 0$ such that $\operatorname{spt} g \subseteq B[0,R]$. It is easy to check that this implies that $\operatorname{spt}(\rho_k * g) \subseteq B[0, R+1]$ for all $k \in \mathbb{N}$. Moreover $\rho_k * g \to g$ uniformly on $B[0, R+1]$, by part (a). Therefore $\rho_k * g \to g$ in $L_p(\mathbb{R}^n)$ as $k \to \infty$.

Now, the denseness of $C_c(\mathbb{R}^n)$ in $L_p(\mathbb{R}^n)$ (which we will accept as a fundamental fact from measure and integration theory) together with Proposition 1.6 implies that $T_k g \to g$ in $L_p(\mathbb{R}^n)$ as $k \to \infty$, for all $g \in L_p(\mathbb{R}^n)$. □

4.4 Corollary. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $1 \leqslant p < \infty$. Then $C_c^\infty(\Omega)$ is dense in $L_p(\Omega)$.*

Proof. Let (ρ_k) be a delta sequence in $C_c^\infty(\mathbb{R}^n)$.

Let $g \in C_c(\Omega)$, and extend g by zero to a function in $C_c(\mathbb{R}^n)$. Then $\rho_k * g \in C^\infty(\mathbb{R}^n)$ for all $k \in \mathbb{N}$, by Lemma 4.1. If $\frac{1}{k} < \operatorname{dist}(\operatorname{spt} g, \mathbb{R}^n \setminus \Omega)$, then $\operatorname{spt}(\rho_k * g) \subseteq \operatorname{spt} g + B[0, 1/k] \subseteq \Omega$ (see Exercise 4.2(a)), and therefore $\rho_k * g \in C_c^\infty(\Omega)$. From Proposition 4.3 we know that $\rho_k * g \to g$ $(k \to \infty)$ in $L_p(\mathbb{R}^n)$. So, we have shown that $C_c^\infty(\Omega)$ is dense in $C_c(\Omega)$ with respect to the L_p-norm.

Now the denseness of $C_c(\Omega)$ in $L_p(\Omega)$ yields the assertion. □

Note that in the above proof, $\operatorname{dist}(\operatorname{spt} g, \mathbb{R}^n \setminus \Omega) = \infty$ in the case $\Omega = \mathbb{R}^n$.

4.1.2 Distributional derivatives

We start with a simple version of integration by parts for functions on an open set $\Omega \subseteq \mathbb{R}^n$. Let $\psi \in C_c^1(\Omega)$ and $j \in \{1, \dots, n\}$. Then

$$\int_\Omega \partial_j \psi(x) \,\mathrm{d}x = 0. \tag{4.1}$$

Indeed, call $\tilde{\psi}$ the extension of ψ to $\mathbb{R}^n$ by zero. Then Fubini's theorem yields

$$\int_\Omega \partial_j \psi(x) \,\mathrm{d}x = \int_{\mathbb{R}^n} \partial_j \tilde{\psi}(x) \,\mathrm{d}x = \int_{\check{x}_j \in \mathbb{R}^{n-1}} \Big(\int_{x_j \in \mathbb{R}} \partial_j \tilde{\psi}(x) \,\mathrm{d}x_j \Big) \,\mathrm{d}\check{x}_j = 0,$$

where we use the notation $\check{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$, and where in the one but last term the inner integral vanishes by the fundamental theorem of calculus. Now let $f \in C^1(\Omega)$, $\varphi \in C_c^\infty(\Omega)$. Then, applying (4.1) to the function φf and observing the product rule of differentiation, one obtains

$$\int_\Omega \partial_j f(x) \varphi(x) \,\mathrm{d}x = -\int_\Omega f(x) \partial_j \varphi(x) \,\mathrm{d}x. \tag{4.2}$$

This equality is fundamental for the following definition of distributional derivatives.

Let $P(\partial) = \sum_{\alpha\in\mathbb{N}_0^n,\,|\alpha|\leqslant k} a_\alpha \partial^\alpha$ be a partial differential operator with constant coefficients $a_\alpha \in \mathbb{K}$ ($|\alpha| \leqslant k$), where $k \in \mathbb{N}$. (An important example is $P(\partial) = \sum_{j=1}^n \partial_j^2 = \Delta$, the Laplace operator.) Let $\Omega \subseteq \mathbb{R}^n$ be open, $f \in C^k(\Omega)$. Then for all "test functions" $\varphi \in C_c^\infty(\Omega)$, repeated application of (4.2) implies

$$\int_\Omega \big(P(\partial)f\big)\varphi \,\mathrm{d}x = \int_\Omega f \sum_{|\alpha|\leqslant k} (-1)^{|\alpha|} a_\alpha \partial^\alpha \varphi \,\mathrm{d}x.$$

Now, let $f \in L_{1,\mathrm{loc}}(\Omega)$. Then we say that $P(\partial)f \in L_{1,\mathrm{loc}}(\Omega)$ if there exists $g \in L_{1,\mathrm{loc}}(\Omega)$ such that

$$\int g\varphi \,\mathrm{d}x = \int f \sum_{|\alpha|\leqslant k} (-1)^{|\alpha|} a_\alpha \partial^\alpha \varphi \,\mathrm{d}x$$

for all $\varphi \in C_c^\infty(\Omega)$, and we say that $P(\partial)f = g$ holds in the **distributional sense**. In particular, if $\partial^\alpha f \in L_{1,\mathrm{loc}}(\Omega)$, then we call $\partial^\alpha f$ a **distributional** (or 'generalised', or 'weak') **derivative** of f.

In order to justify this definition we have to show that $g = P(\partial)f$ is unique as an element of $L_{1,\mathrm{loc}}(\Omega)$. This uniqueness is immediate from the following "fundamental lemma of the calculus of variations".

4.5 Lemma. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $f \in L_{1,\mathrm{loc}}(\Omega)$,*

$$\int f\varphi \,\mathrm{d}x = 0 \qquad (\varphi \in C_c^\infty(\Omega)).$$

Then $f = 0$.

The statement '$f = 0$' means that f is the zero element of $L_{1,\mathrm{loc}}(\Omega)$, i.e., if f is a representative, then $f = 0$ a.e.

Proof of Lemma 4.5. Let $\varphi \in C_c^\infty(\Omega)$. We show that $\varphi f = 0$ a.e. Defining

$$g(x) := \begin{cases} \varphi(x)f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

we have $g \in L_1(\mathbb{R}^n)$.

Let (ρ_k) be a delta sequence in $C_c^\infty(\mathbb{R}^n)$. From Lemma 4.3 we know that $\rho_k * g \to g$ in $L_1(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$, $k \in \mathbb{N}$ we have

$$\rho_k * g(x) = \int_\Omega \rho_k(x-y)\varphi(y)f(y)\,\mathrm{d}y = 0,$$

because $\rho_k(x - \cdot)\varphi \in C_c^\infty(\Omega)$. This shows that $\rho_k * g = 0$, and hence $g = 0$ (as an L_1-function).

From $\varphi f = 0$ a.e. for all $\varphi \in C_c^\infty(\Omega)$ we conclude that $f = 0$ a.e.; see Exercise 4.1. □

4.6 Remark. It follows from Lemma 4.5 and the preceding considerations that the distributional definition of $P(\partial)f$ given above is consistent with the 'classical' definition. To be more precise, let $f \in C^k(\Omega)$ and put $g := P(\partial)f$, computed with classical derivatives. Then $P(\partial)f$ exists in $L_{1,\mathrm{loc}}(\Omega)$ in the distributional sense and has the (unique) continuous representative g. △

In the remainder of this subsection we give more information on the one-dimensional case. The aim is to show the following distributional variant of the fundamental theorem of calculus.

4.7 Proposition. *Let $-\infty \leqslant a < x_0 < b \leqslant \infty$, $f, g \in L_{1,\mathrm{loc}}(a,b)$. Then $f' = g$ in the distributional sense if and only if there exists $c \in \mathbb{K}$ such that*

$$f(x) = c + \int_{x_0}^{x} g(y)\,\mathrm{d}y \qquad (a.e.\ x \in (a,b)).$$

Note that the right-hand side of the previous equality is continuous as a function of x, and that therefore f has a continuous representative.

For the proof of Proposition 4.7 we need a preparatory lemma.

4.8 Lemma. *Let $-\infty \leqslant a < b \leqslant \infty$, $h \in L_{1,\mathrm{loc}}(a,b)$, and assume that $h' = 0$ in the distributional sense. Then there exists $c \in \mathbb{K}$ such that $h = c$.*

Proof. (i) We start with the observation that a function $\psi \in C_{\mathrm{c}}^{\infty}(a,b)$ is the derivative of a function in $C_{\mathrm{c}}^{\infty}(a,b)$ if and only if $\int \psi(x)\,\mathrm{d}x = 0$; in this case one obtains $\int \psi(x)h(x)\,\mathrm{d}x = 0$ since $h' = 0$.

(ii) Let $\rho \in C_{\mathrm{c}}^{\infty}(a,b)$, $\int \rho(x)\,\mathrm{d}x = 1$, and put $c := \int \rho(x)h(x)\,\mathrm{d}x$. For all $\varphi \in C_{\mathrm{c}}^{\infty}(a,b)$ one obtains

$$\begin{aligned}\int \varphi(x)\big(h(x)-c\big)\,\mathrm{d}x &= \int \varphi(x)h(x)\,\mathrm{d}x - \int \varphi(y)\,\mathrm{d}y \int \rho(x)h(x)\,\mathrm{d}x \\ &= \int \Big(\varphi(x) - \int \varphi(y)\,\mathrm{d}y\,\rho(x)\Big)h(x)\,\mathrm{d}x = 0\end{aligned}$$

because of $\int \big(\varphi(x) - \int \varphi(y)\,\mathrm{d}y\,\rho(x)\big)\,\mathrm{d}x = 0$ and step (i). Now the assertion is a consequence of Lemma 4.5. □

Proof of Proposition 4.7. (i) We first show the sufficiency. Let $\varphi \in C_{\mathrm{c}}^{\infty}(a,b)$, and choose $x_1 \in (a, \inf \operatorname{spt}\varphi)$. Then

$$f(x) = c_1 + \int_{x_1}^{x} g(y)\,\mathrm{d}y \qquad (\text{a.e. } x \in (a,b)),$$

with $c_1 := c + \int_{x_0}^{x_1} g(y)\,\mathrm{d}y$. Since $\int_a^b \varphi'(x)\,\mathrm{d}x = 0$, we obtain

$$\begin{aligned}\int_a^b \varphi'(x)f(x)\,\mathrm{d}x &= \int_a^b \varphi'(x)\Big(c_1 + \int_{x_1}^{x} g(y)\,\mathrm{d}y\Big)\,\mathrm{d}x = \iint\limits_{x_1<y<x<b} \varphi'(x)g(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \iint\limits_{x_1<y<x<b} \varphi'(x)g(y)\,\mathrm{d}x\,\mathrm{d}y = -\int_a^b \varphi(y)g(y)\,\mathrm{d}y.\end{aligned}$$

Thus, $f' = g$.

(ii) For the proof of the necessity put

$$h(x) := f(x) - \int_{x_0}^{x} g(y)\,\mathrm{d}y \qquad (a < x < b).$$

Then step (i) implies that $h' = f' - g = 0$ in the distributional sense, so by Lemma 4.8 there exists $c \in \mathbb{K}$ such that $h = c$. □

4.1.3 Definition of $H^1(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$ be open. We define the **Sobolev space**

$$H^1(\Omega) := \big\{ f \in L_2(\Omega);\, \partial_j f \in L_2(\Omega)\ (j \in \{1,\dots,n\}) \big\},$$

with scalar product

$$(f \,|\, g)_{H^1} := (f \,|\, g) + \sum_{j=1}^{n} (\partial_j f \,|\, \partial_j g)$$

(where

$$(f \,|\, g) := \int_\Omega f(x)\overline{g(x)}\,\mathrm{d}x$$

denotes the usual scalar product in $L_2(\Omega)$) and associated norm

$$\|f\|_{H^1} := \Big(\|f\|_2^2 + \sum_{j=1}^{n} \|\partial_j f\|_2^2 \Big)^{1/2}.$$

4.9 Theorem. *The space $H^1(\Omega)$ is a separable Hilbert space.*

For the proof we single out an important property concering the convergence of distributional derivatives, valid in a more general context.

4.10 Lemma. *Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $\alpha \in \mathbb{N}_0^n$. Let (f_k) be a sequence in $L_{1,\mathrm{loc}}(\Omega)$, with $\partial^\alpha f_k \in L_{1,\mathrm{loc}}(\Omega)$ for all $k \in \mathbb{N}$, let $f, g \in L_{1,\mathrm{loc}}(\Omega)$, and assume that $f_k|_K \to f|_K$, $\partial^\alpha f_k|_K \to g|_K$ $(k \to \infty)$ in $L_1(K)$, for all compact sets $K \subseteq \Omega$.*
Then $\partial^\alpha f = g \in L_{1,\mathrm{loc}}(\Omega)$.

Proof. Let $\varphi \in C_\mathrm{c}^\infty(\Omega)$. Then

$$\int \partial^\alpha f_k(x)\varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int f_k(x)\partial^\alpha\varphi(x)\,\mathrm{d}x$$

for all $k \in \mathbb{N}$. Taking $k \to \infty$ we obtain

$$\int g(x)\varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int f(x)\partial^\alpha\varphi(x)\,\mathrm{d}x.$$

This shows the assertion. □

Proof of Theorem 4.9. Clearly $H^1(\Omega)$ is a pre-Hilbert space.

Let $J\colon H^1(\Omega) \to \bigoplus_{j=0}^n L_2(\Omega)$ be defined by $Jf := (f, \partial_1 f, \dots, \partial_n f)$, where $\bigoplus_{j=0}^n L_2(\Omega)$ denotes the orthogonal direct sum. Then clearly J is an isometric operator. Therefore $H^1(\Omega)$ is complete if and only if the range of J is a closed linear subspace of the Hilbert space $\bigoplus_{j=0}^n L_2(\Omega)$. Let (f_k) be a sequence in $H^1(\Omega)$ such that (Jf_k) is convergent in $\bigoplus_{j=0}^n L_2(\Omega)$ to an element $(f^0, \dots, f^n)$. This means that $f_k \to f^0$ and $\partial_j f_k \to f^j$ $(j = 1, \dots, n)$ in $L_2(\Omega)$ as $k \to \infty$. Lemma 4.10 implies $f^j = \partial_j f^0$ $(j = 1, \dots, n)$, so we have shown that $f := f^0 \in H^1(\Omega)$ and that $Jf = (f^0, \dots, f^n)$.

The space $L_2(\Omega)$ is separable, therefore $\bigoplus_{j=0}^n L_2(\Omega)$ is separable, the closed subspace $J(H^1(\Omega))$ of $\bigoplus_{j=0}^n L_2(\Omega)$ is separable, and thus $H^1(\Omega)$ is separable because J is isometric. □

As in Subsection 4.1.2, we give additional information for the one-dimensional case.

4.11 Theorem. *Let $-\infty < a < b < \infty$. Then every $f \in H^1(a,b)$ possesses a representative in $C[a,b]$, and the embedding $H^1(a,b) \hookrightarrow C[a,b]$ thus defined is continuous.*

Proof. By Proposition 4.7, every function $f \in H^1(a,b)$ has a representative of the form $f(x) = c + \int_{x_0}^x f'(y)\,\mathrm{d}y$, with $x_0 \in (a,b)$ and $c \in \mathbb{K}$. Because of $f' \in L_2(a,b) \subseteq L_1(a,b)$, this representative is continuous as a function of $x \in [a,b]$.

Let $f \in H^1(a,b)$, with f chosen as the continuous representative. Then

$$\begin{aligned} \|f\|_{C[a,b]} &\leqslant \inf_{x\in(a,b)} |f(x)| + \int_a^b |f'(x)|\,\mathrm{d}x \leqslant \frac{1}{b-a}\int_a^b |f(x)|\,\mathrm{d}x + \int_a^b |f'(x)|\,\mathrm{d}x \\ &\leqslant (b-a)^{-1/2}\|f\|_2 + (b-a)^{1/2}\|f'\|_2 \leqslant \big((b-a)^{-1/2} + (b-a)^{1/2}\big)\|f\|_{H^1}\,. \end{aligned} \tag{4.3}$$

This inequality shows that the embedding is a bounded operator. □

4.12 Remarks. (a) For $c \in \mathbb{R}$ one also obtains continuous embeddings $H^1(-\infty,c) \hookrightarrow C_0(-\infty,c]$, $H^1(c,\infty) \hookrightarrow C_0[c,\infty)$ and $H^1(-\infty,\infty) \hookrightarrow C_0(-\infty,\infty)$. This is an easy consequence of Theorem 4.11; the important observation is that the norm of the embedding only depends on the length of the interval (a,b).

In fact, the inequality derived in the proof of Theorem 4.11 shows that one obtains

$$\|f\|_\infty \leqslant \big(d^{-1/2} + d^{1/2}\big)\|f\|_{H^1} \qquad (f \in H^1(a,b))$$

if (a,b) contains an interval of length d.

(b) Theorem 4.11 is a simple instance of the Sobolev embedding theorems. △

4.1.4 Denseness properties

Let $\Omega \subseteq \mathbb{R}^n$ be open. For $f \in L_{1,\mathrm{loc}}(\Omega)$ we define the **support** of f by

$$\operatorname{spt} f := \Omega \setminus \bigcup\big\{U \subseteq \Omega;\, U \text{ open},\ f|_U = 0\big\}.$$

(This definition is consistent with the already defined support for continuous functions.) Furthermore we define

$$\begin{aligned} H^1_{\mathrm{c}}(\Omega) &:= \big\{f \in H^1(\Omega);\, \operatorname{spt} f \text{ compact in } \Omega\big\}, \\ H^1_0(\Omega) &:= \overline{H^1_{\mathrm{c}}(\Omega)}^{H^1(\Omega)}. \end{aligned}$$

4.13 Remarks. (a) In a generalised sense, functions in $H^1_0(\Omega)$ 'vanish at the boundary of Ω'. For sets Ω with regular boundary, this will be made more precise in Section 7.2.

(b) For $f \in H^1_0(\Omega)$ the extension to $\mathbb{R}^n$ by zero belongs to $H^1(\mathbb{R}^n)$; see Exercise 4.6. (One first shows this for $f \in H^1_{\mathrm{c}}(\Omega)$ and then uses a denseness argument.) △

4.14 Theorem. (a) $H^1_0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$.

(b) $H^1_0(\Omega) = \overline{C^\infty_{\mathrm{c}}(\Omega)}^{H^1(\Omega)}$.

We need the following auxiliary result.

4.15 Lemma. (a) *Let $\alpha \in \mathbb{N}_0^n$, $f, \partial^\alpha f \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$, $\rho \in C_c^\infty(\mathbb{R}^n)$. Then*

$$\partial^\alpha(\rho * f) = \rho * (\partial^\alpha f).$$

(b) *Let $f \in H^1(\mathbb{R}^n)$, and let (ρ_k) be a delta sequence in $C_c^\infty(\mathbb{R}^n)$. Then $\rho_k * f \to f$ in $H^1(\mathbb{R}^n)$ as $k \to \infty$.*

Proof. (a) Using Lemma 4.1 in the first equality and the definition of the distributional derivative in the third, one obtains

$$\begin{aligned}\partial^\alpha(\rho * f)(x) &= \int \partial^\alpha \rho(x-y) f(y)\,\mathrm{d}y = (-1)^{|\alpha|} \int \partial^\alpha [y \mapsto \rho(x-y)] f(y)\,\mathrm{d}y \\ &= \int \rho(x-y)\,\partial^\alpha f(y)\,\mathrm{d}y = \rho * \partial^\alpha f(x).\end{aligned}$$

(b) From Proposition 4.3(b) we know that $\rho_k * f \to f$ in $L_2(\mathbb{R}^n)$ as $k \to \infty$. Using part (a) we further obtain

$$\partial_j(\rho_k * f) = \rho_k * \partial_j f \to \partial_j f \qquad (k \to \infty)$$

in $L_2(\mathbb{R}^n)$, for $1 \leqslant j \leqslant n$, and the assertion follows. □

Proof of Theorem 4.14. (a) Let $f \in H^1(\mathbb{R}^n)$, $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then from Exercise 4.5(d) we obtain $\partial_j(\varphi f) = \partial_j \varphi f + \varphi \partial_j f \in L_2(\mathbb{R}^n)$ for all $j \in \{1, \ldots, n\}$. Hence $\varphi f \in H_c^1(\mathbb{R}^n)$.

Choose $\psi \in C_c^\infty(\mathbb{R}^n)$, $\psi|_{B(0,1)} = 1$, and put $\psi_k := \psi(\cdot/k)$ $(k \in \mathbb{N})$. Then $\psi_k f \to f$ and $\partial_j(\psi_k f) = \partial_j \psi_k f + \psi_k \partial_j f \to 0 \cdot f + \partial_j f$ in $L_2(\mathbb{R}^n)$ as $k \to \infty$. Thus the functions $\psi_k f \in H_c^1(\mathbb{R}^n)$ approximate f in $H^1(\mathbb{R}^n)$.

(b) The inclusion '$\supseteq$' is trivial. For the inclusion '$\subseteq$' it is sufficient to show that each $f \in H_c^1(\Omega)$ can be approximated by elements of $C_c^\infty(\Omega)$. This, however, is a consequence of Remark 4.13(b) and Lemma 4.15(b). (Observe that the support of $\rho_k * f$ is a compact subset of Ω, for large $k \in \mathbb{N}$; cf. Exercise 4.2(a).) □

4.16 Remark. In general, the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ do not coincide. We show this for the case of bounded $\Omega \neq \varnothing$.

Note that for all $f \in H_0^1(\Omega)$ one has $\int \partial_1 f(x)\,\mathrm{d}x = 0$. This is clear if $f \in C_c^\infty(\Omega)$ and carries over to $H_0^1(\Omega)$ by continuity. Now let $f \in H^1(\Omega)$ be defined by $f(x) := x_1$ $(x \in \Omega)$. Then $\int \partial_1 f(x)\,\mathrm{d}x \neq 0$ and therefore $f \in H^1(\Omega) \setminus H_0^1(\Omega)$. △

4.17 Examples (Right translation semigroups). We come back to Examples 1.7 and describe the generator A, but only for the case $p = 2$.

(a) On $L_2(\mathbb{R})$: It was mentioned in Example 1.16 that $D := C_c^1(\mathbb{R})$ is a core for A, i.e. $A = \overline{A|_D}$, and that $Af = -f'$ for all $f \in D$. Thus $A|_D$, considered as a subspace of $L_2(\mathbb{R}) \times L_2(\mathbb{R})$, is isomorphic to $C_c^1(\mathbb{R})$ as a subspace of $H^1(\mathbb{R})$ (cf. the proof of Theorem 4.9), and one obtains

$$\mathrm{dom}(A) = \overline{C_c^1(\mathbb{R})}^{H^1(\mathbb{R})} = H_0^1(\mathbb{R}) = H^1(\mathbb{R}), \quad Af = -f' \quad (f \in \mathrm{dom}(A)).$$

(b) On $L_2(0,\infty)$: Similarly to part (a) one obtains

$$\operatorname{dom}(A) = \overline{C_c^1(0,\infty)}^{H^1(0,\infty)} = H_0^1(0,\infty), \quad Af = -f' \quad (f \in \operatorname{dom}(A)).$$

(c) On $L_2(-\infty,0)$: In this case, the set $C_c^1(-\infty,0]$ is a subset of $\operatorname{dom}(A)$, $Af = -f'$ for all $f \in C_c^1(-\infty,0]$, and $C_c^1(-\infty,0]$ is invariant under T; hence by Proposition 1.15, $C_c^1(-\infty,0]$ is a core for A. Then as in part (a) we can conclude that

$$\operatorname{dom}(A) = \overline{C_c^1(-\infty,0]}^{H^1(-\infty,0)} = H^1(-\infty,0), \quad Af = -f' \quad (f \in \operatorname{dom}(A)),$$

once we know that $C_c^1(-\infty,0]$ is dense in $H^1(-\infty,0)$.

In order to obtain the denseness statement we first note that the method of the proof of Theorem 4.14(a) shows that $\{f \in H^1(-\infty,0)\,;\, \operatorname{spt} f \text{ bounded}\}$ is dense in $H^1(-\infty,0)$. Now let $a < 0$, $f \in H^1(-\infty,0)$ with $\operatorname{spt} f \subseteq (a,0)$. Then there exists a sequence (φ_n) in $C_c(-\infty,0)$ with $\operatorname{spt}\varphi_n \subseteq (a,0)$ for all $n \in \mathbb{N}$ such that $\varphi_n \to f'$ in $L_2(-\infty,0)$ as $n \to \infty$. Putting

$$f_n(x) := \int_{-\infty}^{x} \varphi_n(y)\,\mathrm{d}y \qquad (x \in (-\infty,0),\ n \in \mathbb{N}),$$

one easily shows that $f_n \in C_c^1(-\infty,0]$ for all $n \in \mathbb{N}$, and $f_n \to f$ in $H^1(-\infty,0)$ as $n \to \infty$.

(d) On $L_2(0,1)$: As in part (c) one first shows that $C_c^1(0,1]$ is a core for A and then obtains

$$\operatorname{dom}(A) = \overline{C_c^1(0,1]}^{H^1(0,1)} = \big\{f \in H^1(0,1)\,;\, f(0) = 0\big\}, \quad Af = -f' \quad (f \in \operatorname{dom}(A)).$$

The required denseness of $C_c^1(0,1]$ is seen as follows. For a function $f \in H^1(0,1)$ with $f(0) = 0$ one chooses a sequence (φ_n) in $C_c(0,1)$ approximating f' in $L_2(0,1)$. Then

$$f_n(x) := \int_{0}^{x} \varphi_n(y)\,\mathrm{d}y \qquad (x \in (0,1),\ n \in \mathbb{N}),$$

defines a sequence (f_n) in $C_c^1(0,1]$ approximating f in $H^1(0,1)$. △

4.2 A variant of the Poisson problem, and the Dirichlet Laplacian

The first aim of this section is to present a variant of the solution of the Poisson equation

$$-\Delta u = f \text{ on } \Omega, \quad u|_{\partial\Omega} = 0,$$

for a given function f on an open set $\Omega \subseteq \mathbb{R}^n$. One might look for a solution u that is twice differentiable on Ω and continuous on the closure. We will not treat the problem in this form but rather weaken the requirements.

More explicitly, the boundary condition $u|_{\partial\Omega} = 0$ will be modified to the requirement that u should belong to $H_0^1(\Omega)$, and the equation itself will only be required to hold in the distributional sense.

In the second part of the section we will establish the connection to m-accretivity and holomorphic semigroups.

4.2.1 The equation $u - \Delta u = f$

4.18 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $f \in L_2(\Omega)$. Then there exists a unique function $u \in H_0^1(\Omega)$ such that $u - \Delta u = f$ in the distributional sense.*

We insert a lemma expressing the distributional equality in another form.

4.19 Lemma. *Let $u \in H^1(\Omega)$, $g \in L_2(\Omega)$. Then $\Delta u = g$ in the distributional sense if and only if*

$$(v \,|\, g)_{L_2(\Omega)} = -\sum_{j=1}^{n} (\partial_j v \,|\, \partial_j u)_{L_2(\Omega)} \qquad (v \in H_0^1(\Omega)). \tag{4.4}$$

Proof. By the definition of distributional derivatives, the equation $\Delta u = g$ is equivalent to

$$(\varphi \,|\, g) = (\Delta\varphi \,|\, u) = -\sum_{j=1}^{n} (\partial_j \varphi \,|\, \partial_j u) \qquad (\varphi \in C_{\mathrm{c}}^{\infty}(\Omega)),$$

i.e. to the validity of equation (4.4) for all $v \in C_{\mathrm{c}}^{\infty}(\Omega)$. As both mappings $H^1(\Omega) \ni v \mapsto (v \,|\, g) \in \mathbb{K}$ and $H^1(\Omega) \ni v \mapsto \sum_{j=1}^{n} (\partial_j v \,|\, \partial_j u) \in \mathbb{K}$ are continuous, the equality of the terms in (4.4) extends to the closure of $C_{\mathrm{c}}^{\infty}(\Omega)$ in $H^1(\Omega)$, i.e. to $H_0^1(\Omega)$ (recall Theorem 4.14(b)). □

Proof of Theorem 4.18. We define a bounded linear functional $\eta\colon H_0^1(\Omega) \to \mathbb{K}$ by

$$\eta(v) := (v \,|\, f)_{L_2(\Omega)} \qquad (v \in H_0^1(\Omega)).$$

Applying the representation theorem of Fréchet–Riesz (see e.g. [Bre11; Theorem 5.5]) we obtain $u \in H_0^1(\Omega)$ such that

$$\eta(v) = (v \,|\, u)_{H_0^1(\Omega)} \qquad (v \in H_0^1(\Omega)).$$

Putting this equation and the definition of η together we obtain

$$(v \,|\, f) = (v \,|\, u)_{H_0^1(\Omega)} = (v \,|\, u) + \sum_{j=1}^{n} (\partial_j v \,|\, \partial_j u) \qquad (v \in H_0^1(\Omega)).$$

Shifting the first term on the right-hand side to the left and applying Lemma 4.19 we conclude that $-\Delta u = f - u$ in the distributional sense.

The uniqueness of u is a consequence of the uniqueness in the Fréchet–Riesz representation theorem. □

4.2.2 The Dirichlet Laplacian

In this subsection we reformulate the result of Subsection 4.2.1 in operator language. As before, let $\Omega \subseteq \mathbb{R}^n$ be open. In what follows the space $L_2(\Omega)$ will be complex.

We define the **Dirichlet Laplacian** Δ_{D} in $L_2(\Omega)$,

$$\Delta_{\mathrm{D}} := \big\{(u, f) \in L_2(\Omega) \times L_2(\Omega);\, u \in H_0^1(\Omega),\ \Delta u = f\big\}.$$

In other words,

$$\begin{aligned} \operatorname{dom}(\Delta_{\mathrm{D}}) &:= \big\{u \in H_0^1(\Omega);\, \Delta u \in L_2(\Omega)\big\}, \\ \Delta_{\mathrm{D}} u &:= \Delta u \qquad (u \in \operatorname{dom}(\Delta_{\mathrm{D}})). \end{aligned}$$

We will show that Δ_{D} generates a contractive holomorphic C_0-semigroup of angle $\pi/2$. The name 'Dirichlet Laplacian' may be somewhat misleading, so we give a short explanation. In principle, 'Dirichlet boundary conditions' are of the form $u|_{\partial\Omega} = \varphi$ for a given function φ defined on $\partial\Omega$. Recall that the membership of u in $H_0^1(\Omega)$ is a version of Dirichlet boundary condition zero. So, 'Dirichlet Laplacian' should be regarded as an abbreviation of 'Laplacian with Dirichlet boundary condition zero'.

4.20 Theorem. *The negative Dirichlet Laplacian $-\Delta_{\mathrm{D}}$ is m-sectorial of angle 0. The operator Δ_{D} is the generator of a contractive holomorphic C_0-semigroup of angle $\pi/2$ on $L_2(\Omega)$.*

Proof. For $u \in \operatorname{dom}(\Delta_{\mathrm{D}})$ an application of Lemma 4.19 yields

$$(-\Delta_{\mathrm{D}} u \,|\, u) = (-\Delta u \,|\, u) = \sum_{j=1}^{n} (\partial_j u \,|\, \partial_j u) \in [0, \infty).$$

This means that $\operatorname{num}(-\Delta_{\mathrm{D}}) \subseteq [0, \infty)$. Also, Theorem 4.18 states that $\operatorname{ran}(I - \Delta_{\mathrm{D}}) = L_2(\Omega)$. As a consequence, $-\Delta_{\mathrm{D}}$ is m-sectorial of angle 0.

Now Theorem 3.20 implies that Δ_{D} generates a contractive holomorphic C_0-semigroup of angle $\pi/2$. □

The semigroup T generated by Δ_{D} describes heat propagation in Ω. In fact, let $u_0 \in L_2(\Omega)$, and put $u(t) := T(t)u_0$ for $t \geqslant 0$. Then $u \in C([0,\infty); L_2(\Omega)) \cap C^\infty(0,\infty; L_2(\Omega))$, $u(t) \in \operatorname{dom}(\Delta_{\mathrm{D}})$ for all $t > 0$, and

$$\begin{aligned} &u'(t) = \Delta u(t), \quad u(t)|_{\partial\Omega} = 0 \qquad (t > 0), \\ &u(0) = u_0\,; \end{aligned}$$

see Remarks 3.11 and 4.13(a). If we consider a body Ω (a bounded open subset of $\mathbb{R}^n$) and $u_0(x)$ as the temperature at $x \in \Omega$ at time 0, then $u(t)(x)$ is the temperature at time $t > 0$ at x. The boundary condition means that the temperature is kept at 0 at the boundary. One expects that $\lim_{t\to\infty} u(t) = 0$. That this is indeed the case will be seen in Exercise 5.2.

The statement that '$-\Delta_{\mathrm{D}}$ is m-sectorial of angle 0' is equivalent to saying that $-\Delta_{\mathrm{D}}$ is an accretive self-adjoint operator; this will be explained in Chapter 6.

Notes

The theory of partial differential equations started in the 18th century, after the invention of Newton's "fluxions" and Leibnitz' ingenious contribution, the introduction of the symbols of differential and integral calculus, used to this day. Up to the beginning of the

20th century, solutions of differential equations were always discussed under the basic assumption that all the derivatives of the solution appearing in the equation are classical derivatives.

It was in the early 20th century, in connection with the development of functional analysis and integration theory, that 'generalised solutions' of problems in differential equation were considered. The pioneering idea appears to be contained in the paper of Beppo Levi [Lev06] from 1906: in his discussion of the Dirichlet problem (in two space dimensions) he used an approximation procedure to find a continuous solution whose generalised first partial derivatives belong to L_2. (For more information on the Dirichlet problem we refer to the Notes of Chapter 7.)

It was the groundbreaking contribution of Sergei Lvovich Sobolev [Sob63] in 1938 to introduce normed spaces of functions with distributional derivatives belonging to L_p and to study embeddings between certain of these spaces. (Our Theorem 4.11 is a particularly simple instance of a "Sobolev embedding theorem".) In the early 1950's, French mathematicians were in search of a name for this kind of spaces. The suggestion to call them 'Beppo Levi spaces' was strongly opposed by Levi himself; as a result, the name "Sobolev spaces" was coined and now is universally accepted. For a detailed account of the above sketchy description of the development in the late 19th and early 20th centuries we refer to [WKK09; Section 10.1] and to [Nau02].

In Section 4.1 we have collected basics of Sobolev spaces as far as we have needed them in this chapter. Additional properties will be presented in subsequent chapters, according to what will be needed and used later. In this book we will mainly need the 'Hilbert-Sobolev spaces' $H^1(\Omega)$ and $H^1_0(\Omega)$, with the exception of Chapter 17 and Appendix D, where we use $W^1_1(\Omega)$. In fact, Sobolev spaces are defined in the more general setting for $k \in \mathbb{N}_0$ and $p \in [1, \infty]$, by

$$W^k_p(\Omega) := \{u \in L_p(\Omega);\, \partial^\alpha u \in L_p(\Omega)\ (\alpha \in \mathbb{N}^n_0,\ |\alpha| \leqslant k)\},$$

where the derivatives $\partial^\alpha u$ are understood in the distributional sense. For more information we refer to [AdFo05]. We mention that one also defines Sobolev spaces of negative order; for an example we refer to the definition of $H^{-1}(\Omega)$ in Section 16.1.

The treatment of the Dirichlet Laplacian as given in this chapter is well-established and can be found in many textbooks on partial differential equations.

Exercises

4.1 Let $\Omega \subseteq \mathbb{R}^n$ be open.

(a) Show that there exists a **standard exhaustion** $(\Omega_k)_{k\in\mathbb{N}}$ of Ω, i.e., Ω_k is open, relatively compact in Ω_{k+1} $(k \in \mathbb{N})$, and $\bigcup_{k\in\mathbb{N}} \Omega_k = \Omega$. (Hint: For $\Omega \neq \mathbb{R}^n$ use

$$\Omega_k := \big\{x \in \Omega;\, |x| < k,\ \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) > \tfrac{1}{k}\big\}.)$$

(b) Let $f \in L_{1,\mathrm{loc}}(\Omega)$, and assume that $f = 0$ locally, i.e., for all $x \in \Omega$ there exists $r > 0$ such that $B(x, r) \subseteq \Omega$ and $f|_{B(x,r)} = 0$. Then $f = 0$. (All '$= 0$' should be interpreted as a.e.)

4.2 (a) Let $f \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$, $\varphi \in C_\mathrm{c}(\mathbb{R}^n)$. Show that $\operatorname{spt}(\varphi * f) \subseteq \operatorname{spt} f + \operatorname{spt} \varphi$. (Hint: Show first that $\operatorname{spt} f + \operatorname{spt} \varphi$ is closed.)

(b) Let $K \subseteq U \subseteq \mathbb{R}^n$, K compact, U open. Show that there exists $\psi \in C_\mathrm{c}^\infty(\mathbb{R}^n)$ with $\operatorname{spt} \psi \subseteq U$ and $\mathbf{1}_K \leqslant \psi \leqslant 1$. (Hint: Note that $\operatorname{dist}(K, \mathbb{R}^n \setminus U) > 0$. Find ψ as the convolution of a suitable function $\rho \in C_\mathrm{c}^\infty(\mathbb{R}^n)$ with a suitable indicator function.)

4.3 (a) Let K be a compact subset of a metric space M, and let $(U_j)_{j=1,\dots,m}$ be a covering of K by open subsets of M. Show that there exists a *shrinking* of $(U_j)_{j=1,\dots,m}$, i.e. an open covering $(V_j)_{j=1,\dots,m}$ of K such that $\overline{V_j} \subseteq U_j$ for all $j = 1, \dots, m$.

Hints: Use the compactness of K to find a finite covering $\big(B(x, r_x)\big)_{x \in F}$ of K, with a finite set $F \subseteq K$ and $r_x > 0$ for all $x \in F$, such that for each $x \in F$ the closed ball $B[x, r_x]$ is contained in one of the U_j. Then put

$$V_j := \bigcup_{x \in F \text{ with } B[x, r_x] \subseteq U_j} B(x, r_x) \qquad (j \in \{1, \dots, m\}).$$

(b) Let $\Omega \subseteq \mathbb{R}^n$ be open, $K \subseteq \Omega$ compact, $(U_j)_{j=1,\dots,m}$ a covering of K by open subsets of Ω. Show that there exists a family $(\chi_j)_{j=1,\dots,m}$ in $C_\mathrm{c}^\infty(\Omega)$, $\chi_j \geqslant 0$ and $\operatorname{spt} \chi_j \subseteq U_j$ for all $j \in \{1, \dots, m\}$, such that $\sum_{j=1}^m \chi_j(x) = 1$ for all $x \in K$. The family $(\chi_j)_{j=1,\dots,m}$ is called a *partition of unity on K subordinate to the covering $(U_j)_{j=1,\dots,m}$ of K*.

Hints: Use part (a) to find a shrinking $(V_j)_{j=1,\dots,m}$ of $(U_j)_{j=1,\dots,m}$ with the additional property that each set V_j is bounded. Construct an m-tuple $(W_j)_{j=1,\dots,m}$ of pairwise disjoint measurable sets such that $\bigcup_{j=1}^m W_j = \bigcup_{j=1}^m V_j$. Then choose $\rho \in C_\mathrm{c}^\infty(\mathbb{R}^n)_+$ with $\int \rho = 1$, $\operatorname{spt} \rho \subseteq B(0, \varepsilon)$ for suitably small $\varepsilon > 0$ and put $\chi_j := \rho * \mathbf{1}_{W_j}$ to obtain the asserted partition of unity.

(c) Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $P(\partial)$ be a partial differential operator with constant coefficients, $f, g \in L_{1,\mathrm{loc}}(\Omega)$, $g = P(\partial) f$ in the distributional sense *locally* on Ω, i.e., for every $x \in \Omega$ there exists an open neighbourhood U_x such that $g|_{U_x} = P(\partial)(f|_{U_x})$ in the distributional sense. Show that $g = P(\partial) f$ in the distributional sense.

Hint: For $\varphi \in C_\mathrm{c}^\infty(\Omega)$, $K := \operatorname{spt} \varphi$ and a finite covering $(U_x)_{x \in F}$ of K by sets as above use a partition of unity as indicated in part (b).

4.4 Let $H \subseteq \mathbb{R}^2$ be the half-plane $H := \{(x_1, x_2);\, x_1 \geqslant 0\}$, and let $f \in L_{1,\mathrm{loc}}(\mathbb{R}^2)$ be defined by $f := \mathbf{1}_H$.

(a) Show that $\int \partial_1 \varphi f \,\mathrm{d}x = -\int_{x_2 \in \mathbb{R}} \varphi(0, x_2) \,\mathrm{d}x_2$ for all $\varphi \in C_\mathrm{c}^\infty(\mathbb{R}^2)$ and that there is no $g \in L_{1,\mathrm{loc}}(\mathbb{R}^2)$ such that $\int \partial_1 \varphi f \,\mathrm{d}x = \int \varphi g \,\mathrm{d}x$ for all $\varphi \in C_\mathrm{c}^\infty(\mathbb{R}^2)$.

(b) Decide which of the partial derivatives $\partial_1 f, \partial_2 f, \partial^{(1,1)} f$ belong to $L_{1,\mathrm{loc}}(\mathbb{R}^2)$.

4.5 Let $\Omega \subseteq \mathbb{R}^n$ be an open set.

(a) Let $\alpha \in \mathbb{N}_0^n$ and $f, \partial^\alpha f \in L_{1,\mathrm{loc}}(\Omega)$. Show that $\operatorname{spt} \partial^\alpha f \subseteq \operatorname{spt} f$. (Hint: Look at the complements of the supports.)

(b) Let $\alpha, \beta \in \mathbb{N}_0^n$ and $f, \partial^\alpha f \in L_{1,\mathrm{loc}}(\Omega)$. Show that then

$$\partial^{\alpha+\beta} f \in L_{1,\mathrm{loc}}(\Omega) \iff \partial^\beta(\partial^\alpha f) \in L_{1,\mathrm{loc}}(\Omega) \implies \partial^{\alpha+\beta} f = \partial^\beta(\partial^\alpha f).$$

(c) Let $f \in L_{1,\mathrm{loc}}(\Omega)$, $\nabla f \in L_{1,\mathrm{loc}}(\Omega; \mathbb{K}^n)$. Show that then

$$\Delta f \in L_{1,\mathrm{loc}}(\Omega) \iff \operatorname{div} \nabla f \in L_{1,\mathrm{loc}}(\Omega) \implies \Delta f = \operatorname{div} \nabla f.$$

(If $w \in L_{1,\mathrm{loc}}(\Omega;\mathbb{K}^n)$ is a vector field, then the distributional divergence $\operatorname{div} w \in L_{1,\mathrm{loc}}(\Omega)$ is defined analogously to the definition of the distributional derivative as in Subsection 4.1.2.)

(c) Let $j \in \{1,\dots,n\}$, $f, \partial_j f \in L_{1,\mathrm{loc}}(\Omega)$, and let $\varphi \in C^\infty(\Omega)$. Show that $\partial_j(\varphi f) = \partial_j\varphi f + \varphi\partial_j f$.

4.6 Let $\Omega \subseteq \mathbb{R}^n$ be open.

(a) Let $f \in H^1_{\mathrm{c}}(\Omega)$, and define $\tilde{f}$ as the extension of f to $\mathbb{R}^n$ by zero. Show that $\tilde{f} \in H^1(\mathbb{R}^n)$. (Hint: Using Exercise 4.2, choose a function $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ with $\operatorname{spt}\psi \subseteq \Omega$ and $\psi = 1$ in a neighbourhood of $\operatorname{spt} f$. With the aid of this function and Exercise 4.5(a), show that $\partial_j\tilde{f}$ is the extension of $\partial_j f$ to $\mathbb{R}^n$ by zero.)

(b) Let $f \in H^1_0(\Omega)$, and define $\tilde{f}$ as the extension of f to $\mathbb{R}^n$ by zero. Show that $\tilde{f} \in H^1(\mathbb{R}^n)$ and that $\partial_j\tilde{f}$ is the extension of $\partial_j f$ to $\mathbb{R}^n$ by zero $(1 \leqslant j \leqslant n)$.

(c) Let $\Omega := (-1,0) \cup (0,1)$. Find a function $f \in H^1(\Omega) \setminus H^1_0(\Omega)$ with the property that the extension $\tilde{f}$ to $\mathbb{R}$ by zero belongs to $H^1(\mathbb{R})$.

4.7 Let $n \geqslant 3$. Show that $H^1(\mathbb{R}^n) = H^1_0(\mathbb{R}^n \setminus \{0\})$. For the more ambitious reader: Show this property for $n = 2$ as well.

Chapter 5

Forms and operators

We now introduce the main object of this book – namely forms in Hilbert spaces. They are very popular in analysis because the Lax–Milgram lemma yields properties of existence and uniqueness which are best adapted for establishing weak solutions of elliptic partial differential equations. Moreover, having the Lumer–Phillips machinery at our disposal, we can go much further and associate holomorphic semigroups with forms.

5.1 Forms: algebraic properties

In this section we introduce forms and put together some algebraic properties. As the domain we consider a vector space V over $\mathbb{K}$.

A **sesquilinear form** on V is a mapping $a\colon V \times V \to \mathbb{K}$ such that

$$\begin{aligned} a(u+v,w) &= a(u,w) + a(v,w), \quad a(\lambda u, w) = \lambda a(u,w), \\ a(u,v+w) &= a(u,v) + a(u,w), \quad a(u,\lambda v) = \overline{\lambda} a(u,v) \end{aligned}$$

for all $u, v, w \in V$, $\lambda \in \mathbb{K}$. (It is common usage to call a a form on V, although a is defined on $V \times V$ – rather than on V.)

If $\mathbb{K} = \mathbb{R}$, then a sesquilinear form is the same as a bilinear form. If $\mathbb{K} = \mathbb{C}$, then a is linear in the first and antilinear in the second argument, i.e., only half of the linearity conditions are fulfilled for the second argument: the form is $1\frac{1}{2}$-linear, or sesquilinear since the Latin 'sesqui' means 'one and a half'.

For simplicity we will mostly use the term **form** instead of sesquilinear form. A form a is called **symmetric** if

$$a(u,v) = \overline{a(v,u)} \qquad (u, v \in V),$$

and a is **accretive** if

$$\operatorname{Re} a(u,u) \geqslant 0 \qquad (u \in V).$$

In the literature, a symmetric form is also called "positive" if it is accretive, but we will not use this terminology.

We will frequently use the notation

$$a(u) := a(u,u) \qquad (u \in V)$$

for the associated quadratic form.

5.1 Remarks. (a) Each form a satisfies the **parallelogram identity**

$$a(u+v)+a(u-v)=2a(u)+2a(v) \qquad (u,v\in V).$$

If a is symmetric, one has

$$\operatorname{Re} a(u,v)=\frac{1}{4}\big(a(u+v)-a(u-v)\big) \qquad (u,v\in V).$$

(b) If $\mathbb{K}=\mathbb{C}$, then each form a satisfies the **polarisation identity**

$$a(u,v)=\frac{1}{4}\Big(a(u+v)-a(u-v)+\mathrm{i}\big(a(u+\mathrm{i}v)-a(u-\mathrm{i}v)\big)\Big) \qquad (u,v\in V),$$

as an elementary computation shows. In particular, the form is determined by its quadratic terms. This in turn implies that a is symmetric if and only if $a(u)\in\mathbb{R}$ for all $u\in V$. (Consider the form $b(u,v):=a(u,v)-\overline{a(v,u)}$ and its quadratic terms $b(u)$.) It follows that a is symmetric and accretive if and only if $a(u)\in[0,\infty)$ for all $u\in V$. Clearly, these two characterisations are only true if $\mathbb{K}=\mathbb{C}$.

(c) If $\mathbb{K}=\mathbb{R}$ and a is symmetric, then part (a) implies the **polarisation identity** in the variant

$$a(u,v)=\frac{1}{4}\big(a(u+v)-a(u-v)\big) \qquad (u,v\in V). \qquad \triangle$$

Now let again $\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$. Recall that a scalar product is a symmetric form a that is positive definite, i.e. $a(u)>0$ for all $u\in V\setminus\{0\}$. For the Cauchy–Schwarz inequality to hold we do not need definiteness. In fact, we even present a version that involves two different symmetric forms. This will be useful later on.

5.2 Proposition (Cauchy–Schwarz inequality). *Let $a,b\colon V\times V\to\mathbb{K}$ be two symmetric forms. Assume that $|a(u)|\leqslant b(u)$ for all $u\in V$. Then*

$$|a(u,v)|\leqslant b(u)^{1/2}b(v)^{1/2} \qquad (u,v\in V). \tag{5.1}$$

Proof. Let $u,v\in V$. In order to show (5.1) we may assume that $a(u,v)\in\mathbb{R}$ (in the complex case replace u by γu with a suitable $\gamma\in\mathbb{C}$, $|\gamma|=1$). Then from Remark 5.1(a) and the hypothesis one obtains

$$|a(u,v)|\leqslant\frac{1}{4}\big(|a(u+v)|+|a(u-v)|\big)\leqslant\frac{1}{4}\big(b(u+v)+b(u-v)\big)=\frac{1}{2}\big(b(u)+b(v)\big).$$

Let $s>b(u)^{1/2}$, $t>b(v)^{1/2}$. Then

$$|a(u,v)|=st\left|a\Big(\frac{1}{s}u,\frac{1}{t}v\Big)\right|\leqslant st\cdot\frac{1}{2}\Big(\frac{b(u)}{s^2}+\frac{b(v)}{t^2}\Big)\leqslant st.$$

Taking the infimum over s and t we obtain (5.1). $\square$

Next we introduce the adjoint form. Let $a\colon V \times V \to \mathbb{K}$ be a form. Then

$$a^*(u,v) := \overline{a(v,u)} \qquad (u,v \in V)$$

defines a form $a^*\colon V \times V \to \mathbb{K}$. Note that a is symmetric if and only if $a = a^*$. In the case of complex scalars, the forms

$$\operatorname{Re} a := \frac{1}{2}(a + a^*) \quad \text{and} \quad \operatorname{Im} a := \frac{1}{2\mathrm{i}}(a - a^*)$$

are symmetric, and

$$a = \operatorname{Re} a + \mathrm{i} \operatorname{Im} a.$$

We call $\operatorname{Re} a$ the **real part** and $\operatorname{Im} a$ the **imaginary part** of a. Observe that $(\operatorname{Re} a)(u) = \operatorname{Re} a(u)$ and $(\operatorname{Im} a)(u) = \operatorname{Im} a(u)$ for all $u \in V$.

There is another algebraic notion – only used for the case $\mathbb{K} = \mathbb{C}$ – that will play an important role. A form $a\colon V \times V \to \mathbb{C}$ is **sectorial** if there exists $\theta \in [0, \pi/2)$ such that $a(u) \in \overline{\Sigma_\theta}$ for all $u \in V$. If we want to specify the angle, we say that a is **sectorial of angle** θ. It is obvious that a form $a\colon V \times V \to \mathbb{C}$ is sectorial if and only if there exists a constant $c \geqslant 0$ such that

$$|\operatorname{Im} a(u)| \leqslant c \operatorname{Re} a(u) \qquad (u \in V).$$

(The angle θ and the constant c are related by $c = \tan\theta$.)

5.2 Representation theorems

We now consider forms whose domain is a Hilbert space V over $\mathbb{K}$. We recall the classical representation theorem of Fréchet–Riesz: If η is a bounded linear functional on V, then there exists a unique $v \in V$ such that

$$\eta(u) = (u \,|\, v)_V \qquad (u \in V).$$

The purpose of this section is to generalise this result.

First of all, it will be natural to use the antidual V^* of V instead of the dual space V'. More precisely, if $\mathbb{K} = \mathbb{R}$, then $V^* = V'$ is the dual space of V, and if $\mathbb{K} = \mathbb{C}$, then we denote by V^* the space of all continuous antilinear functionals. Note that η is antilinear if and only if $\overline{\eta} = \big[u \mapsto \overline{\eta(u)}\big]$ is linear. The space V^* is a Banach space for the norm $\|\eta\|_{V^*} = \sup_{\|v\|_V \leqslant 1} |\eta(v)|$. For $\eta \in V^*$ we will mostly write

$$\langle \eta, v \rangle := \eta(v) \qquad (v \in V).$$

Of course, the theorem of Fréchet–Riesz can be reformulated by saying that for each $\eta \in V^*$ there exists a unique $u \in V$ such that

$$\eta(v) = (u \,|\, v)_V \qquad (v \in V).$$

The mapping $\Phi\colon V \to V^*,\ u \mapsto (u \,|\, \cdot)_V$ is the **Fréchet–Riesz isomorphism**: it is easy to see that Φ is linear and isometric; the Fréchet–Riesz theorem shows that Φ is surjective.

We now aim for the generalisation of the Fréchet–Riesz theorem, the omnipresent Lax–Milgram lemma.

A form $a\colon V \times V \to \mathbb{K}$ is called **bounded** if there exists $M \geqslant 0$ such that

$$|a(u,v)| \leqslant M\|u\|_V\|v\|_V \qquad (u,v \in V).$$

This condition is equivalent to continuity of a; see Exercise 5.1. If a is a bounded form on V, then $a(u,\cdot) \in V^*$ for all $u \in V$, with $\|a(u,\cdot)\| \leqslant M\|u\|_V$, so we obtain a bounded linear operator $\mathcal{A}\colon V \to V^*,\ u \mapsto a(u,\cdot)$ satisfying

$$\langle \mathcal{A}u, v\rangle = a(u,v) \qquad (u,v \in V) \tag{5.2}$$

and $\|\mathcal{A}\|_{\mathcal{L}(V,V^*)} \leqslant M$. The operator $\mathcal{A}$ is called the **Lax–Milgram operator** associated with the form a. If, in addition, a is **coercive**, i.e. if there exists $\alpha > 0$ such that

$$\operatorname{Re} a(u) \geqslant \alpha\|u\|_V^2 \qquad (u \in V), \tag{5.3}$$

then $\mathcal{A}$ is an isomorphism; this is the famous Lax–Milgram lemma shown below.

We first treat the 'operator version' of the lemma, which is based on the following definition. An operator A in V is called **strictly accretive** if there exists $\alpha > 0$ such that

$$\operatorname{Re}(Au \,|\, u)_V \geqslant \alpha\|u\|_V^2 \qquad (u \in \operatorname{dom}(A)) \tag{5.4}$$

(expressed differently, $A - \alpha I$ is accretive), and A is **strictly m-accretive** if A is strictly accretive and m-accretive. Lemma 3.17 implies that A is strictly m-accretive if and only if A is strictly accretive and $\operatorname{ran}(A) = V$.

5.3 Lemma. *Let V be a Hilbert space, and let $A \in \mathcal{L}(V)$ be strictly accretive, with $\alpha > 0$ as in* (5.4)*. Then A is invertible in $\mathcal{L}(V)$, and $\|A^{-1}\| \leqslant \frac{1}{\alpha}$.*

Proof. The operator $A - \alpha I$ is accretive, and Remark 3.18 implies that $A - \alpha I$ is m-accretive. This shows that $A = \alpha I + (A - \alpha I)$ has the asserted properties, by Lemma 3.17. □

5.4 Theorem (Lax–Milgram lemma). *Let V be a Hilbert space, and let $a\colon V \times V \to \mathbb{K}$ be a bounded coercive form. Then the operator $\mathcal{A}\colon V \to V^*$ defined in* (5.2) *is an isomorphism, and $\|\mathcal{A}^{-1}\|_{\mathcal{L}(V^*,V)} \leqslant \frac{1}{\alpha}$ with $\alpha > 0$ from* (5.3)*. In particular, for every $\eta \in V^*$ there exists $u \in V$ such that $\eta(v) = a(u,v)$ for all $v \in V$.*

Proof. Composing $\mathcal{A}$ with the inverse of the Fréchet–Riesz isomorphism $\Phi\colon V \to V^*$ we obtain an operator $\Phi^{-1}\mathcal{A} \in \mathcal{L}(V)$ satisfying

$$\operatorname{Re}\big(\Phi^{-1}\mathcal{A}u \,\big|\, u\big)_V = \operatorname{Re}\langle \mathcal{A}u, u\rangle = \operatorname{Re} a(u,u) \geqslant \alpha\|u\|_V^2 \qquad (u \in V).$$

From Lemma 5.3 we conclude that $\Phi^{-1}\mathcal{A}$ is invertible in $\mathcal{L}(V)$ and that $\|(\Phi^{-1}\mathcal{A})^{-1}\| \leqslant \frac{1}{\alpha}$. As Φ is an isometric isomorphism we obtain the assertions. □

If the form is symmetric, then the Lax–Milgram lemma is the same as the theorem of Fréchet–Riesz. In fact, then a is an equivalent scalar product, i.e. $a(u)^{1/2}$ defines an equivalent norm on V.

5.3 Semigroups by forms

Here we come to the heart of this book: we prove the first generation theorem. With a coercive form we associate an operator that is strictly m-accretive and thus yields a contractive C_0-semigroup, by the Lumer–Phillips theorem, Theorem 3.16. In the complex case the associated operator is m-sectorial and thus yields a contractive holomorphic C_0-semigroup, by Theorem 3.20. Recall that m-accretivity and m-sectoriality of an operator A involve the condition $\operatorname{ran}(I + A) = H$. In applications, showing this range condition often amounts to solving a partial differential equation (usually of elliptic type); see Section 4.2, for instance. If the operator is associated with a form, then the Lax–Milgram lemma does this job, so the range condition is automatically fulfilled.

The setup we describe below is sometimes referred to as the "complete case" in the literature because the form domain V is a Hilbert space. Once we have presented a series of examples in the further development, we will also introduce – in Chapter 12 – the "incomplete case" in which the form domain is just a vector space.

First we will explain how we associate an operator with a form. Let V, H be Hilbert spaces over $\mathbb{K}$, and let $a\colon V \times V \to \mathbb{K}$ be a bounded form. Let $j \in \mathcal{L}(V, H)$ be an operator with dense range. We define

$$A := \big\{(x, y) \in H \times H\,;\, \exists u \in V\colon j(u) = x,\ a(u, v) = (y \,|\, j(v))\ (v \in V)\big\};$$

here and in what follows, scalar products without index are scalar products in H.

5.5 Proposition. *Assume that*

$$u \in V,\ j(u) = 0,\ a(u) = 0 \quad \textit{implies} \quad u = 0. \tag{5.5}$$

(a) *Then the relation A defined above is a linear operator in H. We call A the* ***operator associated with*** *(a, j) and write $A \sim (a, j)$.*

(b) *If a is accretive, then A is accretive.*

(c) *If $\mathbb{K} = \mathbb{C}$ and a is sectorial, then A is sectorial of the same angle as a.*

Proof. (a) It is easy to see that A is a subspace of $H \times H$. Let $(0, y) \in A$. We have to show that $y = 0$. By definition there exists $u \in V$ such that $j(u) = 0$ and $a(u, v) = (y \,|\, j(v))$ for all $v \in V$. In particular, $a(u) = (y \,|\, j(u)) = 0$. Assumption (5.5) implies that $u = 0$. Hence $(y \,|\, j(v)) = 0$ for all $v \in V$. Since j has dense range, it follows that $y = 0$.

(b), (c) If $x \in \operatorname{dom}(A)$, then there exists $u \in V$ such that $j(u) = x$ and $a(u, v) = (Ax \,|\, j(v))$ for all $v \in V$, in particular $a(u) = (Ax \,|\, j(u)) = (Ax \,|\, x)$.

Thus we have shown that the numerical range $\operatorname{num}(A)$ is contained in $\{a(u)\,;\, u \in V\}$, and this proves (b) as well as (c). □

We now prove the first generation theorem for forms. Note that coercivity implies (5.5).

5.6 Theorem (Generation theorem, part 1). *Let $a\colon V \times V \to \mathbb{K}$ be bounded coercive form, and let $j \in \mathcal{L}(V, H)$ have dense range. Let A be the operator associated with (a, j). Then A is strictly m-accretive; in particular, $-A$ generates a contractive C_0-semigroup on H.*

Proof. By hypothesis, there exists $\alpha > 0$ such that $\operatorname{Re} a(u) \geqslant \alpha \|u\|_V^2$ for all $u \in V$. Also, there exists $c > 0$ such that $\|j(u)\|_H \leqslant c\|u\|_V$ for all $u \in V$. Let $x \in \operatorname{dom}(A)$. From the definition of A we know that there exists $u \in V$ such that $j(u) = x$ and $a(u,u) = (Ax \,|\, j(u)) = (Ax \,|\, x)$; hence

$$\operatorname{Re}(Ax \,|\, x) = \operatorname{Re} a(u) \geqslant \alpha \|u\|_V^2 \geqslant \frac{\alpha}{c^2} \|x\|_H^2 .$$

This shows that A is strictly accretive.

For the proof of strict m-accretivity we show that $\operatorname{ran}(A) = H$. Let $y \in H$. Then $\eta(v) := (y \,|\, j(v))$ defines a functional $\eta \in V^*$. By the Lax–Milgram lemma, Theorem 5.4, there exists $u \in V$ such that

$$a(u,v) = (y \,|\, j(v)) \qquad (v \in V).$$

Therefore $x := j(u) \in \operatorname{dom}(A)$ and $Ax = y$.

Now the Lumer–Phillips theorem, Theorem 3.16, implies that $-A$ generates a contractive C_0-semigroup. □

We point out that, in the situation of Theorem 5.6, there exists $\varepsilon > 0$ such that $\|T(t)\| \leqslant \mathrm{e}^{-\varepsilon t}$ for all $t \geqslant 0$; see Exercise 5.2(a).

In an implicit way, the proof of Theorem 5.6 contains a formula for A^{-1}. This formula will be helpful in the proof of Proposition 6.18 and very important in the proof of Theorem 15.2. For this reason we state it explicitly, as follows.

5.7 Proposition. *Let the hypotheses be as in Theorem 5.6. Let $\mathcal{A}\colon V \to V^*$ be the Lax–Milgram operator associated with the form a as in* (5.2)*, and define $k\colon H \to V^*$, $y \mapsto (y \,|\, j(\cdot))$. Then $k \in \mathcal{L}(H, V^*)$, and*

$$A^{-1} = j\mathcal{A}^{-1}k.$$

(Recall from Theorem 5.4 that $\mathcal{A}$ is invertible with inverse in $\mathcal{L}(V^*, V)$.)

Proof. The boundedness of k is seen from

$$|\langle k(y), v\rangle| \leqslant \|y\|_H \|j(v)\|_H \leqslant \|j\| \|y\|_H \|v\|_V \qquad (y \in H,\ v \in V).$$

We return to the notation used in the proof of Theorem 5.6. Starting with $y \in H$ we obtain $\eta = k(y)$, $u = \mathcal{A}^{-1}\eta$, $x = j(u)$. This results in $x = j\mathcal{A}^{-1}k(y)$, and using $Ax = y$ and the invertibility of A we conclude that $A^{-1} = j\mathcal{A}^{-1}k$. □

In the complex case one also obtains results concerning sectoriality.

5.8 Theorem (Generation theorem, part 2). *Let $\mathbb{K} = \mathbb{C}$, let $a\colon V \times V \to \mathbb{C}$ be a bounded coercive form, and let $j \in \mathcal{L}(V, H)$ have dense range. Let A be the operator associated with (a, j). Then the form a is sectorial, and the operator A is m-sectorial, i.e., $-A$ generates a contractive holomorphic C_0-semigroup on H.*

Proof. By assumption there exist $M \geqslant 0$, $\alpha > 0$ such that

$$|a(u,v)| \leqslant M\|u\|_V\|v\|_V, \quad \operatorname{Re} a(u) \geqslant \alpha\|u\|_V^2$$

for all $u, v \in V$. Thus

$$\frac{|\operatorname{Im} a(u)|}{\operatorname{Re} a(u)} \leqslant \frac{M\|u\|_V^2}{\alpha\|u\|_V^2} = \frac{M}{\alpha}$$

for all $u \in V \setminus \{0\}$. This implies that there exists $\theta \in [0, \pi/2)$ such that $a(u) \in \overline{\Sigma_\theta}$ for all $u \in V$, i.e. a is sectorial. The remaining assertions are immediate consequences of Proposition 5.5(c), Theorem 5.6 and Theorem 3.20. □

We give a first example as an illustration.

5.9 Example (Multiplication operators). Let (Ω, μ) be a measure space, and let $m\colon \Omega \to \mathbb{C}$ be measurable. Assume that $w(x) := \operatorname{Re} m(x) \geqslant \delta > 0$ for all $x \in \Omega$ and that there exists $c > 0$ such that

$$|\operatorname{Im} m(x)| \leqslant c \operatorname{Re} m(x) \qquad (x \in \Omega).$$

Let $V := L_2(\Omega, w\mu)$. Then $a(u,v) := \int u\bar{v}m \,\mathrm{d}\mu$ defines a bounded coercive form a on V. Let $H := L_2(\Omega, \mu)$, $j(u) := u$ for all $u \in V$. Then $j \in \mathcal{L}(V, H)$, and $\operatorname{ran}(j) = V$ is dense in H since V is the domain of the maximal multiplication operator induced by the function $\sqrt{w}$; see Exercise 1.6(a). Let $A \sim (a, j)$. Then A is the maximal multiplication operator induced by m, i.e.

$$\begin{aligned} \operatorname{dom}(A) &= \{u \in V\,;\, mu \in L_2(\Omega, \mu)\}, \\ Au &= mu. \end{aligned}$$

Proof. If $u \in V$ is such that $mu \in L_2(\Omega, \mu)$, then

$$a(u,v) = \int u\bar{v}m \,\mathrm{d}\mu = (mu \,|\, v) \qquad (v \in V);$$

hence $u \in \operatorname{dom}(A)$ and $Au = mu$. On the other hand, if $u \in \operatorname{dom}(A)$, then $f := Au \in L_2(\Omega, \mu)$ satisfies

$$\int u\bar{v}m \,\mathrm{d}\mu = a(u,v) = (f \,|\, v) = \int f\bar{v} \,\mathrm{d}\mu$$

for all $v \in V$. In particular, given $n \in \mathbb{N}$, this holds for all $v \in L_2(\Omega, \mu)$ with $[v \neq 0] \subseteq [|m| \leqslant n]$, and it follows that $mu = f$ a.e. on $[|m| \leqslant n]$. Hence from $\Omega = \bigcup_{n\in\mathbb{N}} [|m| \leqslant n]$ we obtain $mu = f$ a.e. □

In many applications the coercivity (accretivity, sectoriality) assumption in the above results can only be reached by 'shifting' the form a. From Section 2.2 we recall the concept of rescaling a C_0-semigroup, which corresponds to 'shifting' the generator: if $-A$ is the generator of a C_0-semigroup T and $\omega \in \mathbb{R}$, then $-(A + \omega)$ generates the semigroup $T_\omega = (\mathrm{e}^{-\omega t}T(t))_{t\geqslant 0}$. One frequently uses the word "quasi" as prefix if something is true after rescaling. (The notation '$A + \omega$' is an abbreviation of '$A + \omega I$'; the ω stands for multiplication by the scalar ω, which is just the operator ωI.)

An operator A is **quasi-accretive** or **quasi-m-accretive** if there exists $\omega \in \mathbb{R}$ such that $A + \omega$ is accretive or m-accretive, respectively.

In the case of a complex Hilbert space H, an operator A in H is **quasi-sectorial** or **quasi-m-sectorial** if there exists $\omega \in \mathbb{R}$ such that $A + \omega$ is sectorial or m-sectorial, respectively. A **quasi-contractive holomorphic semigroup** is a holomorphic semigroup T such that $\|\mathrm{e}^{-\omega z} T(z)\| \leqslant 1$ for all $z \in \Sigma_\theta$, for some $\theta \in (0, \pi/2]$ and some $\omega \in \mathbb{R}$. Thus A is quasi-m-sectorial if and only if $-A$ generates a quasi-contractive holomorphic C_0-semigroup.

Next we explain how to shift on the level of forms.

5.10 Remark. Let $a\colon V \times V \to \mathbb{K}$ be a bounded form, and let $j \in \mathcal{L}(V, H)$ have dense range. Let $\omega \in \mathbb{R}$, and define the shifted form a_ω on V by

$$a_\omega(u, v) := a(u, v) + \omega(j(u) \,|\, j(v)) \qquad (u, v \in V). \tag{5.6}$$

Assume that (5.5) is satisfied. Then clearly the form a_ω satisfies (5.5) as well (with a replaced by a_ω). We show that $A + \omega$ is the operator associated with (a_ω, j).

Let A_ω be the operator associated with (a_ω, j). Let $x, y \in H$. Then for all $u, v \in V$ with $j(u) = x$ we have

$$a_\omega(u, v) = (y \,|\, j(v)) \iff a(u, v) = (y - \omega x \,|\, j(v)).$$

This shows that

$$(x, y) \in A_\omega \iff (x, y - \omega x) \in A \iff (x, y) \in A + \omega I. \qquad \triangle$$

Let $a\colon V \times V \to \mathbb{K}$ be a bounded form, and let $j \in \mathcal{L}(V, H)$ have dense range. In analogy to our terminology for operators we could say that a is *quasi-coercive* (*with respect to* j) if one of the forms a_ω is coercive, i.e., if there exist $\omega \in \mathbb{R}$, $\alpha > 0$ such that

$$\operatorname{Re} a(u) + \omega \|j(u)\|_H^2 \geqslant \alpha \|u\|_V^2 \qquad (u \in V); \tag{5.7}$$

for simplicity we prefer to call a **j-coercive** in this case. (This property has been introduced in [ArEl12b] under the name 'j-elliptic'.) It is obvious that (5.7) implies (5.5); thus the operator A associated with (a, j) is defined by Proposition 5.5.

From Theorems 5.6 and 5.8 we now obtain the following more general generation result.

5.11 Corollary. *Let $j \in \mathcal{L}(V, H)$ have dense range, let $a\colon V \times V \to \mathbb{C}$ be a bounded j-coercive form, and let A be the operator associated with (a, j).*

(a) *Then A is quasi-m-accretive. If additionally a is accretive, then A is m-accretive.*

(b) *If $\mathbb{K} = \mathbb{C}$, then A is quasi-m-sectorial. If additionally a is sectorial, then A is m-sectorial.*

Proof. Define the form a_ω on V by (5.6), with ω as in (5.7). Then a_ω is coercive, and $A + \omega \sim (a_\omega, j)$, by Remark 5.10. Hence the operator $A + \omega$ is m-accretive (by Theorem 5.6), and in the case $\mathbb{K} = \mathbb{C}$ it is m-sectorial (by Theorem 5.8). This shows the first assertions of (a) and (b).

For the second assertion of (a) we recall that Proposition 5.5(b) implies that A is accretive. Applying Lemma 3.17 one concludes that A is m-accretive. For the second assertion of (b) one argues analogously, recalling Proposition 5.5(c). $\square$

Later – in Chapter 8 – we will meet interesting situations in which j is not injective. In most applications, however, j is an embedding; then we call (a, j) an **embedded form**, and we will usually suppress the letter j.

We conclude this section with a brief description of the embedded case. Let H and V be Hilbert spaces, $V \overset{\mathrm{d}}{\hookrightarrow} H$. This is an abbreviation for saying that V is continuously embedded into H (see Remark 3.3(a)) and that V is dense in H. Now let $a\colon V \times V \to \mathbb{K}$ be a bounded form. We say that a is ***H*-coercive** if

$$\operatorname{Re} a(u) + \omega \|u\|_H^2 \geqslant \alpha \|u\|_V^2$$

for all $u \in V$ and some $\alpha > 0$, $\omega \in \mathbb{R}$. (This means that, in the previous terminology, the form a is j-coercive, where $j\colon V \hookrightarrow H$ is the embedding.) In this case the definition of the operator A associated with a (abbreviated by $A \sim a$, without mentioning the given embedding of V into H) reads as follows. For $x, y \in H$ one has

$$x \in \operatorname{dom}(A),\ Ax = y \iff x \in V,\ a(x, v) = (y \,|\, v)_H\ (v \in V).$$

In the case $\mathbb{K} = \mathbb{C}$, this operator is quasi-m-sectorial by Corollary 5.11.

To prevent misunderstanding, we warn the reader that there is conflicting usage of the notion 'H-coercive' in the literature: 'H-coercive form' sometimes stands for what we would call a 'coercive form on H', i.e. a coercive form with domain H (and not V). We also mention that 'H-elliptic' is often used as a synonym for 'H-coercive'.

5.4 The classical Dirichlet form

Let Ω be an open subset of $\mathbb{R}^n$. The **classical Dirichlet form** on $H_0^1(\Omega) \times H_0^1(\Omega)$ is defined by

$$a(u, v) := \int_\Omega \nabla u \cdot \overline{\nabla v}\, \mathrm{d}x = \sum_{j=1}^n \int_\Omega \partial_j u\, \overline{\partial_j v}\, \mathrm{d}x,$$

where $\xi \cdot \eta := \sum_{j=1}^n \xi_j \eta_j$ for $\xi, \eta \in \mathbb{K}^n$ (which for $\xi, \eta \in \mathbb{R}^n$ is the standard scalar product). It is clear that a is bounded; in fact

$$|a(u, v)| \leqslant \|\nabla u\|_{L_2(\Omega)} \|\nabla v\|_{L_2(\Omega)} \leqslant \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

Here the **gradient** ∇u is the transpose of the derivative of u, $\nabla u = (u')^\top = (\partial_1 u, \dots, \partial_n u)^\top$, and $\|\nabla u\|_{L_2(\Omega)} := \big(\sum_{j=1}^n \int_\Omega |\partial_j u|^2\, \mathrm{d}x\big)^{1/2}$; thus $\|u\|_{H^1(\Omega)}^2 = \|u\|_{L_2(\Omega)}^2 + \|\nabla u\|_{L_2(\Omega)}^2$.

We will prove that the Dirichlet form is coercive if Ω is bounded, or more generally, if Ω is **contained in a slab**, i.e., there exist $\beta > 0$ and $j_0 \in \{1, \dots, n\}$ such that $|x_{j_0}| \leqslant \beta$ for all $x \in \Omega$.

5.12 Theorem (Poincaré's inequality). *Assume that Ω is contained in a slab. Then there exists a constant $c_{\mathrm{P}} > 0$ such that*

$$\int_\Omega |u(x)|^2\, \mathrm{d}x \leqslant c_{\mathrm{P}} \int_\Omega |\nabla u(x)|^2\, \mathrm{d}x \qquad (u \in H_0^1(\Omega)).$$

Proof. By Theorem 4.14 it suffices to prove the inequality for all $u \in C_c^\infty(\Omega)$. We may assume that $j_0 = 1$; otherwise we permute the coordinates. Let $\beta > 0$ be such that $|x_1| \leqslant \beta$ for all $x = (x_1, \dots, x_n) \in \Omega$. Let $h \in C^1[-\beta, \beta]$, $h(-\beta) = 0$. Then by Hölder's inequality we estimate

$$\int_{-\beta}^{\beta} |h(x)|^2 \,\mathrm{d}x = \int_{-\beta}^{\beta} \left| \int_{-\beta}^{x} h'(y) \,\mathrm{d}y \right|^2 \mathrm{d}x \leqslant \int_{-\beta}^{\beta} \left(\int_{-\beta}^{x} |h'(y)|^2 \,\mathrm{d}y \right) \left(\int_{-\beta}^{x} 1 \,\mathrm{d}y \right) \mathrm{d}x$$
$$\leqslant (2\beta)^2 \int_{-\beta}^{\beta} |h'(y)|^2 \,\mathrm{d}y.$$

Let $u \in C_c^\infty(\Omega)$. Applying the above estimate to $h(r) = u(r, x_2, \dots, x_n)$ we obtain

$$\int_\Omega |u(x)|^2 \,\mathrm{d}x \leqslant 4\beta^2 \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{-\beta}^{\beta} |\partial_1 u(x_1, \dots, x_n)|^2 \,\mathrm{d}x_1 \cdots \mathrm{d}x_n \leqslant 4\beta^2 \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x. \quad \square$$

In fact, we saw that the constant c_{P} can be chosen as d^2, where $d := 2\beta$ is an upper estimate for the width of Ω. (For bounded domains the best constant can be determined as $c_{\mathrm{P}} = 1/\lambda_1^{\mathrm{D}}$, where λ_1^{D} is the first eigenvalue of $-\Delta_{\mathrm{D}}$; we will come back to this fact in Example 6.20.) First we revisit the Dirichlet Laplacian.

5.13 Example (Dirichlet Laplacian). Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $H := L_2(\Omega)$, $V := H_0^1(\Omega)$ and observe that $V \overset{\mathrm{d}}{\hookrightarrow} H$. As before, define $a \colon V \times V \to \mathbb{C}$ by

$$a(u, v) := \int_\Omega \nabla u \cdot \overline{\nabla v} \,\mathrm{d}x.$$

Then a is bounded and H-coercive. The operator A in H associated with a is given by

$$\begin{aligned} \mathrm{dom}(A) &= \big\{ u \in H_0^1(\Omega) ;\, \Delta u \in L_2(\Omega) \big\}, \\ Au &= -\Delta u, \end{aligned}$$

i.e. $-A$ is the Dirichlet Laplacian Δ_{D} defined in Subsection 4.2.2.

The form a is symmetric and accretive. If Ω is contained in a slab, then a is coercive.

Proof. The inequality $a(u) + 1\,\|u\|_H^2 \geqslant \|u\|_V^2$ $(u \in V)$ – in fact an equality – shows that a is H-coercive. For $u, f \in L_2(\Omega)$ one has $u \in \mathrm{dom}(A)$, $Au = f$ if and only if $u \in H_0^1(\Omega)$ and

$$\int_\Omega \nabla u \cdot \overline{\nabla v} \,\mathrm{d}x = \int_\Omega f \overline{v} \,\mathrm{d}x \qquad (v \in H_0^1(\Omega)).$$

By Lemma 4.19 the latter is equivalent to $-\Delta u = f$ in the distributional sense (i.e. $-\int_\Omega u \overline{\Delta v} \,\mathrm{d}x = \int_\Omega f \overline{v} \,\mathrm{d}x$ for all $v \in C_c^\infty(\Omega)$).

Symmetry and accretivity of a are obvious. Now assume that Ω is contained in a slab. Let $u \in H_0^1(\Omega)$. Then $a(u) \geqslant \frac{1}{c_{\mathrm{P}}} \int_\Omega |u|^2 \,\mathrm{d}x$ by Poincaré's inequality. Thus $a(u) \geqslant \frac{1}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2c_{\mathrm{P}}} \int_\Omega |u|^2 \,\mathrm{d}x \geqslant \alpha \|u\|_{H^1(\Omega)}^2$ where $\alpha = \min\{\frac{1}{2}, \frac{1}{2c_{\mathrm{P}}}\}$. Hence a is coercive. $\square$

Finally we give an example in which j is not the embedding.

5.14 Example (Multiplicative perturbation of Δ_{D}). Let $\Omega \subseteq \mathbb{R}^n$ be an open set which is contained in a slab. Let $m\colon \Omega \to \mathbb{C}$ be measurable, $|m(x)| \geqslant \beta > 0$ for all $x \in \Omega$. Define the operator A in $L_2(\Omega)$ by

$$\begin{aligned} \operatorname{dom}(A) &= \big\{u \in L_2(\Omega);\, mu \in \operatorname{dom}(\Delta_{\mathrm{D}}),\ \overline{m}\Delta(mu) \in L_2(\Omega)\big\}, \\ Au &= -\overline{m}\Delta(mu). \end{aligned}$$

Then A is m-sectorial of angle 0.

Proof. Let $H := L_2(\Omega)$, $V := H_0^1(\Omega)$, $a(u,v) := \int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x$, and let $j \in \mathcal{L}(V,H)$ be given by $j(v) := \frac{1}{m}v$. Then $j(V)$ is dense in $L_2(\Omega)$. Indeed, let $g \in j(V)^\perp$. Then $\int_\Omega v\frac{1}{m}g\,\mathrm{d}x = 0$ for all $v \in C_{\mathrm{c}}^\infty(\Omega)$. Thus $\frac{1}{m}g = 0$ by Lemma 4.5. Consequently, $g = 0$. We have shown that $j(V)^\perp = \{0\}$, which implies $\overline{j(V)} = j(V)^{\perp\perp} = L_2(\Omega)$.

Let A be the operator in $L_2(\Omega)$ associated with (a,j). Then for $u, f \in L_2(\Omega)$ one has $u \in \operatorname{dom}(A)$ and $Au = f$ if and only if there exists $w \in H_0^1(\Omega)$ such that $\frac{w}{m} = u$ and $\int_\Omega \nabla w \cdot \overline{\nabla v}\,\mathrm{d}x = \int_\Omega f\,\overline{(\frac{v}{m})}\,\mathrm{d}x$ for all $v \in H_0^1(\Omega)$. This is equivalent to $mu \in \operatorname{dom}(\Delta_{\mathrm{D}})$ and $-\Delta(mu) = \frac{f}{\overline{m}}$. □

Notes

The approach to forms presented here is the "French approach" following Lions [DaLi92]. As a new ingredient we have introduced the mapping j, following the paper [ArEl12b] by Arendt and ter Elst (see also [ArEl12a]). It will bear fruit when we investigate the Dirichlet-to-Neumann operator (in Chapter 8) and also in Chapter 12, where we consider what is called the "incomplete case" in [ArEl12b] and [ArEl12a].

The Lax–Milgram lemma was proved in 1954 [LaMi54; Theorem 2.1] and has been *the* standard tool for establishing weak solutions since then.

As an interesting bit of history, we note that Hilbert used bilinear forms to treat integral equations in his famous papers from the beginning of the 20th century. His ideas led his students to develop the notion of operators in Hilbert spaces. As a result, operators have taken a more central role, and physical and other problems are formulated with the help of operators. In the 1950's, form methods were developed to solve equations defined by operators. Forms are most appropriate for numerical treatments because a form $a\colon V \times V \to \mathbb{C}$ can easily be restricted to a finite-dimensional subspace $V_m \times V_m$, whereas for operators there might be only few invariant subspaces. The method of finite elements is based on such restrictions.

If a form is symmetric, then solving the associated inhomogeneous equation $\mathcal{A}u = L$ (see below for the notation) is equivalent to a minimisation problem. We want to explain this in more detail. Let V be a real Hilbert space and $a\colon V \times V \to \mathbb{R}$ a symmetric bounded coercive form. Consider the associated Lax–Milgram operator $\mathcal{A}\colon V \to V^*$, given by

$$\langle \mathcal{A}u, v\rangle = a(u,v).$$

Given $L \in V^*$, the Lax–Milgram lemma says that there is a unique $u \in V$ such that $\mathcal{A}u = L$. It turns out that u is the unique element of V minimising the function

$$V \ni w \mapsto \tfrac{1}{2}a(w) - \langle L, w\rangle.$$

Indeed, let $v \in V$. Then

$$\begin{aligned}\tfrac{1}{2}a(u+v) - \langle L, u+v\rangle &= \tfrac{1}{2}a(u) + a(u,v) + \tfrac{1}{2}a(v) - \langle L, u\rangle - \langle L, v\rangle \\ &= \tfrac{1}{2}a(u) + \tfrac{1}{2}a(v) - \langle L, u\rangle > \tfrac{1}{2}a(u) - \langle L, u\rangle\end{aligned}$$

unless $v = 0$.

Because of this argument, form methods are also called 'variational methods': we look at the 'variations' $a(u+v)$ of $a(u)$. It was Johann Peter Gustav Lejeune Dirichlet (1805–1859) who proposed such a variational method for the solution of the Dirichlet problem. We refer to the Notes of Chapter 7 for more information.

Exercises

5.1 (a) Let V be a Hilbert space, and let $a\colon V \times V \to \mathbb{K}$ be a sesquilinear form. Show that the properties

(i) a is bounded,

(ii) there exists $A \in \mathcal{L}(V)$ such that $a(u,v) = (Au \,|\, v)$ for all $u, v \in V$,

(iii) a is continuous

are equivalent.

(b) Show that one has the additional equivalence

(iv) there exists $M \geqslant 0$ such that $|a(u)| \leqslant M\|u\|_V^2$ for all $u \in V$,

if a is symmetric *or* $\mathbb{K} = \mathbb{C}$.

Hint: Use the Cauchy–Schwarz inequality, Proposition 5.2, and in the complex case use the decomposition $a = \operatorname{Re} a + \mathrm{i}\operatorname{Im} a$.

(c) Find an unbounded form $a\colon V \times V \to \mathbb{R}$ such that $a(u,u) = 0$ for all $u \in V$, with the real Hilbert space $V := \ell_2(\mathbb{N};\mathbb{R}^2)$. (This will show that condition (iv) is not equivalent to (i), (ii), (iii) without the additional requirement in part (b).)

Hint: First find a suitable definition of a on the space $\ell_{2,\mathrm{c}}(\mathbb{N};\mathbb{R}^2) \times \ell_{2,\mathrm{c}}(\mathbb{N};\mathbb{R}^2)$, where the index c means that in $(u_n) \in \ell_{2,\mathrm{c}}(\mathbb{N};\mathbb{R}^2)$ only finitely many components are $\neq 0$. In order to achieve the extension of a to $V \times V$, supplement the canonical basis $\check{B}$ of $\ell_{2,\mathrm{c}}(\mathbb{N};\mathbb{R}^2)$ to an (algebraic) basis B of V and define $a(b_1,b_2) := 0$ for $(b_1,b_2) \in (B \times B) \setminus (\check{B} \times \check{B})$.

5.2 (a) Let V, H be Hilbert spaces, let $j \in \mathcal{L}(V,H)$ have dense range, and let $a\colon V \times V \to \mathbb{K}$ be a bounded coercive form. Let $A \sim (a,j)$, and let T be the semigroup generated by $-A$. Show that there exists $\varepsilon > 0$ such that $\|T(t)\| \leqslant \mathrm{e}^{-\varepsilon t}$ for all $t \geqslant 0$. (Hint: Show that $a_{-\varepsilon}(u,v) = a(u,v) - \varepsilon(j(u) \,|\, j(v))_H$ defines an accretive j-coercive form if $\varepsilon > 0$ is small enough.)

(b) Let $\Omega \subseteq \mathbb{R}^n$ be an open set which is contained in a slab. Show that

$$\left\|\mathrm{e}^{t\Delta_{\mathrm{D}}}\right\|_{\mathcal{L}(L_2(\Omega))} \leqslant \mathrm{e}^{-\varepsilon t} \qquad (t \geqslant 0)$$

for some $\varepsilon > 0$. Express $\varepsilon > 0$ in terms of the width of Ω.

5.3 Let H and V be Hilbert spaces, $V \overset{\mathrm{d}}{\hookrightarrow} H$, let $a\colon V \times V \to \mathbb{K}$ be a bounded H-coercive form, and let $A \sim a$.

(a) Show that $\mathrm{dom}(A)$ is dense in V. (Hints: Without loss of generality assume that a is coercive. Then the Lax–Milgram operator $\mathcal{A}^*\colon V \to V^*$, associated with the adjoint form a^*, is surjective, and A is surjective as well. Use these two facts to show that the orthogonal complement of $\mathrm{dom}(A)$ in V equals $\{0\}$.)

(b) Show that a is accretive if (and only if) the C_0-semigroup generated by $-A$ is contractive.

5.4 Let $-\infty < a < b < \infty$. In this exercise we will always use the continuous representative for a function in $H^1(a,b)$; recall Theorem 4.11 for the embedding $H^1(a,b) \hookrightarrow C[a,b]$.

(a) Show that each $u \in H^1(a,b)$ is Hölder continuous with exponent $1/2$, i.e. $|u(t) - u(s)| \leqslant c|t-s|^{1/2}$ for some $c > 0$.

(b) Show that the embedding $H^1(a,b) \hookrightarrow C[a,b]$ is compact (i.e., the unit ball of $H^1(a,b)$ is relatively compact in $C[a,b]$).

(c) Let $H^2(a,b) := \{u \in L_2(a,b)\,;\, u', u'' \in L_2(a,b)\}$. On $H^2(a,b)$ we define the norm by $\|u\|_{H^2} := \big(\|u\|_{L_2}^2 + \|u'\|_{L_2}^2 + \|u''\|_{L_2}^2\big)^{1/2}$. Show that $H^2(a,b)$ is a Hilbert space and that $H^2(a,b) \hookrightarrow C^1[a,b]$ if $C^1[a,b]$ carries the norm $\|u\|_{C^1} = \|u\|_{C[a,b]} + \|u'\|_{C[a,b]}$. (Hint: Recall Exercise 4.5(b).)

5.5 Let $-\infty < a < b < \infty$.

(a) Show that $C^1[a,b]$ is dense in $H^1(a,b)$. (Hint: Given $f \in H^1(a,b)$, approximate f' in $L_2(a,b)$ by a sequence (g_n) in $C_{\mathrm{c}}(a,b)$. With x_0 and c from Proposition 4.7 define $f_n \in C^1[a,b]$ by $f_n(x) := c + \int_{x_0}^x g_n(y)\,\mathrm{d}y$.)

(b) Let $f, g \in H^1(a,b)$. Show $(fg)' = f'g + fg'$ and

$$\int_a^b f'g\,\mathrm{d}x = f(b)g(b) - f(a)g(a) - \int_a^b fg'\,\mathrm{d}x.$$

(Hint: Use Theorem 4.11.)

5.6 Let $-\infty < a < b < \infty$ and $\alpha, \beta > 0$. Define the operator A in $L_2(a,b)$ by

$$\begin{aligned} \mathrm{dom}(A) &= \big\{u \in H^2(a,b)\,;\, -u'(a) + \alpha u(a) = 0,\ u'(b) + \beta u(b) = 0\big\},\\ Au &= -u''. \end{aligned}$$

(See Exercise 5.4(c) for the definition of $H^2(a,b)$ and the existence of $u'(a)$ and $u'(b)$.)

(a) Show that A is m-sectorial. (Hint: Consider the form

$$H^1(a,b) \times H^1(a,b) \ni (u,v) \mapsto \int_a^b u'\overline{v'}\,\mathrm{d}x + \alpha u(a)\overline{v(a)} + \beta u(b)\overline{v(b)}$$

and use Exercise 5.5(b).)

(b) Show that $\|\mathrm{e}^{-tA}\|_{\mathcal{L}(L_2(a,b))} \leqslant \mathrm{e}^{-\varepsilon t}$ $(t \geqslant 0)$ for some $\varepsilon > 0$.

5.7 Let V and H be complex Hilbert spaces, $j \in \mathcal{L}(V,H)$ an operator with dense range, $a\colon V \times V \to \mathbb{C}$ a bounded form satisfying (5.5), $A \sim (a,j)$. Let H_0 denote the real Hilbert

space obtained from H by restricting scalars to $\mathbb{R}$ and defining the modified scalar product in H_0 by

$$(x \,|\, y)_0 = \operatorname{Re}(x \,|\, y) \qquad (x, y \in H_0 = H).$$

(a) Define $a_0 \colon V_0 \times V_0 \to \mathbb{R}$ (where V_0 is the real Hilbert space associated with V) by

$$a_0(u, v) := \operatorname{Re} a(u, v) \qquad (u, v \in V_0).$$

Show that a_0 is a bounded form satisfying (5.5). For $A_0 \sim (a_0, j)$, show that $A_0 = A$.

(b) Assume additionally that a is j-coercive. Show that then a_0 is j-coercive, and that $T_0 = T$, where T and T_0 are the C_0-semigroups generated by $-A$ and $-A_0$, respectively.

Chapter 6

Adjoint operators, and compactness

The main objective of this chapter is to show that, for a bounded open set $\Omega \subseteq \mathbb{R}^n$, the space $L_2(\Omega)$ has an orthonormal basis consisting of eigenfunctions of the Dirichlet Laplacian Δ_{D} on Ω. The proof is based on two crucial facts: $(I - \Delta_{\mathrm{D}})^{-1}$ is a compact self-adjoint operator, and for such operators there exist sufficiently many eigenfunctions. One of the aims of this chapter is to explain all the notions used in the previous sentence.

6.1 Adjoints of operators, and self-adjoint operators

Throughout this section we assume that G and H are Hilbert spaces over $\mathbb{K}$. One of the aims is to prove the following result.

6.1 Theorem. *Let A be an operator in H. Then the following properties are equivalent.*
(i) *A is self-adjoint and accretive.*
(ii) *A is symmetric and m-accretive.*
If H is a complex Hilbert space, then there is the following additional equivalent property.
(iii) *A is m-sectorial of angle 0.*

Evidently, before proceeding to the proof, we first have to explain the notions of self-adjointness and symmetry. The proof will be given at the very end of this section.

Before defining the adjoint of an operator we want to explain the idea behind this notion. If A is an operator from G to H, then the adjoint A^* should be the maximal operator from H to G such that

$$(Ax \,|\, y)_H = (x \,|\, A^* y)_G \qquad \big(x \in \mathrm{dom}(A),\ y \in \mathrm{dom}(A^*)\big).$$

The product space $G \times H$ will be provided with the scalar product

$$\big((x,y) \,\big|\, (x_1,y_1)\big) := (x \,|\, x_1)_G + (y \,|\, y_1)_H \qquad \big((x,y),(x_1,y_1) \in G \times H\big),$$

which makes $G \times H$ a Hilbert space; we will use the notation $G \oplus H$ for this space.

For an operator A from G to H we define the **adjoint**

$$\begin{aligned} A^* &:= \big\{(y,x) \in H \times G;\, \forall x_1 \in \mathrm{dom}(A)\colon (Ax_1 \,|\, y)_H = (x_1 \,|\, x)_G\big\} \\ &= \big\{(y,x) \in H \times G;\, \forall (x_1,y_1) \in A\colon \big((x_1,-y_1) \,\big|\, (x,y)\big) = 0\big\} \\ &= ((-A)^{\perp})^{-1}. \end{aligned}$$

It is obvious from the last equality that A^* is a closed subspace of $H \times G$. We mention that the operator $-A$, as a linear relation, is given by

$$-A = \{(x, -y);\, (x, y) \in A\}.$$

(It is somewhat unfortunate that, in principle, A^* is already defined as the antidual space to the subspace A of $G \oplus H$. We think that this ambiguity will not be a problem.)

6.2 Remarks. Let A be an operator from G to H.

(a) If B is an operator from H to G such that

$$(Ax \,|\, y)_H = (x \,|\, By)_G \qquad \big(x \in \mathrm{dom}(A),\ y \in \mathrm{dom}(B)\big),$$

then it is immediate that $B \subseteq A^*$ and $A \subseteq B^*$.

(b) If A^* is an operator, then the definition implies that $(Ax \,|\, y)_H = (x \,|\, A^*y)_G$ for all $x \in \mathrm{dom}(A)$, $y \in \mathrm{dom}(A^*)$.

(c) If B is an operator from G to H, $A \subseteq B$, then clearly $B^* \subseteq A^*$. △

6.3 Theorem. *Let A be an operator from G to H. Then:*

(a) *A^* is an operator if and only if* $\mathrm{dom}(A)$ *is dense in G.*

(b) *Assume that* $\mathrm{dom}(A)$ *is dense. Then $A^{**} := (A^*)^* = \overline{A}$, and* $\mathrm{dom}(A^*)$ *is dense if and only if A is closable.*

Proof. (a) From the definition of A^* it is easy to see that $\{x \in G;\, (0, x) \in A^*\} = \mathrm{dom}(A)^\perp$. This equality shows the assertion.

(b) It is easy to see that in the expression $((-A)^\perp)^{-1}$ for A^* the order of the operations $A \mapsto -A$, $A \mapsto A^\perp$ and $A \mapsto A^{-1}$ does not matter. It follows that $A^{**} = A^{\perp\perp} = \overline{A}$.

By part (a), $\mathrm{dom}(A^*)$ is dense if and only if $(A^*)^*$ is an operator, and because $A^{**} = \overline{A}$, the latter is equivalent to the closability of A. □

6.4 Example. Define the operator A in $L_2(\mathbb{R})$ by

$$\mathrm{dom}(A) := C_c^\infty(\mathbb{R}), \qquad Af := f' \quad (f \in \mathrm{dom}(A)).$$

For $f, g \in L_2(\mathbb{R})$ we then obtain:

$$\begin{aligned}
(g, f) \in A^* &\iff \forall \varphi \in C_c^\infty(\mathbb{R})\colon (A\varphi \,|\, g) = (\varphi \,|\, f) \\
&\iff \forall \varphi \in C_c^\infty(\mathbb{R})\colon \int g\varphi' = \int f\varphi \\
&\iff f = -g' \text{ in the distributional sense.}
\end{aligned}$$

This shows that $\mathrm{dom}(A^*) = H^1(\mathbb{R})$, $A^*g = -g'$ $(g \in \mathrm{dom}(A^*))$; in particular, $A \subseteq -A^*$.

Now it follows that $\overline{A} = -A^*$. Indeed, the subspace $-A^*$ of $L_2(\mathbb{R}) \oplus L_2(\mathbb{R})$ is isometrically isomorphic to $H^1(\mathbb{R})$, under the mapping $-A^* \ni (f, f') \mapsto f \in H^1(\mathbb{R})$, and $\mathrm{dom}(A) = C_c^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$, by Theorem 4.14. △

An operator A in H is called **symmetric** if $\mathrm{dom}(A)$ is dense and $A \subseteq A^*$; A is called **self-adjoint** if $A = A^*$ (note that then $\mathrm{dom}(A)$ is dense, by Theorem 6.3(a)).

We collect some properties of symmetric operators.

6.5 Remarks. Let A be an operator in H, $\mathrm{dom}(A)$ dense.

(a) Then A is symmetric if and only if $(Ax \,|\, y) = (x \,|\, Ay)$ for all $x, y \in \mathrm{dom}(A)$. This follows immediately from Remark 6.2.

(b) If H is complex, then A is symmetric if and only if $(Ax \,|\, x) \in \mathbb{R}$ for all $x \in \mathrm{dom}(A)$. This follows from Remark 5.1(b), applied to the form $\mathrm{dom}(A) \times \mathrm{dom}(A) \ni (x, y) \mapsto (Ax \,|\, y)$, and part (a) above.

(c) If A is symmetric, then A is closable, and $\overline{A}$ is symmetric. Indeed, from $A \subseteq A^*$ and the fact that A^* is a closed operator one concludes that A is closable and that $\overline{A} \subseteq A^* = \overline{A}^*$ (where the last equality is clear from the definition). In this situation one says that A is **essentially self-adjoint** if $\overline{A}$ is self-adjoint. $\triangle$

Particularly simple examples of self-adjoint operators are those possessing 'sufficiently many' eigenvectors, which we will present next. We note that an eigenvalue of a symmetric operator A is always real: If $0 \neq x \in H$, $Ax = \lambda x$, then $\lambda(x \,|\, x) = (Ax \,|\, x) \in \mathbb{R}$.

6.6 Example (Diagonal self-adjoint operators). Let A be a self-adjoint operator in an infinite-dimensional separable Hilbert space H, and assume that there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ consisting of eigenvectors of A, with corresponding eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$. Then

$$\mathrm{dom}(A) = \Big\{ x \in H \,;\, \sum_{n=1}^{\infty} |\lambda_n|^2 |(x \,|\, e_n)|^2 < \infty \Big\},$$

$$Ax = \sum_{n=1}^{\infty} \lambda_n (x \,|\, e_n)\, e_n \qquad (x \in \mathrm{dom}(A)),$$

and $\mathrm{lin}\{e_n \,;\, n \in \mathbb{N}\}$ is a core for A.

We call A the **diagonal operator** associated with the orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and the sequence $\lambda := (\lambda_n)_{n\in\mathbb{N}}$.

Proof. Using the unitary operator $J\colon H \to \ell_2,\ x \mapsto \big((x \,|\, e_n)\big)_{n\in\mathbb{N}}$, we transform the situation to the case when A is a self-adjoint operator in the Hilbert space ℓ_2, possessing the canonical unit vectors as eigenvectors.

Now we apply Exercise 6.4 to the maximal multiplication operator M_λ in ℓ_2, choosing $\mathbb{N}$ with counting measure as the measure space and the sequence $(\lambda_n)_{n\in\mathbb{N}}$ as the multiplying function m. Then the restriction A_0 of A to $\mathrm{dom}(A_0) := \mathrm{lin}\{e_n \,;\, n \in \mathbb{N}\}$ corresponds to the operator A_0 defined in Exercise 6.4. By part (c) of the exercise we obtain $\overline{A_0} = M_\lambda$; hence $\mathrm{dom}(A_0)$ is a core for M_λ. Because A is closed, we conclude that $M_\lambda \subseteq A$ and, by Remark 6.2(c), $A = A^* \subseteq M_\lambda^* = M_\lambda$. Thus $A = M_\lambda$ is the operator described above. $\square$

6.7 Remark. Assume that in the above example $\lambda_n \geqslant 0$ for all $n \in \mathbb{N}$. Then it follows from Exercise 1.6(b) that $-A$ generates the C_0-semigroup $(M_{\mathrm{e}^{-t\lambda}})_{t\geqslant 0}$, written in the representation of A mentioned at the beginning of the proof given above. Spelling this out in the setting on H, the C_0-semigroup T generated by $-A$ is given by

$$T(t)x = \sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n} (x \,|\, e_n)\, e_n \qquad (x \in H,\ t \geqslant 0). \qquad \triangle$$

In order to connect self-adjointness with m-accretivity we need some preliminary results.

6.8 Lemma. *Let A be an operator from G to H. Then* $\ker(A^*) = \operatorname{ran}(A)^\perp$.

Proof. Let $y \in H$, and recall that $y \in \ker(A^*)$ means $(y, 0) \in A^*$. Hence, by the definition of A^*, $y \in \ker(A^*)$ is equivalent to $(Ax \,|\, y)_H = 0$ for all $x \in \operatorname{dom}(A)$, i.e., to $y \perp \operatorname{ran}(A)$. □

We point out that in the following result, denseness of $\operatorname{dom}(A)$ is not part of the hypotheses.

6.9 Proposition. *Let A be an operator in H, $(Ax \,|\, y) = (x \,|\, Ay)$ for all $x, y \in \operatorname{dom}(A)$, and* $\operatorname{ran}(A) = H$. *Then A is self-adjoint.*

Proof. Remark 6.2(a) shows that $A \subseteq A^*$, and Lemma 6.8 implies that $\ker(A^*) = \operatorname{ran}(A)^\perp = \{0\}$. These two facts, together with $\operatorname{ran}(A) = H$, imply $A^* = A$. (This follows the principle that 'a surjective mapping cannot have a proper injective extension' – which also holds for relations.) □

6.10 Lemma. *Let A be an operator in H,* $\operatorname{dom}(A)$ *dense, and let* $\lambda \in \mathbb{K}$. *Then* $(\lambda I + A)^* = \overline{\lambda} I + A^*$.

Proof. For $x \in \operatorname{dom}(A)$, $y \in \operatorname{dom}(A^*)$ we compute

$$\big((\lambda + A)x \,\big|\, y\big) = \big(x \,\big|\, \overline{\lambda} y\big) + \big(x \,\big|\, A^* y\big) = \big(x \,\big|\, (\overline{\lambda} + A^*)y\big),$$

and using Remark 6.2(a) we obtain $\overline{\lambda} + A^* \subseteq (\lambda + A)^*$. This inclusion also implies that

$$A^* = (-\lambda + (\lambda + A))^* \supseteq -\overline{\lambda} + (\lambda + A)^*,$$

i.e. $\overline{\lambda} + A^* \supseteq (\lambda + A)^*$. The two inclusions prove the assertion. □

We can now prove the main result of this section.

Proof of Theorem 6.1. (i) ⇒ (ii), (iii). Clearly it is sufficient to show that $\operatorname{ran}(I + A) = H$. Recall that $\|(I + A)x\| \geqslant \|x\|$ $(x \in \operatorname{dom}(A))$, by the accretivity of A. As A is closed, this inequality implies that $\operatorname{ran}(I + A)$ is closed, so it remains to show that $\operatorname{ran}(I + A)$ is dense.

By Lemma 6.10 the operator $I + A$ is self-adjoint, and $I + A$ is injective since A is accretive. Now Lemma 6.8 yields $\operatorname{ran}(I + A)^\perp = \ker((I + A)^*) = \ker(I + A) = \{0\}$.

(ii) ⇒ (i). The hypothesis implies that $I + A$ satisfies the conditions in Proposition 6.9, and therefore $I + A$ is self-adjoint. Then Lemma 6.10 shows that A is self-adjoint.

(iii) ⇒ (ii). Being an m-sectorial operator, A is m-accretive. Sectoriality of angle 0 means that $(Ax \,|\, x) \geqslant 0$ for all $x \in \operatorname{dom}(A)$. Applying Remark 6.5(b) one concludes that A is symmetric. □

6.2 Adjoints of forms and operators

In this section we assume that V and H are Hilbert spaces over $\mathbb{K}$ and that $j \in \mathcal{L}(V,H)$ has dense range. Let $a\colon V \times V \to \mathbb{K}$ be a bounded j-coercive form. For the reader's convenience we recall the definition of the operator A associated with a and j,

$$A = \big\{(x,y) \in H \times H\,;\, \exists u \in V\colon j(u) = x,\ a(u,v) = (y \,|\, j(v))\ (v \in V)\big\};$$

see Section 5.3.

The following result shows the close connection between adjoints of forms and operators.

6.11 Theorem. *Let $A \sim (a,j)$, $B \sim (a^*,j)$. Then $B = A^*$.*
If additionally a is symmetric, then A is self-adjoint.

Proof. Without loss of generality we suppose that a is coercive; see Remark 5.10 and Lemma 6.10. Then for $x \in \mathrm{dom}(A)$, $y \in \mathrm{dom}(B)$ there exist $u, v \in V$ such that $j(u) = x$, $j(v) = y$ and

$$(Ax \,|\, y) = (Ax \,|\, j(v)) = a(u,v) = \overline{a^*(v,u)} = \overline{(By \,|\, j(u))} = (x \,|\, By).$$

Hence $B \subseteq A^*$, by Remark 6.2(a).

From Theorem 5.6 we know that $\mathrm{ran}(B) = \mathrm{ran}(A) = H$, and therefore Lemma 6.8 implies that A^* is injective. These properties imply that $B = A^*$ (recall that 'a surjective mapping cannot have a proper injective extension').

If a is symmetric, then $A = B = A^*$. □

6.3 The spectral theorem for compact self-adjoint operators

We recall that, for Banach spaces X, Y, an operator $A\colon X \to Y$ is called **compact** if $A(B_X(0,1))$ is a relatively compact subset of Y, where $B_X(0,1)$ is the open unit ball of X. The set $\mathcal{K}(X,Y)$ of compact operators is a closed subspace of $\mathcal{L}(X,Y)$, and the composition of a compact operator with a bounded operator is compact. The latter is called the **ideal property** of compact operators.

The following result is the spectral theorem for compact self-adjoint operators in a Hilbert space H.

6.12 Theorem (Hilbert). *Let $A \in \mathcal{L}(H)$ be compact and self-adjoint. Then there exist $J \subseteq \mathbb{N}$ and an orthonormal system $(e_n)_{n\in J}$ of eigenvectors of A with corresponding eigenvalues $(\lambda_n)_{n\in J}$ in $\mathbb{R} \setminus \{0\}$, $\lambda_n \to 0$ as $n \to \infty$, such that*

$$Ax = \sum_{n\in J} \lambda_n (x \,|\, e_n)\, e_n \qquad (x \in H).$$

6.13 Remark. Note that in the preceding theorem it follows that $\{e_n\,;\, n \in J\}^\perp = \ker(A)$; in particular, $(e_n)_{n\in J}$ is an orthonormal basis if and only if A is injective. △

For the proof of Theorem 6.12 we need the following two propositions.

6.14 Proposition. *Let $A \in \mathcal{L}(H)$ be self-adjoint. Then*

$$\|A\| = \sup\{|(Ax \,|\, x)|;\ \|x\| \leqslant 1\}.$$

Proof. "$\geqslant$" is obvious from $|(Ax \,|\, x)| \leqslant \|A\|\|x\|^2$.

"$\leqslant$". For $x, y \in H$ we put $a(x,y) := (Ax \,|\, y)$ and $b(x,y) := c(x \,|\, y)$, where $c := \sup\{|a(x)|;\ \|x\| \leqslant 1\}$. Then the Cauchy–Schwarz inequality (Proposition 5.2) implies $|(Ax \,|\, y)| \leqslant c\|x\|\|y\|$ $(x, y \in H)$; therefore $\|A\| \leqslant c$. □

6.15 Proposition. *Let $0 \neq A \in \mathcal{L}(H)$ be compact and self-adjoint. Then $\|A\|$ or $-\|A\|$ is an eigenvalue of A.*

Proof. By Proposition 6.14 there exist a sequence (x_n) in H, $\|x_n\| = 1$ $(n \in \mathbb{N})$, and $\lambda \in \mathbb{R}$ with $|\lambda| = \|A\|$ such that $(Ax_n \,|\, x_n) \to \lambda$. Then

$$\|Ax_n - \lambda x_n\|^2 = \|Ax_n\|^2 - 2\lambda(Ax_n \,|\, x_n) + \lambda^2 \leqslant 2\lambda^2 - 2\lambda(Ax_n \,|\, x_n) \to 0;$$

hence $Ax_n - \lambda x_n \to 0$ $(n \to \infty)$.

The compactness of A implies that there exists a subsequence (x_{n_k}) of (x_n) such that (Ax_{n_k}) is convergent, $Ax_{n_k} \to y$. It follows that

$$x_{n_k} = \frac{1}{\lambda}(\lambda x_{n_k} - Ax_{n_k}) + \frac{1}{\lambda}Ax_{n_k} \to \frac{1}{\lambda}y =: x,$$

$Ax = \lim_{k\to\infty} Ax_{n_k} = \lambda x$ and $\|x\| = 1$. □

Proof of Theorem 6.12. We show the assertion with $J = \mathbb{N}$ or $J = \{1, \dots, N\}$ for some $N \in \mathbb{N}_0$. The orthonormal system $(e_n)_{n\in J}$ and the corresponding family $(\lambda_n)_{n\in J}$ with the properties described in the theorem will be constructed by induction.

Assume that $k \in \mathbb{N}_0$ and that $e_1, \dots, e_k$ and $\lambda_1, \dots, \lambda_k$ are constructed. Let $H_{k+1} := \{e_1, \dots, e_k\}^\perp$. Then $A(H_{k+1}) \subseteq H_{k+1}$; indeed, $x \perp e_n$ implies $(Ax \,|\, e_n) = (x \,|\, Ae_n) = \lambda_n(x \,|\, e_n) = 0$ $(n = 1, \dots, k)$. Then $A_{k+1} := A|_{H_{k+1}}$, considered as an operator in H_{k+1}, is compact. If $A_{k+1} = 0$, then the construction is finished, with $J = \{1, \dots, k\}$. If $A_{k+1} \neq 0$, then Proposition 6.15 implies that there exist $\lambda_{k+1} \in \mathbb{R}$ with $|\lambda_{k+1}| = \|A_{k+1}\|$ and $e_{k+1} \in H_{k+1}$ with $\|e_{k+1}\| = 1$ such that $A_{k+1}e_{k+1} = \lambda_{k+1}e_{k+1}$.

The construction yields $|\lambda_n| = \|A_n\|$ $(n \in J)$, and as $(\|A_n\|)_{n\in J}$ is decreasing, so is $(|\lambda_n|)_{n\in J}$.

If $J = \mathbb{N}$, then $e_n \to 0$ weakly $(n \to \infty)$. It follows that $|\lambda_n| = \|Ae_n\| \to 0$ $(n \to \infty)$, because A is compact; see Exercise 6.2(a).

Finally we show the representation of A. For $x \in H$ we have

$$\Big\|Ax - \sum_{n=1}^{k} \lambda_n(x \,|\, e_n)e_n\Big\| = \Big\|A_{k+1}\Big(x - \sum_{n=1}^{k}(x \,|\, e_n)e_n\Big)\Big\| \leqslant \|A_{k+1}\|\|x\|,$$

which is 0 if $J = \{1, \dots, k\}$ and converges to 0 as $k \to \infty$ if $J = \mathbb{N}$. □

An operator A in a Banach space X is called an **operator with compact resolvent** if there exists $\lambda \in \rho(A)$ such that $R(\lambda, A)$ is a compact operator. It follows from the resolvent equation and the ideal property of compact operators that then $R(\mu, A)$ is compact for all $\mu \in \rho(A)$.

The following theorem is an application of Hilbert's theorem to accretive self-adjoint operators with compact resolvent. It shows that Example 6.6 – with a sequence (λ_n) in $[0, \infty)$ tending to ∞ – is generic for these operators.

6.16 Theorem. *Let A be an accretive self-adjoint operator with compact resolvent in an infinite-dimensional Hilbert space H. Then there exist an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $[0, \infty)$ with $\lim_{n\to\infty} \lambda_n = \infty$ such that A is the associated diagonal operator.*

Proof. It follows from Theorem 6.1 that $(I + A)^{-1}$ exists in $\mathcal{L}(H)$, and by hypothesis this operator is compact. It is easy to see that $(I + A)^{-1}$ is symmetric, hence self-adjoint. Applying Theorem 6.12 to $(I + A)^{-1}$ one obtains an orthonormal system $(e_n)_{n\in J}$ of eigenvectors, with a corresponding family $(\mu_n)_{n\in J}$ of eigenvalues, such that $(I + A)^{-1}x = \sum_{n\in J} \mu_n (x \,|\, e_n) e_n$ for all $x \in H$. This representation together with the injectivity of $(I + A)^{-1}$ implies that $(e_n)_{n\in J}$ is an orthonormal basis; see Remark 6.13. Hence J is countably infinite, without loss of generality $J = \mathbb{N}$.

One easily sees that e_n is an eigenvector of A with eigenvalue $\lambda_n := \mu_n^{-1} - 1$, for all $n \in \mathbb{N}$. The accretivity of A implies

$$\lambda_n = (\lambda_n e_n \,|\, e_n) = (Ae_n \,|\, e_n) \geqslant 0 \qquad (n \in \mathbb{N}).$$

Since $(\mu_n)_{n\in\mathbb{N}}$ is a null sequence, it follows that $\lim_{n\to\infty} \lambda_n = \infty$.

In view of Example 6.6, this completes the proof. □

A sequence (λ_n) in $\mathbb{R}$ with $\lim_{n\to\infty} \lambda_n = \infty$ can always be rearranged to an increasing sequence, and in applications of Theorem 6.16 one usually assumes the sequence of eigenvalues to be increasing. Note that the hypotheses of Theorem 6.16 imply that H is separable.

We now describe how the properties treated above can be obtained in the context of forms. We start by presenting an example illustrating this issue.

6.17 Example (Diagonal forms). Let H be an infinite-dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$, and let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[0, \infty)$. Then

$$V := \Big\{ u \in H \,;\, \sum_{n=1}^{\infty} \lambda_n |(u \,|\, e_n)_H|^2 < \infty \Big\}$$

is a Hilbert space for the scalar product

$$(u \,|\, v)_V := (u \,|\, v)_H + a(u, v),$$

where $a(u, v) := \sum_{n=1}^{\infty} \lambda_n (u \,|\, e_n)_H (e_n \,|\, v)_H$.

With these definitions, the injection j of V into H is continuous, and a is bounded, symmetric and j-coercive. The operator A associated with (a, j) is self-adjoint. It is easy

to see that A is the diagonal operator associated with $(e_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ as defined in Example 6.6.

Assume additionally that $\lim_{n\to\infty}\lambda_n = \infty$. Then the embedding j is compact. Indeed, it is easy to see that the finite rank operators $P_n j$, where P_n denotes the orthogonal projection onto $\operatorname{lin}\{e_1,\dots,e_n\}$, approximate j in the operator norm. By Proposition 6.18 – proved below – we see that A has compact resolvent (which is also easily obtained directly). △

If A is an operator associated with a form, then one has the following convenient condition for A to have compact resolvent.

6.18 Proposition. *Let V, H be Hilbert spaces, let $j \in \mathcal{L}(V,H)$ have dense range, let $a\colon V\times V\to\mathbb{K}$ be a bounded j-coercive form, and let $A\sim(a,j)$. Assume additionally that j is compact.*

Then A has compact resolvent.

Proof. Without loss of generality we may suppose that a is coercive. Then by Proposition 5.7, the inverse of A can be expressed explicitly as $A^{-1} = j\mathcal{A}^{-1}k$. As $\mathcal{A}^{-1}$ and k are bounded operators, we see that A^{-1} is compact. □

We now formulate the consequence of Theorem 6.16 for operators associated with forms.

6.19 Corollary. *Let V, H be Hilbert spaces, let $j \in \mathcal{L}(V,H)$ have dense range, let $a\colon V\times V\to\mathbb{K}$ be a symmetric bounded coercive form, and let A be the operator associated with (a,j). Assume additionally that j is compact.*

Then A is a strictly accretive self-adjoint operator with compact resolvent, and therefore all the conclusions of Theorem 6.16 hold. Moreover, all the eigenvalues of A are strictly positive.

Proof. We know from Theorems 5.6 and 6.11 that A is a strictly m-accretive self-adjoint operator, and it was shown in Proposition 6.18 that A has compact resolvent. The strict m-accretivity implies that 0 belongs to the resolvent set of A, hence cannot be an eigenvalue. □

We finally apply our results to the Dirichlet Laplacian Δ_{D} in $L_2(\Omega)$, for bounded open $\Omega\subseteq\mathbb{R}^n$, as announced in the introduction of this chapter. We will use the compactness of the embedding $H_0^1(\Omega)\hookrightarrow L_2(\Omega)$, which will be proved in the next section.

6.20 Example (Spectral decomposition of the Dirichlet Laplacian). If $\Omega\subseteq\mathbb{R}^n$ is a bounded open set, then Δ_{D} has compact resolvent. There exists an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ of $L_2(\Omega)$ and an increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ in $(0,\infty)$, with $\lim_{k\to\infty}\lambda_k=\infty$, such that $-\Delta_{\mathrm{D}}$ is the associated diagonal operator. In particular, $\varphi_k\in\operatorname{dom}(\Delta_{\mathrm{D}})$ and

$$-\Delta_{\mathrm{D}}\varphi_k = \lambda_k\varphi_k$$

for all $k\in\mathbb{N}$. One has Poincaré's inequality

$$\int_\Omega |u|^2\,\mathrm{d}x \leqslant c_{\mathrm{P}}\int_\Omega |\nabla u|^2\,\mathrm{d}x \qquad (u\in H_0^1(\Omega)),$$

with optimal constant $c_{\mathrm{P}} = \frac{1}{\lambda_1}$.

Proof. From Example 5.13 we know that $-\Delta_{\mathrm{D}}$ is associated with the classical Dirichlet form and $j\colon H_0^1(\Omega) \hookrightarrow L_2(\Omega)$. As the embedding j is compact, by Theorem 6.22, the assertions concerning the sequences (φ_k) and (λ_k) follow from Corollary 6.19.

Now for $u \in \mathrm{dom}(-\Delta_{\mathrm{D}})$ we obtain

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x = (-\Delta_{\mathrm{D}} u \,|\, u) = \sum_k \lambda_k |(u \,|\, \varphi_k)|^2 \geqslant \lambda_1 \|u\|_2^2 \,. \tag{6.1}$$

For $u = \varphi_1$ one has equality. As $\mathrm{dom}(\Delta_{\mathrm{D}})$ contains $C_{\mathrm{c}}^\infty(\Omega)$ and hence is dense in $H_0^1(\Omega)$, the inequality $\int_\Omega |\nabla u|^2 \,\mathrm{d}x \geqslant \lambda_1 \|u\|_2^2$ carries over to all $u \in H_0^1(\Omega)$. □

6.21 Remark. The existence of an orthonormal basis of eigenvectors of $-\Delta_{\mathrm{D}}$ is a highlight and triumph of Hilbert space theory as applied to partial differential equations. There is no way to obtain this kind of result by computing eigenfunctions, even if the boundary is nice.

In the case of an interval in one dimension it is not difficult to compute the eigenfunctions (see Exercise 6.6), and this can be generalised to n-dimensional rectangles. There is also a formula for balls, but this is more complicated and involves Bessel functions. △

6.4 Interlude: compactness of the embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$

The main result of this section is the following special case of the Rellich–Kondrachov theorem.

6.22 Theorem (Rellich–Kondrachov). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Then the embedding $j\colon H_0^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact.*

The proof will be given at the end of the section. We first explain the convolution of functions for a context different from Subsection 4.1.1.

6.23 Proposition. *Let $\rho \in L_1(\mathbb{R}^n)$, and let $1 \leqslant p \leqslant \infty$, $u \in L_p(\mathbb{R}^n)$. Then*

$$\rho * u(x) := \int_{\mathbb{R}^n} \rho(x-y)u(y) \,\mathrm{d}y = \int_{\mathbb{R}^n} \rho(y)u(x-y) \,\mathrm{d}y$$

exists for a.e. $x \in \mathbb{R}^n$, and

$$\|\rho * u\|_p \leqslant \|\rho\|_1 \|u\|_p \,.$$

Proof. The function $\mathbb{R}^{2n} \ni (x,y) \mapsto \rho(x-y)u(y)$ is measurable because the mapping $(x,y) \mapsto (x-y,y)$ is a diffeomorphism. Let $1 \leqslant q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in L_q(\mathbb{R}^n)$ we estimate, using Fubini–Tonelli and Hölder,

$$\begin{aligned} \iint |\rho(y)u(x-y)| \,\mathrm{d}y \, |g(x)| \,\mathrm{d}x &= \int |\rho(y)| \int |u(x-y)||g(x)| \,\mathrm{d}x \,\mathrm{d}y \\ &\leqslant \int |\rho(y)| \|u(\cdot - y)\|_p \|g\|_q \,\mathrm{d}y = \|\rho\|_1 \|u\|_p \|g\|_q \,. \end{aligned}$$

This inequality shows the assertions since $L_q(\mathbb{R}^n)$ is norming for $L_p(\mathbb{R}^n)$. □

The fundamental result in this context is the following theorem.

6.24 Theorem. *Let $\rho \in L_1(\mathbb{R}^n)$, let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $1 \leqslant p \leqslant \infty$. Then the mapping*

$$J\colon L_p(\mathbb{R}^n) \to L_p(\Omega),\ u \mapsto (\rho * u)|_\Omega$$

is compact.

Proof. If $\rho \in C_c(\mathbb{R}^n)$, then we show that

$$F := \big\{(\rho * u)|_{\overline{\Omega}};\, u \in L_p(\mathbb{R}^n),\ \|u\|_p \leqslant 1\big\}$$

is relatively compact in $C(\overline{\Omega})$. Clearly, the set is bounded because $|\rho * u(x)| \leqslant \|\rho\|_q \|u\|_p$, for all $x \in \mathbb{R}^n$ (with $\frac{1}{q} + \frac{1}{p} = 1$, as usual). Moreover, the estimate

$$|\rho * u(x) - \rho * u(y)| \leqslant \int \big|\big(\rho(x-z) - \rho(y-z)\big)u(z)\big|\,\mathrm{d}z \leqslant \|\rho(x-\cdot) - \rho(y-\cdot)\|_q \|u\|_p$$

shows that F is equicontinuous. By the Arzelà-Ascoli theorem it follows that F is relatively compact in $C(\overline{\Omega})$. The embedding $C(\overline{\Omega}) \hookrightarrow L_p(\Omega)$ is continuous, and therefore F is also relatively compact in $L_p(\Omega)$, i.e. J is compact.

Now let $\rho \in L_1(\mathbb{R}^n)$. Then there exists a sequence (ρ_k) in $C_c(\mathbb{R}^n)$ such that $\rho_k \to \rho$ in $L_1(\mathbb{R}^n)$. For all $k \in \mathbb{N}$ the operator

$$J_k\colon L_p(\mathbb{R}^n) \to L_p(\Omega),\ u \mapsto (\rho_k * u)|_\Omega$$

is compact, as shown above. Proposition 6.23 implies that $\|J - J_k\| \leqslant \|\rho - \rho_k\|_1 \to 0$ $(k \to \infty)$, and as $\mathcal{K}(L_p(\mathbb{R}^n), L_p(\Omega))$ is closed in $\mathcal{L}(L_p(\mathbb{R}^n), L_p(\Omega))$ we conclude that J is compact. □

In order to use Theorem 6.24 for the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$, we derive a representation of functions in $C_c^1(\mathbb{R}^n)$ in terms of their derivatives.

6.25 Lemma. *Let $\varphi \in C_c^1(\mathbb{R}^n)$. Then*

$$\varphi(0) = -\frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} \nabla\varphi(y) \cdot \frac{y}{|y|^n}\,\mathrm{d}y,$$

where σ_{n-1} is the $(n-1)$-dimensional volume of the unit sphere $S_{n-1} \subseteq \mathbb{R}^n$.

Proof. For $z \in S_{n-1}$ we have

$$\varphi(0) = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\varphi(tz)\,\mathrm{d}t = -\int_0^\infty \nabla\varphi(tz) \cdot z\,\mathrm{d}t.$$

Integrating this formula over S_{n-1}, provided with the surface measure σ, we obtain

$$\begin{aligned}\sigma_{n-1}\varphi(0) &= -\int_{z\in S_{n-1}} \int_{t=0}^\infty \nabla\varphi(tz) \cdot z\,\mathrm{d}t\,\mathrm{d}\sigma(z) \\ &= -\int_{t=0}^\infty \int_{z\in S_{n-1}} \nabla\varphi(tz) \cdot \frac{tz}{|tz|^n}\,\mathrm{d}\sigma(z)\,t^{n-1}\,\mathrm{d}t = -\int_{\mathbb{R}^n} \nabla\varphi(y) \cdot \frac{y}{|y|^n}\,\mathrm{d}y,\end{aligned}$$

where in the last equality we have used generalised polar coordinates. (We refer to Section 7.1 for the definition of the surface measure for more general sets than S_{n-1}.) □

Proof of Theorem 6.22. In this proof we extend all functions defined on Ω to $\mathbb{R}^n$, by putting them 0 on $\mathbb{R}^n \setminus \Omega$.

Put $R := \operatorname{diam}(\Omega)$. For $\varphi \in C_c^1(\Omega)$, $x \in \Omega$ we obtain, applying Lemma 6.25 to $\varphi(x - \cdot)$,

$$\begin{aligned}\varphi(x) &= \frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} \nabla\varphi(x-y) \cdot \frac{y}{|y|^n} \,\mathrm{d}y \\ &= \frac{1}{\sigma_{n-1}} \int_{B(0,R)} \nabla\varphi(x-y) \cdot \frac{y}{|y|^n} \,\mathrm{d}y = \int_{\mathbb{R}^n} \nabla\varphi(x-y) \cdot \rho(y) \,\mathrm{d}y\end{aligned}$$

with $\rho(y) := \frac{1}{\sigma_{n-1}} \mathbf{1}_{B(0,R)}(y) \frac{y}{|y|^n}$.

Due to the cut-off at R, the (vector-valued) function ρ is integrable. Therefore Theorem 6.24 implies that the mapping

$$J \colon L_2(\mathbb{R}^n; \mathbb{K}^n) \to L_2(\Omega), \ u \mapsto \sum_{j=1}^{n} (\rho_j * u_j)|_\Omega$$

is compact. Obviously the mapping

$$G \colon H_0^1(\Omega) \to L_2(\Omega; \mathbb{K}^n), \ f \mapsto \nabla f$$

is continuous. It follows that $JG \colon H_0^1(\Omega) \to L_2(\Omega)$ is compact.

The formula presented above shows that for $\varphi \in C_c^1(\Omega)$ one has $j(\varphi) = JG\varphi$. As both operators j and JG are continuous and $C_c^1(\Omega)$ is dense in $H_0^1(\Omega)$ one concludes that $j = JG$ is compact. □

Notes

It is difficult to attribute the development of adjoint operators to a source. One of the first more systematic treatments is given in [Neu32b]. The idea of also including linear relations is contained in [Are61]. In Section 14.1 we will treat the adjoint for linear relations and self-adjoint linear relations.

The first version of the spectral theorem for compact self-adjoint operators is contained in [Hil06]. (This paper is also contained in the collection [Hil12].) The diagonal structure of the Dirichlet Laplacian seems to be classical and difficult to attribute; however, the use of compactness methods for this purpose can be attributed to Rellich [Rel30]. In this paper the first version of the Rellich–Kondrachov theorem appeared as well as the application to the Dirichlet Laplacian. The proof we give for the Rellich–Kondrachov theorem is not the one usually found in textbooks. In particular, the authors could not find a version of Theorem 6.24 in the literature.

Exercises

6.1 Let A be a self-adjoint operator in a complex Hilbert space H. Show that $\sigma(A) \subseteq \mathbb{R}$. (Hint: Argue as in the proof of Theorem 6.1.)

6.2 Let X, Y be Banach spaces, $A \in \mathcal{L}(X, Y)$.

(a) Assume that A is a compact operator. Show that A maps weakly convergent sequences in X to convergent sequences in Y.

(b) Assume additionally that X is reflexive, and that A maps weakly convergent sequences in X to convergent sequences in Y. Show that A is compact.

Hints: 1. Recall that a sequence (x_n) in X is called weakly convergent to $x \in X$ if $\eta(x_n) \to \eta(x)$ $(\eta \in X')$, and that, if X is reflexive, every bounded sequence in X contains a weakly convergent subsequence.

2. Recall that $A \in \mathcal{L}(X, Y)$ is also continuous with respect to the weak topologies on X and Y; in particular, if a sequence (x_n) in X is weakly convergent to $x \in X$, then (Ax_n) is weakly convergent to Ax in Y.

6.3 Let G, H be Hilbert spaces, $A \in \mathcal{L}(G, H)$. Show that the following properties are equivalent:

(i) A is compact,

(ii) A^* is compact,

(iii) A^*A is compact.

Hint concerning '(iii) $\Rightarrow$ (i)': Given a sequence (x_n) in G converging weakly to 0, show that $\|Ax_n\|^2 \to 0$; then apply Exercise 6.2.

Note. The equivalence of (i) and (ii) is a special case of "Schauder's theorem": For Banach spaces X, Y, an operator $A \in \mathcal{L}(X, Y)$ is compact if and only if the dual operator $A' \in \mathcal{L}(Y', X')$ is compact.

6.4 Let (Ω, μ) be a measure space, and let $m\colon \Omega \to \mathbb{C}$ be measurable. Let A_0 be defined by

$$\operatorname{dom}(A_0) := \operatorname{lin}\{\mathbf{1}_C\,;\, C \in \mathcal{C}\},$$

with

$$\mathcal{C} := \Big\{C \subseteq \Omega\,;\, C \text{ measurable},\ \mu(C) < \infty,\ \sup_{x \in C} |m(x)| < \infty\Big\},$$

$$A_0 f := mf \qquad (f \in \operatorname{dom}(A_0)).$$

(a) Show that $\operatorname{dom}(A_0)$ is dense in $L_2(\Omega, \mu)$, and that $A_0^* = M_{\overline{m}}$, where $M_{\overline{m}}$ denotes the maximal multiplication operator induced by $\overline{m}$,

$$\operatorname{dom}(M_{\overline{m}}) := \{f \in L_2(\Omega, \mu)\,;\, \overline{m} f \in L_2(\Omega, \mu)\},$$
$$M_{\overline{m}} f := \overline{m} f \qquad (f \in \operatorname{dom}(M_{\overline{m}})).$$

(b) Show further that $M_{\overline{m}}^* = M_m$ (maximal multiplication operator induced by m).

(c) Show that $\overline{A_0} = M_m$, i.e. $\operatorname{dom}(A_0)$ is a core for M_m.

6.5 (a) Let X be a vector space, Y an n-dimensional subspace, Z an $(n-1)$-codimensional subspace (i.e., there exists an $(n-1)$-dimensional subspace Z_0 of X such that $Z \cap Z_0 = \{0\}$, $Z + Z_0 = X$). Show that $Y \cap Z \neq \{0\}$.

(b) Let H be an infinite-dimensional separable Hilbert space, and let A be a self-adjoint diagonal operator in H associated with the form a, described as in Examples 6.6 and 6.17. Assume that the sequence of eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ in $[0, \infty)$ is increasing.

For $x_1, \dots, x_{n-1} \in H$ put

$$I(x_1, \dots, x_{n-1}) := \inf\bigl\{a(x);\, x \in \{x_1, \dots, x_{n-1}\}^\perp \cap V,\ \|x\| = 1\bigr\}.$$

Show the **min-max principle**:

$$\lambda_n = \max\bigl\{I(x_1, \dots, x_{n-1});\, x_1, \dots, x_{n-1} \in H\bigr\} \qquad (n \in \mathbb{N}).$$

(c) Let $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^n$ be bounded open sets, and let Δ_j be the Dirichlet Laplacian in $L_2(\Omega_j)$ and $(\lambda_k^j)_{k\in\mathbb{N}}$ the corresponding increasing sequence of eigenvalues, for $j = 1, 2$.
Show the **domain monotonicity of eigenvalues**: $\lambda_k^1 \geqslant \lambda_k^2$ for all $k \in \mathbb{N}$.

6.6 (a) Let $-\infty \leqslant a < x_0 < b \leqslant \infty$, let $f \in C(a, b)$, and assume that $g_1 := \bigl(f|_{(a,x_0)}\bigr)' \in L_1(a, x_0)$, $g_2 := \bigl(f|_{(x_0,b)}\bigr)' \in L_1(x_0, b)$. Define $g \in L_1(a, b)$ by $g|_{(a,x_0)} := g_1$, $g|_{(x_0,b)} := g_2$. Show that $f' = g$. (Hint: Use Proposition 4.7.)

(b) Let $-\infty < a < b < \infty$. Show that $H_0^1(a, b) = \{f \in H^1(a, b);\, f(a) = f(b) = 0\}$.

Hint: For $f \in H^1(a, b)$ with $f(a) = f(b) = 0$, show that the extension of f to $\mathbb{R}$ by 0 belongs to $H^1(\mathbb{R})$. Then decompose f as $f = g + h$ with $g, h \in H^1(\mathbb{R})$, $\operatorname{spt} g \subseteq [a, b)$, $\operatorname{spt} h \subseteq (a, b]$, and show that $g, h \in H_0^1(a, b)$, using Example 1.7(a).

(c) Compute the orthonormal basis of eigenfunctions and the eigenvalues of $-\Delta_{\mathrm{D}}$ for $\Omega = (0, \pi)$.

(d) Determine the optimal value of the Poincaré constant for the open set $(0, \pi)$ (see Section 5.4).

6.7 Let H be a Hilbert space.

(a) Let $B \in \mathcal{L}(H)$. Show that $\|B^*B\| = \|B\|^2$. (Hint: For the less trivial inequality start with $\|B^*B\| = \sup_{\|x\|,\|y\|\leqslant 1} |(B^*Bx \,|\, y)| \geqslant \sup_{\|x\|\leqslant 1} |(B^*Bx \,|\, x)|$.)

(b) Let T be a bounded C_0-semigroup on H, with generator A. Show that $T(t)$ is self-adjoint for all $t \geqslant 0$ if and only if $-A$ is an accretive self-adjoint operator, and that in this case $\|T(t)\| \leqslant 1$ for all $t \geqslant 0$.

Hint: Use part (a) to show that the boundedness hypothesis together with the self-adjointness of $T(t)$ implies that $\|T(t)\| \leqslant 1$. For the proof of the self-adjointness of A recall Proposition 6.9.

(c) Suppose that H is a complex Hilbert space, and let A be an accretive self-adjoint operator. Show that $-A$ generates a holomorphic C_0-semigroup of angle $\pi/2$, $\|T(z)\| \leqslant 1$ for all $\operatorname{Re} z > 0$.

Chapter 7

Neumann and Robin boundary conditions

So far our study of the Laplacian was restricted to homogeneous Dirichlet boundary conditions. Our aim in this chapter is to investigate Neumann boundary conditions

$$\partial_\nu u = 0 \quad \text{on } \partial\Omega$$

and more generally Robin boundary conditions

$$\partial_\nu u + \beta u = 0 \quad \text{on } \partial\Omega.$$

If we think of heat conduction in a body Ω, then Neumann boundary conditions describe an isolated body, whereas Robin boundary conditions describe when part of the heat is absorbed at the boundary at a rate β.

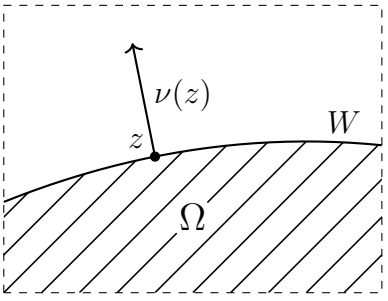
We start with the description of properties of the boundary for an open subset of $\mathbb{R}^n$. The main issue of Section 7.1 will be the discussion of Gauss' theorem and some of its consequences. In an interlude in Section 7.2 we present properties of $H^1(\Omega)$ that will be needed to formulate Neumann and Robin boundary conditions and to derive properties of the corresponding operators.

7.1 Gauss' theorem

Before formulating Gauss' theorem we introduce some notation; in particular we define the notions of open sets with C^1-boundary and of the outer unit normal. Throughout this section, Ω is a bounded open subset of $\mathbb{R}^n$.

Let $W \subseteq \partial\Omega$ be an open subset (of the metric space $\partial\Omega$). We say that W is a **standard C^1-graph** (with respect to Ω) if $W' := \big\{(z_1, \dots, z_{n-1})\,;\, z = (z_1, \dots, z_n) \in W\big\} \subseteq \mathbb{R}^{n-1}$ is open and there exist an open interval $(a, b) \subseteq \mathbb{R}$ and a C^1-function $g\colon W' \to (a, b)$ such that $W = \big\{(y, g(y))\,;\, y \in W'\big\}$, i.e. W is the graph of g, and for every point $(y, t) \in W' \times (a, b)$ one has

$$(y, t) \in \Omega \quad \text{if and only if} \quad t < g(y).$$



It is easy to see that then $(y,t) \notin \overline{\Omega}$ if and only if $t > g(y)$. The set W is called a **C^1-graph** (with respect to Ω) if there exists an orthogonal matrix $B \in \mathbb{R}^{n\times n}$ such that $B(W)$ is a standard C^1-graph with respect to $B(\Omega)$. This means of course that W is a standard C^1-graph with respect to another cartesian coordinate system. We say that Ω has **C^1-boundary** if for each $z \in \partial\Omega$ there exists an open neighbourhood $W \subseteq \partial\Omega$ of z such that W is a C^1-graph with respect to Ω.

Similarly we now define other regularity properties of the boundary.

7.1 Remarks. (C^k-, Lipschitz, continuous boundary)

(a) For $k \in \mathbb{N} \cup \{\infty\}$ we call W a **C^k-graph** if the function $g\colon W' \to (a,b)$ in the definition given above is a C^k-function. We speak of a **Lipschitz graph** if g is Lipschitz continuous (and not necessarily C^1). We call W a **continuous graph** if we merely require that g is continuous.

(b) We say that Ω has **C^k-boundary** (**Lipschitz boundary**, **continuous boundary**), if for each $z \in \partial\Omega$ there exists an open neighbourhood $W \subseteq \partial\Omega$ of z such that W is a C^k-graph (Lipschitz graph, continuous graph). △

In this way we have defined a hierarchy of regularity properties. Continuous boundary is the weakest property we consider – it will suffice for many of our results – and C^∞-boundary is the strongest. If Ω has C^1-boundary, then it also has Lipschitz boundary. Each polygon in $\mathbb{R}^2$ and each convex polyhedron in $\mathbb{R}^3$ has Lipschitz boundary, but not C^1-boundary; so there are good reasons to consider Lipschitz boundary. Nevertheless we will assume C^1-boundary also in many results that would only require Lipschitz boundary, because then things become much easier.

Next we introduce the outer (or exterior) normal of an open set with C^1-boundary. It can be characterised intrinsically by basic geometrical properties as follows; see [ArUr23; Theorem 7.4 and its proof].

7.2 Remark. Assume that Ω has C^1-boundary. Then for each $z \in \partial\Omega$ there is a unique vector $\nu(z) \in \mathbb{R}^n$ satisfying

(i) $|\nu(z)| = 1$;

(ii) if $\gamma \in C^1(-1,1;\mathbb{R}^n)$ is such that $\gamma(0) = z$ and $\gamma(t) \in \partial\Omega$ for all $t \in (-1,1)$, then $\nu(z) \cdot \gamma'(0) = 0$;

(iii) there exists $\varepsilon > 0$ such that $z + t\nu(z) \notin \overline{\Omega}$ (and $z - t\nu(z) \in \Omega$) for all $0 < t < \varepsilon$.

We call $\nu(z)$ the **outer unit normal** at z. Condition (ii) says that $\nu(z)$ is orthogonal to the boundary and (iii) that $\nu(z)$ points out of Ω.

If W is as in the description of a standard C^1-graph, then for $z = (y, g(y)) \in W$ the outer unit normal is given by

$$\nu(z) = \frac{1}{\sqrt{|\nabla g(y)|^2 + 1}} \begin{pmatrix} -\nabla g(y) \\ 1 \end{pmatrix}.$$

The function ν is continuous on $\partial\Omega$ with values in $\mathbb{R}^n$. △

We assume that the reader is acquainted with the integration of functions on $(n-1)$-dimensional C^1-manifolds in $\mathbb{R}^n$. We point out that the boundary $\partial\Omega$ of a bounded open set Ω with C^1-boundary is an $(n-1)$-dimensional C^1-manifold. The **surface measure** on

$\partial\Omega$ will be denoted by σ. The following formula is basic and can be taken as the definition of σ. If Ω is a bounded open set with C^1-boundary, $W \subseteq \partial\Omega$ is a standard C^1-graph, and W', g are as in the definition, then

$$\int_{\partial\Omega} h(z)\,\mathrm{d}\sigma(z) = \int_{W'} h(y, g(y))\sqrt{1 + |\nabla g(y)|^2}\,\mathrm{d}y$$

for all $h \in C(\partial\Omega)$ with support in W. The weight factor in the integral on the right-hand side is such that the $(n-1)$-dimensional Lebesgue measure on W' is transferred to the appropriate Borel measure σ on the $(n-1)$-dimensional manifold $\partial\Omega$. For more information about the surface measure σ we refer to Section A.1; in particular see formula (A.3).

We insert a comment on the notions of **Borel set** and **Borel measure**. Let (X, τ) be a topological space. The σ-algebra $\mathcal{B} \subseteq \mathcal{P}(X)$ (the power set of X) of Borel subsets of X is defined as the σ-algebra generated by τ, i.e. the smallest σ-algebra containing τ. A Borel measure μ on X is a (positive) locally finite measure defined on $\mathcal{B}$, where 'locally finite' means that every point of X possesses an open neighbourhood U with $\mu(U) < \infty$.

We can now formulate Gauss' theorem. By $C^1(\overline{\Omega})$ we denote the space of all functions $u \in C(\overline{\Omega}) \cap C^1(\Omega)$ for which $\partial_j u$ has a continuous extension to $\overline{\Omega}$ for each $j \in \{1, \dots, n\}$; we keep the notation $\partial_j u$ for this extension. (Concerning the notation $C^1(\overline{\Omega})$ we point out Exercise 7.1.)

7.3 Theorem (Gauss). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1-boundary. Then for all $u \in C^1(\overline{\Omega})$, $j \in \{1, \dots, n\}$ one has*

$$\int_\Omega \partial_j u(x)\,\mathrm{d}x = \int_{\partial\Omega} u(z)\nu_j(z)\,\mathrm{d}\sigma(z). \tag{7.1}$$

Here $\nu \in C(\partial\Omega; \mathbb{R}^n)$ is the outer unit normal, $\nu(z) = (\nu_1(z), \dots, \nu_n(z))$.

A proof of Theorem 7.3 is given in Section A.2. We mention that σ is the unique Borel measure on $\partial\Omega$ such that (7.1) holds; see [ArUr23; Section 7.2].

As it should be, the notions introduced above can be interpreted for the case $n = 1$. To get a start, the Lebesgue measure on $\mathbb{R}^0 = \{0\}$ is the counting measure, and all functions $g\colon \mathbb{R}^0 \to \mathbb{R}$ are differentiable with derivative 0. As a consequence, a bounded set $\Omega \subseteq \mathbb{R}$ has C^1-boundary (or continuous boundary, or C^∞-boundary, for that matter) if and only if it is a finite union of bounded open intervals whose closures are pairwise disjoint. The surface measure on $\partial\Omega$, the set of the endpoints of the intervals, is the counting measure.

Specifically, for an interval $\Omega = (a, b)$ we have $\partial\Omega = \{a, b\}$, $\nu(a) = -1$, $\nu(b) = 1$. Applying Gauss' theorem with $u \in C^1[a, b] = C^1(\overline{(a, b)})$ we obtain

$$\int_a^b u'(x)\,\mathrm{d}x = \int_{\{a,b\}} u(z)\nu(z)\,\mathrm{d}\sigma(z) = u(b) - u(a),$$

the second part of the fundamental theorem of calculus.

Next we derive an important consequence of Gauss' theorem. We define $C^2(\overline{\Omega}) := \big\{u \in C^1(\overline{\Omega});\, \partial_j u \in C^1(\overline{\Omega})\ (j = 1, \dots, n)\big\}$. Then for $u \in C^2(\overline{\Omega})$ the functions $\partial_j \partial_k u$ are in $C(\overline{\Omega})$

for all $j, k = 1, \dots, n$. For $u \in C^1(\overline{\Omega})$ the function $\partial_\nu u \colon \partial\Omega \to \mathbb{K}$, given by

$$\partial_\nu u(z) := \nu(z) \cdot \nabla u(z) = \sum_{j=1}^{n} \nu_j(z) \partial_j u(z),$$

is called the **normal derivative** of u. Note that $\partial_\nu u \in C(\partial\Omega)$.

7.4 Corollary (Green's formulas). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1-boundary, and let $u \in C^2(\overline{\Omega})$. Then*

$$\int_\Omega (\Delta u) v \, \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_{\partial\Omega} (\partial_\nu u) v \, \mathrm{d}\sigma \qquad (v \in C^1(\overline{\Omega})), \tag{7.2}$$

$$\int_\Omega (v \Delta u - u \Delta v) \, \mathrm{d}x = \int_{\partial\Omega} (v \partial_\nu u - u \partial_\nu v) \, \mathrm{d}\sigma \qquad (v \in C^2(\overline{\Omega})). \tag{7.3}$$

Proof. By Gauss' theorem one has

$$\int_\Omega (\partial_j^2 u) v \, \mathrm{d}x + \int_\Omega \partial_j u \, \partial_j v \, \mathrm{d}x = \int_\Omega \partial_j \big((\partial_j u) v \big) \, \mathrm{d}x = \int_{\partial\Omega} (\partial_j u) v \, \nu_j \, \mathrm{d}\sigma.$$

Summation over $j = 1, \dots, n$ yields (7.2).

Exchanging u and v in (7.2) and subtracting the result from (7.2) one obtains (7.3). □

In the literature one finds various definitions of C^1-boundary (all equivalent to our definition). The following equivalence is useful for showing that a given set has C^1-boundary; its proof relies on the implicit function theorem. A bounded open set $\Omega \subseteq \mathbb{R}^n$ has C^1-boundary if and only if for each $z \in \partial\Omega$ there exist an open neighbourhood $U \subseteq \mathbb{R}^n$ of z and a C^1-function $h \colon U \to \mathbb{R}$ such that $\Omega \cap U = [h < 0]$ and $\nabla h(z) \neq 0$ for all $z \in \partial\Omega \cap U$. We mention that then the outer unit normal is given by $\nu(z) = \nabla h(z) / |\nabla h(z)|$ for all $z \in \partial\Omega \cap U$.

For instance, for the open unit ball $\Omega := B(0,1)$ in $\mathbb{R}^n$ the formula $\Omega = \{x \in \mathbb{R}^n ; |x|^2 - 1 < 0\}$ immediately shows that Ω has C^1-boundary. (In this case the proof by finding local parametrisations of the boundary is not difficult, but carrying out all the details is somewhat tedious.)

7.2 Interlude: more on $H^1(\Omega)$; denseness, trace and compactness

In this section we present some fundamental properties of the Sobolev space $H^1(\Omega)$ that are needed for our treatment of Neumann and Robin Laplacians in Sections 7.4 and 7.5 below. According to the general philosophy of this book, we give complete proofs of these properties. In a first reading the reader might want to skip some of the more technical proofs and first look at the application of the results.

The first major issue is a denseness property for $H^1(\Omega)$. Denseness results are needed to transfer inequalities or equalities one shows for smooth functions to more general functions.

An example for this procedure is the proof of Poincaré's inequality in Chapter 5; further examples follow in this section.

In the proof of the denseness result as well as in later arguments we need the following property of bounded open sets with continuous boundary.

7.5 Lemma. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with continuous boundary. Then for each $x \in \partial\Omega$ there exist a neighbourhood $W \subseteq \partial\Omega$ of x and $y \in \mathbb{R}^n \setminus \{0\}$ such that*

$$W - (0,1)y \subseteq \Omega, \quad W + (0,1)y \subseteq \mathbb{R}^n \setminus \overline{\Omega}.$$

Proof. There exists an open neighbourhood W_0 of x such that W_0 is a continuous graph with respect to Ω. Without loss of generality we assume that W_0 is a standard continuous graph. Reducing W_0 to a compact neighbourhood W of x one easily sees that the assertion is satisfied for $y = \lambda e_n$ with $\lambda > 0$ sufficiently small and the n-th unit vector e_n. □

In Exercise 7.2(b), the reader is asked to show that Ω having continuous boundary is in fact equivalent to the property stated in Lemma 7.5. We mention that the latter property is – more or less – a reformulation of the 'segment property' in [Ada75; 3.17], which means that also the segment property is equivalent to continuous boundary.

The following lemma contains 'local versions' of Proposition 4.3 and Lemma 4.15(b) that will be needed below. For a bounded open set $\Omega \subseteq \mathbb{R}^n$ we will use the notation

$$C^\infty(\overline{\Omega}) := \big\{u \in C^\infty(\Omega);\, \partial^\alpha u \text{ continuously extendable to } \overline{\Omega}\ (\alpha \in \mathbb{N}_0^n)\big\}.$$

7.6 Lemma. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $u \in L_{1,\mathrm{loc}}(\Omega)$. Let Ω' be a relatively compact open subset of Ω, and let $0 < \delta < \operatorname{dist}(\Omega', \mathbb{R}^n \setminus \Omega)$.*

(a) *Let $\rho \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{spt}\rho \subseteq B[0,\delta]$. Then*

$$\rho * u(x) := \int_\Omega \rho(x-y)u(y)\,\mathrm{d}y$$

is defined for all $x \in \overline{\Omega'}$, and $\rho * u \in C^\infty(\overline{\Omega'})$. If additionally $\partial^\alpha u \in L_{1,\mathrm{loc}}(\Omega)$ for some $\alpha \in \mathbb{N}_0^n$, then $\partial^\alpha(\rho * u) = \rho * \partial^\alpha u$ on Ω'.

(b) *Let (ρ_k) be a δ-sequence in $C_c^\infty(\mathbb{R}^n)$. Then $\rho_k * u \in C^\infty(\overline{\Omega'})$ for all $k \geqslant 1/\delta$, and $\rho_k * u \to u$ in $L_1(\Omega')$ as $1/\delta \leqslant k \to \infty$.*

If $u \in C(\Omega)$, then $\rho_k * u \to u$ in $C(\overline{\Omega'})$ as $1/\delta \leqslant k \to \infty$.

If $u \in H^1(\Omega)$, then $\rho_k * u \to u$ in $H^1(\Omega')$ as $1/\delta \leqslant k \to \infty$.

Proof. The set $\Omega_0 := \Omega' + B(0,\delta)$ is a relatively compact open subset of Ω, and there exists $\varphi_0 \in C_c^\infty(\Omega)$ with $\mathbf{1}_{\Omega_0} \leqslant \varphi_0 \leqslant 1$. Then the function $u_0 := \varphi_0 u$, extended to $\mathbb{R}^n$ by 0, belongs to $L_1(\mathbb{R}^n)$, and $u_0|_{\Omega_0} = u|_{\Omega_0}$.

(a) The properties stated above imply that $\rho * u(x) = \rho * u_0(x)$ for all $x \in \overline{\Omega'}$. Lemma 4.1 shows that $\rho * u_0 \in C^\infty(\mathbb{R}^n)$; hence $\rho * u \in C^\infty(\overline{\Omega'})$. Let $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha u \in L_{1,\mathrm{loc}}(\Omega)$. Then $\partial^\alpha(\rho * u_0) = (\partial^\alpha\rho) * u_0$ by Lemma 4.1, and $(\partial^\alpha\rho) * u_0(x) = \int_{\Omega_0} \partial^\alpha\rho(x-y)u(y)\,\mathrm{d}y$ for all $x \in \Omega'$. Finally, $\int_{\Omega_0} \partial^\alpha\rho(x-y)u(y)\,\mathrm{d}y = \int_{\Omega_0} \rho(x-y)\partial^\alpha u(y)\,\mathrm{d}y$ for all $x \in \Omega'$, by the same computation as in the proof of Lemma 4.15(a).

(b) For $k \geqslant 1/\delta$ one has $\operatorname{spt} \rho_k \subseteq B(0,\delta)$, hence $\rho_k * u \in C^\infty(\overline{\Omega'})$ by part (a) above. Proposition 4.3(b) implies that $\rho_k * u_0 \to u_0$ in $L_1(\mathbb{R}^n)$; hence $(\rho_k * u)|_{\Omega'} = (\rho_k * u_0)|_{\Omega'} \to u_0|_{\Omega'} = u|_{\Omega'}$ in $L_1(\Omega')$ as $k \to \infty$.

If $u \in C(\Omega)$, then applying Proposition 4.3(a) one can carry out the proof analogously.

If $u \in H^1(\Omega)$, then $\varphi_0 u \in H^1_{\mathrm{c}}(\Omega)$, and $u_0 \in H^1(\mathbb{R}^n)$ by Exercise 4.6(a), $\rho_k * u_0 \to u_0$ in $H^1(\mathbb{R}^n)$ by Lemma 4.15(b). The remainder of the proof is as above. □

We now turn to the announced denseness property for $H^1(\Omega)$.

7.7 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with continuous boundary. Then the set*

$$\check{C}^\infty(\Omega) := \{\varphi|_\Omega \,;\, \varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)\}$$

is dense in $H^1(\Omega)$. As a consequence, $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$.

Proof. (i) In this first step we show that for each $x \in \partial\Omega$ there exists an open neighbourhood $U \subseteq \mathbb{R}^n$ such that each $u \in H^1(\Omega)$ with relatively compact support in U can be approximated by functions in $\check{C}^\infty(\Omega)$.

Let $x \in \partial\Omega$. By Lemma 7.5 there exist $y \in \mathbb{R}^n \setminus \{0\}$ and an open neighbourhood $W \subseteq \partial\Omega$ of x such that $W + (-1,0)y \subseteq \Omega$ and $W + (0,1)y \subseteq \mathbb{R}^n \setminus \overline{\Omega}$. Let $u \in H^1(\Omega)$ be such that $\operatorname{spt} u$ is relatively compact in the open(!) set $U := W + (-1,1)y$, and let $\tilde{u}$ denote the extension of u to $\mathbb{R}^n$ by 0. Then $W_u := \operatorname{spt} \tilde{u} \cap \partial\Omega$ is a compact subset of W, $\tilde{u} \in H^1(\mathbb{R}^n \setminus W_u)$ and $\partial_j \tilde{u} = \widetilde{\partial_j u}$ on $\mathbb{R}^n \setminus W_u$ for $j = 1, \ldots, n$; see Exercise 7.3. We shift $\tilde{u}$ 'outwards', putting $u_\tau := \tilde{u}(\cdot - \tau y)$ for $\tau \in (0,1)$; then

$$u_\tau \in H^1\big(\mathbb{R}^n \setminus (W_u + \tau y)\big), \quad \overline{\Omega} \subseteq \mathbb{R}^n \setminus (W_u + \tau y). \tag{7.4}$$

In particular, $u_\tau|_\Omega \in H^1(\Omega)$ and $\partial_j(u_\tau|_\Omega) = \partial_j \tilde{u}(\cdot - \tau y)|_\Omega$ for $j = 1, \ldots, n$. Applying Exercise 7.4(a) we conclude that $u_\tau|_\Omega \to u$ in $H^1(\Omega)$ as $\tau \to 0$. Now let $\tau \in (0,1)$, and let $(\rho_k)_{k\in\mathbb{N}}$ be a δ-sequence in $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$. Then $\rho_k * u_\tau \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$, and in view of (7.4), Lemma 7.6(b) shows that $(\rho_k * u_\tau)|_\Omega \to u_\tau|_\Omega$ in $H^1(\Omega)$ as $k \to \infty$.

(ii) A compactness argument shows that $\partial\Omega$ can be covered by open sets $U_1, \ldots, U_m \subseteq \mathbb{R}^n$ such that for each $k \in \{1, \ldots, m\}$, each function $u \in H^1(\Omega)$ with relatively compact support in U_k can be approximated by functions in $\check{C}^\infty(\Omega)$. Putting $U_0 := \Omega$ we obtain an open covering $(U_k)_{k=0,\ldots,m}$ of $\overline{\Omega}$. There exists a partition of unity $(\varphi_k)_{k=0,\ldots,m}$ in $C^\infty_{\mathrm{c}}(\mathbb{R}^n)_+$ on $\overline{\Omega}$, subordinate to $(U_k)_{k=0,\ldots,m}$; see Exercise 4.3(b).

Let $u \in H^1(\Omega)$. Then $\varphi_k u \in H^1(\Omega)$ for all $k = 0, \ldots, m$, by Exercise 4.5(d). Now, $\varphi_0 u \in H^1_0(\Omega)$ can be approximated by $C^\infty_{\mathrm{c}}(\Omega)$-functions, by Theorem 4.14(b), and $\varphi_1 u, \ldots, \varphi_m u$ can be approximated by $\check{C}^\infty(\Omega)$-functions, by the choice of $U_1, \ldots, U_m$. In consequence, u can be approximated by $\check{C}^\infty(\Omega)$-functions. □

7.8 Remarks. The procedure used in the proof of Theorem 7.7 yields simultaneous approximation with respect to other properties:

(a) If $u \in H^1(\Omega) \cap C(\overline{\Omega})$, then the approximations additionally converge to u in the sup-norm, an observation that will be important in the proof of Theorem 7.10.

Indeed, multiplying u by a function $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ does not effect the continuity property of u. Then one has to apply the last assertion of Lemma 7.6(b) to the approximations used in step (i) of the proof of Theorem 7.7.

(b) Another instance is the property that positive functions in $H^1(\Omega)$ can be approximated by positive functions in $\check{C}^\infty(\Omega)$. In order to see this, one just has to note that the product as well as the convolution of two positive functions is again a positive function. $\triangle$

The following proposition is an important characterisation of $H^1_0(\Omega)$. Its proof uses a method similar to the proof of Theorem 7.7. For a function $u\colon \Omega \to \mathbb{K}$ we will again denote by $\tilde{u}$ the extension of u to $\mathbb{R}^n$ by 0.

7.9 Proposition. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with continuous boundary. Then*

$$H^1_0(\Omega) = \big\{u \in H^1(\Omega)\,;\, \tilde{u} \in H^1(\mathbb{R}^n)\big\}.$$

Proof. The inclusion '$\subseteq$' was noted in Remark 4.13(b). (In fact, this inclusion holds for arbitrary open sets Ω.)

In order to show '$\supseteq$' we assume that $u \in H^1(\Omega)$ is such that $\tilde{u} \in H^1(\mathbb{R}^n)$. Let $U_0, \dots, U_m$ and $\varphi_0, \dots, \varphi_m$ be as in step (ii) of the proof of Theorem 7.7. There exist vectors $y^1, \dots, y^m \in \mathbb{R}^n$ (pointing 'out of Ω') such that $(\varphi_k\tilde{u})(\cdot + \tau y^k)$ has compact support in Ω for $0 < \tau < 1$, $k = 1, \dots, m$. Then $(\varphi_k\tilde{u})(\cdot + \tau y^k)|_\Omega \in H^1_c(\Omega)$ for $0 < \tau < 1$, $k = 1, \dots, m$. We note that $(\varphi_k\tilde{u})(\cdot + \tau y^k) \to \varphi_k\tilde{u}$ in $H^1(\mathbb{R}^n)$ as $\tau \to 0$, for $k = 1, \dots, m$; see Exercise 7.4(b). Since $\varphi_0\tilde{u} \in H^1_c(\Omega)$ as well, the previous considerations show that u can be approximated by functions belonging to $H^1_c(\Omega)$. This implies that $u \in H^1_0(\Omega)$. $\square$

We point out that in the proof given above the functions $\varphi_k\tilde{u}$ are shifted in the direction opposite to the one in the proof of Theorem 7.7, step (i). In the present case the functions are shifted 'inwards' to produce compact support, in the former proof they are shifted 'outwards' in order to produce smoothness on $\overline{\Omega}$.

Next we show that for a set Ω with C^1-boundary one can define a trace mapping $\operatorname{tr}\colon H^1(\Omega) \to L_2(\partial\Omega)$ such that for $u \in C^1(\overline{\Omega})$ one has $\operatorname{tr} u = u|_{\partial\Omega}$. Here and in what follows we understand $L_2(\partial\Omega)$ with respect to the surface measure.

7.10 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1-boundary. Then there exists $c \geqslant 0$ such that*

$$\|u|_{\partial\Omega}\|^2_{L_2(\partial\Omega)} \leqslant c\|u\|_{L_2(\Omega)}\|u\|_{H^1(\Omega)} \tag{7.5}$$

for all $u \in C^1(\overline{\Omega})$.

*There is a unique bounded operator $\operatorname{tr}\colon H^1(\Omega) \to L_2(\partial\Omega)$, called the **trace operator**, such that $\operatorname{tr} u = u|_{\partial\Omega}$ for all $u \in C(\overline{\Omega}) \cap H^1(\Omega)$, and then (7.5) holds for all $u \in H^1(\Omega)$ (with $u|_{\partial\Omega}$ replaced by $\operatorname{tr} u$ on the left-hand side).*

Proof. (i) Let $x \in \partial\Omega$, and let $y := \nu(x)$ be the outer unit normal at x. Since ν is continuous, by Remark 7.2, there exists an open neighbourhood $U \subseteq \mathbb{R}^n$ of x such that $y \cdot \nu(z) \geqslant \frac{1}{2}$ for all $z \in U \cap \partial\Omega$. Let $\varphi \in C^\infty_c(\mathbb{R}^n)$ satisfy $\operatorname{spt}\varphi \subseteq U$, $\varphi \geqslant 0$, and $\varphi = 1$ on a neighbourhood of x. Then $\varphi = 1$ on an open neighbourhood $W_x \subseteq \partial\Omega$ of x.

Let $u \in C^1(\overline{\Omega})$. Then, by Gauss' theorem,

$$\begin{aligned}\frac{1}{2}\int_{W_x} |u|^2\,\mathrm{d}\sigma &\leqslant \int_{\partial\Omega} (\varphi u)\bar{u}\, y\cdot\nu\,\mathrm{d}\sigma = \int_\Omega \big(\nabla(\varphi u)\bar{u} + (\varphi u)\nabla\bar{u}\big)\cdot y\,\mathrm{d}x \\ &\leqslant \|\varphi u\|_{H^1}\|u\|_{L_2} + \|\varphi u\|_{L_2}\|u\|_{H^1} \leqslant c_\varphi\|u\|_{L_2}\|u\|_{H^1}\,,\end{aligned}$$

with a constant $c_\varphi > 0$ only depending on φ.

We have shown that for each $x \in \partial\Omega$ there exist an open neighbourhood $W_x \subseteq \partial\Omega$ and a constant $c_x \geqslant 0$ such that

$$\int_{W_x} |u|^2 \,\mathrm{d}\sigma \leqslant c_x \|u\|_{L_2} \|u\|_{H^1}$$

for all $u \in C^1(\overline{\Omega})$. A standard compactness argument completes the proof of (7.5) for $u \in C^1(\overline{\Omega})$.

(ii) The inequality (7.5) together with the denseness of $C^1(\overline{\Omega})$ in $H^1(\Omega)$ implies that the mapping $u \mapsto u|_{\partial\Omega}$ has a continuous extension $\operatorname{tr}\colon H^1(\Omega) \to L_2(\partial\Omega)$.

(iii) So far we only have shown that $\operatorname{tr} u = u|_{\partial\Omega}$ holds for $u \in C^1(\overline{\Omega})$. In order to show it for $u \in C(\overline{\Omega}) \cap H^1(\Omega)$, we use the remarkable feature of the proof of Theorem 7.7 mentioned in Remark 7.8(a). As explained there, for $u \in C(\overline{\Omega}) \cap H^1(\Omega)$ an approximating sequence (u_k) in $C^1(\overline{\Omega})$ can be chosen that converges to u in $C(\overline{\Omega})$ as well as in $H^1(\Omega)$. For this sequence, $\big(u_k|_{\partial\Omega}\big)$ converges to $\operatorname{tr} u$ in $L_2(\partial\Omega)$ and uniformly to $u|_{\partial\Omega}$, which implies $\operatorname{tr} u = u|_{\partial\Omega}$. □

Sometimes, by abuse of notation, we still write $u|_{\partial\Omega} := \operatorname{tr} u$ for $u \in H^1(\Omega)$. In integrals we will frequently omit the trace notation to make things more readable, writing $\|\operatorname{tr} u\|^2_{L_2(\partial\Omega)} = \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma$ for example.

The trace is compatible with our definition of $H^1_0(\Omega)$ as the following result shows.

7.11 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1-boundary. Then*

$$H^1_0(\Omega) = \big\{u \in H^1(\Omega);\, \operatorname{tr} u = 0\big\}.$$

Proof. The inclusion '$\subseteq$' follows from the continuity of the trace operator and the denseness of $C^\infty_c(\Omega)$ in $H^1_0(\Omega)$.

For the proof of '$\supseteq$' we first note that

$$\int_\Omega \partial_j u\, v \,\mathrm{d}x + \int_\Omega u\, \partial_j v \,\mathrm{d}x = \int_{\partial\Omega} u\, v\, \nu_j \,\mathrm{d}\sigma \tag{7.6}$$

for all $u, v \in H^1(\Omega)$, $j = 1, \dots, n$. This is immediate from Theorem 7.3 if $u, v \in \check{C}^\infty(\Omega)$, and (7.6) carries over to general u, v by Theorems 7.7 and 7.10.

As before, the extension to $\mathbb{R}^n$ by 0 of a function defined on Ω will be denoted by a tilde. Let $u \in H^1(\Omega)$ be such that $\operatorname{tr} u = 0$. Then (7.6) implies

$$\int_{\mathbb{R}^n} \tilde{u}\, \partial_j \varphi \,\mathrm{d}x = \int_\Omega u\, \partial_j \varphi \,\mathrm{d}x = -\int_\Omega \partial_j u\, \varphi \,\mathrm{d}x$$

for all $\varphi \in C^\infty_c(\mathbb{R}^n)$. This shows that $\tilde{u} \in H^1(\mathbb{R}^n)$ (with distributional derivative $\partial_j \tilde{u} = \widetilde{\partial_j u}$ for $j = 1, \dots, n$), and hence $u \in H^1_0(\Omega)$, by Proposition 7.9. □

We close the section by transferring the compactness of the embedding $H^1_0(\Omega) \hookrightarrow L_2(\Omega)$ to $H^1(\Omega)$; for this it suffices to assume continuous boundary.

7.12 Theorem (Rellich–Kondrachov). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with continuous boundary. Then the embedding $j\colon H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact.*

Proof. For each compact set $K \subseteq \Omega$ the exists $\psi \in C_c^\infty(\Omega)$ with $\mathbf{1}_K \leqslant \psi \leqslant 1$. Define $j_\psi\colon H^1(\Omega) \to L_2(\Omega)$ by $j_\psi u := \psi u$. Since j_ψ acts as a bounded operator from $H^1(\Omega)$ to $H_0^1(\Omega)$, Theorem 6.22 implies that j_ψ is compact. Moreover,

$$\|j - j_\psi\|_{\mathcal{L}(H^1(\Omega),L_2(\Omega))} \leqslant \sup\bigl\{ \bigl\|\mathbf{1}_{\Omega\setminus K} u\bigr\|_{L_2(\Omega)};\, u \in C^1(\overline{\Omega}),\ \|u\|_{H^1(\Omega)} \leqslant 1\bigr\}. \tag{7.7}$$

(Recall from Theorem 7.7 that $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$.) We will show that the right-hand side of (7.7) can be made small by an appropriate choice of K; then j is approximated by compact maps, and therefore is compact.

Since Ω has continuous boundary, there exist an open covering of $\partial\Omega$ by continuous graphs $W_1, \dots, W_m$ and vectors $y^1, \dots, y^m \in \mathbb{R}^n$ (pointing 'out of Ω') such that $|y^k| \leqslant 1$, $W_k - (0,2)y^k \subseteq \Omega$ and $W_k + (0,1)y^k \subseteq \mathbb{R}^n \setminus \overline{\Omega}$ for $k = 1, \dots, m$; see Lemma 7.6. Then for all $\varepsilon \in (0,2]$ one obtains

$$U_{k,\varepsilon} := W_k - (0,\varepsilon)y^k \subseteq \Omega \qquad (k = 1, \dots, m),$$

and $K_\varepsilon := \Omega \setminus \bigcup_{k=1}^m U_{k,\varepsilon}$ is a compact subset of Ω (because $\bigcup_{k=1}^m \bigl(W_k + (-\varepsilon,1)y^k\bigr)$ is an open neighbourhood of $\partial\Omega$).

Let $0 < \varepsilon < 1$. We want to estimate the L_2-norm of $u \in C^1(\overline{\Omega})$ on $\Omega \setminus K_\varepsilon$. Let $k \in \{1, \dots, m\}$. Then for $x \in U_{k,\varepsilon}$ we have

$$u(x) = -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\bigl((1-t)u(x - ty^k)\bigr)\,\mathrm{d}t,$$

hence

$$|u(x)|^2 \leqslant \int_0^1 2\bigl(|u(x-ty^k)|^2 + |\nabla u(x-ty^k)|^2|y^k|^2\bigr)\,\mathrm{d}t$$

(note that $x - ty^k \in U_{k,\varepsilon+t} \subseteq \Omega$ for all $t \in [0,1]$). For $z \in W_k$ it follows that

$$\int_0^\varepsilon |u(z-sy^k)|^2\,\mathrm{d}s \leqslant 2\varepsilon \int_0^2 \bigl(|u(z-ty^k)|^2 + |\nabla u(z-ty^k)|^2\bigr)\,\mathrm{d}t. \tag{7.8}$$

We will show below that this implies that

$$\int_{U_{k,\varepsilon}} |u|^2\,\mathrm{d}x \leqslant 2\varepsilon \|u\|^2_{H^1(U_{k,2})}. \tag{7.9}$$

Summing over $k = 1, \dots, m$ we obtain

$$\bigl\|\mathbf{1}_{\Omega\setminus K_\varepsilon} u\bigr\|^2_{L_2(\Omega)} \leqslant \sum_{k=1}^m \int_{U_{k,\varepsilon}} |u|^2\,\mathrm{d}x \leqslant 2\varepsilon m \|u\|^2_{H^1(\Omega)},$$

which shows that for $K = K_\varepsilon$ the right-hand side of (7.7) is $\leqslant 2\varepsilon m$.

For the proof of (7.9) we suppose (without loss of generality) that W_k is a *standard* continuous graph, $W = \bigl\{(x', g(x'));\ x' \in W'\bigr\}$ with an open set $W' \subseteq \mathbb{R}^{n-1}$ and a

continuous function $g\colon W' \to \mathbb{R}$, and that $y^k = \lambda e_n$ with $\lambda = |y^k|$; see Lemma 7.6. Using Fubini's theorem and (7.8) we estimate

$$\begin{aligned}\int_{U_{k,\varepsilon}} |u(x)|^2 \,\mathrm{d}x &= \lambda \int_{x'\in W'} \int_0^\varepsilon |u(x', g(x') - s\lambda)|^2 \,\mathrm{d}s \,\mathrm{d}x' \\ &\leqslant 2\varepsilon\lambda \int_{x'\in W'} \int_0^2 \big(|u|^2 + |\nabla u|^2\big)(x', g(x') - t\lambda) \,\mathrm{d}t \,\mathrm{d}x' = 2\varepsilon \|u\|^2_{H^1(U_{k,2})} . \quad \square\end{aligned}$$

We also refer to [EdEv87; Theorem 4.17] for Theorem 7.12.

7.13 Remark. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1-boundary. We note that (7.5) in combination with Theorem 7.12 shows that the trace operator in Theorem 7.10 is compact. Indeed, if (u_k) is a bounded sequence in $H^1(\Omega)$, then by Theorem 7.12 there exists a subsequence $\big(u_{k_m}\big)$ converging in $L_2(\Omega)$, and then (7.5) implies that $\big(\operatorname{tr} u_{k_m}\big)$ is a Cauchy sequence in $L_2(\partial\Omega)$, hence convergent. $\triangle$

7.3 Weak normal derivative

In this section we define the normal derivative in a weak sense for certain functions in $H^1(\Omega)$, by requiring the validity of Green's formula (7.2). Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1-boundary, and let $u \in H^1(\Omega)$, $\Delta u \in L_2(\Omega)$. We say that $\partial_\nu u \in L_2(\partial\Omega)$ if there exists $h \in L_2(\partial\Omega)$ such that

$$\int_\Omega (\Delta u) v \,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_{\partial\Omega} h v \,\mathrm{d}\sigma \qquad (v \in H^1(\Omega)).$$

In this case we define the **weak normal derivative** of u by $\partial_\nu u := h$. In order to show the uniqueness of h we note that $C^1(\overline{\Omega}) \subseteq H^1(\Omega)$. The Stone–Weierstrass theorem (see Appendix B, Theorem B.2) implies that the set $\big\{\varphi|_{\partial\Omega};\, \varphi \in C^1(\overline{\Omega})\big\}$ is dense in $C(\partial\Omega)$. As $C(\partial\Omega)$ is dense in $L_2(\partial\Omega)$ – see Exercise G.2 – we obtain the uniqueness. (Recall that $L_2(\partial\Omega)$ is understood with respect to the surface measure σ.)

Now let $\Omega \subseteq \mathbb{R}^n$ be any open set, and let $u \in H^1(\Omega)$ be such that $\Delta u \in L_2(\Omega)$. If Ω is bounded and has C^1-boundary, then by the definition given above, $\partial_\nu u = 0$ if and only if

$$\int_\Omega (\Delta u) v \,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = 0 \qquad (v \in H^1(\Omega)). \tag{7.10}$$

It is remarkable that this condition makes sense for an arbitrary open set $\Omega \subseteq \mathbb{R}^n$. Therefore, for a function $u \in H^1(\Omega)$ with $\Delta u \in L_2(\Omega)$ we will write '$\partial_\nu u = 0$' (including the quotes!) if (7.10) holds.

7.4 The Neumann Laplacian

Let $\Omega \subseteq \mathbb{R}^n$ be open (not necessarily bounded). Using the notation '$\partial_\nu u = 0$', introduced in Section 7.3, we define the **Laplacian with Neumann boundary conditions** or

simply **Neumann Laplacian** Δ_{N} in $L_2(\Omega)$ by

$$\begin{aligned}\operatorname{dom}(\Delta_{\mathrm{N}}) &:= \big\{u \in H^1(\Omega);\, \Delta u \in L_2(\Omega),\ \text{`}\partial_\nu u = 0\text{'}\big\},\\ \Delta_{\mathrm{N}} u &:= \Delta u \qquad (u \in \operatorname{dom}(\Delta_{\mathrm{N}})).\end{aligned}$$

7.14 Theorem. *The negative Neumann Laplacian $-\Delta_{\mathrm{N}}$ is self-adjoint and accretive; it is associated with the classical Dirichlet form on $H^1(\Omega)$.*

Proof. Define $a\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{K}$ by $a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x$. Then a is bounded. We consider $H^1(\Omega)$ as a subspace of $L_2(\Omega)$. Since $a(u) + \|u\|^2_{L_2(\Omega)} = \|u\|^2_{H^1(\Omega)}$, the form a is $L_2(\Omega)$-coercive. Moreover, a is symmetric and accretive. Let $A \sim a$. We show that $A = -\Delta_{\mathrm{N}}$; then the assertions follow.

Let $u \in \operatorname{dom}(A)$, $Au = f$. Then by definition $u \in H^1(\Omega)$ and $\int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x = \int_\Omega f\overline{v}\,\mathrm{d}x$ for all $v \in H^1(\Omega)$. Inserting test functions $v \in C_{\mathrm{c}}^\infty(\Omega)$ one obtains $-\Delta u = f$. Thus $\int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x + \int_\Omega (\Delta u)\overline{v}\,\mathrm{d}x = 0$ for all $v \in H^1(\Omega)$, i.e. '$\partial_\nu u = 0$'. We have shown that $A \subseteq -\Delta_{\mathrm{N}}$. Conversely, if $u \in \operatorname{dom}(\Delta_{\mathrm{N}})$ and $-\Delta_{\mathrm{N}} u = f$, then

$$a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x + \int_\Omega (\Delta u)\overline{v}\,\mathrm{d}x + \int_\Omega f\overline{v}\,\mathrm{d}x = \int_\Omega f\overline{v}\,\mathrm{d}x \qquad (v \in H^1(\Omega)).$$

Thus $u \in \operatorname{dom}(A)$ and $Au = f$. □

Applying Theorem 7.12 one concludes that A has compact resolvent if Ω satisfies our weakest regularity property.

7.15 Theorem (Spectral decomposition of the Neumann Laplacian). *If $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with continuous boundary, then Δ_{N} has compact resolvent. There exist an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ of $L_2(\Omega)$ and an increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ in $[0,\infty)$, with $\lambda_1 = 0$ and $\lim_{k\to\infty} \lambda_k = \infty$, such that $-\Delta_{\mathrm{N}}$ is the associated diagonal operator. In particular, $\varphi_k \in \operatorname{dom}(\Delta_{\mathrm{N}})$ and*

$$-\Delta_{\mathrm{N}} \varphi_k = \lambda_k \varphi_k$$

for all $k \in \mathbb{N}$. If Ω is connected, then $\lambda_2 > 0$.

Proof. From Theorem 7.12 (Rellich–Kondrachov) we know that the embedding $j\colon H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact. Therefore Proposition 6.18 – in combination with Theorem 7.14 – implies that Δ_{N} has compact resolvent.

The statement concerning the eigenfunctions and eigenvalues now follows from Theorem 6.16, except for the properties of λ_1 and λ_2. However, it is immediate that $\varphi_1 = \operatorname{vol}_n(\Omega)^{-1/2}\mathbf{1}_\Omega$ is an eigenfunction of $-\Delta_{\mathrm{N}}$ with eigenvalue 0.

Now assume that Ω is connected. If $\varphi \in \operatorname{dom}(\Delta_{\mathrm{N}})$ satisfies $\Delta_{\mathrm{N}}\varphi = 0$, then $\|\nabla\varphi\|_2^2 = (-\Delta_{\mathrm{N}}\varphi \,|\, \varphi) = 0$; hence Lemma 7.16 – proved below – implies that φ is constant. This shows that the eigenspace belonging to the eigenvalue $\lambda_1 = 0$ is one-dimensional. □

The property that for connected Ω one has $\lambda_2 > 0$, together with the formula for the associated semigroup T from Remark 6.7, gives important information concerning the asymptotic behaviour of $T(t)$ as $t \to \infty$.

7.16 Lemma. *Let $\Omega \subseteq \mathbb{R}^n$ be a connected open set, and let $u \in L_{1,\mathrm{loc}}(\Omega)$, $\nabla u = 0$ in the distributional sense. Then u is constant.*

Proof. Let the function $\rho \in C_c^\infty(\mathbb{R}^n)$ have its support in a 'small' neighbourhood of 0. Then far enough away from $\partial\Omega$, the convolution $\rho * u$ is defined and satisfies $\nabla(\rho * u) = \rho * \nabla u = 0$, by Lemma 7.6(a), and therefore is locally constant. Convolving u with a δ-sequence in $C_c^\infty(\mathbb{R}^n)$ one infers that, locally on Ω, u is the limit of constant functions; see Lemma 7.6(b). Thus u has a locally constant representative, which is constant since Ω is connected. □

7.5 The Robin Laplacian

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1-boundary. Given $\beta \in L_\infty(\partial\Omega)$, we define the **Laplacian with Robin boundary conditions** or briefly **Robin Laplacian** Δ_β in $L_2(\Omega)$ by

$$\begin{aligned} \mathrm{dom}(\Delta_\beta) &:= \big\{u \in H^1(\Omega);\, \Delta u \in L_2(\Omega),\ \partial_\nu u + \beta u|_{\partial\Omega} = 0\big\}, \\ \Delta_\beta u &:= \Delta u \qquad (u \in \mathrm{dom}(\Delta_\beta)). \end{aligned}$$

Note that the condition '$\partial_\nu u + \beta u|_{\partial\Omega} = 0$' should be read as '$\partial_\nu u = -\beta \operatorname{tr} u$', in the sense of the definition in Section 7.3.

7.17 Theorem. *Let β be real-valued. Then the operator $-\Delta_\beta$ is self-adjoint and quasi-accretive, with compact resolvent. In particular, Δ_β generates a quasi-contractive C_0-semigroup T_β on $L_2(\Omega)$. If $\beta \geqslant 0$, then $-\Delta_\beta$ is accretive and $\|T_\beta(t)\| \leqslant 1$ for all $t \geqslant 0$.*

Proof. Define the form $a\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{K}$ by $a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x + \int_{\partial\Omega} \beta u \bar{v}\,\mathrm{d}\sigma$. Then $|a(u,v)| \leqslant \|\nabla u\|_2 \|\nabla v\|_2 + \|\beta\|_{L_\infty(\partial\Omega)} \|\operatorname{tr} u\|_{L_2(\partial\Omega)} \|\operatorname{tr} v\|_{L_2(\partial\Omega)}$. Since the trace operator is bounded, it follows that a is bounded.

We consider $H^1(\Omega)$ as a subspace of $L_2(\Omega)$ and show that a is $L_2(\Omega)$-coercive. By Theorem 7.10 and the 'Peter–Paul inequality' (i.e. Young's inequality, $ab \leqslant \frac{1}{2}\big(\gamma a^2 + \frac{1}{\gamma} b^2\big)$ for all $a, b \geqslant 0,\ \gamma > 0$) there exists $c > 0$ such that

$$\Big|\int_{\partial\Omega} \beta |u|^2\,\mathrm{d}\sigma\Big| \leqslant \|\beta\|_{L_\infty(\partial\Omega)} \int_{\partial\Omega} |u|^2\,\mathrm{d}\sigma \leqslant \frac{1}{2}\|u\|_{H^1(\Omega)}^2 + c\|u\|_{L_2(\Omega)}^2$$

for all $u \in H^1(\Omega)$. Therefore

$$a(u) \geqslant \int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{1}{2}\|u\|_{H^1(\Omega)}^2 - c\|u\|_{L_2(\Omega)}^2 = \frac{1}{2}\|u\|_{H^1(\Omega)}^2 - (1+c)\|u\|_{L_2(\Omega)}^2$$

for all $u \in H^1(\Omega)$. This proves that a is $L_2(\Omega)$-coercive.

Let A be the operator associated with a. We show that $A = -\Delta_\beta$. Let $(u, f) \in A$. Then $u \in H^1(\Omega)$ and

$$\int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x + \int_{\partial\Omega} \beta u \bar{v}\,\mathrm{d}\sigma = \int_\Omega f \bar{v}\,\mathrm{d}x \qquad (v \in H^1(\Omega)). \tag{7.11}$$

Taking $v \in C_c^\infty(\Omega)$ we see that $-\Delta u = f$. Replacing f by $-\Delta u$ in (7.11) we find

$$\int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x + \int_\Omega (\Delta u)\bar{v}\,\mathrm{d}x = -\int_{\partial\Omega} \beta u \bar{v}\,\mathrm{d}\sigma \qquad (v \in H^1(\Omega)), \tag{7.12}$$

i.e. $\partial_\nu u = -\beta u|_{\partial\Omega}$. Thus $(u, f) \in -\Delta_\beta$. Conversely, if $u \in \mathrm{dom}(\Delta_\beta)$, then (7.12) holds. Putting $f = -\Delta u$ we obtain (7.11) and thus $(u, f) \in A$.

Since a is symmetric and $L_2(\Omega)$-coercive, A is self-adjoint and quasi-accretive. By Theorem 7.12 (Rellich–Kondrachov) the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact, so the operator A has compact resolvent.

Finally, for $\beta \geqslant 0$ the form a is accretive, and this implies the last assertion of the theorem. □

Notes

In Section 7.1 we partially follow [ArUr23]. It is possible to extend Gauss' theorem to open sets with Lipschitz boundary, for which a suitable surface measure on $\partial\Omega$ is defined analogously. The theorem of Gauss is due to Lagrange in 1792 but has been rediscovered by Carl Friedrich Gauss in 1813, by George Green in 1825, and by Mikhail V. Ostrogradsky in 1831. For this reason one finds it in the literature under these different names. Obviously it can also be written as

$$\int_\Omega \mathrm{div}\, u\,\mathrm{d}x = \int_{\partial\Omega} u \cdot \nu\,\mathrm{d}\sigma,$$

for each vector field $u \in C^1(\overline{\Omega}; \mathbb{R}^n)$. In this form it is frequently called the **divergence theorem**. Physicists and engineers love this version of the theorem because of its immediate interpretation.

Most of the properties of $H^1(\Omega)$ presented in Section 7.2 can be found in the standard literature on Sobolev spaces. For another proof of Theorem 7.11 we refer to [Eva10; Section 5.5, Theorem 2]. For Theorem 7.12 we also refer to [EdEv87; Theorem 4.17].

Victor Gustave Robin (1855–1897) was a French mathematician. (The reader should correctly pronounce the nasal in the second syllable of "Robin".) He was teaching mathematical physics at the Sorbonne in Paris. Not much is known about him since he burnt his manuscripts. But he worked on thermodynamics, and the Russian school introduced the name Robin boundary conditions. In facct, these boundary conditions had already been introduced by Isaac Newton (1643–1727). We refer to [GuAb98b], [GuAb98a] for the interesting history of Gustave Robin and "his" boundary conditions. Neumann boundary conditions carry their name to honour Carl G. Neumann (1832–1925) who was professor at Halle, Basel, Tübingen and Leipzig. He introduced the Neumann series for matrices.

Using the tools presented in this chapter we can now continue the considerations started at the end of the Notes of Chapter 5. In his lectures in Berlin, Dirichlet introduced methods for solving partial differential equations via forms. The most famous is the Dirichlet problem, which is the following. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $g \in C(\partial\Omega)$. The classical **Dirichlet problem** consists in finding a function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ such

that u is harmonic (i.e. $\Delta u = 0$) on Ω, and $u|_{\partial\Omega} = g$. The term "Dirichlet problem" was coined by Bernhard Riemann, in honour of his teacher. And in fact Dirichlet promoted a strategy to its solution. The solution should be the function u with $u|_{\partial\Omega} = g$ having the least energy $\int_\Omega |\nabla u|^2 \,\mathrm{d}x$. On the basis of our knowledge obtained so far, we are able to make this precise, albeit in a slightly different context. Let us assume that Ω has C^1-boundary and that $g\colon \partial\Omega \to \mathbb{R}$ is the trace of some $G \in H^1(\Omega)$, i.e. $g = \operatorname{tr} G$. Then the following holds.

Dirichlet's principle. *There is a unique function $u \in H^1(\Omega)$ such that $\Delta u = 0$ and $\operatorname{tr} u = g$. Moreover, u is the unique minimiser of the function*

$$G + H^1_0(\Omega) = \{v \in H^1(\Omega)\,;\, \operatorname{tr} v = g\} \ni v \mapsto \int_\Omega |\nabla v|^2 \,\mathrm{d}x. \tag{7.13}$$

(See Theorem 7.11 for the equality in (7.13). Weyl's lemma, see Appendix C, implies that the solution u is in fact harmonic and even belongs to $C^\infty(\Omega)$.)

For the proof observe that for $u \in H^1(\Omega)$ one has $\Delta u = 0$ and $\operatorname{tr} u = g$ if and only if $w = G - u \in H^1_0(\Omega)$ and

$$\int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x = \int_\Omega \nabla G \cdot \nabla v \,\mathrm{d}x \qquad (v \in H^1_0(\Omega)), \tag{7.14}$$

and by the Fréchet–Riesz theorem there is a unique $w \in H^1_0(\Omega)$ satisfying (7.14). (Recall that, by Poincaré's inequality, the scalar product $(u \,|\, v)_0 := \int \nabla u \cdot \nabla v \,\mathrm{d}x$ is equivalent to the standard scalar product on $H^1_0(\Omega)$.) Moreover, by the 'variational method' formulated in the Notes of Chapter 5, w is the unique minimiser of the function

$$H^1_0(\Omega) \ni v \mapsto \int_\Omega |\nabla v|^2 \,\mathrm{d}x - 2\int_\Omega \nabla G \cdot \nabla v \,\mathrm{d}x = \int |\nabla(G - v)|^2 \,\mathrm{d}x - \int |\nabla G|^2 \,\mathrm{d}x.$$

This shows that $u = G - w \in G + H^1_0(\Omega)$ is the unique minimiser of (7.13).

In 1906, Hadamard [Had06] showed that even for 'nice' sets Ω there are cases in which the Dirichlet problem cannot be treated by the variational method even though a classical solution exists. More precisely, for $\Omega = B_{\mathbb{R}^2}(0,1)$ he constructed a function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfying $\Delta u = 0$ and $u \notin H^1(\Omega)$; then u is the unique solution corresponding to the boundary function $g = u|_{\partial\Omega} \in C(\partial\Omega)$ (see e.g. [ArUr23; Example 6.67]). This phenomenon does not occur for more regular boundary data g.

The classical Dirichlet problem was finally settled by Oskar Perron [Per23] in 1923. His treatment can be considered as the final point of a development in which Poincaré, Lebesgue, Courant, Lichtenstein, Zaremba and others had contributed. Existence of a solution requires some boundary regularity. In particular, if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, then the Dirichlet problem has a solution $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ for any $g \in C(\partial\Omega)$. We refer to [WKK09; Satz 3.3.10, Satz 3.4.3 and Bemerkung 3.4.4] for a proof of this result; see also [Joh82; Chapter 4, Section 4]. The relation between Perron's and the variational solutions is discussed in [ArDa08a]. For a thorough historical account of the Dirichlet problem and Dirichlet's principle in the 19th and early 20th centuries we refer to [Går79], [WKK09; Section 1.4].

A bounded open set Ω is called **Dirichlet regular** if for each $g \in C(\partial\Omega)$ there exists a (necessarily unique, see Exercise C.3) function $u \in C(\overline{\Omega})$ such that u is harmonic on Ω and $u|_{\partial\Omega} = g$. In this terminology, Perron's result implies that every bounded open set with Lipschitz-boundary is Dirichlet regular. Lebesgue's cusp provides an example of an open bounded set $\Omega \subseteq \mathbb{R}^3$ that has continuous boundary but is not Dirichlet regular (see [CoHi62; Chapter IV, §1.4], [ArDa08b; Section 7]).

Exercises

7.1 In this exercise we illustrate that the space $C^1(\overline{\Omega})$ does not only depend on $\overline{\Omega}$. Let $\Gamma \subseteq [0,1]$ denote Cantor's ternary set, and let $\Omega := (0,1) \setminus \Gamma$. Then $\overline{\Omega} = [0,1] = \overline{(0,1)}$. Let $u\colon [0,1] \to [0,1]$ be the "Cantor function", the increasing function generating the usual singular continuous measure on Γ; to wit,

$$\begin{aligned} &u(0) = 0, \qquad u = 1/2 \ \text{ on } \ (1/3, 2/3), \qquad u(1) = 1, \\ &u = 1/4 \ \text{ on } \ (1/9, 2/9), \qquad u = 3/4 \ \text{ on } \ (7/9, 8/9), \\ &u = 1/8 \ \text{ on } \ (1/27, 2/27), \qquad \text{etc.} \end{aligned}$$

and u extended to $[0,1]$ by continuity. Show that then $u \in C^1(\overline{\Omega}) \setminus C^1(\overline{(0,1)})$.

7.2 (a) Let M be a compact metric space, and let $g\colon M \to \mathbb{R}$ be such that the graph $\{(x, g(x));\, x \in M\}$ of g is compact. Show that g is continuous.

(b) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Complete the proof of the equivalence of the two properties

(i) Ω has continuous boundary;

(ii) for each $x \in \partial\Omega$ there exist a neighbourhood $W \subseteq \partial\Omega$ of x and a vector $y \in \mathbb{R}^n$ such that $W - (0,1)y \subseteq \Omega$ and $W + (0,1)y \subseteq \mathbb{R}^n \setminus \overline{\Omega}$.

(The implication '(i) $\Rightarrow$ (ii)' was shown in Lemma 7.5.)

Hint: Transform property (ii) to the 'standard situation' in which y is a positive multiple of the n-th unit vector e_n; show that then W is the graph of a function g and apply part (a). (The point is that the function g with graph W is not continuous a priori, but will be continuous automatically, by part (a).)

7.3 Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $u \in H^1(\Omega)$, $\tilde{u}$ the extension of u to $\mathbb{R}^n$ by zero.

(a) Show that $\operatorname{spt} \tilde{u} = \overline{\operatorname{spt} u}$.

(b) Define $W_u := \operatorname{spt} \tilde{u} \cap \partial\Omega$. Show that $\tilde{u} \in H^1(\mathbb{R}^n \setminus W_u)$, $\partial_j \tilde{u} = \widetilde{\partial_j u}$ on $\mathbb{R}^n \setminus W_u$ $(j = 1, \dots, n)$. (Hint: Use Exercise 4.3(b).)

7.4 (a) Let $1 \leqslant p < \infty$. For $z \in \mathbb{R}^n$ define $T_z \in \mathcal{L}(L_p(\mathbb{R}^n))$ by $T_z f := f(\cdot - z)$ $(f \in L_p(\mathbb{R}^n))$. Show that the function $\mathbb{R}^n \ni z \mapsto T_z \in \mathcal{L}(L_p(\mathbb{R}^n))$ is strongly continuous.

Hint: Show that $z \mapsto T_z\varphi \in L_p(\mathbb{R}^n)$ is continuous for all $\varphi \in C_c(\mathbb{R}^n)$, and use the denseness of $C_c(\mathbb{R}^n)$ in $L_p(\mathbb{R}^n)$.

(b) Let $u \in H^1(\mathbb{R}^n)$. Show that $u(\cdot - z) \to u$ in $H^1(\mathbb{R}^n)$ as $z \to 0$.

7.5 Let $\Omega \subseteq \mathbb{R}^n$ be a connected bounded open set with C^1-boundary. Let $0 < \beta \in L_\infty(\partial\Omega)$ (i.e. $0 \leqslant \beta \in L_\infty(\partial\Omega) \setminus \{0\}$). Denote by T_β the C_0-semigroup generated by the Robin Laplacian Δ_β. Show that

$$\|T_\beta(t)\| \leqslant \mathrm{e}^{-\varepsilon t} \qquad (t \geqslant 0)$$

for some $\varepsilon > 0$. (Hint: Use Lemma 7.16.)

7.6 Let $\Omega \subseteq \mathbb{R}^n$ be a C^1-domain with a hole, i.e., we assume that there exist bounded open sets $\widetilde{\Omega}$ and ω with C^1-boundary such that $\overline{\omega} \subseteq \widetilde{\Omega}$ and $\Omega = \widetilde{\Omega} \setminus \overline{\omega}$. Then we may write $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 := \partial\widetilde{\Omega}$ and $\Gamma_2 := \partial\omega$. Let $\beta \in L_\infty(\Gamma_2)$ be real-valued. Define the Laplacian with Robin boundary condition $\partial_\nu u + \beta u = 0$ on Γ_2 and Dirichlet boundary condition zero on Γ_1. Show that it is a self-adjoint operator. (It is part of the task to give a proper definition of the indicated boundary conditions and to find the 'translation' into a suitable form.)

7.7 Let $a\colon V \times V \to \mathbb{C}$ be a symmetric bounded form that is j-coercive, where V, H are complex Hilbert spaces and $j \in \mathcal{L}(V, H)$ has dense range. Let $b\colon V \times V \to \mathbb{C}$ be a bounded form and assume that there exists $c \geqslant 0$ such that

$$|b(u)| \leqslant c\|u\|_V \|j(u)\|_H \qquad (u \in V).$$

(a) Show that $a + b\colon V \times V \to \mathbb{C}$ is j-coercive.

(b) Denote by A the (quasi-m-sectorial) operator associated with $(a + b, j)$. Show that the numerical range $\mathrm{num}(A)$ is contained in the region 'surrounded' by a parabola with vertex on the real axis and opened in the direction of the positive real axis.

(c) Show that A is quasi-m-sectorial of any angle $\varphi < \pi/2$ and that $-A$ generates a holomorphic semigroup of angle $\pi/2$. (Hint: Look at the operator $A + \omega$, for arbitrarily large ω, and remember rescaled semigroups.)

7.8 Let $\mathbb{K} = \mathbb{C}$, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1-boundary. Let $\beta \in L_\infty(\partial\Omega)$ (not necessarily real-valued), and let Δ_β be the Robin Laplacian in $L_2(\partial\Omega)$.

(a) Show that $-\Delta_\beta$ has the properties described in parts (b) and (c) of Exercise 7.7.

(b) Suppose that $\mathrm{Re}\,\beta \geqslant 0$. Show that $-\Delta_\beta$ is accretive, and $\|T_\beta(t)\| \leqslant 1$ $(t \geqslant 0)$.

Chapter 8

The Dirichlet-to-Neumann operator

The Dirichlet-to-Neumann operator plays an important role in the theory of inverse problems. For instance, from measurements of electrical currents at the surface of the human body one wishes to determine conductivity inside the body. But the Dirichlet-to-Neumann operator also plays a big role in many other parts of analysis. Here we use form methods to show that it is a self-adjoint operator in $L_2(\partial\Omega)$. It becomes important that our setting allows the mapping j to be non-injective: throughout this chapter j will be the trace operator.

8.1 The Dirichlet-to-Neumann operator for the Laplacian

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1-boundary. We use the classical Dirichlet form

$$a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\, \mathrm{d}x \qquad (u, v \in H^1(\Omega)). \tag{8.1}$$

If we choose the canonical injection of $H^1(\Omega)$ into $L_2(\Omega)$, the associated operator is the Neumann Laplacian. Here we will choose as j the trace operator from $H^1(\Omega)$ to $L_2(\partial\Omega)$, introduced in Theorem 7.10; then j has dense range, as noted in Section 7.3. We will show that the form a is j-coercive; thus (a, j) is associated with a self-adjoint operator in $L_2(\partial\Omega)$. It turns out that this is the Dirichlet-to-Neumann operator D_0 which maps the 'Dirichlet data' $u|_{\partial\Omega} \in L_2(\partial\Omega)$ of a harmonic function $u \in H^1(\Omega)$ to the 'Neumann data' $\partial_\nu u \in L_2(\partial\Omega)$; see Theorem 8.3 below. (A function u on Ω is called **harmonic** if it is twice continuously differentiable and $\Delta u = 0$.)

For the proof of the j-coercivity of a we need an auxiliary result, which is a version of what sometimes comes under the heading "Ehrling's lemma". We state it for the general case of Banach spaces; in our context it will only be needed for Hilbert spaces.

8.1 Lemma. *Let X, Y, Z be Banach spaces, X reflexive, $K \in \mathcal{L}(X,Y)$ compact and $S \in \mathcal{L}(X,Z)$ injective. Then for all $\varepsilon > 0$ there exists $c_\varepsilon \geqslant 0$ such that*

$$\|Kx\|_Y \leqslant \varepsilon \|x\|_X + c_\varepsilon \|Sx\|_Z \qquad (x \in X).$$

Proof. For a contradiction, assume that there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $x_n \in X$ with

$$\|Kx_n\|_Y > \varepsilon \|x_n\|_X + n\|Sx_n\|_Z$$

and $\|x_n\|_X = 1$. Passing to a subsequence we may assume that $x_n \to x$ weakly. Then $Sx_n \to Sx$ weakly; see the hints in Exercise 6.2. The inequality implies that $\|Sx_n\|_Z \to 0$, hence $Sx = 0$; so $x = 0$ because S is injective. Since K is compact it follows that $Kx_n \to 0$ in norm; see Exercise 6.2(a). But $\|Kx_n\|_Y > \varepsilon$ for all $n \in \mathbb{N}$, a contradiction. □

8.2 Proposition. *With $j = \operatorname{tr}$, the classical Dirichlet form a is j-coercive.*

Proof. Define $S\colon H^1(\Omega) \to L_2(\Omega;\mathbb{K}^n) \oplus L_2(\partial\Omega)$ by

$$Su := (\nabla u, \operatorname{tr} u).$$

We show that S is injective. If $Su = 0$, then $\nabla u = 0$, and therefore Lemma 7.16 implies that u is constant on each connected component ω of Ω. This constant is zero because $\operatorname{tr} u = 0$; therefore $u = 0$.

As the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact, by Theorem 7.12 (Rellich–Kondrachov), the application of Lemma 8.1 yields a constant $c \geqslant 0$ such that for all $u \in H^1(\Omega)$ one has

$$\begin{aligned}\int_\Omega |u|^2 \,\mathrm{d}x &\leqslant \frac{1}{2}\|u\|^2_{H^1(\Omega)} + c\|Su\|^2 \\ &= \frac{1}{2}\int_\Omega |u|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + c\int_\Omega |\nabla u|^2 \,\mathrm{d}x + c\int_{\partial\Omega} |\operatorname{tr} u|^2 \,\mathrm{d}\sigma.\end{aligned}$$

Adding $\frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{2}\int_\Omega |u|^2 \,\mathrm{d}x$ to this inequality we obtain

$$\frac{1}{2}\|u\|^2_{H^1(\Omega)} \leqslant (c+1)a(u) + c\|\operatorname{tr} u\|^2_{L_2(\partial\Omega)} \qquad (u \in H^1(\Omega)), \tag{8.2}$$

and it follows that a is j-coercive. □

We now come back to the Dirichlet-to-Neumann operator.

8.3 Theorem. *Let j be the trace operator, and let a be the classical Dirichlet form* (8.1)*. Then the operator D_0 in $L_2(\partial\Omega)$ associated with (a, j) is described by*

$$D_0 = \big\{(g,h) \in L_2(\partial\Omega) \times L_2(\partial\Omega)\,;\, \exists u \in H^1(\Omega)\colon \Delta u = 0,\ u|_{\partial\Omega} = g,\ \partial_\nu u = h\big\}.$$

The operator D_0 is self-adjoint and accretive and has compact resolvent. We call D_0 the ***Dirichlet-to-Neumann operator*** *(with respect to Δ).*

The attentive reader may have noticed a discrepancy between the description of the Dirichlet-to-Neumann operater in the first paragraph of the present section, where it is stated that the Dirichlet data of a *harmonic* function u are mapped to the Neumann data, and the definition of D_0 given above, where u is merely required to satisfy $\Delta u = 0$ in the distributional sense. This discrepancy is resolved by Weyl's lemma, for which we refer to Appendix C.

The action of the operator D_0 can be described as follows. Given $g \in L_2(\partial\Omega)$ one seeks a solution $u_g \in H^1(\Omega)$ of the Dirichlet problem $\Delta u = 0$, $\operatorname{tr} u = g$. As mentioned in the Notes of Chapter 7, a solution u_g exists for all $g \in \operatorname{ran}(\operatorname{tr})$. If this solution satisfies $\partial_\nu u_g \in L_2(\partial\Omega)$, then $g \in \operatorname{dom}(D_0)$ and $D_0 g = \partial_\nu u_g$.

Proof of Theorem 8.3. Let $(g,h) \in D_0$. Since $D_0 \sim (a,j)$, there exists $u \in H^1(\Omega)$ such that $u|_{\partial\Omega} = j(u) = g$ and

$$\int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x = a(u,v) = (h \,|\, j(v)) = \int_{\partial\Omega} h\bar{v}\,\mathrm{d}\sigma \tag{8.3}$$

for all $v \in H^1(\Omega)$. Employing this equality with $v \in C_c^\infty(\Omega)$ we obtain $-\Delta u = 0$. Thus, adding $\int (\Delta u)\bar{v}\,\mathrm{d}x = 0$ in (8.3) we find that

$$\int_\Omega (\Delta u)\bar{v}\,\mathrm{d}x + \int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x = \int_{\partial\Omega} h\bar{v}\,\mathrm{d}\sigma \qquad (v \in H^1(\Omega)).$$

Hence $\partial_\nu u = h$ by our definition of the weak normal derivative. This shows "$\subseteq$" in the asserted equality for D_0.

In order to prove the converse inclusion let $u \in H^1(\Omega)$ satisfy $\Delta u = 0$ and $h := \partial_\nu u \in L_2(\partial\Omega)$, and put $g := u|_{\partial\Omega}$. Then

$$a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x + \int_\Omega (\Delta u)\bar{v}\,\mathrm{d}x = \int_{\partial\Omega} h\bar{v}\,\mathrm{d}\sigma = (h \,|\, j(v))_{L_2(\partial\Omega)} \qquad (v \in H^1(\Omega));$$

consequently $(g,h) = (j(u),h) \in D_0$.

The symmetry and accretivity of a imply that D_0 is self-adjoint and accretive. Finally, it was shown in Remark 7.13 that $\mathrm{tr}\colon H^1(\Omega) \to L_2(\partial\Omega)$ is compact; hence D_0 has compact resolvent, by Proposition 6.18. □

Our next aim is to define Dirichlet-to-Neumann operators with respect to more general Dirichlet problems. The following interlude contains preparatory material for this treatment.

8.2 Interlude: the Fredholm alternative in Hilbert space

We need a detail from the spectral theory of compact operators which we formulate and prove only for operators in Hilbert spaces. It will be used in the proof of Proposition 8.16.

8.4 Theorem (Fredholm alternative). *Let H be a Hilbert space, $K \in \mathcal{L}(H)$ compact. Then the operator $I + K$ is injective if and only if it is surjective, and in this case $I + K$ is invertible in $\mathcal{L}(H)$.*

Proof. (i) In this step we treat the case when $\dim \mathrm{ran}(K) < \infty$. We define $H_1 := \ker(K)$, $H_2 := \ker(K)^\perp$ and denote by P_1, P_2 the orthogonal projections onto H_1, H_2, respectively. Observe that $\dim H_2 < \infty$. (Indeed, from $K = \sum_{j=1}^n (\cdot \,|\, x_j)y_j$ one obtains $\ker(K) \supseteq \{x_1,\dots,x_n\}^\perp$, and therefore $\ker(K)^\perp \subseteq \mathrm{lin}\{x_1,\dots,x_n\}$.) On the orthogonal sum $H_1 \oplus H_2$ the operator $I + K$ can be written as the operator matrix

$$I + K = \begin{pmatrix} I_1 & P_1 K|_{H_2} \\ 0 & I_2 + P_2 K|_{H_2} \end{pmatrix},$$

where I_1, I_2 are the identity operators in H_1, H_2, respectively. It follows from Exercise 8.3 that $I + K$ is injective/surjective/invertible in $\mathcal{L}(H)$ if and only if the corresponding property holds for $I_2 + P_2 K|_{H_2}$. Thus the assertions follow from (finite-dimensional) linear algebra.

(ii) Now assume that $\operatorname{ran}(K)$ is not finite-dimensional. Note that $K(B_H[0,1])$ is a relatively compact set in a metric space, and as such is separable; therefore $\operatorname{ran}(K)$ is separable. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $\overline{\operatorname{ran}(K)}$, and let P_n be the orthogonal projection onto $\operatorname{lin}\{e_1, \dots, e_n\}$. Then $P_n x \to x$ $(n \to \infty)$ for all $x \in \overline{\operatorname{ran}(K)}$. Hence $P_n \to I$ uniformly on the compact set $\overline{K(B_H[0,1])}$, by Exercise 3.2(b), and this implies that $P_n K \to K$ in the operator norm.

Therefore K can be written as a sum $K = K_1 + K_2$, where $\|K_1\| < 1$ and $\operatorname{ran}(K_2)$ is finite-dimensional. Then $I + K_1$ is invertible in $\mathcal{L}(H)$ (Neumann series), and thus one can write $(I + K_1)^{-1}(I + K) = I + F$, with the finite rank operator $F = (I + K_1)^{-1} K_2$. Hence, $I + K$ is injective/surjective/invertible in $\mathcal{L}(H)$ if and only if the corresponding property holds for $I + F$. Thus the assertions follow from step (i). □

Theorem 8.4 carries the name 'Fredholm alternative' because the only *alternative* to $I + K$ being invertible is that $I + K$ is neither injective *nor* surjective.

8.5 Remark. Let G, H be Hilbert spaces, and let $B, K \in \mathcal{L}(G, H)$, B invertible, K compact. Then the operator $B + K$ is injective if and only if it is surjective. This follows from $B + K = B(I + B^{-1}K)$ and Theorem 8.4, applied to the compact operator $B^{-1}K \in \mathcal{L}(G)$. △

8.3 Quasi-m-accretive and self-adjoint operators via essentially coercive forms

Let V, H be Hilbert spaces, and let $a\colon V \times V \to \mathbb{K}$ be a bounded form. Let $j \in \mathcal{L}(V, H)$ have dense range. Throughout this section we assume that

$$u \in \ker(j),\ a(u, v) = 0 \text{ for all } v \in \ker(j) \ \text{ implies } \ u = 0. \tag{8.4}$$

This assumption is slightly weaker than (5.5).

As in Section 5.3 we define

$$A := \big\{(x, y) \in H \times H\,;\, \exists u \in V\colon j(u) = x,\ a(u, v) = (y \,|\, j(v))\ (v \in V)\big\}.$$

Given $\omega \in \mathbb{K}$, we denote by a_ω the shifted form defined by $a_\omega(u, v) := a(u, v) + \omega(j(u) \,|\, j(v))$.

8.6 Proposition. *Assume that* (8.4) *is satisfied. Then the relation A defined above is an operator. As before, we call A the operator associated with (a, j). For $\omega \in \mathbb{K}$, the form a_ω satisfies* (8.4)*, and $A + \omega$ is the operator associated with (a_ω, j). If* (8.4) *is also satisfied for a^*, and B is the operator associated with (a^*, j), then $B \subseteq A^*$ and $A \subseteq B^*$. In particular, if a is symmetric and* $\operatorname{dom}(A)$ *is dense, then A is symmetric.*

The proof is delegated to Exercise 8.4. We will be working with the space

$$V_j(a) := \big\{u \in V\,;\, a(u,v) = 0 \ (v \in \ker(j))\big\} = \bigcap_{v\in\ker(j)} \ker a(\cdot, v). \tag{8.5}$$

8.7 Remarks. (a) Condition (8.4) is equivalent to $V_j(a) \cap \ker(j) = \{0\}$.

(b) $V_j(a)$ is a closed subspace of V. One might think of $V_j(a)$ as the 'orthogonal complement' of $\ker(j)$ with respect to a; cf. Proposition 8.16.

(c) It is easy to see that $V_j(a_\omega) = V_j(a)$ for all $\omega \in \mathbb{K}$.

(d) If $(x, y) \in A$ and $u \in V$ are as in the definition of A, then $u \in V_j(a)$. Indeed, for all $v \in \ker(j)$ one obtains $a(u, v) = (y \,|\, j(v)) = 0$.

(e) Assume that the Lax–Milgram operator $\mathcal{A}\colon V \to V^*$ given by $\mathcal{A}u := a(u, \cdot)$ is surjective. Then A is surjective. Indeed, for all $y \in H$ one has $(y \,|\, j(\cdot)) \in V^*$, hence $(y \,|\, j(\cdot)) = \mathcal{A}u$ for some $u \in V$; then $y = Aj(u) \in \operatorname{ran}(A)$. △

8.8 Example. We determine the space $V_j(a)$ for the case of the Dirichlet-to-Neumann operator in $L_2(\partial\Omega)$ associated with (a, j), where $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with C^1-boundary, a the classical Dirichlet form on $V = H^1(\Omega)$, and $j = \operatorname{tr}\colon H^1(\Omega) \to L_2(\partial\Omega)$.

Then $\ker(j) = H^1_0(\Omega)$, by Theorem 7.11, and an element $u \in H^1(\Omega)$ satisfies

$$0 = a(u, v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \qquad (v \in \ker(j) = H^1_0(\Omega))$$

if and only if $\Delta u = 0$, by Lemma 4.19. Hence $V_j(a) = \{u \in H^1(\Omega)\,;\, \Delta u = 0\}$. △

In the next lemma we draw an important conclusion from parts (d) and (e) of Remarks 8.7. Afterwards we will indicate a condition under which the assumptions of the lemma are satisfied.

8.9 Lemma. *In addition to* (8.4) *assume that the restriction of a to $V_j(a)$ is coercive and that $\mathcal{A}$ is surjective. Then A is strictly m-accretive. If $\mathbb{K} = \mathbb{C}$, then A is m-sectorial.*

Proof. For $x \in \operatorname{dom}(A)$ and $u \in V$ with $j(u) = x$ as in the definition of A one has $u \in V_j(a)$ by Remark 8.7(d). Thus by the coercivity assumption one obtains

$$\operatorname{Re}(Ax \,|\, x) = \operatorname{Re}(Ax \,|\, j(u)) = \operatorname{Re} a(u, u) \geqslant \alpha\|u\|_V^2 \geqslant \frac{\alpha}{c^2}\|x\|_H^2\,,$$

with $c > 0$ such that $c \geqslant \|j\|$ and some $\alpha > 0$ (not depending on x). This shows that A is strictly accretive. (The previous argument is essentially the same as in the proof of Theorem 5.6.) Now the surjectivity of A – see Remark 8.7(e) – implies that A is strictly m-accretive.

Now let $\mathbb{K} = \mathbb{C}$. As seen above one has

$$\{(Ax \,|\, x)\,;\, x \in \operatorname{dom}(A)\} \subseteq \{a(u)\,;\, u \in V_j(a)\}.$$

Since $\check{a} := a|_{V_j(a)\times V_j(a)}$ is coercive, Theorem 5.8 shows that $\check{a}$ is sectorial, so the above inclusion implies that A is sectorial. Then from the m-accretivity of A one concludes that A is m-sectorial. □

We call the form a **essentially coercive** if there exist a Hilbert space $\widetilde{H}$ and a compact operator $\tilde{\jmath}\colon V \to \widetilde{H}$ such that a is $\tilde{\jmath}$-coercive, i.e.

$$\operatorname{Re} a(u) + \tilde{\omega}\|\tilde{\jmath}(u)\|_{\widetilde{H}}^2 \geqslant \tilde{\alpha}\|u\|_V^2 \qquad (u \in V), \tag{8.6}$$

with some $\tilde{\omega}, \tilde{\alpha} > 0$. Note that the operator $\mathcal{K} \in \mathcal{L}(V, V^*)$, defined by

$$\langle \mathcal{K}u, v\rangle := (\tilde{\jmath}(u) \,|\, \tilde{\jmath}(v))_{\widetilde{H}} \qquad (u, v \in V), \tag{8.7}$$

is compact. Indeed, the operator

$$k\colon \widetilde{H} \to V^*,\ x \mapsto (x \,|\, \tilde{\jmath}(\cdot))_{\widetilde{H}}$$

belongs to $\mathcal{L}(\widetilde{H}, V^*)$; hence $\mathcal{K} = k \circ \tilde{\jmath}$ is compact, by the ideal property of compact operators.

The property 'essentially coercive' has been introduced in [AEKS14] under the name 'compactly elliptic'. Our terminology is adapted from [ArCh20], where an equivalent property is called 'essentially positive-coercive'; see [ArCh20; Theorem 4.4]. (The background for 'positive' is that in this paper more general versions of coercivity are under consideration.) We refer to Exercise 8.5 for further equivalent properties.

Theorem 8.11 below states that for essentially coercive a the operator associated with (a, j) is quasi-m-accretive. To prove this result we will use Lemma 8.9 and the following two properties of essentially coercive forms.

8.10 Lemma. *Assume that* (8.4) *is satisfied and that a is essentially coercive.*

(a) *Then there exist $\omega \geqslant 0$, $\alpha > 0$ such that*

$$\operatorname{Re} a(u) + \omega\|j(u)\|_H^2 \geqslant \alpha\|u\|_V^2 \qquad (u \in V_j(a)). \tag{8.8}$$

(This means that the restriction of a_ω to $V_j(a)$ is coercive).

(b) *With ω from* (a)*, the operator $\mathcal{A}_\omega\colon V \to V^*$ defined by $\mathcal{A}_\omega u := a_\omega(u, \cdot)$ is an isomorphism.*

Proof. (a) By Remark 8.7(a), j is injective on $V_j(a)$. Therefore Lemma 8.1 implies

$$\tilde{\omega}\|\tilde{\jmath}(u)\|_{\widetilde{H}}^2 \leqslant \frac{\tilde{\alpha}}{2}\|u\|_V^2 + \omega\|j(u)\|_H^2 \qquad (u \in V_j(a)),$$

with $\tilde{\alpha} > 0$ from (8.6) and some $\omega \geqslant 0$. From (8.6) we then obtain

$$\operatorname{Re} a(u) + \omega\|j(u)\|_H^2 \geqslant \frac{\tilde{\alpha}}{2}\|u\|_V^2 \qquad (u \in V_j(a)).$$

(b) If $\mathcal{A}_\omega u = 0$, then $a(u, v) = \langle \mathcal{A}_\omega u, v\rangle - \omega(j(u) \,|\, j(v)) = 0$ for all $v \in \ker(j)$, i.e. $u \in V_j(a)$; hence $\alpha\|u\|_V^2 \leqslant \operatorname{Re}\langle \mathcal{A}_\omega u, u\rangle = 0$ by (8.8). Thus $\mathcal{A}_\omega$ is injective.

We define a bounded form $b_\omega\colon V \times V \to \mathbb{K}$ by

$$b_\omega(u, v) := a_\omega(u, v) + \tilde{\omega}(\tilde{\jmath}(u) \,|\, \tilde{\jmath}(v))_{\widetilde{H}}\,;$$

then b_ω is coercive by (8.6) since $\omega \geqslant 0$. The Lax–Milgram lemma, Theorem 5.4, implies that the operator $\mathcal{B}_\omega \colon V \to V^*$, $u \mapsto b_\omega(u, \cdot)$ is an isomorphism. Moreover $\mathcal{A}_\omega = \mathcal{B}_\omega - \tilde{\omega}\mathcal{K}$, with the compact operator $\mathcal{K} \in \mathcal{L}(V, V^*)$ from (8.7). As $\mathcal{A}_\omega$ is injective, the 'Fredholm alternative' (in the guise of Remark 8.5) shows that $\mathcal{A}_\omega$ is an isomorphism. □

We now show that the operator associated with (a, j) is quasi-m-accretive.

8.11 Theorem. *Let $a \colon V \times V \to \mathbb{K}$ be a bounded form, and let $j \in \mathcal{L}(V, H)$ have dense range. Assume that (8.4) is satisfied and that a is essentially coercive. Let A be the operator associated with (a, j).*

(a) *Then A is quasi-m-accretive.*

(b) *If $\mathbb{K} = \mathbb{C}$, then A is quasi-m-sectorial.*

(c) *If a is symmetric, then A is self-adjoint.*

(d) *If j is compact, then A has compact resolvent.*

Proof. (a) By Lemma 8.10 there exists $\omega \geqslant 0$ such that $a_\omega|_{V_j(a) \times V_j(a)}$ is coercive and $\mathcal{A}_\omega$ is surjective. Recall from Proposition 8.6 that a_ω is associated with the operator $A + \omega$. Now Lemma 8.9 shows that $A + \omega$ is strictly m-accretive, hence A is quasi-m-accretive.

(b) In this case Lemma 8.9 implies that $A + \omega$ is m-sectorial, hence A is quasi-m-sectorial.

(c) If a is symmetric, then A is symmetric, by Proposition 8.6. Since A is quasi-m-accretive, Theorem 6.1 implies that A is self-adjoint.

(d) We show that $(A + \omega)^{-1} = j\mathcal{A}_\omega^{-1}k$, with $k \colon H \to V^*$, $y \mapsto (y \,|\, j(\cdot))$; then A has compact resolvent. By definition, $(y, x) \in (A + \omega)^{-1}$ if and only if there exists $u \in V$ such that $j(u) = x$ and $\mathcal{A}_\omega u = (y \,|\, j(\cdot)) = k(y)$, and this is equivalent to $x = j\mathcal{A}_\omega^{-1}k(y)$. (The previous argument is essentially the same as in the proof of Proposition 5.7; see also the proof of Proposition 6.18.) □

8.4 The Dirichlet-to-Neumann operator with respect to $\Delta + m$

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1-boundary, and let $m \in L_\infty(\Omega)$ be real-valued. We want to study the Dirichlet-to-Neumann operator D_m with respect to '$(\Delta + m)$-harmonic' functions. More precisely, we define D_m in $L_2(\partial\Omega)$ by requiring that for $g, h \in L_2(\partial\Omega)$ one has $g \in \mathrm{dom}(D_m)$ and $D_m g = h$ if there exists a solution $u \in H^1(\Omega)$ of $\Delta u + mu = 0$, $u|_{\partial\Omega} = g$ such that $\partial_\nu u = h$. We will show that D_m is a self-adjoint operator if

$$0 \notin \sigma(\Delta_{\mathrm{D}} + m), \tag{8.9}$$

which we will suppose throughout. Here $\Delta_{\mathrm{D}} + m$ is the Dirichlet Laplacian perturbed by the bounded operator of multiplication by the function m. We observe that $\Delta_{\mathrm{D}} + m$ is self-adjoint and has compact resolvent; see Exercise 8.6.

As in Section 8.1 we choose $H = L_2(\partial\Omega)$, $V = H^1(\Omega)$ and as $j \colon H^1(\Omega) \to L_2(\partial\Omega)$ the trace operator. According to the intended setup we now define the form $a \colon V \times V \to \mathbb{C}$ by

$$a(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} \, \mathrm{d}x - \int_\Omega mu\bar{v} \, \mathrm{d}x \qquad (u, v \in H^1(\Omega)).$$

8.12 Remark. In general, the form a is not j-coercive. For example, let $\lambda > \lambda_1$, where λ_1 is the first Dirichlet eigenvalue, and put $m := \lambda$. Let $u \in H_0^1(\Omega)$ be an eigenfunction of $-\Delta_{\mathrm{D}}$ corresponding to λ_1. Then $\int_\Omega |\nabla u|^2 \,\mathrm{d}x = \lambda_1 \int_\Omega |u|^2 \,\mathrm{d}x$, so

$$a(u) + \omega \|j(u)\|_{L_2(\partial\Omega)}^2 = \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \lambda \int_\Omega |u|^2 \,\mathrm{d}x = (\lambda_1 - \lambda) \int_\Omega |u|^2 \,\mathrm{d}x < 0$$

for all $\omega \geqslant 0$. Thus the form is not j-coercive. (Note that (8.9) is satisfied if λ is not an eigenvalue of Δ_{D}.) $\triangle$

Since the form a is not necessarily j-coercive, the theory developed in Chapter 5 is not applicable. However, we can apply the results of the previous section.

8.13 Theorem. *Suppose that* (8.9) *holds. Then*

$$D_m := \big\{(g,h) \in L_2(\partial\Omega) \times L_2(\partial\Omega);\, \exists u \in H^1(\Omega) \colon \Delta u + mu = 0,\ u|_{\partial\Omega} = g,\ \partial_\nu u = h\big\}$$

is a quasi-accretive self-adjoint operator with compact resolvent.

Proof. Let a be the form defined above, and let $j \in \mathcal{L}(H^1(\Omega), L_2(\partial\Omega))$ be the trace operator. We first show that condition (8.4) is satisfied. We recall from Theorem 7.11 that $\ker(j) = H_0^1(\Omega)$. Let $u \in \ker(j)$ be such that $a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v} \,\mathrm{d}x - \int_\Omega mu\bar{v} \,\mathrm{d}x = 0$ for all $v \in \ker(j) = H_0^1(\Omega)$. Then $u \in \mathrm{dom}(\Delta_{\mathrm{D}} + m)$ and $\Delta_{\mathrm{D}} u + mu = 0$, by the definition of Δ_{D}. This implies $u = 0$ since $0 \notin \sigma(\Delta_{\mathrm{D}} + m)$ by our assumption.

In order to show that a is essentially coercive, we choose $\widetilde{H} := L_2(\Omega)$ and as $\tilde{\jmath}$ the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$. Then

$$a(u) + (\|m\|_\infty + 1)\|\tilde{\jmath}(u)\|_2^2 = \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega m|u|^2 \,\mathrm{d}x + (\|m\|_\infty + 1)\|u\|_2^2 \geqslant \|u\|_{H^1}^2$$

for all $u \in H^1(\Omega)$, and from Theorem 7.12 we know that $\tilde{\jmath}$ is compact.

Let A be the operator associated with (a, j). By Theorem 8.11, parts (a) and (c), A is quasi-accretive and self-adjoint. As j is compact, by Remark 7.13, Theorem 8.11(d) implies that A has compact resolvent.

We show that $A = D_m$. If $(g,h) \in A$, then there exists $u \in H^1(\Omega)$ such that $u|_{\partial\Omega} = g$ and

$$\int_\Omega \nabla u \cdot \overline{\nabla v} \,\mathrm{d}x - \int_\Omega mu\bar{v} \,\mathrm{d}x = \int_{\partial\Omega} h\bar{v} \,\mathrm{d}\sigma \qquad (v \in H^1(\Omega)). \tag{8.10}$$

Inserting test functions $v \in C_{\mathrm{c}}^\infty(\Omega)$ we obtain $-\Delta u - mu = 0$. Plugging $mu = -\Delta u$ into (8.10) we deduce that $\partial_\nu u = h$. Thus $(g,h) \in D_m$. Conversely, if $(g,h) \in D_m$, then there exists $u \in H^1(\Omega)$ such that $\Delta u + mu = 0$, $u|_{\partial\Omega} = g$ and $\partial_\nu u = h$. Thus, by the definition of the weak normal derivative,

$$\int_{\partial\Omega} h\bar{v} \,\mathrm{d}\sigma = \int_\Omega \nabla u \cdot \overline{\nabla v} \,\mathrm{d}x + \int_\Omega (\Delta u)\bar{v} \,\mathrm{d}x = \int_\Omega \nabla u \cdot \overline{\nabla v} \,\mathrm{d}x - \int_\Omega mu\bar{v} \,\mathrm{d}x = a(u,v)$$

for all $v \in H^1(\Omega)$, hence $(g,h) \in A$. $\square$

8.5 Decomposition of the form domain

In this section we continue the analysis started in Section 8.3. Throughout the section let V, H be Hilbert spaces, and let $j \in \mathcal{L}(V,H)$ have dense range. Let $a\colon V \times V \to \mathbb{K}$ be a bounded essentially coercive form satisfying (8.4).

Note that a coercive form is automatically essentially coercive (and satisfies (8.4)). Thus, the following results are also valid – and interesting! – for coercive forms. It will become apparent that the properties treated in this section are invariant under shifting the form, and therefore they hold for bounded j-coercive forms as well.

In the applications in this chapter the space $V_j(a)$ is only a 'small' subspace of V because j is not injective; see for instance Example 8.8. We now show that the associated operator does not change if one restricts both a and j to $V_j(a)$.

8.14 Proposition. *Let $V_j(a)$ be defined by* (8.5)*. Then $j|_{V_j(a)}$ is injective and has dense range, and the operator A associated with (a,j) is also associated with $(a|_{V_j(a)\times V_j(a)}, j|_{V_j(a)})$.*

Proof. The injectivity of $j|_{V_j(a)}$ is is clear from Remark 8.7(a). Let $\check{a} := a|_{V_j(a)\times V_j(a)}$ and $\check{\jmath} := j|_{V_j(a)}$. From Remark 8.7(d) we obtain $j(V_j(a)) \supseteq \mathrm{dom}(A)$, and as A is densely defined, we conclude that $\check{\jmath}$ has dense range. Lemma 8.10(a) shows that the form $\check{a}$ is $\check{\jmath}$-coercive; hence the operator $\check{A}$ associated with $(\check{a},\check{\jmath})$ is quasi-m-accretive, by the results of Chapter 5.

Let $(x,y) \in A$. Then there exists $u \in V_j(a)$ such that $a(u,v) = (y \,|\, j(v))$ for all $v \in V$, hence a fortiori for all $v \in V_j(a)$, and this shows that $(x,y) \in \check{A}$. Thus we have shown $A \subseteq \check{A}$. As both operators A and $\check{A}$ are quasi-m-accretive, we conclude equality (by the well-known reasoning that a surjective mapping cannot have a proper injective extension). □

8.15 Remark. Proposition 8.14 implies that the operator A associated with (a,j) can also be obtained by an embedded form (i.e. by a form $(\check{a},\check{\jmath})$ for which $\check{\jmath}$ is injective). In the next result we show that there is even more structure. △

Let X be a normed space, and let $X = X_1 \oplus X_2$ be the algebraic direct sum of two subspaces X_1, X_2, i.e. $X = X_1 + X_2$ and $X_1 \cap X_2 = \{0\}$. If the norm $\|\cdot\|$ on X is equivalent to the norm $\|\cdot\|_1$ defined by

$$\|x_1 + x_2\|_1 := \|x_1\| + \|x_2\| \qquad (x_1 \in X_1,\ x_2 \in X_2), \tag{8.11}$$

then X is called the **topological direct sum** of X_1, X_2. The equivalence of the norms is equivalent to the property that the projection from X onto X_1 along X_2 is a bounded operator; see Exercise 8.7(a).

We emphasise that in the present section we use the symbol '$\oplus$' to denote the *algebraic* direct sum of vector spaces. This is in contrast to the remainder of the book, where it is always used for the *orthogonal* direct sum of Hilbert spaces.

8.16 Proposition. *Assume that* (8.4) *is satisfied and that a is essentially coercive. Then $V = V_j(a) \oplus \ker(j)$ is a topological direct sum.*

In the proof we will apply the following lemma which has a more abstract flavour.

8.17 Lemma. *Let $V_1, V_2 \subseteq V$ be closed subspaces, and let $V_1^\circ := \{\eta \in V^*;\ \eta|_{V_1} = 0\}$. Assume that $T\colon V_1^\circ \to V_2^*$, $\eta \mapsto \eta|_{V_2}$ is an isomorphism. Then $V = V_1 \oplus V_2$ is a topological direct sum.*

Proof. Let $x \in V_1$, $y \in V_2$. Since $(y \,|\, \cdot) \in V_2^*$, there exists $\eta \in V_1^\circ$ such that $\eta|_{V_2} = (y \,|\, \cdot)$ and $\|\eta\| \leqslant c\|y\|$, where c is the norm of the inverse of the isomorphism T. Then $\|y\|^2 = \eta(y) = \eta(x+y) \leqslant c\|y\|\|x+y\|$ and hence $\|y\| \leqslant c\|x+y\|$. This inequality shows that $V_1 \cap V_2 = \{0\}$ and that the projection from $V_1 + V_2$ onto V_2 along V_1 is a bounded operator. Therefore, $V_1 + V_2$ is the topological direct sum of V_1, V_2 and in particular a closed subspace of V; see Exercise 8.7(b).

Now let $\eta \in V^*$, $\eta|_{V_1+V_2} = 0$. Then $\eta \in V_1^\circ$ and $T\eta = \eta|_{V_2} = 0$; thus $\eta = 0$. This shows that $V_1 + V_2$ is dense in V, and hence $V = V_1 \oplus V_2$. □

Proof of Proposition 8.16. Let $\mathcal{A}_\omega$ be defined as in Lemma 8.10(b). Recall from Remark 8.7(c) that $V_j(a_\omega) = V_j(a)$; hence

$$V_j(a) = \{u \in V;\ \langle \mathcal{A}_\omega u, v\rangle = a_\omega(u,v) = 0 \ (v \in \ker(j))\} = \big\{u \in V;\ \mathcal{A}_\omega u \in \ker(j)^\circ\big\}.$$

Since $\mathcal{A}_\omega$ is an isomorphism, by Lemma 8.10(b), it follows that $\mathcal{A}_\omega|_{V_j(a)}\colon V_j(a) \to \ker(j)^\circ$ is an isomorphism.

The form $a_\omega|_{V_j(a)\times V_j(a)}$ is coercive, by Lemma 8.10(a). Thus, the associated Lax–Milgram operator $\check{\mathcal{A}}_\omega\colon V_j(a) \to V_j(a)^*$ is an isomorphism. Note that $\check{\mathcal{A}}_\omega$ is given by $\check{\mathcal{A}}_\omega u = a_\omega(u,\cdot)|_{V_j(a)} = T\mathcal{A}_\omega u$ for all $u \in V_j(a)$, with $T\colon \ker(j)^\circ \to V_j(a)^*$, $\eta \mapsto \eta|_{V_j(a)}$. Therefore $\check{\mathcal{A}}_\omega = T(\mathcal{A}_\omega|_{V_j(a)})$. Since $\check{\mathcal{A}}_\omega$ and $\mathcal{A}_\omega|_{V_j(a)}\colon V_j(a) \to \ker(j)^\circ$ are isomorphisms, we infer that T is an isomorphism, and the assertion follows from Lemma 8.17. □

8.18 Example. We come back to the Dirichlet-to-Neumann operator treated in Example 8.8 and give another explanation for the decomposition $H^1(\Omega) = V_j(a) \oplus H_0^1(\Omega)$ in this case.

From inequality (8.2) in the proof of Proposition 8.2 we obtain the equivalence of the norms belonging to the scalar products $(\cdot \,|\, \cdot)_{H^1(\Omega)}$ and

$$(u \,|\, v)_\sim := \int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x + \int_{\partial\Omega} \operatorname{tr} u\, \overline{\operatorname{tr} v}\,\mathrm{d}\sigma \qquad (u, v \in H^1(\Omega)).$$

For $u \in H^1(\Omega)$, $v \in H_0^1(\Omega)$ one obviously has

$$(u \,|\, v)_\sim = \int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x + 0 = a(u,v).$$

Therefore $V_j(a)$ is the orthogonal complement of $H_0^1(\Omega)$ with respect to $(\cdot \,|\, \cdot)_\sim$; this confirms that $(H^1(\Omega), \|\cdot\|_{H^1(\Omega)})$ is the topological direct sum of $V_j(a)$ and $H_0^1(\Omega)$. In Exercise 8.8 the reader is asked to show that the decomposition $H^1(\Omega) = V_j(a) \oplus H_0^1(\Omega)$ is not orthogonal in $H^1(\Omega)$.

Finally, let $P_1\colon H^1(\Omega) \to V_j(a)$ be the projection corresponding to the decomposition $H^1(\Omega) = V_j(a) \oplus H_0^1(\Omega)$, and recall that $V_j(a) = \{u \in H^1(\Omega);\ \Delta u = 0\}$. Then, given $u \in H^1(\Omega)$, the function $u_1 := P_1 u$ satisfies $\Delta u_1 = 0$ and $\operatorname{tr} u_1 = \operatorname{tr} u$, i.e., u_1 is the solution of the Dirichlet problem mentioned in the Notes of Chapter 7. △

Notes

A large part of the material in this chapter is adapted from [AEKS14]. The main result of Section 8.3, Theorem 8.11, goes beyond this paper and is due to H. Vogt. A different proof of Theorem 8.11 can be given by means of results in [Sau13].

In the investigation of the Dirichlet-to-Neumann operator D_m it is possible to avoid assumption (8.9). However, if $0 \in \sigma(\Delta_{\mathrm{D}} + m)$, then D_m is no longer a self-adjoint operator but rather a self-adjoint linear relation; we refer to Section 14.1 for this notion. In the complex case, the resolvent $(\mathrm{i}s - D_m)^{-1}$ is a bounded operator on $L_2(\partial\Omega)$, and the mapping $L_\infty(\Omega) \ni m \mapsto (\mathrm{i}s - D_m)^{-1} \in \mathcal{L}(L_2(\partial\Omega))$ is continuous; see [AEKS14; Theorem 7.3]. This gives valuable information on the stability of the inverse problem which interests engineers and medical doctors likewise.

In fact, one of the most famous inverse problems, the Calderón problem, can be formulated in terms of the Dirichlet-to-Neumann operator with respect to $\Delta - m$. It was Alberto Calderón (1920-1998) who encountered this problem when he worked as an electrical engineer for Yacimientos Petrolíferos Fiscales, the state oil company of Argentina. Calderón worked on *Electrical Impedance Tomography*, a method which can be used to detect oil, but also to find a tumor in the lungs. It consists in determining the electrical conductivity of a medium by making voltage and current measurements at the boundary of a medium. The mathematical problem Calderón formulates is the following.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and let $\gamma\colon \Omega \to (0, \infty)$ be a bounded measurable function with strictly positive lower bound. One should think of Ω as a body with 'electrical conductivity' γ. If one applies a 'voltage' function $g\colon \partial\Omega \to \mathbb{R}$ at the boundary, then the induced 'electric potential' $u\colon \Omega \to \mathbb{R}$ satisfies the Dirichlet problem

$$\begin{aligned} \operatorname{div}(\gamma \nabla u) &= 0 \qquad \text{on } \Omega, \\ u &= g \qquad \text{on } \partial\Omega, \end{aligned}$$

and one measures the 'current' $\gamma \partial_\nu u$ through the surface. The voltage-to-current map $g \mapsto \gamma \partial_\nu u$ is just the Dirichlet-to-Neumann operator Λ_γ,

$$\Lambda_\gamma g = \gamma \partial_\nu u.$$

In other words, Λ_γ maps the 'Dirichlet data' $u|_{\partial\Omega}$ of the 'div$(\gamma\nabla)$-harmonic' function u to the 'Neumann data' $\gamma\partial_\nu u$.

The inverse problem Calderón formulates is "to decide whether γ is uniquely determined by Λ_γ and to calculate γ in terms of Λ_γ, if γ is indeed determined by Λ_γ"; see [Cal06] for a reprint of his paper from 1980. (Calderón does not formulate the problem in terms of the Dirichlet-to-Neumann operator, but rather in terms of a quadratic form; we refer to [Uhl09; p. 2] for the equivalence of Calderón's and the above formulation in terms of Λ_γ.)

We add a few comments on the notation used above. For the definition of the distributional divergence of a vector field we refer to Exercise 4.5(c) and to the paragraph preceding Remark 11.3. The weak normal derivative $\gamma\partial_\nu u$, or expressed differently, the weak normal trace of the vector field $\gamma\nabla u$ at $\partial\Omega$, is defined similarly as in Section 7.3.

In the remaining discussion we will only deal with the first of the two issues Calderón states, the uniqueness question. Assuming suitable regularity of γ, one can transform this

problem into a uniqueness problem for the Dirichlet-to-Neumann operator D_m treated in Section 8.4, where m is related to the conductivity γ in a one-to-one way, as described in [Uhl09; Section 5]. Let $m_1, m_2 \in L_\infty(\Omega)$ and consider the associated Dirichlet-to-Neumann operators

$$D_{m_j} := \big\{(g,h) \in L_2(\partial\Omega) \times L_2(\partial\Omega)\,;\, \exists u \in H^1(\Omega)\colon \Delta u + m_j u = 0,\ u|_{\partial\Omega} = g,\ \partial_\nu u = h\big\},$$

for $j = 1, 2$; see Theorem 8.13. Then, analogously to the above Calderón uniqueness problem, one can ask: if $D_{m_1} = D_{m_2}$, does it follow that $m_1 = m_2$?

In this latter form, the problem has been solved in [KrUh14; Theorem 1.1], where the uniqueness is proved in a very general context. The original uniqueness problem of Calderón has been solved by Sylvester and Uhlmann [SyUh87; Theorem 0.1], for Ω with smooth boundary and conductivities γ in $C^\infty(\overline{\Omega})$, and by Haberman and Tataru [HaTa13; Theorem 1.1], for Ω with Lipschitz boundary and conductivities in $C^1(\overline{\Omega})$. For more information we refer to [Uhl09] from which this account is adapted and where also further references can be found. We mention that in real-world applications, partial measurements at the boundary are of course more realistic, and much research goes on in this direction; see [KeSa14].

Exercises

8.1 Let $a, b \in \mathbb{R}$, $a < b$.

(a) Compute the Dirichlet-to-Neumann operator D_0 for $\Omega = (a, b)$, and compute the C_0-semigroup generated by $-D_0$.

(b) For $a = -1$, $b = 1$, interpret the result in the light of Exercise 8.2.

8.2 Let $U_n := B_{\mathbb{R}^n}(0, 1)$ be the open unit ball in $\mathbb{R}^n$, $S_{n-1} := \partial U_n$ the unit sphere. The following facts can be used for the solution of this exercise: for each $\varphi \in C(S_{n-1})$ there exists a unique solution $u \in C(\overline{U_n})$ of the Dirichlet problem,

$$u|_{S_{n-1}} = \varphi, \quad u|_{U_n} \text{ harmonic.}$$

(We mention that the solution can be written down explicitly with the aid of the Poisson kernel, but this will not be needed for solving the exercise.) The solution satisfies $u|_{U_n} \in C^\infty(U_n)$ and $\|u\|_\infty \leqslant \|\varphi\|_\infty$. Writing $G\varphi := u$ one obtains $G \in \mathcal{L}(C(S_{n-1}), C(\overline{U_n}))$.

Define $T(t) \in \mathcal{L}(C(S_{n-1}))$ by

$$T(t)\varphi(z) := u(\mathrm{e}^{-t}z) \qquad (z \in S_{n-1},\ t \geqslant 0)$$

(with φ and $u = G\varphi$ as above).

(a) Show that T is a C_0-semigroup of contractions on $C(S_{n-1})$.

(b) Let A be the generator of T. Show that $D := \bigcup_{t>0} \operatorname{ran}(T(t))$ is a core for A, and that $A\varphi = -\partial_\nu(G\varphi)$ for all $\varphi \in D$.

(c) Put $A_{\min} := A|_D$. Show that $-A_{\min}$ is a restriction of the Dirichlet-to-Neumann operator D_0 in $L_2(S_{n-1})$, and that $D_0 = \overline{-A_{\min}}$ (where $A_{\min}$ is interpreted as an operator in $L_2(S_{n-1})$). Conclude that T extends to a C_0-semigroup T_2 of contractions on $L_2(S_{n-1})$, and that $-D_0$ is the generator of T_2.

(We refer to [Lax02; Section 36.2] for this exercise.)

8.3 Let H_1, H_2 be Hilbert spaces, $H := H_1 \oplus H_2$.

(a) Let $A_1 \in \mathcal{L}(H_1)$, $A_2 \in \mathcal{L}(H_2)$ be invertible in $\mathcal{L}(H_1)$, $\mathcal{L}(H_2)$, respectively, and let $B \in \mathcal{L}(H_2, H_1)$. Show that the operator matrix $\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$ is invertible in $\mathcal{L}(H)$, and compute its inverse.

(b) Let $A \in \mathcal{L}(H_2, H_1)$, $B \in \mathcal{L}(H_2)$. Show that $\begin{pmatrix} I_1 & A \\ 0 & B \end{pmatrix} \begin{pmatrix} I_1 & -A \\ 0 & I_2 \end{pmatrix} = \begin{pmatrix} I_1 & 0 \\ 0 & B \end{pmatrix}$, where I_1, I_2 are the identity operators in H_1, H_2, respectively. Conclude that $\begin{pmatrix} I_1 & A \\ 0 & B \end{pmatrix}$ is injective/surjective/invertible in $\mathcal{L}(H)$ if and only if the corresponding property holds for B.

8.4 Prove Proposition 8.6.

8.5 Let a be a bounded form on a Hilbert space V. Show that the following properties are equivalent:

(i) a is essentially coercive;

(ii) there exists $\alpha > 0$ such that for all sequences (u_n) in V one has

$$\|u_n\| = 1 \ (n \in \mathbb{N}),\ u_n \to 0 \text{ weakly } \implies \liminf \operatorname{Re} a(u_n) \geqslant \alpha;$$

(iii) there exist $\alpha > 0$ and a finite-dimensional subspace $V_1 \subseteq V$ such that

$$\operatorname{Re} a(u) \geqslant \alpha \|u\|_V^2 \qquad (u \in V_1^\perp).$$

Hint concerning '(ii) $\Rightarrow$ (iii)': Assume that (iii) does not hold and construct an orthonormal sequence (u_n) in V such that $\limsup \operatorname{Re} a(u_n) \leqslant 0$.

Hint concerning '(iii) $\Rightarrow$ (i)': For $u \in V_1^\perp$, $v \in V_1$ write $a(u+v) = a(u) + a(u,v) + a(v,u) + a(v)$ and use the Peter–Paul inequality to show that there exists $\omega > 0$ (not depending on u, v) such that $\operatorname{Re} a(u+v) \geqslant \frac{\alpha}{2}\|u+v\|_V^2 - \omega\|v\|_V^2$. Then take the orthogonal projection onto V_1 as the compact operator $\tilde{\jmath}$ in the definition of 'essentially coercive'.

8.6 (a) Let G, H be Hilbert spaces. Let A be a densely defined operator from G to H, $B \in \mathcal{L}(G, H)$. Show that $(A+B)^* = A^* + B^*$.

(b) Let $\Omega \subseteq \mathbb{R}^n$ be open, $m \in L_\infty(\Omega)$ real-valued. Show: $-(\Delta_{\mathrm{D}} + m)$ is self-adjoint and quasi-accretive. If Ω is bounded, then $\Delta_{\mathrm{D}} + m$ has compact resolvent.

8.7 Let X be a normed space, and suppose that $X = X_1 \oplus X_2$ is the algebraic direct sum of two subspaces X_1, X_2.

(a) Show that X is the topological direct sum of X_1, X_2 if and only if the projection from X onto X_1 along X_2 is a bounded operator.

(b) Suppose that $X = X_1 \oplus X_2$ is a topological direct sum and that X_1 and X_2 are Banach spaces. Show that then X is a Banach space.

8.8 Let $\varnothing \neq \Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1-boundary. Show that the decomposition $H^1(\Omega) = V_j(a) \oplus H_0^1(\Omega)$ from Example 8.18 is not an orthogonal sum. (Hint: Choose $0 \neq u_0 \in C_{\mathrm{c}}^2(\Omega)_+$, and decompose the function $u := 1 + u_0$.)

Chapter 9

Invariance of closed convex sets

In this chapter we investigate criteria for a closed convex set to be invariant under a semigroup. To begin with, we present criteria involving properties of the generator. Applying these criteria to the Dirichlet and Neumann Laplacians one realises that further properies of H^1-functions are needed; these will be provided in an interlude on lattice properties of H^1. In the last section we present criteria involving properties of forms that have a wide range of applications. Examples include the Robin Laplacian and the Dirichlet-to-Neumann operator, for which we refer to Exercises 9.4 and 9.5. In Chapter 11 we will apply the criteria to semigroups generated by elliptic operators.

9.1 Invariance for semigroups

Let T be a C_0-semigroup on a Banach space X over $\mathbb{K}$, with generator A. Our aim is to characterise when a closed convex subset C of X is invariant under the semigroup T, i.e. $T(t)(C) \subseteq C$ for all $t \geqslant 0$. This means that, for an initial value in C, the solution of the corresponding Cauchy problem remains in C for all $t \geqslant 0$. First we show that invariance under T is equivalent to invariance under the resolvent.

9.1 Proposition. *Let $C \subseteq X$ be a closed convex set. Then the following properties are equivalent.*

(i) *C is invariant under T.*

(ii) *There exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A)$ and $\lambda R(\lambda, A)(C) \subseteq C$ for all $\lambda > \omega$.*

Noting that $\lambda R(\lambda, A) = (I - \frac{1}{\lambda}A)^{-1}$ we see that condition (ii) can be expressed equivalently by requiring that there exists $r_0 > 0$ such that $\{1/r\,;\, 0 < r < r_0\} \subseteq \rho(A)$ and $(I - rA)^{-1}(C) \subseteq C$ for all $0 < r < r_0$. It is this version of condition (ii) that will mostly be used below.

We insert a fact concerning integration that is needed in the proof of the implication '(i) $\Rightarrow$ (ii)'. It should be understood as a statement on generalised convex combinations.

9.2 Lemma. *Let C be a closed convex subset of a Banach space X, $a, b \in \mathbb{R}$, $a < b$. Let $u\colon [a, b] \to C$ be continuous and $\varphi \in C[a, b]$, $\varphi \geqslant 0$, $\int_a^b \varphi(t)\,\mathrm{d}t = 1$.*

Then $\int_a^b \varphi(t)u(t)\,\mathrm{d}t \in C$.

Proof. For simplicity of notation (and without loss of generality) we assume that $[a,b] = [0,1]$. For $n \in \mathbb{N}$ we put

$$u_n := u(0)\mathbf{1}_{\{0\}} + \sum_{k=1}^{n} u(k/n)\mathbf{1}_{((k-1)/n,k/n]}\,.$$

Then $\|\varphi u_n - \varphi u\|_\infty \to 0$ $(n \to \infty)$; hence $\int_0^1 \varphi u_n \,\mathrm{d}t \to \int_0^1 \varphi u \,\mathrm{d}t$. Moreover $\int_0^1 \varphi u_n \,\mathrm{d}t = \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \varphi(t)\,\mathrm{d}t\, u(k/n) \in C$, as a convex combination of elements of C. As C is closed we obtain the assertion. □

Proof of Proposition 9.1. (i) ⇒ (ii). Let $\omega \in \mathbb{R}$, $M \geqslant 0$ be such that $\|T(t)\| \leqslant M\mathrm{e}^{\omega t}$ for all $t \geqslant 0$, and let $\lambda > \omega$. Then $\lambda R(\lambda, A) = \int_0^\infty \lambda \mathrm{e}^{-\lambda t} T(t)\,\mathrm{d}t$ (strong improper integral). Let $x \in C$. For $\tau > 0$ we obtain $(1-\mathrm{e}^{-\lambda\tau})^{-1}\int_0^\tau \lambda \mathrm{e}^{-\lambda t} T(t)x\,\mathrm{d}t \in C$, by Lemma 9.2. Letting $\tau \to \infty$ we conclude that $\lambda R(\lambda, A)x \in C$.

(ii) ⇒ (i). This follows from the exponential formula (Theorem 2.12):

$$T(t)x = \lim_{n\to\infty} \Big(I - \frac{t}{n}A\Big)^{-n} x \in C \qquad (t \geqslant 0,\ x \in C).$$ □

In order to motivate why one is interested in the invariance of closed convex sets, we indicate several examples.

9.3 Remarks. Let (Ω, μ) be a measure space, $H := L_2(\mu; \mathbb{K})$.

(a) Let $C \subseteq L_2(\mu)$ be the **positive cone**, $C := L_2(\mu)_+ := \{u \in L_2(\mu)\,;\, u \geqslant 0\}$. Clearly C is a closed convex subset of $L_2(\mu)$. An operator $S \in \mathcal{L}(H)$ leaves C invariant if and only if S is **positive**, i.e. $Su \geqslant 0$ for all $u \geqslant 0$.

(b) Let $\mathbb{K} = \mathbb{C}$, and let $C := L_2(\mu; \mathbb{R})$ be the closed convex subset of real-valued functions. An operator $S \in \mathcal{L}(H)$ leaves C invariant if and only if S is **real**, i.e. Su is real-valued for all real-valued u.

(c) Let $C := \{u \in L_2(\mu)\,;\, \|u\|_\infty \leqslant 1\}$. Then C is convex and closed, and $S \in \mathcal{L}(H)$ leaves C invariant if and only if S is $\boldsymbol{L_\infty}$**-contractive**, i.e. $\|Su\|_\infty \leqslant \|u\|_\infty$ for all $u \in L_2 \cap L_\infty(\mu)$.

(d) Let $C := \{u \in L_2(\mu)\,;\, u \leqslant 1\}$; then C is convex and closed. We show that $S \in \mathcal{L}(H)$ leaves C invariant if and only if S is **sub-Markovian**, i.e. S is positive and L_∞-contractive.

Indeed, assume that $S(C) \subseteq C$. If $u \in L_2(\mu)_+$, then $-\alpha u \leqslant 1$ and therefore $-\alpha Su \leqslant 1$, for all $\alpha \geqslant 0$, and this implies that $Su \geqslant 0$. Hence S is a positive operator; in particular, S is real. Now let $u \in L_2 \cap L_\infty(\mu)$, $\|u\|_\infty \leqslant 1$. Then for all $\gamma \in \mathbb{K}$ with $|\gamma| = 1$ one obtains

$$\mathrm{Re}(\gamma Su) = \mathrm{Re}\big(S(\gamma u)\big) = S\big(\mathrm{Re}(\gamma u)\big) \leqslant 1. \tag{9.1}$$

(Note that in the complex case, for $v \in L_2(\mu)$, one has $\mathrm{Re}\, Sv = \mathrm{Re}\, S(\mathrm{Re}\, v + \mathrm{i}\,\mathrm{Im}\, v) = \mathrm{Re}\big(S(\mathrm{Re}\, v) + \mathrm{i}S(\mathrm{Im}\, v)\big) = S(\mathrm{Re}\, v)$, because $S(\mathrm{Re}\, v)$ and $S(\mathrm{Im}\, v)$ are real.) Using (9.1) for γ from a countable dense subset of the unit circle, one concludes that $\|Su\|_\infty \leqslant 1$.

Conversely, if S is positive and L_∞-contractive, then $u \leqslant 1$ implies $Su = Su^+ - Su^- \leqslant Su^+ \leqslant 1$. Here, $u^+ := u \vee 0$ is the **positive part** of u, and the **negative part** of u is defined by $u^- := (-u)^+ = u^+ - u$. (Notice that the negative part is positive.) △

Concerning notation, the symbol '$\vee$' used above denotes 'supremum', i.e., for functions $u, v\colon \Omega \to \mathbb{R}$ the supremum is given by $(u \vee v)(x) = \max\{u(x), v(x)\}$ $(x \in \Omega)$. Similarly, we will use the symbol '$\wedge$' to denote 'infimum', $(u \wedge v)(x) = \min\{u(x), v(x)\}$ $(x \in \Omega)$.

9.4 Remark. Concerning the notion of 'positive' operators we mention that this property is sometimes called 'positivity preserving'. This is because in the literature 'positive' is often used as a synonym for 'accretive', in the context of symmetric operators in a Hilbert space. △

We are looking for another characterisation of T-invariance of a closed convex set, involving more directly the generator A instead of its resolvent. This is possible in Hilbert spaces. Let H be a Hilbert space over $\mathbb{K}$, and let $\varnothing \neq C \subseteq H$ be convex and closed. We denote by $P_C\colon H \to C$ the **minimising projection** from H onto C, which maps $x \in H$ to its best approximation $P_C x$ in C, i.e., $P_C x$ is the unique element of C satisfying

$$\|x - P_C x\| = \inf\{\|x - y\|\,;\, y \in C\}.$$

We will often use the fact that $P_C x$ is the unique element of C such that

$$\operatorname{Re}(y - P_C x \,|\, x - P_C x) \leqslant 0 \qquad (y \in C); \tag{9.2}$$

see Exercise 9.2(a) (or [Bre11; Section 5.1]). Geometrically, this means that the vectors in the scalar product in (9.2) form an obtuse angle; if C has a tangent hyperplane at $P_C x$, then $x - P_C x$ is orthogonal to this hyperplane. The mapping P_C is a contraction; see Exercise 9.2(b). In particular, P_C is continuous. Clearly the minimising projection P_C satisfies $P_C \circ P_C = P_C$; so it deserves the name 'projection'. We could not find a commonly accepted name for this mapping in the literature. One should keep in mind that, in general, P_C is not a linear operator.

9.5 Remark. We illustrate the minimising projection for the closed convex subsets of $L_2(\mu)$ treated in Remarks 9.3. All the statements are immediate consequences of Exercise 9.3.

For the positive cone $C = L_2(\mu)_+$ the minimising projection is given by $P_C u = (\operatorname{Re} u)^+$.

For $\mathbb{K} = \mathbb{C}$ and $C = L_2(\mu; \mathbb{R})$ one has $P_C u = \operatorname{Re} u$.

For the set $C = \{u \in L_2(\mu)\,;\, \|u\|_\infty \leqslant 1\}$ one finds $P_C u = (\operatorname{sgn} u)(|u| \wedge 1)$.

For $C = \{u \in L_2(\mu)\,;\, u \leqslant 1\}$ one obtains $P_C u = (\operatorname{Re} u) \wedge 1$.

In the description given above we have used the **signum function** $\operatorname{sgn}\colon \mathbb{K} \to \mathbb{K}$, which is defined as $\operatorname{sgn} \alpha := \frac{\alpha}{|\alpha|}$ if $0 \neq \alpha \in \mathbb{K}$, and $\operatorname{sgn} 0 := 0$. △

The following result has a geometric appeal. Its assumption (9.3) expresses that the 'driving term' $Au(t)$ in the equation $u'(t) = Au(t)$ always points 'sufficiently' from $u(t)$ towards C. (For $\omega \leqslant 0$ this is quite intuitive. If $\omega > 0$, one can interpret that it is more and more true the closer $u(t)$ is to C.)

9.6 Proposition. *Let T be a C_0-semigroup on a Hilbert space H, with generator A. Let $\varnothing \neq C \subseteq H$ be a closed convex set, and denote by $P := P_C$ the minimising projection. Assume that there exists $\omega \in \mathbb{R}$ such that*

$$\operatorname{Re}(Ax \,|\, x - Px) \leqslant \omega \|x - Px\|^2 \qquad (x \in \operatorname{dom}(A)). \tag{9.3}$$

Then C is invariant under T.

Proof. Because of Proposition 9.1 we only have to show that $(I - rA)^{-1}(C) \subseteq C$ for small $r > 0$. (Observe that $(I - rA)^{-1} \in \mathcal{L}(H)$ for small $r > 0$.) Without loss of generality we assume $\omega > 0$. Let $0 < r < 1/\omega$ and let $x \in \mathrm{dom}(A)$ be such that $(I - rA)x \in C$. We have to show that $x \in C$. Applying (9.2) with $y = (I - rA)x \in C$ we obtain

$$\mathrm{Re}\big((I - rA)x - Px \,\big|\, x - Px\big) \leqslant 0.$$

Thus

$$\begin{aligned}\|x - Px\|^2 &= \mathrm{Re}\big(rAx + (I - rA)x - Px \,\big|\, x - Px\big)\\ &\leqslant r\,\mathrm{Re}(Ax \,|\, x - Px) \leqslant r\omega\|x - Px\|^2.\end{aligned}$$

Using $r\omega < 1$ we conclude that $\|x - Px\| = 0$, $x = Px \in C$. □

The converse of Proposition 9.6 holds for quasi-contractive semigroups.

9.7 Proposition. *Let H, T, A have the same properties as before, and assume that T is quasi-contractive, i.e., there exists $\omega \in \mathbb{R}$ such that $\|T(t)\| \leqslant \mathrm{e}^{\omega t}$ for all $t \geqslant 0$. Let $\varnothing \neq C \subseteq H$ be a closed convex set, and assume that C is invariant under T.*

Then (9.3) *holds with the minimising projection $P := P_C$.*

Proof. Let $x \in \mathrm{dom}(A)$. Then (9.2) implies $\mathrm{Re}\big(T(t)Px - Px \,\big|\, x - Px\big) \leqslant 0$, and this inequality can be rewritten as $0 \leqslant \mathrm{Re}\big(-T(t)Px + Px \,\big|\, x - Px\big)$. One then obtains

$$\begin{aligned}\mathrm{Re}\big(T(t)x - x \,\big|\, x - Px\big) &\leqslant \mathrm{Re}\big(T(t)(x - Px) - (x - Px) \,\big|\, x - Px\big)\\ &\leqslant (\mathrm{e}^{\omega t} - 1)\|x - Px\|^2.\end{aligned}$$

Dividing by t and taking the limit $t \to 0+$ one concludes that

$$\mathrm{Re}(Ax \,|\, x - Px) \leqslant \omega\|x - Px\|^2.$$ □

For the case of contractive C_0-semigroups we summarise the results of Propositions 9.6 and 9.7 as an equivalence.

9.8 Corollary. *Let T be a contractive C_0-semigroup on a Hilbert space H, with generator A, and let $C \neq \varnothing$ be a closed convex subset of H. Then C is invariant under T if and only if*

$$\mathrm{Re}(Ax \,|\, x - Px) \leqslant 0 \qquad (x \in \mathrm{dom}(A)), \tag{9.4}$$

where $P := P_C$ is the minimising projection.

We specify Corollary 9.8 for the case when $H = L_2(\mu; \mathbb{R})$ and $C = L_2(\mu)_+$ is the positive cone; see Remark 9.3(a). Then $Pu = u^+$ for all $u \in H$ (see Remark 9.5), and condition (9.4) becomes

$$(Au \,|\, -u^-) = (Au \,|\, u - Pu) \leqslant 0 \qquad (u \in \mathrm{dom}(A)). \tag{9.5}$$

Replacing u by $-u$ transforms (9.5) into

$$(Au \,|\, u^+) = -\big(A(-u) \,\big|\, (-u)^-\big) \leqslant 0 \qquad (u \in \mathrm{dom}(A)). \tag{9.6}$$

An operator A in $L_2(\mu; \mathbb{R})$ satisfying (9.6) is called **dispersive**, and A is **m-dispersive** if additionally $\mathrm{ran}(I - A) = L_2(\mu; \mathbb{R})$. A semigroup T on $L_2(\mu)$ is called **positive** if $T(t)$ is positive for all $t \geqslant 0$.

We now obtain the following 'positive version' of the Lumer–Phillips theorem.

9.9 Corollary. *Let A be an operator in $L_2(\mu;\mathbb{R})$. Then A is the generator of a positive contractive C_0-semigroup if and only if A is m-dispersive.*

Proof. If A satisfies (9.6) – hence also (9.5) –, then $(-Au\,|\,u) = -(Au\,|\,u^+) - (Au\,|\,-u^-) \geqslant 0$ for all $u \in \mathrm{dom}(A)$, i.e. $-A$ is accretive. Now the assertion is an immediate consequence of Corollary 9.8 and Theorem 3.16 (Lumer–Phillips). □

9.2 Application to Laplacians

We recall that the Dirichlet Laplacian Δ_{D} in $L_2(\Omega)$ is associated with the classical Dirichlet form on $V = H_0^1(\Omega)$. We also recall that the Neumann Laplacian Δ_{N} in $L_2(\Omega)$ is associated with the classical Dirichlet form on $V = H^1(\Omega)$. (See Example 5.13 and Theorem 7.14.)

9.10 Example. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then the C_0-semigroup on $L_2(\Omega)$ generated by Δ_{D} is **sub-Markovian**, i.e. $\mathrm{e}^{t\Delta_{\mathrm{D}}}$ is sub-Markovian for all $t \geqslant 0$. This will be shown below.

In the complex case, the semigroup being sub-Markovian means in particular that it leaves $L_2(\Omega;\mathbb{R})$ invariant. For the proof of this property one could employ Corollary 9.8. However, it is also instructive to use another reasoning, on the basis of the following proposition.

9.11 Proposition. *Let $X_1 \hookrightarrow X_2$ be Banach spaces. For $j = 1, 2$ let T_j be a C_0-semigroup on X_j with generator A_j, and assume that $A_1 \subseteq A_2$. Then X_1 is invariant under T_2, and $T_2(t)|_{X_1} = T_1(t)$ for all $t \geqslant 0$.*

Proof. Let $x \in \mathrm{dom}(A_1)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} T_1(t)x = A_1 T_1(t)x = A_2 T_1(t)x \qquad (t \geqslant 0)$$

shows that $T_1(t)x = T_2(t)x$ for all $t \geqslant 0$, by Theorem 1.13(a). From the denseness of $\mathrm{dom}(A_1)$ in X_1 and extension by continuity one obtains $T_2(t)|_{X_1} = T_1(t)$ for all $t \geqslant 0$. □

Proof of the assertion in Example 9.10. (i) We recall that $\Delta_{\mathrm{D},\mathbb{K}} := \Delta_{\mathrm{D}}$ is a generator in $L_2(\Omega;\mathbb{K})$. We interpret $L_2(\Omega;\mathbb{C})$ as a Banach space over $\mathbb{R}$ and use Proposition 9.11 with $X_1 := L_2(\Omega;\mathbb{R})$, $X_2 := L_2(\Omega;\mathbb{C})$. Then $\Delta_{\mathrm{D},\mathbb{R}} \subseteq \Delta_{\mathrm{D},\mathbb{C}}$ implies that $\big(T_2(t) = \mathrm{e}^{t\Delta_{\mathrm{D},\mathbb{C}}}\big)_{t\geqslant 0}$ leaves $X_1 = L_2(\Omega;\mathbb{R})$ invariant.

(ii) We now restrict $\mathrm{e}^{t\Delta_{\mathrm{D}}}$ a priori to $L_2(\Omega;\mathbb{R})$, and we show that the (closed convex) set $C := \{u \in L_2(\Omega;\mathbb{R})\,;\, u \leqslant 1\}$ is invariant under $(\mathrm{e}^{t\Delta_{\mathrm{D}}})_{t\geqslant 0}$. Then by Remark 9.3(d), $\mathrm{e}^{t\Delta_{\mathrm{D}}}$ is sub-Markovian for all $t \geqslant 0$. The minimising projection onto C is given by $Pu = u \wedge 1$; see Remark 9.5. We have to show that

$$\mathrm{Re}(\Delta_{\mathrm{D}} u\,|\,u - u \wedge 1) \leqslant 0$$

for all $u \in \mathrm{dom}(\Delta_{\mathrm{D}})$.

We need to know that $u \wedge 1 \in H^1_0(\Omega)$ and $\nabla(u \wedge 1) = \mathbf{1}_{[u<1]}\nabla u$. These properties will be shown in the next section (and the present example should serve as a motivation for this treatment); see Theorem 9.16. Accepting these properties we obtain

$$(\Delta_{\mathrm{D}} u \,|\, u - u \wedge 1) = -\big(\nabla u \,\big|\, \mathbf{1}_{[u\geqslant 1]}\nabla u\big) = -\int_{[u\geqslant 1]} |\nabla u|^2 \,\mathrm{d}x \leqslant 0.$$

Now the application of Corollary 9.8 yields the invariance of C. □

We note that the same arguments show that the Neumann Laplacian generates a sub-Markovian C_0-semigroup on $L_2(\Omega)$. We refer to Exercise 9.4 for the discussion of invariance properties for the Robin Laplacian.

9.3 Interlude: lattice properties of $H^1(\Omega)$

Throughout this section we will use $\mathbb{K} = \mathbb{R}$, i.e. all the functions will be real-valued. We start with a warm-up.

9.12 Lemma. *Let $-\infty \leqslant a < b \leqslant \infty$, and let $u \in C^1(a,b)$. Then $\partial|u| = (\operatorname{sgn} u)u'$ in the distributional sense; in particular, $\partial|u| = 0$ on $[u = 0]$.*

Proof. We use a sequence (F_k) of functions $F_k \in C^\infty(\mathbb{R};\mathbb{R})$ with $F_k(t) = |t| - \frac{1}{k}$ for $|t| \geqslant \frac{2}{k}$, $F_k(t) = 0$ for $|t| \leqslant \frac{1}{2k}$ and $|F_k'(t)| \leqslant 1$ for all $t \in \mathbb{R}$. (F_1 can be obtained as a convolution of $t \mapsto (|t| - 1)^+$ with a suitable C^∞_{c}-function, and then $F_k(t) := \frac{1}{k}F_1(kt)$ $(t \in \mathbb{R})$.)

By the chain rule we have $(F_k \circ u)' = (F_k' \circ u)u'$. Then $F_k \circ u \to |u|$, uniformly on compact subsets of (a,b), and $(F_k \circ u)' \to (\operatorname{sgn} u)u'$ pointwise on (a,b). Moreover $|(F_k \circ u)'| \leqslant |u'|$ for all $k \in \mathbb{N}$. Therefore $(F_k \circ u)' \to (\operatorname{sgn} u)u'$ in L_1 on compact subsets of (a,b). This implies $\partial|u| = (\operatorname{sgn} u)u'$, by Lemma 4.10.

The last statement is a consequence of $\operatorname{sgn} u = 0$ on $[u = 0]$. □

Our aim is to show similar properties in more general situations. The first point is that the chain rule also holds for distributional derivatives.

In the remainder of this section let $\Omega \subseteq \mathbb{R}^n$ be an open set.

9.13 Proposition. *Let $F \in C^1(\mathbb{R})$, $\|F'\|_\infty \leqslant 1$, $j \in \{1,\dots,n\}$, $u, \partial_j u \in L_{1,\mathrm{loc}}(\Omega)$. Then $F \circ u \in L_{1,\mathrm{loc}}(\Omega)$, $\partial_j(F \circ u) = (F' \circ u)\partial_j u$.*

Proof. Without loss of generality we assume that $F(0) = 0$; then $|F(t)| \leqslant |t|$ for all $t \in \mathbb{R}$. Being the distributional derivative of a function is a local property (see Exercise 4.3(b)), and therefore (after suitable multiplication by a C^∞_{c}-function) it is sufficient to treat the case when $\Omega = \mathbb{R}^n$ and $u, \partial_j u \in L_1(\mathbb{R}^n)$.

Let $(\rho_k)_{k\in\mathbb{N}}$ be a δ-sequence in $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$. Then $u_k := \rho_k * u \to u$, $\partial_j u_k = \rho_k * \partial_j u \to \partial_j u$ in $L_1(\mathbb{R}^n)$, and for a suitable subsequence (ρ_{k_m}) these convergences also hold a.e. as well as dominated, in the sense that there exists $h \in L_1(\mathbb{R}^n)$ such that $|u_{k_m}|, |\partial_j u_{k_m}| \leqslant h$ for all $m \in \mathbb{N}$. As $u_k \in C^\infty(\mathbb{R}^n)$, one has $F \circ u_{k_m} \in C^1(\mathbb{R}^n)$, and

$$F \circ u_{k_m} \to F \circ u, \quad \partial_j(F \circ u_{k_m}) = (F' \circ u_{k_m})\partial_j u_{k_m} \to (F' \circ u)\partial_j u \quad \text{a.e.},$$

since F and F' are continuous. Furthermore, $|F \circ u_{k_m}| \leqslant |u_{k_m}| \leqslant h$, $|\partial_j(F \circ u_{k_m})| \leqslant |\partial_j u_{k_m}| \leqslant h$, by the hypotheses on F and on the subsequence, and therefore $F \circ u_{k_m} \to F \circ u$, $\partial_j(F \circ u_{k_m}) \to (F' \circ u)\partial_j u$ in $L_1(\mathbb{R}^n)$. This implies that $\partial_j(F \circ u) = (F' \circ u)\partial_j u$, by Lemma 4.10. □

Next, we extend the chain rule of Proposition 9.13 to more general composition functions F.

9.14 Proposition. *Let $F \in C(\mathbb{R})$, and assume that there exist a function $G\colon \mathbb{R} \to \mathbb{R}$ and a sequence (F_k) in $C^1(\mathbb{R})$ with $\|F_k'\|_\infty \leqslant 1$ $(k \in \mathbb{N})$, and $F_k \to F$, $F_k' \to G$ pointwise $(k \to \infty)$. Let $j \in \{1, \dots, n\}$, $u, \partial_j u \in L_{1,\mathrm{loc}}(\Omega)$.*

Then $F \circ u \in L_{1,\mathrm{loc}}(\Omega)$, $\partial_j(F \circ u) = (G \circ u)\partial_j u$.

Proof. From Proposition 9.13 we know that $F_k \circ u \in L_{1,\mathrm{loc}}(\Omega)$, $\partial_j(F_k \circ u) = (F_k' \circ u)\partial_j u$. Without loss of generality we may assume that $F(0) = 0$ and that $F_k(0) = 0$ for all $k \in \mathbb{N}$, which implies that the functions $|F_k \circ u|$ are dominated by $|u|$. Then, applying the dominated convergence theorem on compact subsets of Ω and using Lemma 4.10 one obtains the assertions. □

9.15 Corollary. *Let $j \in \{1, \dots, n\}$, $u, \partial_j u \in L_{1,\mathrm{loc}}(\Omega)$. Then $u^+, u \wedge 1 \in L_{1,\mathrm{loc}}(\Omega)$, $\partial_j(u^+) = \mathbf{1}_{[u>0]}\partial_j u$, $\partial_j(u \wedge 1) = \mathbf{1}_{[u<1]}\partial_j u$.*

Proof. Similarly to the construction of the sequence (F_k) in the proof of Lemma 9.12 one can construct a sequence (F_k) converging to $F(t) := t^+$, with the properties mentioned in Proposition 9.14 and such that $F_k' \to \mathbf{1}_{(0,\infty)}$ pointwise. Then Proposition 9.14 implies the assertion for u^+. The reasoning for $u \wedge 1$ is analogous. □

A **vector sublattice** V of $L_2(\Omega)$ is a subspace with the property that $u, v \in V$ implies that $u \vee v, u \wedge v \in V$. A subspace V is a vector sublattice if and only if for all $u \in V$ one has $u^+ \in V$ or $|u| \in V$; this equivalence follows from the relations

$$|u| = u^+ - u^-, \quad u^+ = \tfrac{1}{2}(u + |u|) \qquad (u \in L_2(\Omega)),$$
$$u \vee v = \tfrac{1}{2}(u + v + |u - v|), \quad u \wedge v = \tfrac{1}{2}(u + v - |u - v|) \qquad (u, v \in L_2(\Omega)).$$

The vector sublattice V is called **Stonean** if $u \wedge 1 \in V$ for all $u \in V$.

In this terminology, the following result says in particular that $H^1(\Omega)$ and $H_0^1(\Omega)$ are Stonean vector sublattices of $L_2(\mu)$.

9.16 Theorem. *Let $u \in H^1(\Omega; \mathbb{R})$. Then $u^+, u \wedge 1 \in H^1(\Omega)$, $\nabla u^+ = \mathbf{1}_{[u>0]}\nabla u$, $\nabla(u \wedge 1) = \mathbf{1}_{[u<1]}\nabla u$.*

If $u \in H_0^1(\Omega; \mathbb{R})$, then $u^+, u \wedge 1 \in H_0^1(\Omega)$.

Proof. It was shown in Corollary 9.15 that the indicated derivatives for u^+ and $u \wedge 1$ are the distributional derivatives. As they belong to $L_2(\Omega; \mathbb{R}^n)$, the first part of the theorem is proved.

Now let $u \in H_0^1(\Omega)$. There exists a sequence (u_k) in $H_c^1(\Omega)$ such that $u_k \to u$ in $H^1(\Omega)$. Then (u_k^+) is a bounded sequence in $H_c^1(\Omega) \subseteq H_0^1(\Omega)$, by the first part of the theorem, and $u_k^+ \to u^+$ in $L_2(\Omega)$. Thus, Remark 9.17 below shows that $u^+ \in H_0^1(\Omega)$. The argument for $u \wedge 1$ is similar. □

9.17 Remark. Let V, H be Hilbert spaces, $V \hookrightarrow H$. Let (v_k) be a bounded sequence in V that is weakly convergent in H to $u \in H$. Then $u \in V$, and $v_k \to u$ weakly in V.

Indeed, there exist $v \in V$ and a subsequence (v_{k_m}), $v_{k_m} \to v$ weakly in V. Then $v_{k_m} \to v$ weakly in H as well; hence $v = u$. A standard sub-subsequence argument shows that $v_k \to u$ weakly in V; see Exercise 9.7. △

9.4 Invariance described by forms

In this section we transform the invariance criteria obtained in Propositions 9.6 and 9.7 to conditions on forms instead of operators. This means that we only treat C_0-semigroups associated with forms. In this case the C_0-semigroup is quasi-contractive, and the condition (9.3) is equivalent to the invariance of C under the semigroup.

We restrict our treatment to the case of embedded forms, i.e., we assume that V is a Hilbert space that is densely embedded into H and that $a\colon V \times V \to \mathbb{K}$ is a bounded H-coercive form. We recall that H-coercivity means that there exist $\omega \in \mathbb{R}$, $\alpha > 0$ such that

$$\operatorname{Re} a(u) + \omega\|u\|_H^2 \geqslant \alpha\|u\|_V^2 \qquad (u \in V). \tag{9.7}$$

Let $A \sim a$. It follows from Proposition 5.5(b) that $-A$ is the generator of a C_0-semigroup T; note the change of the sign in the generator with respect to Section 9.1.

The notation used above will be fixed troughout this section. Coming back to invariance, let $\varnothing \neq C \subseteq H$ be a closed convex set, and let $P := P_C$ be the minimising projection.

9.18 Proposition. *Let C be invariant under T. Then $P(V) \subseteq V$.*

9.19 Remark. At the first glance, this property might seem rather unexpected, because the elements of V have some quality (or 'regularity'), and it is surprising that this quality is preserved under P. To make the point, the elements of the domain of the generator will not be mapped to the domain of the generator, in general. △

For the proof we single out a technical detail, which will be useful in several of the subsequent proofs.

9.20 Lemma. *Let $(u_n), (v_n)$ be sequences in V, $u_n \to u$ in H, (v_n) bounded in V, and*

$$\operatorname{Re} a(u_n, v_n - u_n) \geqslant 0 \qquad (n \in \mathbb{N}).$$

Then $u \in V$, and $u_n \to u$ weakly in V.

Proof. Without loss of generality assume that $\omega \geqslant 0$. Using (9.7) we estimate

$$\begin{aligned} \alpha\|u_n\|_V^2 &\leqslant \operatorname{Re} a(u_n, u_n) + \omega\|u_n\|_H^2 \leqslant \operatorname{Re} a(u_n, v_n) + \omega\|u_n\|_H^2 \\ &\leqslant M\|u_n\|_V\|v_n\|_V + \omega c\|u_n\|_V\|u_n\|_H\,, \end{aligned}$$

where M denotes the bound of a and c the norm of the embedding $V \hookrightarrow H$. This inequality implies that (u_n) is bounded in V, and by Remark 9.17 it follows that $u_n \to u$ weakly in V. □

Proof of Proposition 9.18. Let $u \in V$. By (9.2) we obtain

$$\operatorname{Re}(Pu - y \,|\, u - y) = \operatorname{Re}(Pu - y \,|\, u - Pu) + \|Pu - y\|^2 \geqslant 0 \qquad (y \in C). \tag{9.8}$$

For $r > 0$ small enough (say, $0 < r < r_0 < \infty$) we define $R_r := (I + rA)^{-1}$. Then $AR_r = \frac{1}{r}(I + rA - I)R_r = \frac{1}{r}(I - R_r)$, so

$$a(R_r x, v) = (AR_r x \,|\, v) = \frac{1}{r}(x - R_r x \,|\, v) \qquad (x \in H,\ v \in V). \tag{9.9}$$

Now $R_r Pu \in C$, by Proposition 9.1. Using (9.9) and applying (9.8) with $y = R_r Pu$, we obtain

$$\operatorname{Re} a(R_r Pu, u - R_r Pu) = \frac{1}{r} \operatorname{Re}(Pu - R_r Pu \,|\, u - R_r Pu) \geqslant 0.$$

Since $R_r Pu \to Pu$ in H as $r \to 0$ (by Lemma 2.10(a); recall also Remark 2.13(b)), Lemma 9.20 implies that $Pu \in V$. □

We insert an auxiliary result that will be used in the proof of the next theorem.

9.21 Lemma. (a) *As in the proof of Proposition 9.18 we define $R_r := (I + rA)^{-1}$ for $r \in (0, r_0)$, with suitable $r_0 > 0$. Then $R_r u \to u$ $(r \to 0)$ weakly in V for all $u \in V$.*

(b) *Let (u_n) be a sequence converging weakly in V to u. Then $a(u_n, v) \to a(u, v)$ for all $v \in V$.*

Proof. (a) Let $u \in V$. By (9.9) we obtain

$$a(R_r u, u - R_r u) = \frac{1}{r}(u - R_r u \,|\, u - R_r u) \geqslant 0.$$

Since $R_r u \to u$ $(r \to 0)$ in H, Lemma 9.20 implies that $R_r u \to u$ weakly in V as $r \to 0$.

(b) follows since $a(\cdot, v) \in V'$. □

We now come to the fundamental result characterising invariance by properties of the form. The inequality (9.11) which appears below has already been commented upon in the paragraph before Proposition 9.6. In order to give a geometrical interpretation to (9.10) we note that, loosely speaking, the expression $a(Pu, u - Pu)$ can be understood as $(A(Pu) \,|\, u - Pu)$ (except that Pu does not necessarily lie in $\operatorname{dom}(A)$). So, the condition gives information on the driving term $-Au(t)$ in the equation $u' = -Au$, whenever $u(t)$ is the image Pu of some $u \in H \setminus C$: in these points, the driving term 'points towards C'.

9.22 Theorem. *Under the previous assumptions the following properties are equivalent.*

(i) *C is invariant under T.*

(ii) *$P(V) \subseteq V$, and*

$$\operatorname{Re} a(Pu, u - Pu) \geqslant 0 \tag{9.10}$$

for all $u \in V$.

(iii) *There exists a dense subset D of V such that $P(D) \subseteq V$, and (9.10) holds for all $u \in D$.*

(iv) *$P(V) \subseteq V$, and*

$$\operatorname{Re} a(u, u - Pu) \geqslant -\omega \|u - Pu\|^2 \qquad (u \in V), \tag{9.11}$$

for some $\omega \in \mathbb{R}$.

Proof. (i) $\Rightarrow$ (ii). $P(V) \subseteq V$ was shown in Proposition 9.18. Let $u \in V$. Then for $0 < r < r_0$ we have

$$\operatorname{Re} a(R_r P u, u - Pu) = \frac{1}{r} \operatorname{Re}(Pu - R_r P u \,|\, u - Pu) \geqslant 0$$

by (9.9) and (9.2), and from Lemma 9.21 we obtain

$$\operatorname{Re} a(Pu, u - Pu) \geqslant 0.$$

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (ii). Let $u \in V$. There exists a sequence (u_n) in D such that $u_n \to u$ in V as $n \to \infty$. By hypothesis we have

$$\operatorname{Re} a(Pu_n, u_n - Pu_n) \geqslant 0 \qquad (n \in \mathbb{N}). \tag{9.12}$$

From the continuity of $P\colon H \to H$ we obtain $Pu_n \to Pu$ in H, and therefore Lemma 9.20 implies that $Pu \in V$ and that $Pu_n \to Pu$ weakly in V.

In order to show (9.10) we first observe that an equivalent norm on V is given by

$$\|v\|_a := \big(\operatorname{Re} a(v) + \omega \|v\|_H^2\big)^{1/2} \qquad (v \in V),$$

with ω from (9.7). It follows that for the weakly convergent sequence (Pu_n) in V one has $\|Pu\|_a \leqslant \liminf_{n\to\infty} \|Pu_n\|_a$. Using (9.12) we obtain

$$\begin{aligned} \operatorname{Re} a(Pu) + \omega \|Pu\|_H^2 &\leqslant \liminf_{n\to\infty} \big(\operatorname{Re} a(Pu_n) + \omega \|Pu_n\|_H^2\big) \\ &\leqslant \liminf_{n\to\infty} \operatorname{Re} a(Pu_n, u_n) + \omega \|Pu\|_H^2 . \end{aligned}$$

Since $a(Pu_n, u_n) = a(Pu_n, u_n - u) + a(Pu_n, u) \to 0 + a(Pu, u)$ as $n \to \infty$ (recall Lemma 9.21(b)), we conclude that $\operatorname{Re} a(Pu, u - Pu) = \operatorname{Re} a(Pu, u) - \operatorname{Re} a(Pu) \geqslant 0$.

(ii) $\Rightarrow$ (iv) follows from the identity

$$a(u, u - Pu) = a(Pu, u - Pu) + a(u - Pu).$$

(iv) $\Rightarrow$ (i). Because of $a(u, u - Pu) = (Au \,|\, u - Pu)$ $(u \in \operatorname{dom}(A))$, condition (9.11) implies (9.3) for the generator $-A$ of T. Then the assertion follows from Proposition 9.6. □

9.23 Example. We come back to Example 9.10. Let us again show step (ii) of its proof, i.e., that the C_0-semigroup on $L_2(\Omega; \mathbb{R})$ generated by the Dirichlet or Neumann Laplacian is sub-Markovian. We check property (ii) of Theorem 9.22 for the convex set $C = \{u \in L_2(\Omega; \mathbb{R})\,;\, u \leqslant 1\}$ and the minimising projection $Pu = u \wedge 1$: Theorem 9.16 implies that P leaves $V = H_0^1(\Omega)$ (and also $V = H^1(\Omega)$) invariant, and

$$a(Pu, u - Pu) = \int \nabla(u \wedge 1) \cdot (\nabla u - \nabla(u \wedge 1)) \,\mathrm{d}x = \int \mathbf{1}_{[u<1]} \nabla u \cdot \mathbf{1}_{[u\geqslant 1]} \nabla u \,\mathrm{d}x = 0$$

shows that property (ii) is satisfied. Hence C is invariant by Theorem 9.22. △

9.24 Remark. We emphasise that in Theorem 9.22 the associated semigroup is not assumed to be contractive. If (as in Example 9.23) the convex set C is not a cone, then the quasi-contractive case cannot be reduced to the contractive case by rescaling.

An example for an application to a non-contractive semigroup can be found in Exercise 9.6. △

Notes

Clearly, it is of fundamental interest to ask for criteria that describe when certain sets are invariant under the time evolution of a system, and questions of this kind have a long history, in particular in the finite-dimensional case, for linear and nonlinear problems.

In the seminal papers [BeDe58], [BeDe59], Beurling and Deny investigated such questions in infinite-dimensional spaces. Phillips studied positive contractions semigroups in the fundamental paper [Phi62]. Skipping a lot of history we refer to Kunita [Kun70] whose treatment included non-contractive non-symmetric semigroups. We also refer the reader to the more recent papers [Ouh96] and [MVV05] as well as to Ouhabaz' book [Ouh05] and to the literature mentioned in these sources. The invariance criterion in Theorem 9.22 is taken from [MVV05; Theorem 2.1]. An investigation on invariance for nonlinear evolution equations can be found in [Bar96].

The lattice properties of the Sobolev spaces $H_0^1(\Omega)$ and $H^1(\Omega)$ were developed in the 70's; see the paper of Marcus and Mizel [MaMi79], which also includes earlier references. Meanwhile, these properties can be found in several books on Sobolev spaces or partial differential equations. We refer to [EdEv87; VI.2] for a more general chain rule than presented in this chapter.

Exercises

9.1 Let (Ω, μ) be a measure space, $1 \leqslant p \leqslant \infty$, and let $A \in \mathcal{L}(L_p(\mu))$ be a positive operator, i.e. $Au \geqslant 0$ for all $u \in L_p(\mu)$ with $u \geqslant 0$.

(a) Show that $|Au| \leqslant A|u|$ for all $u \in L_p(\mu)$.

(b) Show that

$$\|A\| = \sup\big\{\|Au\|_p;\, u \in L_p(\mu),\ u \geqslant 0,\ \|u\|_p \leqslant 1\big\}.$$

9.2 Let H be a Hilbert space, $\varnothing \neq C \subseteq H$ a closed convex subset, and let P_C be the minimising projection onto C.

(a) Let $x \in H$, $z \in C$. Show that $z = P_C x$ if and only if

$$\operatorname{Re}(y - z \,|\, x - z) \leqslant 0 \qquad (y \in C);$$

see (9.2). Hints: For '$\Rightarrow$' take $y \in C$ and look at the derivative at 0 of the function $[0,1) \ni t \mapsto \|x - (z + t(y-z))\|^2$. For '$\Leftarrow$' take $y \in C$ and compare $\|x - y\| = \|(x-z) + (z-y)\|$ with $\|x - z\|$.

(b) Use part (a) to show that P_C is a contraction, i.e. $\|P_C x - P_C y\| \leqslant \|x - y\|$ for all $x, y \in H$.

9.3 Let (Ω, μ) be a measure space, $\check{C} \subseteq \mathbb{K}$ a closed convex set, $0 \in \check{C}$; let $\check{P}\colon \mathbb{K} \to \check{C}$ be the minimising projection. Show that

$$C := \big\{u \in L_2(\mu);\, u(x) \in \check{C} \text{ for } \mu\text{-a.e. } x\big\} \neq \varnothing$$

is convex and closed.

Show that the minimising projection $P\colon L_2(\mu) \to C$ is given by $(Pu)(x) = \check{P}(u(x))$ $(x \in \Omega)$.

9.4 Let Δ_β be the Robin Laplacian from Section 7.5.

(a) Let β be real-valued. Show that the C_0-semigroup generated by Δ_β is positive.

(b) Let $\beta \geqslant 0$. Show that the C_0-semigroup generated by Δ_β is sub-Markovian.

9.5 (a) Let H, V, a, j be as in Proposition 5.5, and assume that minus the operator associated with (a, j) is a generator. Let $\varnothing \neq C \subseteq H$ be convex and closed, P the minimising projection onto C. Let $\hat{P}\colon V \to V$ be a mapping satisfying $Pj = j\hat{P}$, and assume that

$$\operatorname{Re} a(u, u - \hat{P}u) \geqslant 0 \qquad (u \in V).$$

Show that C is invariant under the C_0-semigroup associated with (a, j). (Hint: Use Proposition 9.6.)

(b) Let H, V, a, j be as in Corollary 5.11, and let C, P be as in part (a). Show that C is invariant under the C_0-semigroup associated with (a, j) if and only if there exists a mapping $\hat{P}\colon V \to V$ such that $Pj = j\hat{P}$ and

$$\operatorname{Re} a(\hat{P}u, u - \hat{P}u) \geqslant 0 \qquad (u \in V).$$

Hint concerning 'only if': Apply Theorem 9.22, using Proposition 8.14 and the decomposition of the form domain V given in Proposition 8.16; assume without loss of generality that $V_j(a) \subseteq H$, $j(u) = u$ for all $u \in V_j(a)$, and put $\hat{P} := Pj$.

(c) Show that the C_0-semigroup generated by the Dirichlet-to-Neumann operator from Section 8.1 is sub-Markovian. (Hint: Show that the mapping $u \mapsto u \wedge 1$ on $H^1(\Omega)$ is consistent with the trace operator.)

9.6 Let $\Omega \subseteq \mathbb{R}^n$ be open, let $b \in L_\infty(\Omega; \mathbb{R}^n)$, and define the operator A in $L_2(\Omega)$ by

$$\operatorname{dom}(A) := \big\{u \in H_0^1(\Omega);\ -\Delta u \in L_2(\Omega)\big\},$$
$$Au := -\Delta u + b \cdot \nabla u \qquad (u \in \operatorname{dom}(A)).$$

(a) Show that A is associated with an H-coercive form on V, with $V := H_0^1(\Omega) \hookrightarrow H := L_2(\Omega)$ (and therefore $-A$ generates a quasi-contractive C_0-semigroup). Show that the semigroup generated by $-A$ is holomorphic of angle $\pi/2$ if $\mathbb{K} = \mathbb{C}$ (Hint: Use Exercises 7.7 and 7.8).

(b) Show that the C_0-semigroup generated by $-A$ is sub-Markovian.

9.7 Let Ω be a topological space, let $x \in \Omega$, and let (x_n) be a sequence in Ω. Assume that each subsequence of (x_n) contains a subsequence converging to x. Show that then $x_n \to x$ as $n \to \infty$. (This fact will be referred to as the *(standard) sub-subsequence argument.*)

9.8 As in Theorem 9.22, let V and H be Hilbert spaces, $V \overset{\mathrm{d}}{\hookrightarrow} H$, $a\colon V \times V \to \mathbb{K}$ an H-coercive form, but assume additionally that $C = H_1$ is a (closed) subspace of H. Let $P \in \mathcal{L}(H)$ be the orthogonal projection onto H_1.

(a) Show that H_1 is invariant under T if and only if $P(V) \subseteq V$ and $a(u, v) = 0$ for all $u \in P(V)$, $v \in (I - P)(V)$. (Hint: For the necessity show first that $\operatorname{Re} a(u, v) = 0$ for u, v as specified; consider $u \pm v$.)

(b) Now assume that H_1 is invariant under T, and denote by T_1 the restriction of T to H_1; then T_1 is a C_0-semigroup on H_1. Show that $a_1 := a|_{V_1 \times V_1}$ is H_1-coercive, where $V_1 := V \cap H_1 = P(V)$. With $A_1 \sim a_1$, show that $A_1 = A \cap (H_1 \times H_1)$ and that $-A_1$ is the generator of T_1. (Hint: Use Proposition 9.11.)

Chapter 10

Interpolation of holomorphic semigroups

In the first section of this chapter we will present an extremely powerful tool of functional analysis, important in many areas: complex interpolation. It should be looked upon as the surprising fact that (elementary) complex methods are a useful tool for deriving inequalities. The main result is the Stein interpolation theorem; as a particular case one also obtains the famous and important Riesz–Thorin interpolation theorem.

For us, the important consequence will be that a holomorphic semigroup on some L_{p_1} that is bounded on some L_{p_0} for real times can be 'interpolated' holomorphically to other L_p-spaces. In the last section we demonstrate the interplay of invariance, interpolation and duality in applications to C_0-semigroups on L_2.

10.1 Interlude: the Stein interpolation theorem

Throughout this section the scalar field will be $\mathbb{K} = \mathbb{C}$.

10.1.1 The three lines theorem

In this subsection we prove a version of the maximum principle for holomorphic functions on an unbounded set. First we recall the maximum principle.

If $\Omega \subseteq \mathbb{C}$ is a bounded open set, and $h\colon \overline{\Omega} \to \mathbb{C}$ is continuous and holomorphic on Ω, then $\|h\|_{\overline{\Omega}} \leqslant \|h\|_{\partial\Omega}$. This is an easy consequence of Cauchy's integral formula. Here and in what follows we denote by $\|\cdot\|_M$ the supremum norm taken over the set M.

In this and the next section the set $S \subseteq \mathbb{C}$ will be the open strip

$$S := \{z \in \mathbb{C};\, 0 < \operatorname{Re} z < 1\}.$$

10.1 Proposition. *Let $h\colon \overline{S} \to \mathbb{C}$ be continuous, bounded, and holomorphic on S. Then*

$$\|h\|_{\overline{S}} \leqslant \|h\|_{\partial S}\,.$$

Proof. Let $n \in \mathbb{N}$. Then the function $\psi_n(z) := \frac{n}{z+n}$ is continuous on $\overline{S}$ and holomorphic on S. With $S_k := \{z \in S;\, |\operatorname{Im} z| < k\}$, for $k \in \mathbb{N}$, we have $\lim_{k\to\infty} \|\psi_n h\|_{S\setminus S_k} = 0$ since h is bounded. Using the maximum principle we conclude that

$$\|\psi_n h\|_{\overline{S}} \leqslant \|\psi_n h\|_{\overline{S}\setminus S_k} \to \|\psi_n h\|_{\partial S} \qquad (k \to \infty);$$

hence $\|\psi_n h\|_{\overline{S}} \leqslant \|\psi_n h\|_{\partial S} \leqslant \|h\|_{\partial S}$. Letting $n \to \infty$ we obtain the assertion. □

10.2 Remark. Taking more astute functions ψ_n one may weaken the assumption that h is bounded; it suffices that $|h(z)| \leqslant \mathrm{e}^{c\mathrm{e}^{\alpha|\mathrm{Im}\, z|}}$ with $c > 0$ and $0 < \alpha < \pi$ (see Exercise 10.1). $\triangle$

The following result is a refinement of Proposition 10.1 in which one distiguishes between the suprema of h at $[\mathrm{Re} = 0]$ and at $[\mathrm{Re} = 1]$.

10.3 Theorem (Three lines theorem). *Let h be as in Proposition 10.1. Then*

$$\|h\|_{[\mathrm{Re}=\tau]} \leqslant \|h\|_{[\mathrm{Re}=0]}^{1-\tau} \|h\|_{[\mathrm{Re}=1]}^{\tau}$$

for all $0 \leqslant \tau \leqslant 1$.

Proof. Let $b_j > \|h\|_{[\mathrm{Re}=j]}$ for $j = 0, 1$, and let $\tau \in [0, 1]$. We apply Proposition 10.1 to the function $z \mapsto \left(\frac{b_0}{b_1}\right)^z h(z)$ and obtain

$$\left(\frac{b_0}{b_1}\right)^{\tau} \|h\|_{[\mathrm{Re}=\tau]} \leqslant \left\| z \mapsto \left(\frac{b_0}{b_1}\right)^z h(z) \right\|_{\partial S} \leqslant \max\left\{ b_0, \left(\frac{b_0}{b_1}\right)^1 b_1 \right\} = b_0\,;$$

hence $\|h\|_{[\mathrm{Re}=\tau]} \leqslant b_0^{1-\tau} b_1^{\tau}$. Taking the infima over b_1, b_2 one obtains the assertion. $\square$

10.4 Remark. It follows that $\tau \mapsto \|h\|_{[\mathrm{Re}=\tau]}$ is a log-convex function, i.e. $\tau \mapsto \ln \|h\|_{[\mathrm{Re}=\tau]}$ is convex. $\triangle$

10.1.2 The Stein interpolation theorem

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $\mathcal{A}_{\mathrm{c}} \subseteq \{A \in \mathcal{A}\,;\, \mu(A) < \infty\}$ be a $\cap$-stable collection of subsets of Ω, with the property that the space of **simple functions** over $\mathcal{A}_{\mathrm{c}}$,

$$S(\mathcal{A}_{\mathrm{c}}) := \mathrm{lin}\{\mathbf{1}_A\,;\, A \in \mathcal{A}_{\mathrm{c}}\},$$

is dense in $L_1(\mu)$. (The index 'c' should be remindful of 'compact': if $\Omega \subseteq \mathbb{R}^n$ is an open set and μ is the Lebesgue measure, then one can choose $\mathcal{A}_{\mathrm{c}}$ as the collection of compact subsets of Ω.)

10.5 Remarks. (a) The $\cap$-stability of $\mathcal{A}_{\mathrm{c}}$ implies that the product uv of two elements $u, v \in S(\mathcal{A}_{\mathrm{c}})$ belongs to $S(\mathcal{A}_{\mathrm{c}})$; hence $S(\mathcal{A}_{\mathrm{c}})$ is an algebra.

Let $\mathcal{R}$ be the ring of subsets generated by $\mathcal{A}_{\mathrm{c}}$; then $S(\mathcal{A}_{\mathrm{c}}) = S(\mathcal{R})$. Indeed, clearly $S(\mathcal{A}_{\mathrm{c}}) \subseteq S(\mathcal{R})$. On the other hand one shows – see Exercise 10.2(a) – that $\mathbf{1}_A \in S(\mathcal{A}_{\mathrm{c}})$ for all $A \in \mathcal{R}$; hence $S(\mathcal{R}) \subseteq S(\mathcal{A}_{\mathrm{c}})$. In Exercise 10.2(b) the reader is asked to show that every function $u \in S(\mathcal{R})$ has a 'disjoint representation' $u = \sum_{A \in \mathcal{F}} c_A \mathbf{1}_A$, with a finite collection $\mathcal{F}$ of pairwise disjoint elements of $\mathcal{R}$ and suitable $c_A \in \mathbb{C}$ $(A \in \mathcal{F})$. As an important consequence one obtains $\varphi \circ u \in S(\mathcal{A}_{\mathrm{c}})$ for any function $\varphi\colon \mathbb{C} \to \mathbb{C}$ and all $u \in S(\mathcal{A}_{\mathrm{c}})$.

(b) A standard situation in measure theory would be to assume that $\mathcal{A}_{\mathrm{c}}$ is a $\cap$-stable generator of $\mathcal{A}$ consisting of sets of finite measure. Then $S(\mathcal{A}_{\mathrm{c}})$ is dense in $L_1(\mu)$. $\triangle$

It will be important to know that $S(\mathcal{A}_{\mathrm{c}})$ is dense not only in $L_1(\mu)$ but also in $L_p(\mu)$, for all $p \in (1, \infty)$.

10.6 Lemma. *With the notation introduced above, the set $S(\mathcal{A}_c)$ is dense in $L_p(\mu)$ for all $p \in [1,\infty)$. More strongly, if $p_1, p_2 \in [1,\infty)$ and $u \in L_{p_1} \cap L_{p_2}(\mu)$, then there exists a sequence (u_n) in $S(\mathcal{A}_c)$ converging to u in $L_{p_1}(\mu)$ as well as in $L_{p_2}(\mu)$.*

Proof. (i) Let $p \in [1,\infty)$, $u \in L_p(\mu)$. Then $u\mathbf{1}_{[1/n \leqslant |u| \leqslant n]} \to u$ in $L_p(\mu)$ as $n \to \infty$. Hence it is sufficient to show that each function $v \in L_1 \cap L_\infty(\mu)$ belongs to the closure of $S(\mathcal{A}_c)$ in $L_p(\mu)$.

(ii) Let $v \in L_1 \cap L_\infty(\mu)$. Since $S(\mathcal{A}_c)$ is dense in $L_1(\mu)$, there exist a sequence $(\tilde{v}_k)$ in $S(\mathcal{A}_c)$ and a function $\tilde{v} \in L_1(\mu)$ such that $\tilde{v}_k \to v$ μ-a.e. as $k \to \infty$, and $|\tilde{v}_k| \leqslant \tilde{v}$ for all $k \in \mathbb{N}$. For $k \in \mathbb{N}$ we put $v_k := \tilde{v}_k \mathbf{1}_{[|\tilde{v}_k| \leqslant \|v\|_\infty + 1]}$; then $v_k \in S(\mathcal{A}_c)$, by Remark 10.5(a). Moreover, $|v_k| \leqslant \tilde{v} \wedge (\|v\|_\infty + 1) \in L_p(\mu)$ for all $k \in \mathbb{N}$ and $v_k \to v$ μ-a.e., hence $v_k \to v$ in $L_p(\mu)$ as $k \to \infty$.

(iii) Concerning the second statement we note that for $u \in L_{p_1} \cap L_{p_2}(\mu)$ the sequence defined in step (i) converges to u in $L_{p_1}(\mu)$ and in $L_{p_2}(\mu)$. Similarly, for $v \in L_1 \cap L_\infty(\mu)$, the sequence (v_n) constructed in step (ii) converges in $L_{p_1}(\mu)$ and in $L_{p_2}(\mu)$. □

We will use the notation

$$L_{1,\mathrm{loc}}(\mathcal{A}_c) := \big\{u\colon \Omega \to \mathbb{C};\ u \text{ measurable},\ \mathbf{1}_A u \in L_1(\mu)\ (A \in \mathcal{A}_c),\ [u \neq 0]\ \sigma\text{-finite}\big\},$$

where we understand the elements of $L_{1,\mathrm{loc}}(\mathcal{A}_c)$ as equivalence classes of a.e. equal functions. With these conventions one has $uv \in L_1(\mu)$ for all $u \in S(\mathcal{A}_c)$, $v \in L_{1,\mathrm{loc}}(\mathcal{A}_c)$. For completeness we recall that a set $A \in \mathcal{A}$ is called σ-finite if there exists a sequence (A_n) in $\mathcal{A}$, with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$, such that $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$. We note that $L_p(\mu) \subseteq L_{1,\mathrm{loc}}(\mathcal{A}_c)$ for all $p \in [1,\infty)$.

Let the strip $S \subseteq \mathbb{C}$ be defined as in Subsection 10.1.1, and let $p_0, p_1, q_0, q_1 \in [1,\infty]$, $M_0, M_1 \geqslant 0$. For $\tau \in (0,1)$ we denote

$$\frac{1}{p_\tau} := \frac{1-\tau}{p_0} + \frac{\tau}{p_1}, \quad \frac{1}{q_\tau} := \frac{1-\tau}{q_0} + \frac{\tau}{q_1}, \quad M_\tau := M_0^{1-\tau} M_1^{\tau}.$$

Finally, let $L(S(\mathcal{A}_c), L_{1,\mathrm{loc}}(\mathcal{A}_c))$ denote the linear operators from $S(\mathcal{A}_c)$ to $L_{1,\mathrm{loc}}(\mathcal{A}_c)$ (without continuity requirement), and let $\Phi\colon \overline{S} \to L(S(\mathcal{A}_c), L_{1,\mathrm{loc}}(\mathcal{A}_c))$ be a mapping satisfying the following two conditions.

(St1) $\|\Phi(j+\mathrm{i}s)u\|_{q_j} \leqslant M_j \|u\|_{p_j}$ for all $u \in S(\mathcal{A}_c)$, all $s \in \mathbb{R}$ and $j = 0, 1$. (The estimate means, in particular, that $\Phi(j+\mathrm{i}s)u \in L_{q_j}(\mu)$ for all the indicated terms.)

(St2) For all $A, B \in \mathcal{A}_c$ the function $\overline{S} \ni z \mapsto \int (\Phi(z)\mathbf{1}_A)\mathbf{1}_B \,\mathrm{d}\mu$ is continuous and bounded, and its restriction to S is holomorphic.

We can now state the Stein interpolation theorem, the main result of this section.

10.7 Theorem (Stein). *In the context described above it follows that*

$$\|\Phi(\tau+\mathrm{i}s)u\|_{q_\tau} \leqslant M_\tau \|u\|_{p_\tau}$$

for all $u \in S(\mathcal{A}_c)$, $s \in \mathbb{R}$ and $\tau \in (0,1)$.

Before proceeding to the proof we apply the theorem to the important special case of a constant function Φ.

10.8 Corollary (Riesz–Thorin). *Let the notation be as above, and let B be a linear operator from $S(\mathcal{A}_c)$ to $L_{1,\mathrm{loc}}(\mathcal{A}_c)$ such that $\|Bu\|_{q_j} \leqslant M_j \|u\|_{p_j}$ $(u \in S(\mathcal{A}_c),\ j = 0, 1)$.*

Then for all $\tau \in (0,1)$ one has $\|Bu\|_{q_\tau} \leqslant M_\tau \|u\|_{p_\tau}$ $(u \in S(\mathcal{A}_c))$.

As an application we recall Example 9.10, where it was shown that the operators $\mathrm{e}^{t\Delta_D}$ are sub-Markovian. Since we know that $\mathrm{e}^{t\Delta_D}$ is contractive in $L_2(\Omega)$, we conclude from Corollary 10.8 that $\mathrm{e}^{t\Delta_D}$ is contractive in $L_p(\Omega)$ for all $p \in [2, \infty]$.

The following fact will be needed in the proof of Theorem 10.7.

10.9 Lemma. *Let the notation be as above. Let $p, p' \in [1, \infty]$, $\frac{1}{p} + \frac{1}{p'} = 1$, $u \in L_{1,\mathrm{loc}}(\mathcal{A}_c)$, and assume that there exists $c \geqslant 0$ such that*

$$\left| \int uv \, \mathrm{d}\mu \right| \leqslant c \|v\|_{p'} \qquad (v \in S(\mathcal{A}_c)). \tag{10.1}$$

Then $u \in L_p(\mu)$, $\|u\|_p \leqslant c$.

Proof. (i) In the first step we show that (10.1) also holds with $|u|$ in place of u. Let $v \in S(\mathcal{A}_c)$. Then $\overline{\operatorname{sgn} u}\, \mathbf{1}_{[v \neq 0]} \in L_1(\mu)$, so there exists a sequence (w_k) in $S(\mathcal{A}_c)$ such that $w_k \to \overline{\operatorname{sgn} u}\, \mathbf{1}_{[v \neq 0]}$ μ-a.e. Then $v_k := v \operatorname{sgn} w_k \in S(\mathcal{A}_c)$ by Remark 10.5(a), $|uv_k| \leqslant |uv| \in L_1(\mu)$ for all $k \in \mathbb{N}$, and $uv_k = uv \operatorname{sgn} w_k \to |u|v$ μ-a.e., hence

$$\left| \int |u| v \, \mathrm{d}\mu \right| = \lim_{k \to \infty} \left| \int uv_k \, \mathrm{d}\mu \right| \leqslant c \sup_{k \in \mathbb{N}} \|v_k\|_{p'} \leqslant c \|v\|_{p'} .$$

(ii) In view of step (i) we now suppose, without loss of generality, that $u \geqslant 0$.

In the case $p = 1$ let $A \in \mathcal{A}$ be a set of finite measure. We choose a sequence (w_k) in $S(\mathcal{A}_c)$ such that $w_k \to \mathbf{1}_A$ μ-a.e. and put $v_k := \mathbf{1}_{[w_k \neq 0]}$. Then (v_k) is a sequence in $S(\mathcal{A}_c)_+$ with $\mathbf{1}_A \leqslant \liminf_{k \to \infty} v_k$, and Fatou's lemma implies

$$\int_A u \, \mathrm{d}\mu \leqslant \int u \liminf_{k \to \infty} v_k \, \mathrm{d}\mu \leqslant \liminf_{k \to \infty} \int uv_k \, \mathrm{d}\mu \leqslant c.$$

As $[u \neq 0]$ is σ-finite, the case $p = 1$ is settled.

In the case $p \in (1, \infty]$ let $v \in L_{p'}(\mu)_+$. There exists a sequence $(\tilde{v}_k)$ in $S(\mathcal{A}_c)$ such that $\tilde{v}_k \to v$ in $L_{p'}(\mu)$ and μ-a.e. as $k \to \infty$. Putting $v_k := (\operatorname{Re} \tilde{v}_k)^+$ we obtain a sequence (v_k) in $S(\mathcal{A}_c)_+$, still converging to v in $L_{p'}(\mu)$ and μ-a.e. Hence Fatou's lemma yields

$$\int uv \, \mathrm{d}\mu = \int u \liminf_{k \to \infty} v_k \, \mathrm{d}\mu \leqslant \liminf_{k \to \infty} \int uv_k \, \mathrm{d}\mu \leqslant c \lim_{k \to \infty} \|v_k\|_{p'} = c \|v\|_{p'} .$$

In step (iii) we show that this inequality implies the assertion for $p \in (1, \infty]$.

(iii) Let $p \in (1, \infty)$. Since $[u \neq 0]$ is σ-finite, u can be approximated pointwise by an increasing sequence (u_k) in $L_p(\mu)_+$. The inequality proved in step (ii) yields

$$\|u_k\|_p^p \leqslant \int uu_k^{p-1} \, \mathrm{d}\mu \leqslant c \big\| u_k^{p-1} \big\|_{p'} = c \|u_k\|_p^{p-1};$$

hence $\|u_k\|_p \leqslant c$ for all $k \in \mathbb{N}$, and consequently $\|u\|_p \leqslant c$.

In the case $p = \infty$ we obtain $\int u \mathbf{1}_A \, \mathrm{d}\mu \leqslant c \|\mathbf{1}_A\|_1$ for all $A \in \mathcal{A}$ with $\mu(A) < \infty$, and hence $\int_A (u - c) \, \mathrm{d}\mu \leqslant 0$. It follows that $u \leqslant c$ a.e. because $[u \neq 0]$ is σ-finite. □

Proof of Theorem 10.7. In order to motivate the subsequent computations we first explain why it is sufficient to show the inequality

$$\left|\int (\Phi(\tau)u)v\,\mathrm{d}\mu\right| \leqslant M_\tau \|u\|_{p_\tau}\|v\|_{q'_\tau} \qquad (u,v\in S(\mathcal{A}_\mathrm{c}),\ \tau\in(0,1)) \tag{10.2}$$

(where $\frac{1}{q_\tau}+\frac{1}{q'_\tau}=1$). In view of Lemma 10.9, this inequality implies the assertion for $s=0$. If $s\in\mathbb{R}$, then the previous consideration can be applied to the function $z\mapsto\Phi(z+\mathrm{i}s)$, and this shows the assertion for general s.

For the proof of (10.2) let $\tau\in(0,1)$, $u,v\in S(\mathcal{A}_\mathrm{c})$, without loss of generality $\|u\|_{p_\tau}=\|v\|_{q'_\tau}=1$. For $z\in\overline{S}$ put $\alpha(z):=\frac{1-z}{p_0}+\frac{z}{p_1}$, $\beta(z):=\frac{1-z}{q'_0}+\frac{z}{q'_1}$,

$$F(z):=\begin{cases}|u|^{\alpha(z)p_\tau}\operatorname{sgn}u & \text{if } p_\tau<\infty,\\ u & \text{if } p_\tau=\infty,\end{cases}$$

$$G(z):=\begin{cases}|v|^{\beta(z)q'_\tau}\operatorname{sgn}v & \text{if } q'_\tau<\infty,\\ v & \text{if } q'_\tau=\infty;\end{cases}$$

then $F(\tau)=u$, $G(\tau)=v$. Observe that $F(z),G(z)\in S(\mathcal{A}_\mathrm{c})$ for all $z\in\overline{S}$; cf. Remark 10.5(a).

Finally we define

$$h(z):=\int\big(\Phi(z)F(z)\big)G(z)\,\mathrm{d}\mu \qquad (z\in\overline{S}).$$

Then h is continuous, bounded, and holomorphic on S. In fact, since u and v have 'disjoint representations' as in Remark 10.5(a), it suffices to show these properties for the case $u=c\mathbf{1}_A$, $v=d\mathbf{1}_B$, with $c,d\in\mathbb{C}$, $A,B\in\mathcal{A}_\mathrm{c}$. If $p_\tau,q'_\tau<\infty$, then

$$h(z)=|c|^{\alpha(z)p_\tau}(\operatorname{sgn}c)|d|^{\beta(z)q'_\tau}(\operatorname{sgn}d)\int(\Phi(z)\mathbf{1}_A)\mathbf{1}_B\,\mathrm{d}\mu \qquad (z\in\overline{S}),$$

and this function has the required properties, by condition (St2); for the boundedness of h note that $\operatorname{Re}\alpha(z),\operatorname{Re}\beta(z)\in[0,1]$ for all $z\in\overline{S}$. An analogous – easier – argument applies if one or both of p_τ and q'_τ are equal to ∞.

The definition of F is such that $\|F(\sigma+\mathrm{i}s)\|_{p_\sigma}=1$ for all $\sigma\in[0,1]$, $s\in\mathbb{R}$. Indeed, note that $\operatorname{Re}\alpha(\sigma+\mathrm{i}s)=\frac{1}{p_\sigma}$. If $p_\sigma<\infty$, then $p_\tau<\infty$, and therefore

$$\|F(\sigma+\mathrm{i}s)\|_{p_\sigma}^{p_\sigma}=\int\big||u|^{\alpha(\sigma+\mathrm{i}s)p_\tau}\big|^{p_\sigma}\,\mathrm{d}\mu=\|u\|_{p_\tau}^{p_\tau}=1.$$

If $p_\sigma=\infty$, then $\operatorname{Re}\alpha(\sigma+\mathrm{i}s)=0$, and therefore

$$\|F(\sigma+\mathrm{i}s)\|_{p_\sigma}=\big\||u|^0\mathbf{1}_{[u\neq0]}\big\|_\infty=1.$$

An analogous observation applies to G. By condition (St1) it follows that

$$\|\Phi(\mathrm{i}s)F(\mathrm{i}s)\|_{q_0}\leqslant M_0\|F(\mathrm{i}s)\|_{p_0}=M_0\,,$$

$$|h(\mathrm{i}s)|=\left|\int(\Phi(\mathrm{i}s)F(\mathrm{i}s))G(\mathrm{i}s)\,\mathrm{d}\mu\right|\leqslant\|\Phi(\mathrm{i}s)F(\mathrm{i}s)\|_{q_0}\|G(\mathrm{i}s)\|_{q'_0}\leqslant M_0$$

for all $s\in\mathbb{R}$. In the same way one obtains $|h(1+\mathrm{i}s)|\leqslant M_1$ for all $s\in\mathbb{R}$.

At this point we can apply Theorem 10.3 and conclude that $\left|\int(\Phi(\tau)u)v\,\mathrm{d}\mu\right|=|h(\tau)|\leqslant M_\tau$, thus completing the proof of (10.2). □

10.2 Interpolation of semigroups

As in Section 10.1, the scalar field in this section will be $\mathbb{K} = \mathbb{C}$.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $p_1 \in [1, \infty)$, $\theta \in (0, \pi/2]$, let T be a bounded holomorphic C_0-semigroup of angle θ on $L_{p_1}(\mu)$, and put $M_1 := \sup_{z \in \Sigma_\theta} \|T(z)\|_{\mathcal{L}(L_{p_1}(\mu))}$. Let $p_0 \in [1, \infty]$, $p_0 \neq p_1$, and assume that $T|_{[0,\infty)}$ is L_{p_0}-bounded; by this we mean that there exists $M_0 \geqslant 0$ such that

$$\|T(t)u\|_{p_0} \leqslant M_0 \|u\|_{p_0} \qquad (u \in L_{p_1} \cap L_{p_0}(\mu),\ t \geqslant 0).$$

The following result on interpolation of holomorphic semigroups is our main application of the Stein interpolation theorem.

10.10 Theorem. *Let the hypotheses be as above, let $\tau \in (0,1)$, and put $\theta_\tau := \tau\theta$, $\frac{1}{p_\tau} := \frac{1-\tau}{p_0} + \frac{\tau}{p_1}$, $M_\tau := M_0^{1-\tau} M_1^\tau$.*

Then for all $z \in \Sigma_{\theta_\tau}$ the operator $T(z)|_{L_{p_1} \cap L_{p_\tau}(\mu)}$ extends (uniquely) to an operator $T_\tau(z) \in \mathcal{L}(L_{p_\tau}(\mu))$, and T_τ is a bounded holomorphic C_0-semigroup of angle θ_τ, $\|T_\tau(z)\| \leqslant M_\tau$ for all $z \in \Sigma_{\theta_\tau}$.

Proof. We use the notation $S(\mathcal{A}_{\mathrm{c}}), L_{1,\mathrm{loc}}(\mathcal{A}_{\mathrm{c}})$ from the beginning of Subsection 10.1.2, with $\mathcal{A}_{\mathrm{c}} := \{A \in \mathcal{A};\ \mu(A) < \infty\}$.

(i) The essential part of the proof is to show the boundedness of $T(z)|_{S(\mathcal{A}_{\mathrm{c}})}$ with respect to the L_{p_τ}-norm, as follows.

Let $\theta' \in (0, \theta)$. With the 'semi-sector' $\Sigma'_{\theta'} := \{z \in \mathbb{C} \setminus \{0\};\ 0 \leqslant \operatorname{Arg} z \leqslant \theta'\}$, the function $\psi\colon \overline{S} \to \Sigma'_{\theta'}$, $z \mapsto \mathrm{e}^{\mathrm{i}\theta' z}$ is continuous, bijective, and holomorphic on S. For $\sigma \in [0,1]$, it maps the line $[\mathrm{Re} = \sigma]$ onto the ray $\{r\mathrm{e}^{\mathrm{i}\theta'\sigma};\ r > 0\}$. From these properties it follows that the function $\Phi := T \circ \psi\colon \overline{S} \to L(S(\mathcal{A}_{\mathrm{c}}), L_{1,\mathrm{loc}}(\mathcal{A}_{\mathrm{c}}))$ satisfies the hypotheses of Theorem 10.7 with $q_0 = p_0$, $q_1 = p_1$. We just comment on the hypothesis (St2): for $A \in \mathcal{A}_{\mathrm{c}}$ the function $\overline{S} \ni z \mapsto T(\psi(z))\mathbf{1}_A \in L_{p_1}(\mu)$ is continuous, bounded, and holomorphic on S; hence for all $B \in \mathcal{A}_{\mathrm{c}}$ the function $\overline{S} \ni z \mapsto \int \big(T(\psi(z))\mathbf{1}_A\big)\mathbf{1}_B \,\mathrm{d}\mu$ has the corresponding properties.

For every $s \in \mathbb{R}$, Theorem 10.7 implies that $T(\psi(\tau + \mathrm{i}s))|_{S(\mathcal{A}_{\mathrm{c}})}$ is bounded with respect to the L_{p_τ}-norm, with norm $\leqslant M_\tau$. In other words, the L_{p_τ}-norm of $T(z)|_{S(\mathcal{A}_{\mathrm{c}})}$ is bounded by M_τ, for all z in the ray $\{r\mathrm{e}^{\mathrm{i}\theta'\tau};\ r > 0\}$. Note that the union of these rays, for $\theta' \in (0, \theta)$, is equal to the open semi-sector $\overset{\circ}{\Sigma}{}'_{\theta_\tau}$.

For the complementary open semi-sector $\big\{\overline{z};\ z \in \overset{\circ}{\Sigma}{}'_{\theta_\tau}\big\}$ the reasoning is analogous, and for $z \geqslant 0$ the boundedness statement follows from Corollary 10.8.

(ii) Let $z \in \Sigma_{\theta_\tau}$. As $S(\mathcal{A}_{\mathrm{c}})$ is a dense subspace of $L_{p_\tau}(\mu)$, the operator $T(z)|_{S(\mathcal{A}_{\mathrm{c}})}$ has a unique extension $T_\tau(z) \in \mathcal{L}(L_{p_\tau}(\mu))$. The operators $T(z)$ and $T_\tau(z)$ coincide on $L_{p_1} \cap L_{p_\tau}(\mu)$ since for all $u \in L_{p_1} \cap L_{p_\tau}(\mu)$ there exists a sequence (u_n) in $S(\mathcal{A}_{\mathrm{c}})$ such that $u_n \to u$ in $L_{p_1}(\mu)$ as well as in $L_{p_\tau}(\mu)$, by Lemma 10.6.

In order to show that $\Sigma_{\theta_\tau} \ni z \mapsto T_\tau(z)$ is holomorphic we use the results of Section 3.1. For $u, v \in S(\mathcal{A}_{\mathrm{c}})$ the function $\Sigma_{\theta_\tau} \ni z \mapsto \int (T(z)u)v \,\mathrm{d}\mu$ is holomorphic. As $S(\mathcal{A}_{\mathrm{c}})$ is dense in $L_{p_\tau}(\mu)$ and in $L_{p'_\tau}(\mu)$ and $T_\tau\colon \Sigma_{\theta_\tau} \to \mathcal{L}(L_{p_\tau}(\mu))$ is bounded, Theorem 3.4 implies that $z \mapsto T_\tau(z) \in \mathcal{L}(L_{p_\tau}(\mu))$ is holomorphic on Σ_{θ_τ}.

It remains to show that T_τ is strongly continuous at 0. We use Hölder's inequality

$$\|v\|_{p_\tau} \leqslant \|v\|_{p_0}^{1-\tau} \|v\|_{p_1}^{\tau} \qquad \big(v \in L_{p_0} \cap L_{p_1}(\mu)\big)$$

and obtain $\|T(t)u - u\|_{p_\tau} \to 0$ as $t \to 0$ for all $u \in S(\mathcal{A}_c)$, because $\{\|T(t)u\|_{p_0}\,;\, t \geqslant 0\}$ is bounded and $T(\cdot)u$ is continuous at 0 in $L_{p_1}(\mu)$. Then the combination of Lemma 1.5 and Proposition 3.9 implies that T_τ is strongly continuous at 0. □

10.3 Adjoint semigroups

In this section we insert some information on adjoint semigroups. Let T a C_0-semigroup on a Hilbert space H. Then clearly $T^* := (T(t)^*)_{t\geqslant 0}$ is a one-parameter semigroup on H. It turns out that T^* is strongly continuous, but this is not obvious; we will show it by looking at the generator of T. However, for simplicity we will restrict our treatment to the case of quasi-contractive semigroups.

10.11 Theorem. *Let H be a Hilbert space, let T be a quasi-contractive C_0-semigroup on H, and let A be its generator. Then A^* is the generator of a quasi-contractive C_0-semigroup, and the generated C_0-semigroup is T^* as defined above.*

Proof. By rescaling we can reduce the situation to the case in which T is contractive. Then Theorem 2.7 implies that $(0,\infty) \subseteq \rho(A)$ and that $\|(\lambda - A)^{-1}\| \leqslant \frac{1}{\lambda}$ for all $\lambda > 0$.

As A is closed, A^* is densely defined; see Theorem 6.3(b). Similarly as in the proof of Theorem 6.3(b) one obtains $(\lambda - A^*)^{-1} = ((\lambda - A)^*)^{-1} = ((\lambda - A)^{-1})^*$ for all $\lambda > 0$, and it follows that $\lambda \in \rho(A^*)$ and $\|(\lambda - A^*)^{-1}\| \leqslant \frac{1}{\lambda}$. Therefore the Hille–Yosida theorem, Theorem 2.9, implies that A^* generates a contractive C_0-semigroup.

From the exponential formula, Theorem 2.12, we conclude that the C_0-semigroup generated by A^* is the adjoint semigroup T^*. □

10.12 Remarks. (a) As a particular consequence of Theorem 10.11 we conclude: the semigroup T is **self-adjoint**, i.e. $T(t)$ is self-adjoint for all $t \geqslant 0$, if and only if its generator A is self-adjoint. See also Exercise 6.7.

(b) If $\mathbb{K} = \mathbb{C}$, and the semigroup T in Theorem 10.11 is holomorphic of some angle $\theta \in (0, \pi/2]$, then T^* (defined as the adjoint of $T|_{[0,\infty)}$) has a holomorphic extension to the sector Σ_θ. This extension is given by

$$T^*(z) := T(\bar{z})^* \qquad (z \in \Sigma_\theta). \tag{10.3}$$

Indeed, it is not difficult to show that T^*, defined by (10.3), is holomorphic. Hence T^* is a holomorphic C_0-semigroup. △

10.13 Remarks. (a) Using the general Hille–Yosida generation theorem (see Exercise 2.4) one also obtains Theorem 10.11 for general C_0-semigroups on H.

(b) If X is a Banach space and T is a C_0-semigroup on X, then it is not generally true that $T'(t) := T(t)'$ $(t \geqslant 0)$ defines a C_0-semigroup on X', where $T(t)' \in \mathcal{L}(X')$ is the dual operator. It is true, however, if X is reflexive. This follows from the fact that any weakly continuous one-parameter semigroup is strongly continuous; see [EnNa00; Theorem I.5.8]. △

10.4 Applications of invariance criteria and interpolation

Throughout this section let $(\Omega, \mathcal{A}, \mu)$ be a measure space.

We start our treatment with a criterion for a semigroup on a complex L_2-space to be real. A C_0-semigroup T on $L_2(\mu;\mathbb{C})$ is called **real** if $T(t)$ is a real operator for all $t \geqslant 0$.

10.14 Proposition. *Let V be a complex Hilbert space, $V \overset{\mathrm{d}}{\hookrightarrow} H := L_2(\mu;\mathbb{C})$, and let $a\colon V \times V \to \mathbb{R}$ be a bounded H-coercive form. Let A be the operator associated with a, and let T be the C_0-semigroup generated by $-A$. Then T is real if and only if $\operatorname{Re} u \in V$ for all $u \in V$ and $a(u,v) \in \mathbb{R}$ for all real $u, v \in V$.*

Now assume that T is real, and let $H_1 := L_2(\mu;\mathbb{R})$, $V_1 := V \cap H_1$ and $a_1 := a|_{V_1 \times V_1}$. Then $V_1 \overset{\mathrm{d}}{\hookrightarrow} H_1$, and a_1 is H_1-coercive; let $A_1 \sim a_1$. Then $A_1 = A \cap (H_1 \times H_1)$, and $-A_1$ is the generator of the restriction T_1 of T to H_1.

Proof. In order to show the first assertion we recall that the minimising projection P onto the closed convex set $L_2(\mu;\mathbb{R})$ is given by $Pu = \operatorname{Re} u$ ($u \in L_2(\mu;\mathbb{C})$). From the equivalence '(i) $\Leftrightarrow$ (ii)' in Theorem 9.22 we know that $L_2(\mu;\mathbb{R})$ is invariant under T if and only if $\operatorname{Re} u \in V$ for all $u \in V$ and for all real $u, v \in V$ one has

$$0 \leqslant \operatorname{Re} a\big(P(u \pm \mathrm{i}v), (I-P)(u \pm \mathrm{i}v)\big) = \operatorname{Re} a(u, \pm \mathrm{i}v) = \pm \operatorname{Im} a(u,v),$$

i.e. $a(u,v) \in \mathbb{R}$.

For the second part we show that $A_1 \subseteq A$. Indeed, if $(u,f) \in A_1$, then $u \in V_1 \subseteq V$, and for all $v \in V_1$ we have $a(u,v) = (f \mid v)$. From $V = V_1 + \mathrm{i}V_1$ we conclude that $a(u,v) = (f \mid v)$ for all $v \in V$, and thus $(u,f) \in A$. Interpreting $L_2(\mu;\mathbb{C})$ as a real Banach space and invoking Proposition 9.11 we obtain the assertion. (Incidentally, this argument also proves the sufficiency in the first assertion.) □

An operator $S \in \mathcal{L}(L_2(\mu))$ is called ***L*$_1$-contractive** if $\|Su\|_1 \leqslant \|u\|_1$ for all $u \in L_2 \cap L_1(\mu)$, and S is called **substochastic** if S is positive and L_1-contractive. The same terminology will be used for semigroups if all the semigroup operators satisfy the corresponding property.

In view of Proposition 10.14 (and for simplicity) we state the following criteria only for real-valued L_2-spaces.

10.15 Theorem (Beurling–Deny). *Let V be a real Hilbert space, $V \overset{\mathrm{d}}{\hookrightarrow} H := L_2(\mu;\mathbb{R})$, and let $a\colon V \times V \to \mathbb{R}$ be a bounded H-coercive form. Let A be the operator associated with a, and let T be the C_0-semigroup generated by $-A$. Then one has the following properties.*

(a) *T is positive if and only if $u^+ \in V$, $a(u^+, u^-) \leqslant 0$ for all $u \in V$.*

(b) *T is sub-Markovian if and only if $u \wedge 1 \in V$, $a(u \wedge 1, (u-1)^+) \geqslant 0$ for all $u \in V$.*

(c) *T is substochastic if and only if $u \wedge 1 \in V$, $a((u-1)^+, u \wedge 1) \geqslant 0$ for all $u \in V$.*

For the proof of part (c) we need an auxiliary result.

10.16 Lemma. *Let $S \in \mathcal{L}(L_2(\mu))$. Then S is L_∞-contractive if and only if S^* is L_1-contractive, and S is sub-Markovian if and only if S^* is substochastic.*

Proof. Let S be L_∞-contractive. Let $v \in L_2 \cap L_1(\mu)$, $\|v\|_1 \leqslant 1$. Then

$$\left| \int u \overline{S^* v} \, \mathrm{d}\mu \right| = \left| \int (Su) \bar{v} \, \mathrm{d}\mu \right| \leqslant 1 \qquad \big(u \in L_2 \cap L_\infty(\mu),\ \|u\|_\infty \leqslant 1\big),$$

and from Lemma 10.9, applied with $\mathcal{A}_{\mathrm{c}} := \{A \in \mathcal{A};\ \mu(A) < \infty\}$, we conclude that $\|S^* v\|_1 \leqslant 1$. The converse statement is proved in the same way.

For the second statement it remains to observe that S is positive if and only if S^* is positive. This equivalence is valid (regardless of whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) because $f \in L_2 \cap L_\infty(\mu)$ is positive if and only if $\int g f \, \mathrm{d}\mu \geqslant 0$ for all $g \in L_2 \cap L_1(\mu)_+$, and similarly with exchanged roles of $L_2 \cap L_1(\mu)$ and $L_2 \cap L_\infty(\mu)$. □

Proof of Theorem 10.15. In view of Remarks 9.3 and 9.5, the statements (a) and (b) are easy consequences of '(i) ⇔ (ii)' in Theorem 9.22. For (a) we note that $u - u^+ = -u^-$, and for (b) we note that $u - u \wedge 1 = (u-1)^+$.

For part (c) we recall that $-A^*$ is the generator of the C_0-semigroup T^*, by Theorem 10.11, and that A^* is associated with the form a^*, by Theorem 6.11. Lemma 10.16 implies that T is substochastic if and only if T^* is sub-Markovian, or equivalently, $u \wedge 1 \in V$ and $a((u-1)^+, u \wedge 1) = a^*(u \wedge 1, (u-1)^+) \geqslant 0$ for all $u \in V$, by part (b). □

10.17 Remarks. (a) The equivalences formulated in Theorem 10.15 are the **Beurling–Deny criteria**. A form a satisfying the conditions in parts (b) and (c) of Theorem 10.15 is called a (**non-symmetric**) **Dirichlet form**.

(b) In view of condition (iii) of Theorem 9.22 one could also formulate the conditions in Theorem 10.15 with a dense subset of V. △

10.18 Theorem. *Let T be a C_0-semigroup on $L_2(\mu)$.*

(a) *Assume that T is sub-Markovian and substochastic. Then for all $p \in [1, \infty)$ the operators $T(t)|_{L_2 \cap L_p(\mu)}$ extend to operators $T_p(t) \in \mathcal{L}(L_p(\mu))$, and T_p thus defined is a contractive C_0-semigroup on $L_p(\mu)$. For $p, q \in [1, \infty)$ the semigroups T_p, T_q are **consistent**, i.e. $T_p(t)|_{L_p \cap L_q(\mu)} = T_q(t)|_{L_p \cap L_q(\mu)}$ for all $t \geqslant 0$.*

(b) *Assume that T is self-adjoint and sub-Markovian. Then the assertions of* (a) *hold. If $\mathbb{K} = \mathbb{C}$, then for all $p \in (1, \infty)$ the semigroup T_p extends to a contractive holomorphic C_0-semigroup of angle*

$$\theta_p = \begin{cases} \big(1 - \frac{1}{p}\big)\pi & \text{if } 1 < p < 2, \\ \frac{1}{p}\pi & \text{if } 2 \leqslant p < \infty. \end{cases}$$

Proof. (a) Let $1 \leqslant p < \infty$. For every $t > 0$, Exercise 10.4 (or Corollary 10.8, if $\mathbb{K} = \mathbb{C}$) implies that $T(t)|_{L_1 \cap L_\infty(\mu)}$ extends to a contractive operator $T_p(t)$ on $L_p(\mu)$. It is standard to show that T_p is a one-parameter semigroup. The strong continuity of T_p at 0 is obtained as follows. If $f \in L_p \cap L_2(\mu)$ and (t_n) is a null sequence in $(0, \infty)$, then $T(t_n) f \to f$ in $L_2(\mu)$, so for a subsequence one has $T(t_{n_k}) f \to f$ a.e. Now the contractivity of T_p in combination with Lemma 10.19, proved subsequently, implies that $T_p(t_{n_k}) f \to f$ in $L_p(\mu)$. Applying a standard sub-subsequence argument – see Exercise 9.7 – one obtains $T_p(t) f \to f$ in $L_p(\mu)$ as $t \to 0+$. Lemma 1.5 concludes the argument.

The consistency is shown as in step (ii) of the proof of Theorem 10.10.

(b) From $T(t)^* = T(t)$ for all $t \geqslant 0$ and Lemma 10.16 it follows that T is also sub-stochastic. Thus part (a) is applicable. Next, the generator A of T is self-adjoint by Remark 10.12(a), and as T is contractive, $-A$ is accretive.

Now let $\mathbb{K} = \mathbb{C}$. Then $-A$ is m-sectorial of angle 0, by Theorem 6.1. Hence A is the generator of a contractive holomorphic C_0-semigroup of angle $\pi/2$; see Theorem 3.20. In view of Theorem 10.10 this implies the remaining assertions. □

10.19 Lemma. *Let $1 \leqslant p < \infty$ and $f \in L_p(\mu)$. Let (f_n) be a sequence in $L_p(\mu)$ such that $f_n \to f$ a.e. and $\limsup_{n\to\infty} \|f_n\|_p \leqslant \|f\|_p$. Then $f_n \to f$ in $L_p(\mu)$.*

Proof. For $n \in \mathbb{N}$ put $\check{f}_n := \operatorname{sgn} f_n \, (|f| \wedge |f_n|)$. Then $\check{f}_n \to f$ in $L_p(\mu)$ by the dominated convergence theorem. Moreover $|f_n| = |\check{f}_n| + |f_n - \check{f}_n|$, hence $|f_n|^p \geqslant |\check{f}_n|^p + |f_n - \check{f}_n|^p$ for all $n \in \mathbb{N}$. Therefore $\|f_n - \check{f}_n\|_p^p \leqslant \|f_n\|_p^p - \|\check{f}_n\|_p^p$, which implies $\limsup \|f_n - \check{f}_n\|_p^p \leqslant 0$, and it follows that $f_n = (f_n - \check{f}_n) + \check{f}_n \to f$ in $L_p(\mu)$. □

Notes

The three lines theorem is generally attributed to Hadamard. In fact, in [Had96] Hadamard announced a variant, the 'three circles theorem'; the well-established version of the three lines theorem was first stated and proved by Doetsch [Doe20]. The Stein interpolation theorem, essentially in the form presented here, is contained in [Ste56]. We refer to this paper for some history of the Riesz–Thorin convexity theorem, finally proved by Thorin by the complex variable method, which initiated a whole new branch of functional analysis. In fact, the paper [Ste56] can be considered as the start of interpolation theory, for which we refer to the seminal paper of Calderón [Cal64] as well as to the monographs [BeLö76], [Lun18].

The application of invariance and interpolation as described in Section 10.4 is well-established in the theory of semigroups for diffusion equations, Schrödinger semigroups and related topics. The proof of Theorem 10.18(b) via interpolation appears natural and elegant; however, it does not yield the optimal angle of holomorphy for the L_p-semigroup. Indeed, Liskevich and Perelmuter showed in [LiPe95] that one obtains the larger angle $\arccos|1 - \frac{2}{p}|$ if one exploits more directly that the semigroup is sub-Markovian. There seems to be no way to obtain the larger angle via interpolation; an example in [Voi96] shows that the angle from [LiPe95] is optimal. It came as a big surprise when Kriegler [Kri11] showed that extension to a contractive holomorphic C_0-semigroup on L_p of angle $\arccos|1 - \frac{2}{p}|$ is still valid if one merely assumes that T is self-adjoint and L_∞-contractive (and not necessarily positive). We refer to [HKV16] for an elementary proof of Kriegler's result.

Exercises

10.1 For this exercise let S be the strip

$$S := \{z \in \mathbb{C};\ -1/2 < \operatorname{Re} z < 1/2\}$$

(of width 1). Let $h\colon \overline{S} \to \mathbb{C}$ be continuous and holomorphic on S. Assume that h is bounded on ∂S and that there exist $c > 0$ and $\alpha \in (0, \pi)$ such that

$$|h(z)| \leqslant \mathrm{e}^{c\mathrm{e}^{\alpha|\operatorname{Im} z|}} \qquad (z \in S).$$

Show that h is bounded by $\|h\|_{\partial S}$. (Hint: Use $\psi_n(z) := \mathrm{e}^{-\frac{1}{n}(\mathrm{e}^{\mathrm{i}\beta z}+\mathrm{e}^{-\mathrm{i}\beta z})}$, with $\alpha < \beta < \pi$.)

10.2 Let Ω be a set, $\mathcal{A}_\mathrm{c}$ a $\cap$-stable collection of subsets of Ω (i.e. $A \cap B \in \mathcal{A}_\mathrm{c}$ for all $A, B \in \mathcal{A}_\mathrm{c}$), and let $\mathcal{R}$ be the ring of subsets generated by $\mathcal{A}_\mathrm{c}$ (i.e. $\mathcal{R}$ is the smallest ring containing $\mathcal{A}_\mathrm{c}$).

(a) Show that $\mathbf{1}_A \in S(\mathcal{A}_\mathrm{c})$ for all $A \in \mathcal{R}$.

Hint: Show first that the collection $\{A \subseteq \Omega;\ \mathbf{1}_A \in S(\mathcal{A}_\mathrm{c})\}$ is a ring. (Recall from Remark 10.5(a) that $S(\mathcal{A}_\mathrm{c})$ is an algebra.)

(b) Show that every function $u \in S(\mathcal{R})$ can be written as described in Remark 10.5(a).

10.3 Let (Ω, μ) be a measure space. Show that

$$\big(\|\tfrac{1}{2}(f+g)\|_p^p + \|\tfrac{1}{2}(f-g)\|_p^p\big)^{1/p} \leqslant 2^{-1/p}\big(\|f\|_p^p + \|g\|_p^p\big)^{1/p} \tag{10.4}$$

for all $f, g \in L_p(\mu)$, $2 \leqslant p \leqslant \infty$.

Hint: Use the mapping

$$T\colon L_p(\mu) \times L_p(\mu) \to L_p(\mu) \times L_p(\mu),\ (f,g) \mapsto \big(\tfrac{1}{2}(f+g), \tfrac{1}{2}(f-g)\big).$$

Compute the norm of T for $p = 2$ and for $p = \infty$ and use the Riesz–Thorin interpolation theorem. (The inequality (10.4) is one of Clarkson's inequalities; see Adams [Ada75; Theorem 2.28]. The other inequalities of Clarkson involve p and the conjugate exponent p' and can also be obtained by interpolation, but this is more complicated. Clarkson's inequalities can be used to show that L_p-spaces, for $1 < p < \infty$, are uniformly convex – see [Ada75; Corollary 2.29] –, which in turn implies that these spaces are reflexive.)

10.4 (a) Let $p \in (1, \infty)$, $r \in [0, \infty)$. Show that

$$r = \inf_{\alpha \in (0,\infty)\cap\mathbb{Q}} \big(\tfrac{1}{p}\alpha^{1-p}r^p + (1 - \tfrac{1}{p})\alpha\big).$$

(b) Let (Ω, μ) be a measure space, and let $S \in \mathcal{L}(L_2(\mu))$ be sub-Markovian and substochastic. Show that S is L_p-contractive for all $p \in (1, \infty)$. (The case $\mathbb{K} = \mathbb{C}$ is already covered by Corollary 10.8, but not the case $\mathbb{K} = \mathbb{R}$!)

Hint: Using (a) twice show first that $S|u| \leqslant \big(S|u|^p\big)^{1/p}$ for simple functions.

(c) Let (Ω, μ) be a measure space, let $S \in \mathcal{L}(L_2(\mu))$ be sub-Markovian, and assume that there exists $c > 0$ such that $\frac{1}{c}S$ is substochastic. Show that S "extrapolates" to an operator $S_p \in \mathcal{L}(L_p(\mu))$ with $\|S_p\| \leqslant c^{1/p}$, for all $1 < p < \infty$.

10.5 (Continuation of Exercise 9.6) Let the hypotheses be as in Exercise 9.6. Assume additionally that $b \in C^1(\Omega; \mathbb{R}^n)$ and that $\omega := \sup_{x\in\Omega} \operatorname{div} b(x) < \infty$.

(a) Show that $\|T(t)u\|_1 \leqslant \mathrm{e}^{\omega t}\|u\|_1$ $(u \in L_2 \cap L_1(\Omega),\ t \geqslant 0)$, where T is the C_0-semigroup generated by the operator $-A$.

Hint: Use $C_c^1(\Omega)$ as the dense subset of $V = H_0^1(\Omega)$ for the application of the invariance criterion to the semigroup $\big(\mathrm{e}^{-\omega t}T(t)\big)_{t\geqslant 0}$. Observe that on $C_c^1(\Omega)$ one can transform $(b\cdot\nabla u\,|\,v)$ – using integration by parts – into an expression in which u appears without a derivative.

(b) Compute estimates for $\|T_p(t)\|$ in terms of ω for $t \geqslant 0,\ 1 \leqslant p < \infty$, where T_p is the interpolated semigroup on $L_p(\Omega)$, analogous to Theorem 10.18(b).

Chapter 11

Elliptic operators

Elliptic operators with measurable coefficients are a classical topic in partial differential equations. They have realisations under diverse boundary conditions that generate semigroups, which results in well-posedness of parabolic initial boundary value problems. Form methods are very efficient to treat these problems and to derive properties of the solutions of the equations. It will be seen that many of the topics presented so far enter our treatment; in particular the properties of the Sobolev space $H^1(\Omega)$ and the Beurling–Deny criteria will play a decisive role. The latter lead to positivity and to sub-Markovian and substochastic behaviour of the generated semigroups.

In order to achieve these goals it turns out that additional lattice properties of $H^1(\Omega)$ – treated in an interlude – are needed.

11.1 Perturbation of bounded forms

Versions of the following perturbation result have already appeared in the proof of Theorem 7.17 and in Exercise 7.7(a).

11.1 Lemma. *Let V, H be Hilbert spaces, $V \overset{\mathrm{d}}{\hookrightarrow} H$. Let $a\colon V \times V \to \mathbb{K}$ be a bounded H-coercive form. Let $b\colon V \times V \to \mathbb{K}$ be a bounded form, and assume that there exists $M \geqslant 0$ such that*

$$|b(u)| \leqslant M\|u\|_V\|u\|_H \qquad (u \in V).$$

Then $a + b\colon V \times V \to \mathbb{K}$ is H-coercive.

Proof. By the H-coercivity of a there exist $\omega \in \mathbb{R}$ and $\alpha > 0$ such that

$$\operatorname{Re} a(u) + \omega\|u\|_H^2 \geqslant \alpha\|u\|_V^2$$

for all $u \in V$.

By the Peter–Paul inequality one has

$$\begin{aligned}\operatorname{Re} a(u) + \operatorname{Re} b(u) + \omega\|u\|_H^2 &\geqslant \alpha\|u\|_V^2 - M\|u\|_V\|u\|_H \\ &\geqslant \alpha\|u\|_V^2 - \frac{1}{2}\Big(\alpha\|u\|_V^2 + \frac{1}{\alpha}M^2\|u\|_H^2\Big).\end{aligned}$$

This implies

$$\operatorname{Re}\big(a(u) + b(u)\big) + \Big(\omega + \frac{M^2}{2\alpha}\Big)\|u\|_H^2 \geqslant \frac{\alpha}{2}\|u\|_V^2 \qquad (u \in V). \qquad \square$$

11.2 Elliptic operators

Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $a_{jk} \in L_\infty(\Omega)$ $(j,k = 1,\dots,n)$ be coefficient functions satisfying the **ellipticity condition**

$$\operatorname{Re} \sum_{j,k=1}^{n} a_{jk}(x)\xi_k\overline{\xi_j} \geqslant \alpha|\xi|^2 \qquad (\xi \in \mathbb{K}^n) \tag{11.1}$$

for a.e. $x \in \Omega$, with $\alpha > 0$, and let $b_j, c_j \in L_\infty(\Omega)$ $(j = 1,\dots,n)$, $d \in L_\infty(\Omega)$. Our aim is to define operators in $L_2(\Omega)$ corresponding to the "elliptic operator in divergence form" $\mathcal{A}$ written formally as

$$\begin{aligned} \mathcal{A}u &= -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \sum_{j=1}^{n}\big(b_j\partial_j u - \partial_j(c_j u)\big) + du \\ &= -\operatorname{div}\big((a_{jk})\nabla u\big) + b\cdot\nabla u - \operatorname{div}(cu) + du. \end{aligned} \tag{11.2}$$

The first order terms in $\mathcal{A}$ will sometimes be called the **drift terms**. We consider the form

$$a\colon H^1(\Omega)\times H^1(\Omega) \to \mathbb{K}$$

given by

$$a(u,v) = \int_\Omega \Big(\sum_{j,k=1}^{n} a_{jk}\partial_k u\overline{\partial_j v} + \sum_{j=1}^{n}(b_j\partial_j u\,\overline{v} + c_j u\overline{\partial_j v}) + du\overline{v}\Big)\,\mathrm{d}x. \tag{11.3}$$

11.2 Proposition. *The form a is bounded and $L_2(\Omega)$-coercive.*

Proof. Let the form $a_0\colon H^1(\Omega)\times H^1(\Omega)\to\mathbb{K}$ be given by

$$a_0(u,v) = \int_\Omega \sum_{j,k=1}^{n} a_{jk}\partial_k u\,\overline{\partial_j v}\,\mathrm{d}x.$$

Then the boundedness of the coefficients a_{jk} implies that a_0 is bounded. By the ellipticity condition (11.1) one obtains $\operatorname{Re} a_0(u) + \alpha\|u\|_{L_2}^2 \geqslant \alpha\|\nabla u\|_{L_2}^2 + \alpha\|u\|_{L_2}^2 = \alpha\|u\|_{H^1}^2$ for all $u \in H^1(\Omega)$. Thus a_0 is $L_2(\Omega)$-coercive.

Define $a_1\colon H^1(\Omega)\times H^1(\Omega)\to\mathbb{K}$ by

$$a_1(u,v) = \int_\Omega \Big(\sum_{j=1}^{n}(b_j\partial_j u\,\overline{v} + c_j u\overline{\partial_j v}) + du\overline{v}\Big)\,\mathrm{d}x;$$

then $a = a_0 + a_1$.

The boundedness of the coefficient functions b_j, c_j and d implies that a_1 is bounded. It also implies that there exists $M \geqslant 0$ such that

$$\begin{aligned} |a_1(u)| &\leqslant M\int_\Omega |\nabla u||u|\,\mathrm{d}x + \|d\|_{L_\infty}\|u\|_{L_2}^2 \\ &\leqslant M\|\nabla u\|_{L_2}\|u\|_{L_2} + \|d\|_{L_\infty}\|u\|_{L_2}^2 \\ &\leqslant (M + \|d\|_{L_\infty})\|u\|_{H^1}\|u\|_{L_2} \end{aligned}$$

for all $u \in H^1(\Omega)$. Now the claim follows from Lemma 11.1. □

Coming back to (11.2), we now give this formula a precise meaning, using the distributional divergence of a vector field $w \in L_{1,\mathrm{loc}}(\Omega;\mathbb{K}^n)$. We say that $\operatorname{div} w \in L_{1,\mathrm{loc}}(\Omega)$ if there exists $g \in L_{1,\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega w \cdot \nabla\varphi \,\mathrm{d}x = -\int_\Omega g\varphi \,\mathrm{d}x$$

for all test functions $\varphi \in C_\mathrm{c}^\infty(\Omega)$, and we say that $\operatorname{div} w = g$ **in the distributional sense**. (This definition was already mentioned in Exercise 4.5(c).) Then the maximal operator A_max in $L_2(\Omega)$ is defined by

$$A_\mathrm{max} u := -\operatorname{div}\big((a_{jk})\nabla u + cu\big) + b\cdot\nabla u + du$$

for $u \in H^1(\Omega)$ with $\operatorname{div}\big((a_{jk})\nabla u + cu\big) \in L_2(\Omega)$. In terms of the form a, the maximal operator can be written as

$$A_\mathrm{max} = \big\{(u,f) \in H^1(\Omega)\times L_2(\Omega);\, a(u,v) = (f\,|\,v)\ (v \in C_\mathrm{c}^\infty(\Omega))\big\}.$$

Next we define suitable restrictions of A_max. Let V be a closed subspace of $H^1(\Omega)$ containing $H_0^1(\Omega)$. Then the restriction of the form a to $V\times V$ is bounded and $L_2(\Omega)$-coercive, by Proposition 11.2. Denote by A_V the operator associated with $a|_{V\times V}$. Then $-A_V$ generates a C_0-semigroup T_V on $L_2(\Omega)$, and T_V is holomorphic if $\mathbb{K} = \mathbb{C}$; see Section 5.3. Note that the operator A_V is a restriction of A_max because $C_\mathrm{c}^\infty(\Omega) \subseteq H_0^1(\Omega)$.

11.3 Remark. Let $V \subseteq H_0^1(\Omega)$ be as above. For $(u,f) \in A_\mathrm{max}$ to be in A_V it is needed that

(i) $u \in V$ (and not merely $u \in H^1(\Omega)$),

(ii) $a(u,v) = (f\,|\,v)_{L_2}$ for all $v \in V$ (and not merely for $v \in C_\mathrm{c}^\infty(\Omega)$).

These conditions can be interpreted as boundary conditions. △

11.4 Examples. (a) Let us first consider the case $V = H_0^1(\Omega)$. Then $A_{H_0^1}$ is just the restriction of A_max to $\operatorname{dom}(A_\mathrm{max}) \cap H_0^1(\Omega)$. We write $A_\mathrm{D} := A_{H_0^1}$ and call A_D the realisation of the elliptic operator $\mathcal{A}$ with **Dirichlet boundary conditions**. We define $T_\mathrm{D} := T_{H_0^1}$.

(b) Next we consider $V = H^1(\Omega)$. We define $T_\mathrm{N} := T_{H^1}$ and call $A_\mathrm{N} := A_{H^1}$ the realisation of the elliptic operator $\mathcal{A}$ with (generalised) **Neumann boundary conditions**. However, we will see that it is not the normal derivative of u that is 0 at the boundary, but rather the **conormal derivative**

$$\nu\cdot\big((a_{jk})\nabla u + cu\big)\big|_{\partial\Omega},$$

which is defined in terms of the coefficients of $\mathcal{A}$.

Clearly, the vector field $w := (a_{jk})\nabla u + cu \in L_2(\Omega;\mathbb{K}^n)$ cannot simply be restricted to $\partial\Omega$, so we first have to explain what is meant by $\nu\cdot w|_{\partial\Omega} = 0$. If Ω is bounded and has C^1-boundary, and $w \in C^1(\overline{\Omega};\mathbb{K}^n)$, then for all $\varphi \in C^1(\overline{\Omega})$ we obtain

$$\int_\Omega w\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega (\operatorname{div} w)\varphi\,\mathrm{d}x = \int_\Omega \operatorname{div}(w\varphi)\,\mathrm{d}x = \int_{\partial\Omega} \nu\cdot w\varphi\,\mathrm{d}\sigma,$$

by Gauss' theorem; note that the right-hand side vanishes for all φ if and only if $\nu \cdot w|_{\partial\Omega} = 0$. Thus, for an arbitrary open set $\Omega \subseteq \mathbb{R}^n$ and $w \in L_2(\Omega;\mathbb{K}^n)$ satisfying $\operatorname{div} w \in L_2(\Omega)$ we will write $\nu \cdot w|_{\partial\Omega} = 0$ if

$$\int_\Omega w \cdot \nabla v \,\mathrm{d}x + \int_\Omega (\operatorname{div} w) v \,\mathrm{d}x = 0 \qquad (v \in H^1(\Omega)).$$

Now let $(u,f) \in A_{\max}$, and recall that A_{N} is a restriction of $A_{\max}$. For (u,f) to be in A_{N}, only condition (ii) of Remark 11.3 plays a role. Thus, $(u,f) \in A_{\mathrm{N}}$ if and only if

$$\int_\Omega \big((a_{jk})\nabla u + cu\big) \cdot \overline{\nabla v} \,\mathrm{d}x = \int_\Omega (f - b\cdot\nabla u - du)\overline{v} \,\mathrm{d}x \qquad (v \in H^1(\Omega)),$$

and since $f = A_{\max} u$, the latter is equivalent to

$$\int_\Omega \big((a_{jk})\nabla u + cu\big) \cdot \overline{\nabla v} \,\mathrm{d}x = -\int_\Omega \operatorname{div}\big((a_{jk})\nabla u + cu\big)\overline{v} \,\mathrm{d}x \qquad (v \in H^1(\Omega)).$$

Thus, in terms of the above definition, we obtain

$$A_{\mathrm{N}} = \big\{(u,f) \in A_{\max};\, \nu \cdot \big((a_{jk})\nabla u + cu\big)\big|_{\partial\Omega} = 0\big\}.$$

(c) There are other possible choices of V. For example, assume that Ω is bounded and has C^1-boundary, and let $\Gamma \subseteq \partial\Omega$ be a Borel set. Then we define

$$V = \big\{u \in H^1(\Omega);\, \operatorname{tr} u = 0 \text{ a.e. on } \Gamma\big\}.$$

We call A_V the realisation of the elliptic operator $\mathcal{A}$ with **mixed boundary conditions** (with the interpretation 'Dirichlet on Γ, Neumann on $\partial\Omega \setminus \Gamma$'). △

11.5 Remarks. Assume that the coefficient matrix $(a_{jk}(x))$ is self-adjoint, i.e. $a_{jk}(x) = \overline{a_{kj}(x)}$ $(j,k = 1,\dots,n)$ for a.e. $x \in \Omega$. Then the ellipticity condition (11.1) says that the smallest eigenvalue of the matrix $(a_{jk}(x))$ is $\geqslant \alpha$ for a.e. $x \in \Omega$. The properties asserted in the following statements hold for any closed subspace V of $H^1(\Omega)$ containing $H_0^1(\Omega)$.

(a) In the complex case, the semigroup T_V generated by $-A_V$ is holomorphic of angle $\pi/2$. This follows from Exercise 7.7(c), applied with the decomposition $a = a_0 + a_1$ from Proposition 11.2; note that $a_0|_{V\times V}$ is symmetric by the assumption on (a_{jk}).

(b) If $b_j = \overline{c_j}$ $(j = 1,\dots,n)$ and d is real-valued, then a is symmetric, and the operator A_V is self-adjoint. △

We conclude this section with comments on the interplay between the real and complex cases.

11.6 Remarks. Assume that all the coefficient functions a_{jk}, b_j, c_j, d are real-valued. Then the ellipticity condition (11.1) needs to be checked for $\xi \in \mathbb{R}^n$ only; see Exercise 11.1.

(a) Let $\mathbb{K} = \mathbb{C}$, and let V be a closed subspace of $H^1(\Omega)$ containing $H_0^1(\Omega)$ with the property that $\operatorname{Re} u \in V$ for all $u \in V$. Then Proposition 10.14 implies that T_V is real. Moreover, with $V_{\mathbb{R}} = V \cap L_2(\Omega;\mathbb{R})$, the restriction of a to $V_{\mathbb{R}} \times V_{\mathbb{R}}$ is bounded and $L_2(\Omega;\mathbb{R})$-coercive, and the operator associated with $a|_{V_{\mathbb{R}}\times V_{\mathbb{R}}}$ is minus the generator of the restriction of T_V to $L_2(\Omega;\mathbb{R})$.

(b) Conversely, if V is a closed subspace of $H^1(\Omega;\mathbb{R})$ containing $H_0^1(\Omega;\mathbb{R})$, then $V_{\mathbb{C}} = V + \mathrm{i}V \subseteq H^1(\Omega;\mathbb{C})$ is a space as considered in (a). △

In order to study further properties of the semigroup on $L_2(\Omega;\mathbb{R})$ generated by an elliptic operator with real coefficients we need additional properties of the Sobolev space $H^1(\Omega)$.

11.3 Interlude: Further lattice properties of $H^1(\Omega)$

Throughout this section the scalar field will be $\mathbb{K} = \mathbb{R}$. Let $\Omega \subseteq \mathbb{R}^n$ be open. In Section 9.3 we have seen that $H^1(\Omega)$ is a vector sublattice of $L_2(\Omega)$. More precisely, if $u \in H^1(\Omega)$, then Theorem 9.16 implies that $u^+, u^- = (-u)^+, |u| = u^+ + u^- \in H^1(\Omega)$ and $\partial_j u^+ = \mathbf{1}_{[u>0]}\partial_j u$, $\partial_j u^- = -\mathbf{1}_{[u<0]}\partial_j u$, $\partial_j |u| = \partial_j u^+ + \partial_j u^- = (\operatorname{sgn} u)\,\partial_j u$. It follows that

$$\||u|\|_{H^1} = \|u\|_{H^1} \qquad (u \in H^1(\Omega)) \tag{11.4}$$

since $\partial_j u = \partial_j u^+ - \partial_j u^- = \mathbf{1}_{[u\neq 0]}\partial_j u$ $(j = 1, \dots, n)$. Incidentally, this last equality implies that $\mathbf{1}_{[u=0]}\partial_j u = 0$, i.e., $\partial_j u = 0$ a.e. on $[u = 0]$ ("Stampacchia's Lemma").

Next we show that the lattice operations are continuous.

11.7 Proposition. (a) *The mapping $H^1(\Omega) \ni u \mapsto |u| \in H^1(\Omega)$ is continuous.*

(b) *The mappings $(u, v) \mapsto u \wedge v$ and $(u, v) \mapsto u \vee v$ are continuous from $H^1(\Omega) \times H^1(\Omega)$ to $H^1(\Omega)$. In particular, the mappings $H^1(\Omega) \ni u \mapsto u^+, u^- \in H^1(\Omega)$ are continuous.*

Proof. (a) Let (u_k) be a sequence in $H^1(\Omega)$, $u_k \to u$ in $H^1(\Omega)$. Then $|u_k| \to |u|$ in $L_2(\Omega)$, and (11.4) implies that $(|u_k|)$ is bounded in $H^1(\Omega)$. By Remark 9.17 it follows that $|u_k| \to |u|$ weakly in $H^1(\Omega)$.

From (11.4) we also obtain

$$\||u_k|\|_{H^1} = \|u_k\|_{H^1} \to \|u\|_{H^1} = \||u|\|_{H^1} \qquad (k \to \infty),$$

from which we conclude that $|u_k| \to |u|$ in $H^1(\Omega)$; see the subsequent Remark 11.8.

(b) follows from $u \wedge v = \frac{1}{2}(u + v - |u - v|)$, $u \vee v = \frac{1}{2}(u + v + |u - v|)$ and part (a). □

11.8 Remark. Let H be a Hilbert space, (u_k) a sequence in H, $u_k \to u$ weakly in H, and $\|u_k\| \to \|u\|$ as $k \to \infty$. Then

$$\|u_k - u\|^2 = \|u_k\|^2 + \|u\|^2 - 2\operatorname{Re}(u_k \,|\, u) \to 0;$$

so $u_k \to u$ in H as $k \to \infty$. △

For a function $u\colon \Omega \to \mathbb{R}$ we denote by $\tilde{u}\colon \mathbb{R}^n \to \mathbb{R}$ the extension of u by 0,

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

From Subsection 4.1.4 we recall the definitions

$$H^1_{\mathrm{c}}(\Omega) := \big\{ f \in H^1(\Omega);\, \operatorname{spt} f \text{ compact in } \Omega \big\},$$
$$H^1_0(\Omega) := \overline{H^1_{\mathrm{c}}(\Omega)}^{H^1(\Omega)}$$

and the denseness property $H^1_0(\Omega) = \overline{C^\infty_{\mathrm{c}}(\Omega)}^{H^1(\Omega)}$. We also recall that for $u \in H^1_0(\Omega)$ one has $\tilde{u} \in H^1(\mathbb{R}^n)$ and $\partial_j \tilde{u} = \widetilde{\partial_j u}$ $(j = 1, \dots, n)$; see Remark 4.13(b) and Exercise 4.6(b). We already know that $H^1_0(\Omega)$ is a vector sublattice of $H^1(\Omega)$; see Theorem 9.16. (A **vector**

sublattice of $H^1(\Omega)$ is a subspace V with the property that $u^+ \in V$ for all $u \in V$, analogously to the definition of vector sublattices of $L_2(\Omega)$.)

We now show that $H^1_0(\Omega)$ is even an ideal in $H^1(\Omega)$ (in the sense of vector lattices). An **order ideal** (or 'lattice ideal') V in $H^1(\Omega)$ is a subspace with the property that $u \in V$, $v \in H^1(\Omega)$, $|v| \leqslant |u|$ imply that $v \in V$. Note that then V is a vector sublattice (because $|u| \in V$ for all $u \in V$). On the other hand, if V is a vector sublattice with the property that for all $u \in V$, $v \in H^1(\Omega)$ with $0 \leqslant v \leqslant u$ one has $v \in V$, then V is an order ideal.

11.9 Proposition. *The space $H^1_0(\Omega)$ is an order ideal in $H^1(\Omega)$.*

Proof. Let $u \in H^1_0(\Omega)$, $v \in H^1(\Omega)$, $0 \leqslant v \leqslant u$. There exists a sequence (u_k) in $H^1_c(\Omega)$ such that $u_k \to u$ in $H^1(\Omega)$. It follows from Proposition 11.7 that $u_k \wedge v \to u \wedge v = v$ in $H^1(\Omega)$ as $k \to \infty$. Since $u_k \wedge v \in H^1_c(\Omega)$ we conclude that $v \in H^1_0(\Omega)$. □

11.10 Proposition. *Assume that Ω is bounded and has C^1-boundary. Let $\Gamma \subseteq \partial\Omega$ be a Borel set, and let V be as in Example 11.4(c). Then V is a closed order ideal in $H^1(\Omega)$.*

Proof. The linearity and continuity of the trace operator $\operatorname{tr}\colon H^1(\Omega) \to L_2(\partial\Omega)$ imply that V is a closed subspace.

For the proof that V is a vector sublattice of $H^1(\Omega)$, the important additional observation is that the trace operator is also a lattice homomorphism, i.e. $\operatorname{tr}(u \vee v) = \operatorname{tr} u \vee \operatorname{tr} v$ for all $u, v \in H^1(\Omega)$. This property is clear for $u, v \in H^1(\Omega) \cap C(\overline{\Omega})$ and carries over to $H^1(\Omega)$ by denseness and continuity; see Theorems 7.7 and 7.10 as well as Proposition 11.7(b). Therefore, if $u \in V$, then $\operatorname{tr}(u^+)|_\Gamma = (\operatorname{tr} u)^+|_\Gamma = 0$, i.e. $u^+ \in V$.

Finally, let $u \in V$, $v \in H^1(\Omega)$, $0 \leqslant v \leqslant u$. Then $0 \leqslant \operatorname{tr} v|_\Gamma \leqslant \operatorname{tr} u|_\Gamma = 0$. This shows that $v \in V$. □

We will also need the following denseness properties.

11.11 Lemma. *The set $C^\infty_c(\Omega)_+ := \{\varphi \in C^\infty_c(\Omega)\,;\, \varphi \geqslant 0\}$ is dense in $H^1_0(\Omega)_+ := \{u \in H^1_0(\Omega)\,;\, u \geqslant 0\}$.*

Proof. (i) First we show that $H^1_c(\Omega)_+ := \{u \in H^1_c(\Omega)\,;\, u \geqslant 0\}$ is dense in $H^1_0(\Omega)_+$. Let $u \in H^1_0(\Omega)_+$. There exists a sequence (u_k) in $H^1_c(\Omega)$ such that $u_k \to u$. Then clearly $u_k^+ \in H^1_c(\Omega)_+$ for all $k \in \mathbb{N}$, and Proposition 11.7 implies that $u_k^+ \to u^+ = u$.

(ii) Let $u \in H^1_c(\Omega)_+$. Let (ρ_k) be a δ-sequence in $C^\infty_c(\mathbb{R}^n)$. It was shown in the proof of Theorem 4.14(b) that then $(\rho_k * \tilde{u})|_\Omega \in C^\infty_c(\Omega)$ for large k, and $(\rho_k * \tilde{u})|_\Omega \to u$ in $H^1(\Omega)$. Clearly $\rho_k * \tilde{u} \geqslant 0$ for all $k \in \mathbb{N}$, and this shows that $u \in \overline{C^\infty_c(\Omega)_+}^{H^1(\Omega)}$. □

11.12 Proposition. *Assume that Ω is bounded and has continuous boundary. Then $\check{C}^\infty(\Omega)_+ := \big\{u \in \check{C}^\infty(\Omega)\,;\, u \geqslant 0\big\}$ is dense in $H^1(\Omega)_+ := \{u \in H^1(\Omega)\,;\, u \geqslant 0\}$.*

Proof. This property follows from the proof of Theorem 7.7, as explained in Remark 7.8(b). □

11.4 Elliptic operators with real coefficients

Here we continue with the setting of Section 11.2, assuming in addition that $\mathbb{K} = \mathbb{R}$. In particular, $\Omega \subseteq \mathbb{R}^n$ is an open set, the coefficients $a_{jk}, b_j, c_j, d \in L_\infty(\Omega)$ are all real-valued, and we assume that the ellipticity condition (11.1) is satisfied. Furthermore we recall the form a on $H^1(\Omega)$ from (11.2), as well as the operator A_V associated with $a|_{V\times V}$, for a closed subspace V of $H^1(\Omega)$ containing $H^1_0(\Omega)$, the operator $A_{\mathrm{D}} = A_{H^1_0}$ with Dirichlet boundary conditions, the operator $A_{\mathrm{N}} = A_{H^1}$ with Neumann boundary conditions, and the associated C_0-semigroups T_V, T_{D} and T_{N}.

11.13 Proposition. *Let V be a closed vector sublattice of $H^1(\Omega)$ containing $H^1_0(\Omega)$. Then the semigroup T_V generated by $-A_V$ is positive. In particular, T_{D} and T_{N} are positive.*

Proof. Let $u \in V$; then $u^+ \in V$. Theorem 9.16 implies that $\partial_j u^+ = \mathbf{1}_{[u>0]}\partial_j u$, $\partial_j u^- = -\mathbf{1}_{[u<0]}\partial_j u$, and therefore $\partial_k u^+ \partial_j u^- = 0$, $\partial_j u^+ u^- = 0$, $u^+ \partial_j u^- = 0$, $u^+ u^- = 0$, for all $j, k = 1, \dots, n$. Thus $a(u^+, u^-) = 0$. Now it follows from Theorem 10.15(a) that T_V is positive.

The assertion for T_{D} and T_{N} is a consequence of Theorem 9.16. □

Proposition 11.13 also implies that the semigroup is positive for mixed boundary conditions; see Proposition 11.10.

Next we show additional properties for Dirichlet boundary conditions.

11.14 Theorem. (a) *Assume that $c = (c_1, \dots, c_n)$ satisfies $\operatorname{div} c \in L_{1,\mathrm{loc}}(\Omega)$ and $\operatorname{div} c \leqslant d$. Then T_{D} is sub-Markovian.*

(b) *Let $b = (b_1, \dots, b_n)$ satisfy $\operatorname{div} b \in L_{1,\mathrm{loc}}(\Omega)$ and $\operatorname{div} b \leqslant d$. Then T_{D} is substochastic.*

Proof. (a) Let $u \in H^1_0(\Omega)$. Then $u \wedge 1 \in H^1_0(\Omega)$ and $\partial_j(u \wedge 1) = \mathbf{1}_{[u<1]}\partial_j u$, by Theorem 9.16. Since $u = u \wedge 1 + (u-1)^+$, it follows that $(u-1)^+ \in H^1_0(\Omega)$ and $\partial_j (u-1)^+ = \mathbf{1}_{[u\geqslant 1]}\partial_j u$. Thus $\partial_k(u \wedge 1)\partial_j (u-1)^+ = 0$ and $\partial_j (u \wedge 1)(u-1)^+ = 0$. It follows that

$$\begin{aligned} a(u \wedge 1, (u-1)^+) &= \int_\Omega \Big(\sum_{j=1}^n c_j (u \wedge 1) \partial_j (u-1)^+ + d(u \wedge 1)(u-1)^+ \Big) \,\mathrm{d}x \\ &= \int_\Omega \Big(\sum_{j=1}^n c_j \partial_j (u-1)^+ + d(u-1)^+ \Big) \,\mathrm{d}x \end{aligned}$$

(where the last equality holds because $u \wedge 1 = 1$ on $[u \geqslant 1]$). From the hypotheses we obtain

$$\int_\Omega \Big(\sum_{j=1}^n c_j \partial_j \varphi + d\varphi \Big) \,\mathrm{d}x = \int_\Omega (-\operatorname{div} c + d)\varphi \,\mathrm{d}x \geqslant 0$$

for all $0 \leqslant \varphi \in C_{\mathrm{c}}^\infty(\Omega)$. Since $(u-1)^+$ can be approximated by positive test functions, by Lemma 11.11, it follows that $a(u \wedge 1, (u-1)^+) \geqslant 0$. Now Theorem 10.15(b) implies that T_{D} is sub-Markovian.

(b) The proof is analogous to (a) and uses Theorem 10.15(c). □

We now show a similar result for boundary conditions that are defined by more general spaces V. Because of the boundary terms we need stronger regularity assumptions on the coefficients c and d than in Theorem 11.14.

11.15 Theorem. *Assume that Ω is bounded and has C^1-boundary, and let V be a Stonean sublattice of $H^1(\Omega)$ containing $H^1_0(\Omega)$.*

(a) *If $c \in C^1(\overline{\Omega}; \mathbb{R}^n)$, $\operatorname{div} c \leqslant d$ on Ω and $c \cdot \nu \geqslant 0$ on $\partial\Omega$, then T_V is sub-Markovian. (As before, $\nu(z) = (\nu_1(z), \dots, \nu_n(z))$ denotes the outer unit normal at $z \in \partial\Omega$.)*

(b) *If $b \in C^1(\overline{\Omega}; \mathbb{R}^n)$, $\operatorname{div} b \leqslant d$ on Ω and $b \cdot \nu \geqslant 0$ on $\partial\Omega$, then T_V is substochastic.*

Proof. (a) As in the proof of Theorem 11.14 one has

$$a(u \wedge 1, (u-1)^+) = \int_\Omega \Big(\sum_{j=1}^n c_j \partial_j (u-1)^+ + d(u-1)^+ \Big) \, \mathrm{d}x \qquad (u \in V).$$

For $0 \leqslant \varphi \in C^1(\overline{\Omega})$ we now use Gauss' theorem (Theorem 7.3) to obtain

$$\int_\Omega \Big(\sum_{j=1}^n c_j \partial_j \varphi + d\varphi \Big) \, \mathrm{d}x = \int_\Omega \Big(-\sum_{j=1}^n \partial_j c_j + d \Big) \varphi \, \mathrm{d}x + \int_{\partial\Omega} \sum_{j=1}^n \nu_j c_j \varphi \, \mathrm{d}\sigma \geqslant 0.$$

By approximation – applying Proposition 11.12 – we deduce that $a(u \wedge 1, (u-1)^+) \geqslant 0$.

The proof of (b) is analogous. □

11.16 Remarks. Let Ω and V be as in Theorem 11.15, and assume that $\mathbf{1}_\Omega \in V$.

(a) If in Theorem 11.15(a) one has the equalities

$$\operatorname{div} c = d \quad \text{on } \Omega, \qquad c \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

then $T_V(t)\mathbf{1}_\Omega = \mathbf{1}_\Omega$ for all $t \geqslant 0$. This means that T_V is not only sub-Markovian but **Markovian**.

Indeed, the proof of Theorem 11.15 shows that $a(\mathbf{1}_\Omega, v) = 0$ for all $v \in C^1(\overline{\Omega})$ and hence for all $v \in H^1(\Omega)$, by Proposition 11.12. It follows that $\mathbf{1}_\Omega \in \operatorname{dom}(A_V)$ and $A_V \mathbf{1}_\Omega = 0$, and then Theorem 1.13(a) implies the assertion.

(b) Similarly, if in Theorem 11.15(b) one has the equalities

$$\operatorname{div} b = d \quad \text{on } \Omega, \qquad b \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

then T_V is not only substochastic but **stochastic**, i.e. $\|T_V(t)u\|_1 = \|u\|_1$ for all $0 \leqslant u \in L_2 \cap L_1(\Omega)$, $t \geqslant 0$. See Exercise 11.4. △

11.17 Remarks. (a) In the situation of Theorem 11.14(a), the C_0-semigroup T_{D} gives rise to a consistent family $\big(T_{\mathrm{D},p}\big)_{2 \leqslant p < \infty}$ of C_0-semigroups $T_{\mathrm{D},p}$ on $L_p(\Omega)$. The proof is analogous to the proof of Theorem 10.18(a). Similarly, in Theorem 11.14(b) one obtains a consistent family of C_0-semigroups $\big(T_{\mathrm{D},p}\big)_{1 \leqslant p \leqslant 2}$.

The corresponding comment applies to Theorem 11.15.

(b) Deviating from the initial announcement of this section that only $\mathbb{K} = \mathbb{R}$ is treated, we include a comment on the complex case. It was mentioned in Section 11.2 that the C_0-semigroup T_V, with a suitable $H^1_0(\Omega) \subseteq V \subseteq H^1(\Omega)$, is holomorphic. Therefore Theorem 10.10 implies that the C_0-semigroups discussed in part (a) extend to holomorphic semigroups on $L_p(\Omega)$ if $p \neq 1$. △

11.5 Domination

We use the assumptions and notation from Section 11.2, as recalled at the beginning of Section 11.4, and throughout we assume that $\mathbb{K} = \mathbb{R}$.

We already know from Theorem 11.13 that the semigroups T_{D}, T_{N} and T_V are positive if V is a sublattice of $H^1(\Omega)$ containing $H^1_0(\Omega)$. Here we investigate the order relation between these semigroups.

11.18 Theorem. *Let $V \subseteq W$ be closed vector sublattices of $H^1(\Omega)$ containing $H^1_0(\Omega)$, and suppose that V is an order ideal in W. Then $0 \leqslant T_V(t) \leqslant T_W(t)$ for all $t \geqslant 0$, i.e. $0 \leqslant T_V(t)f \leqslant T_W(t)f$ for all $f \in L_2(\Omega)_+$.*

Proof. Because of the exponential formula, Theorem 2.12, it suffices to show the domination property for the resolvents; i.e., for large λ we have to show that

$$(\lambda + A_V)^{-1} f \leqslant (\lambda + A_W)^{-1} f \qquad (0 \leqslant f \in L_2(\Omega)).$$

Adding λ to the coefficient d we may assume $\lambda = 0$ and also that the form a is coercive.

Let $0 \leqslant f \in L_2(\Omega)$ and put $u_1 = A_V^{-1} f$, $u_2 = A_W^{-1} f$. Then $u_1 \in V_+ := V \cap L_2(\Omega)_+$ and $u_2 \in W_+ := W \cap L_2(\Omega)_+$ by the positivity of T_V, T_W and Proposition 9.1. Moreover

$$\begin{aligned} a(u_1, v) &= (f \,|\, v)_{L_2} \qquad (v \in V), \\ a(u_2, v) &= (f \,|\, v)_{L_2} \qquad (v \in W), \end{aligned}$$

and thus $a(u_1 - u_2, v) = 0$ for all $v \in V$. We want to take $v := (u_1 - u_2)^+$, but at first it is only clear that $(u_1 - u_2)^+ \in W$. Observing that $0 \leqslant (u_1 - u_2)^+ \leqslant u_1$ we obtain $v = (u_1 - u_2)^+ \in V$ by the ideal property of V. Hence we have $a(u_1 - u_2, (u_1 - u_2)^+) = 0$. But $a(w^-, w^+) = a((-w)^+, (-w)^-) = 0$ for all $w \in H^1(\Omega)$ as we saw in the proof of Proposition 11.13. It follows that $a((u_1 - u_2)^+) = 0$, which implies $(u_1 - u_2)^+ = 0$ by our coercivity assumption. Consequently, $u_1 \leqslant u_2$. We have shown that $A_V^{-1} \leqslant A_W^{-1}$. □

Now the following conclusion is immediate from Theorem 11.18 and the ideal property of $H^1_0(\Omega)$, Proposition 11.9.

11.19 Corollary. *Let V be a closed vector sublattice of $H^1(\Omega)$ containing $H^1_0(\Omega)$. Then $T_{\mathrm{D}}(t) \leqslant T_V(t)$ for all $t \geqslant 0$.*

If additionally V is a closed order ideal in $H^1(\Omega)$, then $T_V(t) \leqslant T_{\mathrm{N}}(t)$ for all $t \geqslant 0$.

We mention that closed order ideals in $H^1(\Omega)$ are characterised by Stollmann [Sto93]. If A_V is the elliptic operator with mixed boundary conditions – see Example 11.4(c) –, then V is an order ideal in $H^1(\Omega)$, by Proposition 11.10, and so $T_V(t) \leqslant T_{\mathrm{N}}(t)$ for all $t \geqslant 0$.

Finally we want to prove domain monotonicity for Dirichlet boundary conditions. Let $\Omega_1 \subseteq \Omega$ be open. We consider the semigroup T_{D} on $L_2(\Omega)$. But we may also restrict the coefficients to Ω_1 and consider the semigroup T^1_{D} on $L_2(\Omega_1)$. We identify $L_2(\Omega_1)$ with a subspace of $L_2(\Omega)$ by extending functions in $L_2(\Omega_1)$ by 0 on $\Omega \setminus \Omega_1$.

11.20 Theorem. *One has $T^1_{\mathrm{D}}(t)f \leqslant T_{\mathrm{D}}(t)f$ for all $0 \leqslant f \in L_2(\Omega_1)$, $t \geqslant 0$.*

Proof. By the exponential formula it suffices to show that $(\lambda + A_{\mathrm{D}}^1)^{-1} f \leqslant (\lambda + A_{\mathrm{D}})^{-1} f$ on Ω_1 for large enough λ and $0 \leqslant f \in L_2(\Omega_1)$. As in the proof of Theorem 11.18 we may assume that the form a is coercive and that $\lambda = 0$.

Let $0 \leqslant f \in L_2(\Omega_1)$ and put $u_1 := (A_{\mathrm{D}}^1)^{-1} f$, $u_2 := (A_{\mathrm{D}})^{-1} f$. Then $u_1 \in H_0^1(\Omega_1)_+$, $u_2 \in H_0^1(\Omega)_+$ and

$$a(u_1, v) = (f \mid v)_{L_2(\Omega_1)} \qquad (v \in H_0^1(\Omega_1)),$$
$$a(u_2, v) = (f \mid v)_{L_2(\Omega)} \qquad (v \in H_0^1(\Omega)).$$

Observe that for $v \in H_0^1(\Omega_1)$ the extension $\tilde{v}$ to Ω belongs to $H_0^1(\Omega)$ and that $\partial_j \tilde{v} = \widetilde{\partial_j v}$ for all $j = 1, \dots, n$ (see the paragraph below Remark 11.8). In this sense we will consider $H_0^1(\Omega_1)$ as embedded into $H_0^1(\Omega)$ and omit the tilde. The above formulas imply $a(u_1 - u_2, v) = 0$ for all $v \in H_0^1(\Omega_1)$. Clearly $0 \leqslant (u_1 - u_2)^+ \leqslant u_1$, and it follows from Proposition 11.9 that $v := (u_1 - u_2)^+ \in H_0^1(\Omega_1)$. Hence $a(u_1 - u_2, (u_1 - u_2)^+) = 0$. Since $a((u_1 - u_2)^-, (u_1 - u_2)^+) = 0$ it follows that $a((u_1 - u_2)^+) = 0$ and hence $(u_1 - u_2)^+ = 0$ by the coercivity assumption. Thus $u_1 \leqslant u_2$. □

Notes

The application of the Beurling–Deny criteria to elliptic operators has a long history. In the beginning, mostly symmetric operators were studied, in particular in connection with the heat equation with potential ("Schrödinger semigroups"); see for instance [ReSi78], [Dav89]. The application to non-symmetric operators seems to start with [MaRo92], [Ouh92], [Ouh96]; see [Ouh05] for a more recent presentation. Domination is also considered in the above references. It is interesting that the conditions for domination given in Exercise 11.5 are also necessary; see [MVV05; Corollary 4.3]. We refer to [MVV05] and the literature cited there for the treatment of more general domination results, which are then treated in the context of invariance criteria.

Perturbations as in Lemma 11.1 play an important role for the second order equation (cosine functions), see [ABHN11; Chapter 7 and Section 3.14] as well as the corresponding notes.

We note that the treatment of second order elliptic operators by forms is particularly effective for operators in divergence form, as written in (11.2). This terminology concerns the second order part of the operator. Transforming an expression $\sum_{j,k=1}^n a_{jk} \partial_j \partial_k u$ into divergence form would require differentiability properties of a_{jk} and produce first order terms (called drift terms).

In the context of Section 11.4, it is a remarkable fact that the semigroup operators $T_{\mathrm{D}}(t)$ always extend to bounded operators $T_{\mathrm{D},p}(t)$ on $L_p(\Omega)$, and that $T_{\mathrm{D},p}$ thus defined is a C_0-semigroup on $L_p(\Omega)$, for all $p \in [1, \infty)$. The same is true for the semigroup T_{N} if Ω is bounded and has C^1-boundary (or, less restrictively, satisfies an interior cone condition). This follows from the results of [Dan00; Section 6], where it is shown that the semigroup operators have integral kernels satisfying Gaussian estimates. If $\mathbb{K} = \mathbb{C}$, then it follows that the semigroups $T_{\mathrm{D},p}$ and $T_{\mathrm{N},p}$ extend to holomorphic C_0-semigroups on $L_p(\Omega)$, with an angle not depending on $p \in [1, \infty)$ (see [Hie96; Theorem 2.3]). In particular, if the

coefficient matrices $(a_{jk}(x))$ are self-adjoint, then all the semigroups are holomorphic of angle $\pi/2$; cf. Remark 11.5(a). This is in contrast to Theorem 10.10, where the angle is obtained by interpolation and depends on p.

Exercises

11.1 Let $(a_{jk}) \in \mathbb{R}^{n\times n}$, $\alpha > 0$ be such that

$$\sum_{j,k=1}^{n} a_{jk}\xi_k\xi_j \geqslant \alpha|\xi|^2$$

for all $\xi \in \mathbb{R}^n$. Show that

$$\operatorname{Re}\sum_{j,k=1}^{n} a_{jk}\xi_k\overline{\xi_j} \geqslant \alpha|\xi|^2$$

for all $\xi \in \mathbb{C}^n$.

11.2 Let $\Omega \subseteq \mathbb{R}^2$ be open, $(a_{jk}) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $b = c = 0$, $d = 0$.

(a) Show that $A_{\mathrm{D}} = -\Delta_{\mathrm{D}}$.

(b) Assume that Ω is bounded and has C^1-boundary. Find the conormal derivative corresponding to A_{N}; cf. Example 11.4(b). Find Ω such that $A_{\mathrm{N}} \neq -\Delta_{\mathrm{N}}$. Is $A_{\mathrm{N}} \neq -\Delta_{\mathrm{N}}$ for any bounded open set $\Omega \neq \varnothing$ with C^1-boundary?

11.3 Let a_{jk}, b_j, c_j, d be as in Section 11.2.

(a) Assume additionally that $b \in C^1_{\mathrm{b}}(\Omega;\mathbb{K}^n)$ (bounded derivatives!). Let the formal elliptic operators $\mathcal{A}_1, \mathcal{A}_2$, in the sense of (11.2), be defined by

$$\mathcal{A}_1 u := -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + b\cdot(\nabla u), \quad \mathcal{A}_2 u := -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \operatorname{div}(bu) - (\operatorname{div} b)u.$$

Show that $A_{1,\mathrm{D}} = A_{2,\mathrm{D}}$.

(b) Assume additionally that $b, c \in C^1_{\mathrm{b}}(\Omega;\mathbb{K}^n)$, $c = b$, and let $\mathcal{A}$ be defined by

$$\mathcal{A}u := -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + b\cdot(\nabla u) - \operatorname{div}(cu).$$

Show that A_{D} is associated with a formal elliptic operator without drift terms.

11.4 Prove Remark 11.16(b).

11.5 Let $\mathbb{K} = \mathbb{R}$, (Ω, μ) a measure space, $H := L_2(\mu)$, V, W Hilbert spaces, $V \overset{\mathrm{d}}{\hookrightarrow} H$, $W \overset{\mathrm{d}}{\hookrightarrow} H$, and let $a\colon V\times V \to \mathbb{R}$, $b\colon W\times W \to \mathbb{R}$ be bounded bilinear forms, both H-coercive. Denote by A the operator associated with a and by B the operator associated with b. Assume that the semigroups T generated by $-A$ and S generated by $-B$ are both positive. (Recall that then V and W are sublattices of $L_2(\mu)$, by Theorem 10.15(a).) Assume that

(i) V is an order ideal in W;

(ii) $a(u,v) \geqslant b(u,v)$ for all $0 \leqslant u, v \in V$.

Show that $T(t) \leqslant S(t)$ $(t \geqslant 0)$. (Hint: Proceed as in the proof of Theorem 11.18.)

Chapter 12

Sectorial forms

In this chapter we study sectorial forms $a\colon V \times V \to \mathbb{C}$ in the general situation when V is just a vector space (and not necessarily a Hilbert space). As in Chapter 5, $j\colon V \to H$ is a linear mapping with dense range in a Hilbert space H. In Section 12.1 we describe how to associate an m-sectorial operator in H with the form (a, j) in this general setting. If the form is closed (see Section 12.1 for the definition), then the approach presented here is equivalent to the approach via j-coercive forms. In Section 12.4 we give an alternative description of the associated operator, in the non-closed case. Two examples are given that illustrate the theory: the Robin Laplacian and the Dirichlet-to-Neumann operator are revisited, but now on rough domains.

12.1 Operators associated with sectorial forms

As defined in Chapter 5, a form a is a sesquilinear mapping $a\colon V \times V \to \mathbb{K}$, where V is a $\mathbb{K}$-vector space. In contrast to previous sections we do not assume V to carry any additional structure. For our purposes it will be convenient to simply call V the **domain** of a and to put $\operatorname{dom}(a) := V$. Evidently, this is a misuse of the symbol 'dom' since the domain of a, in the usual sense, is the cartesian product $V \times V$. However, this notation is useful, has a long history, and should not lead to confusion.

Let $\mathbb{K} = \mathbb{C}$, and let $a\colon \operatorname{dom}(a) \times \operatorname{dom}(a) \to \mathbb{C}$ be a form. We recall that, by definition, a is sectorial if there exists $\theta \in [0, \pi/2)$ such that $a(u) \in \overline{\Sigma_\theta}$ for all $u \in \operatorname{dom}(a)$. This is the same as saying that there exists $c \geqslant 0$ such that

$$|\operatorname{Im} a(u)| \leqslant c \operatorname{Re} a(u) \tag{12.1}$$

for all $u \in \operatorname{dom}(a)$. We further recall that $a = \operatorname{Re} a + \mathrm{i} \operatorname{Im} a$ with the symmetric forms $\operatorname{Re} a, \operatorname{Im} a\colon \operatorname{dom}(a) \times \operatorname{dom}(a) \to \mathbb{C}$, and that $\operatorname{Re} a(u) = (\operatorname{Re} a)(u)$ and $\operatorname{Im} a(u) = (\operatorname{Im} a)(u)$ for all $u \in \operatorname{dom}(a)$. Using Proposition 5.2 we see that (12.1) implies the inequality

$$|a(u, v)| \leqslant (1 + c)(\operatorname{Re} a(u))^{1/2} (\operatorname{Re} a(v))^{1/2}. \tag{12.2}$$

This is a sort of intrinsic continuity of a which we will use throughout.

Let H be a complex Hilbert space. If $a\colon \operatorname{dom}(a) \times \operatorname{dom}(a) \to \mathbb{C}$ is a sectorial form and $j\colon \operatorname{dom}(a) \to H$ is linear, then for brevity we call the couple (a, j) a **sectorial form in H**. In the special case when a is symmetric and accretive, we call (a, j) an **accretive symmetric form in H**. We say that (a, j) is **densely defined** if j has dense range.

12.1 Remark. In our applications in the last two sections of the present chapter we will consider accretive symmetric forms in *real* Hilbert spaces. In this situation the key inequality (12.2) holds with $c = 0$. A careful inspection of our treatment shows that the results of the chapter depend on (12.2) rather than on (12.1), hence they remain valid in real Hilbert spaces if one replaces 'sectorial form' by 'accretive symmetric form', 'sectorial operator' by 'accretive symmetric operator' and 'm-sectorial operator' by 'accretive self-adjoint operator'. (Note that sectorial forms in a complex Hilbert space are accretive.) △

12.2 Proposition. *Let (a, j) be a densely defined sectorial form in H. Then*

$$A_0 := \big\{(x, y) \in H \times H\,;\, \exists u \in \mathrm{dom}(a)\colon j(u) = x,\ a(u, v) = (y \,|\, j(v))_H\ (v \in \mathrm{dom}(a))\big\} \tag{12.3}$$

defines a sectorial operator in H.

Proof. (i) Let $(0, y) \in A_0$. We show that $y = 0$; then it follows that A_0 is an operator. By definition there exists $u \in \mathrm{dom}(a)$ such that $j(u) = 0$ and $a(u, v) = (y \,|\, j(v))_H$ for all $v \in \mathrm{dom}(a)$. In particular, $a(u) = 0$. By (12.2) this implies that $a(u, v) = 0$ for all $v \in \mathrm{dom}(a)$. Consequently $(y \,|\, j(v))_H = 0$ for all $v \in \mathrm{dom}(a)$. Since j has dense range it follows that $y = 0$.

(ii) Let $x \in \mathrm{dom}(A_0)$. Then there exists $u \in \mathrm{dom}(a)$ such that $j(u) = x$ and $a(u, v) = (A_0 x \,|\, j(v))_H$ for all $v \in \mathrm{dom}(a)$. In particular, $(A_0 x \,|\, x) = a(u)$. Thus A_0 is sectorial. □

In general, the operator A_0 is not m-sectorial (see Exercise 12.3), but below we will construct an m-sectorial extension. The idea is to use a completion procedure that enables us to apply the generation theorems of Chapter 5. The resulting operator will be discussed in Theorem 12.4.

Let (a, j) be a densely defined sectorial form in H. Then

$$(u \,|\, v)_{a,j} := (\mathrm{Re}\, a)(u, v) + (j(u) \,|\, j(v))_H \tag{12.4}$$

defines a semi-inner product, i.e. a symmetric, accretive sesquilinear form on $\mathrm{dom}(a)$. Thus

$$\|u\|_{a,j} := \sqrt{(u \,|\, u)_{a,j}} = \big(\mathrm{Re}\, a(u) + \|j(u)\|_H^2\big)^{1/2}$$

defines a semi-norm on $\mathrm{dom}(a)$. (We will suppress 'j' in the index if $\mathrm{dom}(a) \subseteq H$ and j is the natural embedding.)

12.3 Remark (Completion of a semi-inner product space). Let E be a vector space over $\mathbb{K}$ and $(\cdot \,|\, \cdot)$ a **semi-inner product** on E (i.e. the form $(\cdot \,|\, \cdot)$ is symmetric and accretive), with associated semi-norm $\|\cdot\|$.

(a) There exist a Hilbert space $\widetilde{E}$ and a linear mapping $q\colon E \to \widetilde{E}$ that

(i) is **isometric**, i.e. $(q(u) \,|\, q(v))_{\widetilde{E}} = (u \,|\, v)$ for all $u, v \in E$, and

(ii) has dense range.

Indeed, note that $F := \{u \in E\,;\, \|u\| = 0\}$ is a subspace of E, and let $q\colon E \to E/F =: G$ denote the quotient map. Since $|(u \,|\, v)| \leqslant \|u\|\|v\|$, by Proposition 5.2, $\|u\| = 0$ implies $(u \,|\, v) = 0$ for all $v \in E$. As a consequence, putting $(q(u) \,|\, q(v))_G = (u \,|\, v)$ one obtains a

well-defined scalar product on G that makes q isometric. We define $\widetilde{E}$ as the completion of the pre-Hilbert space G.

The space $\widetilde{E}$ is unique up to unitary equivalence. We call the pair $(\widetilde{E}, q)$ the **completion** of E.

(b) The mapping q is injective if and only if $(\cdot \,|\, \cdot)$ is positive definite, and q is surjective if and only if $(E, \|\cdot\|)$ is complete. If $(E, \|\cdot\|)$ is complete, but $(\cdot \,|\, \cdot)$ is not positive definite, then the completion $(\widetilde{E}, q) = (E/F, q)$ is different from E. (Recall that a semi-normed space is called complete if every Cauchy sequence is convergent.)

(c) If H is a Hilbert space and $j\colon E \to H$ is a bounded linear operator, then there exists a unique bounded linear operator $\tilde{\jmath}\colon \widetilde{E} \to H$ such that $\tilde{\jmath} \circ q = j$. Indeed, there exists a unique operator $\tilde{\jmath}_0\colon E/F \to H$ such that $\tilde{\jmath}_0 \circ q = j$, and this (bounded) operator has the asserted extension $\tilde{\jmath}$. Similarly, if $a\colon E \times E \to \mathbb{K}$ is a bounded form (i.e. $|a(u,v)| \leqslant M\|u\|\|v\|$ for all $u, v \in E$), then there exists a uniquely determined bounded form $\tilde{a}\colon \widetilde{E} \times \widetilde{E} \to \mathbb{K}$ such that $\tilde{a}(q(u), q(v)) = a(u,v)$ for all $u, v \in E$. $\triangle$

We now construct the m-sectorial operator associated with a sectorial form. Let H be a complex Hilbert space, and let (a, j) be a densely defined sectorial form in H. Let (V, q) denote the completion of $(\mathrm{dom}(a), (\cdot \,|\, \cdot)_{a,j})$, and let $\tilde{\jmath}\colon V \to H$ and $\tilde{a}\colon V \times V \to \mathbb{C}$ be as explained in Remark 12.3(c). Observe that $\tilde{a}$ is sectorial and that

$$\mathrm{Re}\, \tilde{a}(q(u)) + \|\tilde{\jmath}(q(u))\|_H^2 = \mathrm{Re}\, a(u) + \|j(u)\|_H^2 = \|u\|_{a,j}^2 = \|q(u)\|_V^2$$

for all $u \in \mathrm{dom}(a)$. As $\mathrm{ran}(q)$ is dense in V, and $\tilde{a}$ as well as q are continuous, we conclude that $\mathrm{Re}\, \tilde{a}(v) + \|\tilde{\jmath}(v)\|_H^2 = \|v\|_V^2$ for all $v \in V$. This equality shows that $\tilde{a}$ is $\tilde{\jmath}$-coercive. Let A be the m-sectorial operator associated with $(\tilde{a}, \tilde{\jmath})$ according to Corollary 5.11; we call A the **operator associated with** (a, j), and we write $A \sim (a, j)$. We recall that

$$A = \big\{(x, y) \in H \times H\,;\, \exists u \in V\colon \tilde{\jmath}(u) = x,\ \tilde{a}(u, v) = (y \,|\, \tilde{\jmath}(v))_H \ (v \in V)\big\}. \tag{12.5}$$

For another description of A we refer to Theorem 12.10.

We summarise the properties of the operator A in the next theorem. A sectorial form (a, j) in a complex Hilbert space is called **closed** if the semi-inner product space $(\mathrm{dom}(a), (\cdot \,|\, \cdot)_{a,j})$ is complete.

12.4 Theorem. *Let (a, j) be a densely defined sectorial form in a complex Hilbert space H. Then the m-sectorial operator $A \sim (a, j)$ in* (12.5) *is an extension of the operator A_0 defined in* (12.3)*. If the form a is symmetric, then A is self-adjoint. If (a, j) is closed, then $A = A_0$.*

Proof. If $(x, y) \in A_0$, then there exists $u \in \mathrm{dom}(a)$ such that $j(u) = x$ and $a(u, v) = (y \,|\, j(v))$ for all $v \in \mathrm{dom}(a)$. It follows that $\tilde{\jmath}(q(u)) = x$ and

$$\tilde{a}(q(u), \tilde{v}) = (y \,|\, \tilde{\jmath}(\tilde{v})) \tag{12.6}$$

for all $\tilde{v} \in \mathrm{ran}(q)$. The denseness of $\mathrm{ran}(q)$ in V and the continuity of $\tilde{a}$ and $\tilde{\jmath}$ imply that (12.6) carries over to all $\tilde{v} \in V$, and this shows that $(x, y) \in A$.

Again by denseness and continuity, the symmetry of a carries over to $\tilde{a}$, and then the self-adjointness of A follows from Theorem 6.11.

If (a, j) is closed, then q is surjective. Let $(x, y) \in A$, and let $\tilde{u} \in V$ be such that $\tilde{\jmath}(\tilde{u}) = x$ and $\tilde{a}(\tilde{u}, \tilde{v}) = (y \,|\, \tilde{\jmath}(\tilde{v}))$ for all $\tilde{v} \in V$. There exists $u \in \mathrm{dom}(a)$ such that $\tilde{u} = q(u)$. Then

$$a(u, v) = \tilde{a}(\tilde{u}, q(v)) = \big(y \,\big|\, \tilde{\jmath}(q(v))\big) = (y \,|\, j(v))$$

for all $v \in \mathrm{dom}(a)$, and because of $j(u) = \tilde{\jmath}(q(u)) = x$ it follows that $(x, y) \in A_0$. □

The assertions of Theorem 12.4 remain valid – with appropriate changes – if (a, j) is merely a **quasi-sectorial form** in H, i.e., if instead of a being sectorial one assumes that there exists $\omega \in \mathbb{R}$ such that the shifted form

$$a_\omega(u, v) := a(u, v) + \omega(j(u) \,|\, j(v))_H \qquad (u, v \in \mathrm{dom}(a))$$

is sectorial. The number $-\omega$ is called a **vertex** of (a, j), and (a, j) is **closed** if (a_ω, j) is closed. Of course, in the completion procedure described above one has to work with the semi-inner product $(\cdot \,|\, \cdot)_{a_\omega, j}$ instead of $(\cdot \,|\, \cdot)_{a,j}$. Then the resulting form $\tilde{a}$ satisfies $\mathrm{Re}\, \tilde{a}(v) + (\omega + 1)\|\tilde{\jmath}(v)\|_H^2 = \|v\|_V^2$ for all $v \in V$, and hence is $\tilde{\jmath}$-coercive. This implies that the operator A given by (12.5) is quasi-m-sectorial; as before we call A the operator associated with (a, j). Observe that A does not depend on the choice of the vertex $-\omega$ because different values of ω result in equivalent semi-norms $\|\cdot\|_{a_\omega, j}$ on $\mathrm{dom}(a)$.

We introduce one more notion that will be used below. For a quasi-sectorial form (a, j) with vertex $-\omega$, a subspace D of $\mathrm{dom}(a)$ is called a **core for** (a, j) if D is dense in $(\mathrm{dom}(a), \|\cdot\|_{a_\omega, j})$. In view of Remark 12.1, this notion makes also sense for accretive symmetric forms in real Hilbert spaces.

12.5 Example. If (a, j) is a quasi-sectorial form and D is a core for a, then it is immediate from the definitions that the operator associated with $(a|_{D \times D}, j|_D)$ is the same as the one associated with (a, j).

For instance, if Ω is an open subset of $\mathbb{R}^n$, then the operator associated with the restriction of the classical Dirchlet form to $C_c^\infty(\Omega)$ is the negative Dirichlet Laplacian. △

12.6 Remark. In Theorems 5.6 and 5.8, Corollary 5.11 and Theorem 12.4 we have formulated generation results for m-accretive operators under different hypotheses. Looking more closely, one can see that Theorem 5.6 is the basic result, which is then applied in different situations. In this context we also mention the generation result Theorem 8.11, which is proved by similar arguments as Theorem 5.6, with the Fredholm alternative as an additional argument. △

12.2 The Friedrichs extension

The main result of this section is that every densely defined sectorial operator has an m-sectorial extension. This extension is associated with a suitably constructed closed form, and an important "by-product" of the construction will be that every m-sectorial operator is associated with a unique embedded closed sectorial form.

As before let H be a complex Hilbert space. We consider the situation in which a is a sectorial form whose domain is a subspace of H and $j \colon \mathrm{dom}(a) \hookrightarrow H$ is the embedding.

This means that (a, j) is an embedded form; we will call a a **form in H** and drop the letter j in our notation. If $\mathrm{dom}(a)$ is dense and a is closed, then by Theorem 12.4 the associated m-sectorial operator is given by

$$A = \big\{(u, y) \in \mathrm{dom}(a) \times H\,;\, a(u, v) = (y \,|\, v)_H \ (v \in \mathrm{dom}(a))\big\}. \tag{12.7}$$

Here, a being closed means that $\mathrm{dom}(a)$ is complete for the norm $\|\cdot\|_a$ given by

$$\|u\|_a^2 = \mathrm{Re}\, a(u) + \|u\|_H^2\,.$$

12.7 Theorem (Friedrichs extension). *Let B be a densely defined sectorial operator in H. Then there exists a unique densely defined embedded closed sectorial form a in H such that* $\mathrm{dom}(B) \subseteq \mathrm{dom}(a)$, $\mathrm{dom}(B)$ *is a core for a, and*

$$a(u, v) = (Bu \,|\, v)_H \qquad \big(u \in \mathrm{dom}(B),\ v \in \mathrm{dom}(a)\big).$$

Let $A \sim a$. Then $B \subseteq A$.

The operator A is called the **Friedrichs extension** of B. By the proof given below the theorem also holds for accretive symmetric operators in real Hilbert spaces. In that case the a is an accretive symmetric form, and A is an accretive self-adjoint operator.

Proof of Theorem 12.7. The uniqueness of a follows from the following general fact: if a_1 and a_2 are two embedded closed sectorial forms in H with a common core $D \subseteq \mathrm{dom}(a_1) \cap \mathrm{dom}(a_2)$, then $a_1|_{D\times D} = a_2|_{D\times D}$ implies $a_1 = a_2$.

For the existence define $b\colon \mathrm{dom}(B) \times \mathrm{dom}(B) \to \mathbb{C}$ by $b(u, v) := (Bu \,|\, v)_H$. Then b is densely defined and sectorial. We use the embedding $j\colon \mathrm{dom}(b) \hookrightarrow H$ and the scalar product $(\cdot \,|\, \cdot)_b$ analogous to (12.4) on $\mathrm{dom}(b) = \mathrm{dom}(B)$. Let (V, q) be the completion of $(\mathrm{dom}(b), (\cdot \,|\, \cdot)_b)$; then q is injective (cf. Remark 12.3(b)). We consider $\mathrm{dom}(b)$ as a subset of V and suppress the letter q in our notation.

We now show that $\tilde{\jmath} \in \mathcal{L}(V, H)$ – from Remark 12.3(c) – is injective. Let $u \in V$, $\tilde{\jmath}(u) = 0$. There exists a sequence (u_n) in $\mathrm{dom}(b)$ such that $u_n \to u$ in V. Then $u_n = \tilde{\jmath}(u_n) \to \tilde{\jmath}(u) = 0$ in H. Using (12.2) one obtains

$$\begin{aligned}\mathrm{Re}\, b(u_n) &= \mathrm{Re}\, b(u_n, u_n - u_k) + \mathrm{Re}\, b(u_n, u_k)\\ &\leqslant (1 + c)\|u_n\|_b \|u_n - u_k\|_b + |(Bu_n \,|\, u_k)_H| \qquad (k, n \in \mathbb{N}).\end{aligned}$$

Letting $k \to \infty$ we deduce that $\mathrm{Re}\, b(u_n) \leqslant (1+c)\|u_n\|_V \|u_n - u\|_V$ for all $n \in \mathbb{N}$. It follows that $\|u_n\|_V^2 = \mathrm{Re}\, b(u_n) + \|u_n\|_H^2 \to 0$ as $n \to \infty$, $\|u\|_V = \lim_{n\to\infty} \|u_n\|_V = 0$.

Because $\tilde{\jmath}$ is injective we can consider V as a subspace of H and $a := \tilde{b}$ – from Remark 12.3(c) – as an embedded form (with $\mathrm{dom}(a) = V$). Then a is a closed sectorial form in H with the required properties, and it follows from (12.7) that the associated operator A is an extension of B. □

12.8 Corollary. *Let A be an m-sectorial operator in H. Then there exists a unique densely defined embedded closed sectorial form a in H such that A is associated with a.*

Proof. The existence follows from Theorem 12.7 since m-sectorial operators do not have proper m-sectorial extensions.

In order to prove uniqueness let a be a form with the asserted properties. Then $\mathrm{dom}(A) \subseteq \mathrm{dom}(a)$ and $a(u,v) = (Au \,|\, v)$ for all $u, v \in \mathrm{dom}(A)$. Moreover $\mathrm{dom}(A)$ is dense in $(\mathrm{dom}(a), \|\cdot\|_a)$ by Exercise 5.3(a) (or Lemma 9.21(a)). Therefore the uniqueness follows from the uniqueness in Theorem 12.7. □

Just like Theorem 12.7, Corollary 12.8 also holds for accretive symmetric operators in real Hilbert spaces.

12.3 Sectorial versus coercive

We start by giving a short summary of the different types of forms that we have introduced so far. In Section 5.3 we discussed bounded j-coercive forms on V and their associated m-accretive operators in H (which are m-sectorial if $\mathbb{K} = \mathbb{C}$). Here, both V and H are Hilbert spaces. In the special case when $V \subseteq H$ and $j\colon V \hookrightarrow H$ is the embedding, the form was called H-coercive. In Section 8.3 we investigated the related concept of essentially coercive forms. We recall from Proposition 8.14 that it is always possible to perform a reduction to the embedded case.

In Section 12.1 we have introduced the notion of a (quasi-)sectorial form (a, j) in H. The new aspect of this notion is that the domain of a is not supposed to be a Hilbert space; the corresponding semi-inner product $(\cdot \,|\, \cdot)_{a,j}$ need not be positive definite, nor does $(\mathrm{dom}(a), \|\cdot\|_{a,j})$ have to be complete. The construction of the associated operator in H by completion is basically a reduction to the case of j-coercive forms. The point in this approach is that the norm $\|\cdot\|_V$, needed for the generation theorems of Section 5, is defined intrinsically by the form itself and the mapping j. In the special case of closed forms the associated operator is given by the same formula as for j-coercive forms, by the last assertion of Theorem 12.4.

In the following remark we summarise the relations between the notions '(closed) quasi-sectorial form' and 'bounded j-coercive form', for $\mathbb{K} = \mathbb{C}$.

12.9 Remarks. (a) Let V be a Hilbert space and $j \in \mathcal{L}(V, H)$. Let $a\colon V \times V \to \mathbb{C}$ be a bounded j-coercive form, say

$$|a(u,v)| \leqslant M\|u\|_V\|v\|_V\,, \quad \mathrm{Re}\, a(u) + \omega\|j(u)\|_H^2 \geqslant \alpha\|u\|_V^2 \qquad (u, v \in \mathrm{dom}(a)).$$

Then a_ω is sectorial (see Theorem 5.8) and the norm $\|\cdot\|_{a_\omega,j}$ is equivalent to the given norm $\|\cdot\|_V$ on V. It follows that (a, j) is a closed quasi-sectorial form in H.

(b) Conversely, let (a, j) be a closed quasi-sectorial form in H, with vertex $-\omega$, and assume that $(\cdot \,|\, \cdot)_{a_\omega,j}$ is positive definite. Then $V := \mathrm{dom}(a)$ is a Hilbert space with scalar product $(\cdot \,|\, \cdot)_{a_\omega,j}$, and a is a bounded j-coercive form on V.

(c) Let (a, j) be a quasi-sectorial form in H, with vertex $-\omega$. Then the completion (V, q) of $(\mathrm{dom}(a), (\cdot \,|\, \cdot)_{a_\omega,j})$ is a Hilbert space for the norm given by $\|u\|_V := \|u\|_{\tilde{a}_\omega,\tilde{\jmath}} = (\mathrm{Re}\,\tilde{a}(u) + (\omega+1)\|\tilde{\jmath}(u)\|_H^2)^{1/2}$. The form $\tilde{a}$ is bounded for this norm and $\tilde{\jmath}$-coercive.

If additionally the form (a, j) is closed, then the construction simplifies, because the completion V of $\mathrm{dom}(a)$ is isomorphic to $\mathrm{dom}(a)/\{u \in \mathrm{dom}(a)\,;\, \|u\|_{a_\omega,j} = 0\}$, $q\colon \mathrm{dom}(a) \to V$ is the quotient map, and $\tilde{a}$ is given by

$$\tilde{a}(q(u), q(v)) = a(u,v) \qquad (u, v \in \mathrm{dom}(a)).$$ △

12.4 More on the non-closed case

In Section 12.1 we have defined the operator associated with a densely defined sectorial form that is not necessarily closed; see (12.5). In (12.8) we give a more direct description of this operator that avoids the reference to completion; it will be applied to concrete examples in Sections 12.5 and 12.6. It is surprising that, out of the purely algebraic condition of sectoriality, by the approximation formula (12.8) we obtain the generator of a holomorphic C_0-semigroup.

12.10 Theorem (Approximation formula). *Let H be a complex Hilbert space, and let (a, j) be a densely defined sectorial form in H. Then for the operator $A \sim (a, j)$ one has*

$$\begin{aligned} A = \big\{ (x,y) \in H \times H \,;\, &\text{there exists } (u_k) \text{ in } \mathrm{dom}(a) \text{ such that} \\ &\text{(i) } j(u_k) \to x \text{ as } k \to \infty, \\ &\text{(ii) } \lim_{k,\ell\to\infty} \mathrm{Re}\, a(u_k - u_\ell) = 0, \\ &\text{(iii) } a(u_k, v) \to (y \,|\, j(v))_H \ (v \in \mathrm{dom}(a)) \big\}. \end{aligned} \tag{12.8}$$

In this description, property (ii) *can be replaced by*
(ii′) $\sup_{k\in\mathbb{N}} \mathrm{Re}\, a(u_k) < \infty$.

Proof. We recall the notation used in the definition (12.5) of A: (V, q) is the completion of $(\mathrm{dom}(a), (\cdot \,|\, \cdot)_{a,j})$, and $\tilde{\jmath}\colon V \to H$ and $\tilde{a}\colon V \times V \to \mathbb{C}$ are as explained in Remark 12.3(c).

Let $(x, y) \in A$. Then there exists $w \in V$ such that $\tilde{\jmath}(w) = x$ and $\tilde{a}(w, v) = (y \,|\, \tilde{\jmath}(v))_H$ for all $v \in V$. Since q has dense range there exists a sequence (u_k) in $\mathrm{dom}(a)$ such that $q(u_k) \to w$ in V. By the continuity of $\tilde{\jmath}$ it follows that $j(u_k) = \tilde{\jmath}(q(u_k)) \to \tilde{\jmath}(w) = x$. Since q is isometric, we obtain

$$\mathrm{Re}\, a(u_k - u_\ell) + \|j(u_k) - j(u_\ell)\|_H^2 = \|u_k - u_\ell\|_a^2 = \|q(u_k) - q(u_\ell)\|_V^2 \to 0$$

as $k, \ell \to \infty$. Finally, for $v \in \mathrm{dom}(a)$ we have

$$a(u_k, v) = \tilde{a}(q(u_k), q(v)) \to \tilde{a}(w, q(v)) = (y \,|\, \tilde{\jmath}(q(v)))_H = (y \,|\, j(v))_H \,.$$

This shows the existence of a sequence (u_k) satisfying (i), (ii) and (iii) – and hence also the weaker property (ii′).

Conversely, let $(x, y) \in H \times H$ be such that there exists a sequence (u_k) in $\mathrm{dom}(a)$ satisfying (i), (ii′) and (iii). It follows from (i) together with (ii′) that $\sup_{k\in\mathbb{N}} \|u_k\|_{a,j} < \infty$. Thus, taking a subsequence if necessary, we can assume that $(q(u_k))$ converges weakly to some $w \in V$. Hence $j(u_k) = \tilde{\jmath}(q(u_k)) \to \tilde{\jmath}(w)$ weakly in H, and $\tilde{\jmath}(w) = x$ by (i). Property (iii) implies

$$\tilde{a}(w, q(v)) = \lim_{k\to\infty} \tilde{a}(q(u_k), q(v)) = \lim_{k\to\infty} a(u_k, v) = (y \,|\, j(v))_H = \big(y \,\big|\, \tilde{\jmath}(q(v))\big)_H$$

for all $v \in \mathrm{dom}(a)$. Since q has dense range and $\tilde{a}, \tilde{\jmath}$ are continuous, it follows that $\tilde{a}(w, \tilde{v}) = (y \,|\, \tilde{\jmath}(\tilde{v}))_H$ for all $\tilde{v} \in V$. This implies that $(x, y) \in A$. □

With the same proof one obtains the following real version of Theorem 12.10.

12.11 Theorem. *Let (a, j) be a densely defined accretive symmetric form in a real Hilbert space H. Define the linear relation A by* (12.8) *(where* (ii) *or* (ii′) *can be used). Then A is the accretive self-adjoint operator associated with (a, j).*

12.12 Remarks. Here we assume that a is an embedded densely defined sectorial form in H. Let $j\colon \mathrm{dom}(a) \hookrightarrow H$ denote the embedding.

(a) Let $(\tilde{a}, \tilde{\jmath})$ be the 'extension' of (a, j) as described in Remark 12.3(c). We call the form a **closable** if $\tilde{\jmath}$ is injective. Then we may identify $\mathrm{dom}(\tilde{a})$ with a subspace of H and consider $\tilde{a}$ as an embedded form.

An example of a closable form is the form b from the proof of Theorem 12.7 which leads to the Friedrichs extension.

(b) We have seen that we may associate an m-sectorial operator A with the form a, no matter whether a is closable or not. This means that in our context we can simply forget about the notion of closability.

From Corollary 12.8 we know that there exists a unique embedded closed sectorial form $\bar{a}$ in H that is associated with A. We refer to part (c) for a (somewhat involved) description of $\bar{a}$ in terms of a; we will not use this description in our subsequent results.

(c) Using the results of Section 8.3, we can describe $\bar{a}$ as follows; in particular it will turn out that $\mathrm{dom}(a)$ is a core for $\bar{a}$.

Let (V, q) be the completion of $(\mathrm{dom}(a), (\cdot \mid \cdot)_a)$; then q is injective by Remark 12.3(b). The application of Proposition 8.16 implies that $V = V_{\tilde{\jmath}}(\tilde{a}) \oplus \ker(\tilde{\jmath})$ is a topological direct sum, where

$$V_{\tilde{\jmath}}(\tilde{a}) = \big\{u \in V\,;\, \tilde{a}(u, v) = 0 \ (v \in \ker(\tilde{\jmath}))\big\}.$$

If we put $\check{\tilde{a}} := \tilde{a}|_{V_{\tilde{\jmath}}(\tilde{a}) \times V_{\tilde{\jmath}}(\tilde{a})}$ and $\check{\tilde{\jmath}} := \tilde{\jmath}|_{V_{\tilde{\jmath}}(\tilde{a})}$, then the operator A associated with (a, j) is also associated with the form $(\check{\tilde{a}}, \check{\tilde{\jmath}})$, by Proposition 8.14, and $\check{\tilde{\jmath}}$ is injective. Therefore the 'transported' form $\bar{a}$,

$$\bar{a}(\check{\tilde{\jmath}}(u), \check{\tilde{\jmath}}(v)) := \tilde{a}(u, v) \qquad (u, v \in V_{\tilde{\jmath}}(\tilde{a}))$$

is the unique embedded closed sectorial form associated with A.

Let P, Q denote the projections corresponding to the direct sum $V = V_{\tilde{\jmath}}(\tilde{a}) \oplus \ker(\tilde{\jmath})$. For $u \in \mathrm{dom}(a)$ one has

$$u = j(u) = \tilde{\jmath}(q(u)) = \tilde{\jmath}((P + Q)q(u)) = \tilde{\jmath}(Pq(u)),$$

and this shows that $\mathrm{dom}(a) = \tilde{\jmath}(Pq(\mathrm{dom}(a)) \subseteq \tilde{\jmath}(V_{\tilde{\jmath}}(\tilde{a})) = \mathrm{dom}(\bar{a})$ and that

$$\bar{a}(u, v) = \tilde{a}(Pq(u), Pq(v)) \qquad (u, v \in \mathrm{dom}(a)).$$

Moreover, the denseness of $q(\mathrm{dom}(a))$ in V implies that $Pq(\mathrm{dom}(a))$ is dense in $V_{\tilde{\jmath}}(\tilde{a})$, i.e. $Pq(\mathrm{dom}(a))$ is a core for $\check{\tilde{a}}$; hence $\mathrm{dom}(a)$ is a core for $\bar{a}$.

If additionally the form a is symmetric, then it turns out that $\bar{a}|_{\mathrm{dom}(a) \times \mathrm{dom}(a)}$ is the 'regular part' of a described in [Sim78; Section 2].

(d) We add a comment on the terminology. The term 'closed form' (in the situation of embedded forms) was forged by Kato – see, for instance, [Kat80; VI, § 1.3] – in a vague analogy to 'closed operator'. However, whereas closed operators *are* closed in the product topology, for forms there is no visible closed set. Note that the definition of 'closable' – see [Kat80; VI, § 1.4] – requires the associated closed form to be an embedded form. △

12.5 The Robin Laplacian for rough domains

Throughout this section the scalar field will be $\mathbb{K} = \mathbb{R}$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. On $\partial\Omega$ we consider the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}_{n-1}$; we refer to Section D.3 for the definition of Hausdorff measures and some of their properties. Here we mention that the n-dimensional Hausdorff measure on $\mathbb{R}^n$ coincides with the n-dimensional Lebesgue measure. Moreover, if Ω has C^1-boundary, then $\mathcal{H}_{n-1}$ coincides with the surface measure σ. For both of these properties we refer to Appendix E.

We will denote by $L_2(\partial\Omega)$ the space of L_2-functions with respect to the Hausdorff measure $\mathcal{H}_{n-1}$; according to the previous paragraph, the notation $L_2(\partial\Omega)$ is consistent with the notation from Chapter 7. In the present section we assume that $\mathcal{H}_{n-1}(\partial\Omega) < \infty$.

We define the **trace** tr as the closure of the operator $u \mapsto u|_{\partial\Omega} \colon C(\overline{\Omega}) \cap H^1(\Omega) \to L_2(\partial\Omega)$ in $H^1(\Omega) \times L_2(\partial\Omega)$,

$$\operatorname{tr} := \big\{(u,\varphi) \in H^1(\Omega) \times L_2(\partial\Omega)\,;\ \text{there exists } (u_k) \text{ in } C(\overline{\Omega}) \cap H^1(\Omega) \text{ such that } u_k \to u \text{ in } H^1(\Omega),\ u_k|_{\partial\Omega} \to \varphi \text{ in } L_2(\partial\Omega)\big\}.$$

For $u \in H^1(\Omega)$ we denote $\operatorname{tr} u := \{\varphi \in L_2(\partial\Omega)\,;\ (u,\varphi) \in \operatorname{tr}\}$. In general, the set $\operatorname{tr} u$ may consist of more than one element (and may also be empty); see Exercise 12.8. But if Ω has Lipschitz boundary, then $\operatorname{tr} u$ is a singleton for each $u \in H^1(\Omega)$, i.e. tr is an operator. For the case of C^1-boundary this has been shown in Theorem 7.10, for the case of Lipschitz boundary we refer to [Alt85; A 5.7].

Let $u \in H^1(\Omega)$ be such that $\Delta u \in L_2(\Omega)$. We say that $\partial_\nu u \in L_2(\partial\Omega)$ if there exists $h \in L_2(\partial\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega (\Delta u) v \,\mathrm{d}x = \int_{\partial\Omega} h v \,\mathrm{d}\mathcal{H}_{n-1} \tag{12.9}$$

for all $v \in C(\overline{\Omega}) \cap H^1(\Omega)$; in this case we define the **weak normal derivative** $\partial_\nu u := h$. The uniqueness of $\partial_\nu u$ is obtained from the denseness of $\big\{v|_{\partial\Omega}\,;\ v \in C^1(\overline{\Omega})\big\}$ in $L_2(\partial\Omega)$ as in Section 7.3.

12.13 Remarks. (a) The definition of the weak normal derivative given above is an extension of the definition in Section 7.3. Note that in the present situation there may exist points of $\partial\Omega$ for which there is no outer normal ν.

(b) The reader should be warned that the definition of $\partial_\nu u$ given above is not consistent with our definition of '$\partial_\nu u = 0$' (in quotes!) in Section 7.3. In fact, '$\partial_\nu u = 0$' means that (12.9) holds with $h = 0$ for all $v \in H^1(\Omega)$ (and not just for $v \in C(\overline{\Omega}) \cap H^1(\Omega)$); see (7.10). Thus, '$\partial_\nu u = 0$' implies $\partial_\nu u = 0$. The converse is true if Ω is such that $C(\overline{\Omega}) \cap H^1(\Omega)$ is dense in $H^1(\Omega)$. We refer to Exercise 12.6 for an example showing that in general $\partial_\nu u = 0$ does not imply '$\partial_\nu u = 0$'. △

Let $\beta \in L_\infty(\partial\Omega)$, $\inf \beta > 0$. Under our present general hypotheses, the **Robin Laplacian** is defined by

$$\Delta_\beta := \big\{(u,f) \in H^1(\Omega) \times L_2(\Omega)\,;\ \Delta u = f,\ \exists \varphi \in \operatorname{tr} u \colon \partial_\nu u = -\beta\varphi\big\}.$$

12.14 Theorem. *The operator $-\Delta_\beta$ is an accretive self-adjoint operator in $L_2(\Omega)$.*

Proof. Put $\operatorname{dom}(a) := C(\overline{\Omega}) \cap H^1(\Omega)$ and

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \beta uv \,\mathrm{d}\mathcal{H}_{n-1}\,.$$

Then a is densely defined in $L_2(\Omega)$, symmetric and accretive. Let A be the operator associated with a; see Section 12.1. By Remark **??**, A is self-adjoint and accretive. We will use Theorem 12.10 (with the embedding $j\colon \operatorname{dom}(a) \hookrightarrow L_2(\Omega)$) and Remark **??** to show that $-\Delta_\beta = A$.

'$A \subseteq -\Delta_\beta$'. Let $(u,f) \in A$. Then there exists a sequence (u_k) in $\operatorname{dom}(a)$ such that
(i) $u_k \to u$ in $L_2(\Omega)$,
(ii) $\lim_{k,\ell\to\infty} a(u_k - u_\ell) = 0$,
(iii) $\int_\Omega \nabla u_k \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \beta u_k v \,\mathrm{d}\mathcal{H}_{n-1} \to \int_\Omega fv \,\mathrm{d}x$ for all $v \in \operatorname{dom}(a)$.
One concludes from (i) and (ii) that $u \in H^1(\Omega)$ and $u_k \to u$ in $H^1(\Omega)$ and – using the hypothesis $\inf \beta > 0$ – that $\varphi := \lim_{k\to\infty} u_k|_{\partial\Omega}$ exists in $L_2(\partial\Omega)$. Then $\varphi \in \operatorname{tr} u$. Now (iii) implies that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \beta\varphi v \,\mathrm{d}\mathcal{H}_{n-1} = \int_\Omega fv \,\mathrm{d}x$$

for all $v \in \operatorname{dom}(a)$. Taking $v \in C_c^\infty(\Omega)$ we obtain $-\Delta u = f$. Thus

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega (\Delta u) v \,\mathrm{d}x = -\int_{\partial\Omega} \beta\varphi v \,\mathrm{d}\mathcal{H}_{n-1}$$

for all $v \in \operatorname{dom}(a)$. This means that $\partial_\nu u = -\beta\varphi$, by our definition (12.9).

'$-\Delta_\beta \subseteq A$'. Let $(u,f) \in -\Delta_\beta$. Then $u \in H^1(\Omega)$, $f = -\Delta u$, and there exists $\varphi \in \operatorname{tr} u$ such that $\partial_\nu u = -\beta\varphi$. By the definition of $\operatorname{tr} u$, there exists a sequence (u_k) in $\operatorname{dom}(a)$ such that $u_k \to u$ in $H^1(\Omega)$ and $u_k|_{\partial\Omega} \to \varphi$ in $L_2(\partial\Omega)$ as $k \to \infty$. Thus (i) and (ii) hold, and

$$a(u_k, v) = \int_\Omega \nabla u_k \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \beta u_k v \,\mathrm{d}\mathcal{H}_{n-1} \to \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \beta\varphi v \,\mathrm{d}\mathcal{H}_{n-1} = \int_\Omega fv \,\mathrm{d}x$$

for all $v \in \operatorname{dom}(a)$ since $-\Delta u = f$ and $\partial_\nu u = -\beta\varphi$. It follows that $(u,f) \in A$, by Theorem 12.10. □

Actually it is not necessary to call on Theorem 12.10 for the proof of Theorem 12.14. We add a proof that merely relies on the generation results of Section 5.3.

Second proof of Theorem 12.14. Let V be the closure of $\big\{(u, u|_{\partial\Omega});\, u \in C(\overline{\Omega}) \cap H^1(\Omega)\big\}$ in $H^1(\Omega) \oplus L_2(\partial\Omega)$. (This just means that V is the linear relation tr introduced above.) Put

$$a\big((u,\varphi),(v,\psi)\big) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \beta\varphi\psi \,\mathrm{d}\mathcal{H}_{n-1} \qquad ((u,\varphi),(v,\psi) \in V),$$

and define $j\colon V \to L_2(\Omega)$ by $j(u,\varphi) := u$. Then a is a bounded j-coercive form on V because $\inf \beta > 0$, and j has dense range. Since a is symmetric and accretive, by Theorem 5.6 the associated operator A is self-adjoint and accretive.

Let $(u,f)\in A$. Then there exists $\varphi\in L_2(\partial\Omega)$ such that $(u,\varphi)\in V$ (so $\varphi\in\operatorname{tr}u$) and

$$\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x+\int_{\partial\Omega}\beta\varphi\psi\,\mathrm{d}\mathcal{H}_{n-1}=\int_\Omega fv\,\mathrm{d}x \tag{12.10}$$

for all $(v,\psi)\in V$. Employing (12.10) for all $(v,0)$ with $v\in C_c^\infty(\Omega)$ we obtain $\Delta u=-f$. Then using (12.10) once more for all $(v,v|_{\partial\Omega})$ with $v\in C(\overline{\Omega})\cap H^1(\Omega)$, we conclude that $\partial_\nu u=-\beta\varphi\in L_2(\partial\Omega)$.

Conversely, let $(u,f)\in-\Delta_\beta$, i.e., $u\in H^1(\Omega)$, $f=-\Delta u$, and there exists $\varphi\in\operatorname{tr}u$ such that $\partial_\nu u=-\beta\varphi$. Then (12.10) holds for all $v\in C(\overline{\Omega})\cap H^1(\Omega)$, with $\psi=v|_{\partial\Omega}$. By denseness, (12.10) carries over to all $(v,\psi)\in V$; hence $(u,f)\in A$. □

12.6 The Dirichlet-to-Neumann operator for rough domains

Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set. We keep the definitions concerning $\mathcal{H}_{n-1}$, tr and ∂_ν from Section 12.5. Again we use $\mathbb{K}=\mathbb{R}$ and assume that $\mathcal{H}_{n-1}(\partial\Omega)<\infty$. We now define the **Dirichlet-to-Neumann operator** D_0.

12.15 Theorem. *The linear relation*

$$D_0:=\big\{(g,h)\in L_2(\partial\Omega)\times L_2(\partial\Omega);\,\exists u\in H^1(\Omega)\colon\Delta u=0,\ g\in\operatorname{tr}u,\ h=\partial_\nu u\big\}$$

is an accretive self-adjoint operator in $L_2(\partial\Omega)$.

It is part of the assertion of Theorem 12.15 that D_0 is an operator and not just a relation. This is remarkable because, given $g\in L_2(\partial\Omega)$, there may exist several functions $u\in H^1(\Omega)$ such that $\Delta u=0$ and $g=\operatorname{tr}u$, due to roughness of the boundary of Ω (see the Notes of this chapter). D_0 being an operator means that the weak normal derivatives $\partial_\nu u$ of these functions must coincide if they exist in $L_2(\partial\Omega)$.

For the proof of Theorem 12.15 we need a remarkable inequality due to Maz'ya. There exists a constant $c>0$ such that

$$\|u\|_{L_2(\Omega)}^2\leqslant c\Big(\int_\Omega|\nabla u|^2\,\mathrm{d}x+\int_{\partial\Omega}|u|^2\,\mathrm{d}\mathcal{H}_{n-1}\Big) \tag{12.11}$$

for all $u\in C(\overline{\Omega})\cap H^1(\Omega)$. We refer to Corollary D.10 for the proof of this inequality and to Remarks 12.17 and D.11(a) for further comments.

Proof of Theorem 12.15. Put $\operatorname{dom}(a):=C(\overline{\Omega})\cap H^1(\Omega)$ and $a(u,v):=\int_\Omega\nabla u\cdot\nabla v\,\mathrm{d}x$. Let $j\colon\operatorname{dom}(a)\to L_2(\partial\Omega)$ be given by $j(u):=u|_{\partial\Omega}$. Then (a,j) is an accretive symmetric form in $L_2(\partial\Omega)$, and (a,j) is densely defined because $C^1(\overline{\Omega})\subseteq\operatorname{dom}(a)$; recall Section 7.3 for the denseness of $\big\{v|_{\partial\Omega};\,v\in C^1(\overline{\Omega})\big\}$ in $L_2(\partial\Omega)$. Let A be the operator associated with (a,j); see Section 12.1. By Theorem 12.11 A is self-adjoint and accretive; we use Theorem 12.11 to show that $D_0=A$.

Let $(g,h)\in A$. Then there exists a sequence (u_k) in $C(\overline{\Omega})\cap H^1(\Omega)$ such that $u_k|_{\partial\Omega}\to g$ in $L_2(\partial\Omega)$, $\lim_{k,\ell\to\infty}\int_\Omega|\nabla(u_k-u_\ell)|^2\,\mathrm{d}x=0$ and $\lim_{k\to\infty}a(u_k,v)=\int_{\partial\Omega}hv\,\mathrm{d}\mathcal{H}_{n-1}$ for all

$v \in C(\overline{\Omega}) \cap H^1(\Omega)$. Now inequality (12.11) implies that (u_k) is a Cauchy sequence in $H^1(\Omega)$; let $u \in H^1(\Omega)$ be its limit. Then $g \in \operatorname{tr} u$ and

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_{\partial\Omega} hv \,\mathrm{d}\mathcal{H}_{n-1} \qquad (v \in C(\overline{\Omega}) \cap H^1(\Omega)).$$

Taking test functions $v \in C_c^\infty(\Omega)$ one obtains $\Delta u = 0$. Thus

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega (\Delta u) v \,\mathrm{d}x = \int_{\partial\Omega} hv \,\mathrm{d}\mathcal{H}_{n-1}$$

for all $v \in C(\overline{\Omega}) \cap H^1(\Omega)$. Hence $\partial_\nu u = h$ by the definition in Section 12.5, and we have shown that $A \subseteq D_0$.

Conversely, let $(g, h) \in D_0$. Then there exists $u \in H^1(\Omega)$ such that $g \in \operatorname{tr} u$, $\Delta u = 0$, $\partial_\nu u = h$. Hence, there exists a sequence (u_k) in $\operatorname{dom}(a)$ such that $u_k|_{\partial\Omega} \to g$ in $L_2(\partial\Omega)$ and $u_k \to u$ in $H^1(\Omega)$. Then $\lim_{k,\ell\to\infty} a(u_k - u_\ell) = 0$ and

$$a(u_k, v) = \int_\Omega \nabla u_k \cdot \nabla v \,\mathrm{d}x \to \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega (\Delta u) v \,\mathrm{d}x = \int_{\partial\Omega} hv \,\mathrm{d}\mathcal{H}_{n-1}$$

for all $v \in \operatorname{dom}(a)$. This shows that $(g, h) \in A$. □

12.16 Remark. One can prove Theorem 12.15 without using the results of Sections 12.1 to 12.4, similarly as in our second proof of Theorem 12.14. Again one puts $V := \operatorname{tr}$, and one defines the form a on V by

$$a\big((u,\varphi),(v,\psi)\big) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \qquad \big((u,\varphi),(v,\psi) \in V\big)$$

and $j\colon V \to L_2(\partial\Omega)$ by $j\big((u,\varphi)\big) := \varphi$. We point out that then the Maz'ya inequality (12.11) is responsible for the j-coercivity of the form a:

$$\begin{aligned} a\big((u,\varphi)\big) + \|j(u,\varphi)\|^2 &= \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_{\partial\Omega} |\varphi|^2 \,\mathrm{d}\mathcal{H}_{n-1} \\ &\geqslant \tfrac{1}{2c}\|u\|_{L_2(\Omega)}^2 + \tfrac{1}{2}\|\nabla u\|_2^2 + \tfrac{1}{2}\int_{\partial\Omega} |\varphi|^2 \,\mathrm{d}\mathcal{H}_{n-1} \end{aligned}$$

for all $(u, \varphi) = (u, u|_{\partial\Omega})$ with $u \in C^1(\overline{\Omega}) \cap H^1(\Omega)$, and then by denseness for all $(u, \varphi) \in V$.

The reader is asked to carry out the details in Exercise 12.9. △

12.17 Remark. If Ω has C^1-boundary, then we can easily show a version of (12.11) at this point. Indeed, in the proof of Proposition 8.2 we have shown the inequality

$$\int_\Omega |u|^2 \,\mathrm{d}x \leqslant \frac{1}{2}\int_\Omega |u|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + c\int_\Omega |\nabla u|^2 \,\mathrm{d}x + c\int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma,$$

and this implies

$$\|u\|_2^2 \leqslant (2c+1)\int_\Omega |\nabla u|^2 \,\mathrm{d}x + 2c\int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma.$$

Note, however, that the case of C^1-boundary has already been treated in Section 8.1; the purpose of the present section is to treat open sets with rough boundary. (The attentive reader will have noticed that above we have shown (12.11) with σ in place of $\mathcal{H}_{n-1}$. In Theorem E.3 we show that in fact σ and $\mathcal{H}_{n-1}$ coincide on $\partial\Omega$.)

Inequality (12.11) is a consequence of the following remarkable stronger inequality due to Maz'ya. There exists a constant $c(n)$, only depending on the dimension n, such that

$$\|u\|_{L_q(\Omega)} \leqslant c(n)\Big(\int_\Omega |\nabla u|\,\mathrm{d}x + \int_{\partial\Omega} |u|\,\mathrm{d}\mathcal{H}_{n-1}\Big), \tag{12.12}$$

for all $u \in C(\overline{\Omega}) \cap W_1^1(\Omega)$, where $q = \frac{n}{n-1}$; see [Maz11; Example 5.6.2/1 and Theorem 5.6.3]. We refer to Section D.1 for the definition of the Sobolev space $W_1^1(\Omega)$ and to Remark D.11(a) for comments on (12.12) and its proof. △

Notes

The treatment of sectorial forms in the setting of this chapter is adapted from [ArEl12b]. In that paper, our description of the operator A in Theorem 12.10 is taken as the *definition* of the operator associated with a non-closed sectorial form a. The applications treated in Sections 12.5 and 12.6 are also taken from [ArEl12b]. More information on the Dirichlet-to-Neumann operator is contained in [ArEl11].

For semi-bounded symmetric operators, the Friedrichs extension is contained in [Fri34]; see also Freudenthal [Fre36]. The Friedrichs extension for sectorial operators is due to Kato [Kat80; VI, § 2.3].

We now give some further information concerning the trace, as it was defined in Section 12.5. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and assume that $\mathcal{H}_{n-1}(\partial\Omega) < \infty$. Recall that the linear relation tr in $H^1(\Omega) \times L_2(\partial\Omega)$ is defined as the closure of the operator $u \mapsto u|_{\partial\Omega} \colon C(\overline{\Omega}) \cap H^1(\Omega) \to L_2(\partial\Omega)$. (For $u \in H^1(\Omega)$, the set $\operatorname{tr} u$ is sometimes called the 'approximative trace' of u.)

An intriguing question is whether $\operatorname{tr} 0 = \{0\}$, or equivalently, whether $\operatorname{tr} u$ is a singleton for all $u \in \operatorname{dom}(\mathrm{tr})$. In general, the answer is "no", as one sees from Exercise 12.8(b); for other examples see [ArWa03; Example 4.2], [ArEl11; Example 4.4], [Sau16]. Recent positive results are due to Sauter [Sau16], who investigates this question systematically. If $\Omega \subseteq \mathbb{R}$ is a bounded open set with continuous boundary, then $\operatorname{tr} u$ has at most one element for all $u \in H^1(\Omega)$; see [Sau16; Theorem 4.11]. Surprisingly, in dimension $n = 2$ the trace is unique for arbitrary connected bounded open sets Ω; see [Sau16; Corollary 5.4].

Another interesting question is whether the space $H_0^1(\Omega)$ can be characterized by means of the approximative trace, for a bounded open set $\Omega \subseteq \mathbb{R}^n$ with $\mathcal{H}_{n-1}(\partial\Omega) < \infty$. This is true if Ω has continuous boundary: then it follows from [Sau13; Corollary 7.48] in combination with the result mentioned in the previous paragraph that $H_0^1(\Omega) = \{u \in H^1(\Omega)\,;\, \operatorname{tr} u = \{0\}\}$; this generalises Theorem 7.11.

More generally (oder "In general"??) the inclusion $H_0^1(\Omega) \subseteq \{u \in H^1(\Omega)\,;\, 0 \in \operatorname{tr} u\}$ is obvious. It turns out that the converse (oder "opposite"??) inclusion holds if and only if the Dirichlet problem $\Delta u = 0$, $g \in \operatorname{tr} u$ has a unique solution $u \in H^1(\Omega)$ for all

$g \in \operatorname{ran}(\mathrm{tr})$; see [AES22; Theorem 5.2]. An example for non-uniqueness can be found in [AES22; Example 5.3], with the set $\Omega = B(0,2) \setminus (S \times \{0\}) \subseteq \mathbb{R}^2$, where $S \subseteq \mathbb{R}$ is the usual middle third Cantor set.

We give a short proof of the equivalence described/mentioned/stated in the previous paragraph. Let $H(\Omega) := \{u \in H^1(\Omega)\,;\, \Delta u = 0\}$. Then the mapping $J\colon H(\Omega) \to H^1(\Omega)/H^1_0(\Omega)$, $u \mapsto u + H^1_0(\Omega)$ is bijective, by Exercise 12.1. The equality of the spaces $H^1_0(\Omega)$ and $\{u \in H^1(\Omega)\,;\, 0 \in \operatorname{tr} u\}$ can be rephrased as the property that the mapping $\tilde{\mathrm{tr}}\colon H^1(\Omega)/H^1_0(\Omega) \to \operatorname{ran}(\mathrm{tr})$, $u + H^1_0(\Omega) \mapsto \operatorname{tr} u$ is bijective. Thus, equality of the spaces holds if and only if $\mathrm{tr}\colon H(\Omega) \to \operatorname{ran}(\mathrm{tr})$ is bijective, and the latter means precisely uniqueness of solutions of the Dirichlet problem. —-

Mit dem letzten Absatz bin ich bisher nur so mittelzufrieden. Ich bin jedenfalls gespannt, wie Du diesen Beweis findest! Er benutzt eine Aufgabe, die Wolfgang vorgeschlagen hatte, wenn auch nicht in der Form. Das sollte Exercise 7.9 werden. Wir hatten sowieso vor, den Beweis für die Äquivalenz in den Notes zu machen, und Wolfgang hatte vor ein paar Wochen einen Vorschlag gemacht, aber von dem waren beide nicht überzeugt.

Wolfgangs Vorschlag für die Exercise 7.9 sah so aus: —- Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Let $u \in H^1(\Omega)$. Show that there exist unique $u_1 \in H^1(\Omega)$, $u_0 \in H^1_0(\Omega)$ such that $\Delta u_1 = 0$ and $u = u_1 + u_0$. (Hint: Proof of Dirichlet's principle in the Notes of Chapter **??**.)

Ich würde gern folgendes draus machen: —-

12.1 Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Show that for any $v \in H^1(\Omega)$ there exist a unique function $u \in H^1(\Omega)$ such that $\Delta u = 0$, $v - u \in H^1_0(\Omega)$. Express the existence and uniqueness of u in terms of properties of the mapping $J\colon H(\Omega) \to H^1(\Omega)/H^1_0(\Omega)$, $u \mapsto u + H^1_0(\Omega)$. (Hint: Proof of Dirichlet's principle in the Notes of this chapter.)

—- Von der Formulierung bin ich auch hier bisher nur so mittelbegeistert. Die Aufgabe ist mit den Notes eigentlich trivial, aber irgendwie finde ich es ganz nett, die Leserin aufzufordern, sich zu überlegen, was das für die Abbildung J bedeutet. Was meinst Du?

Another interesting result, following from [Sau13; Corollary 7.48] in combination with the result mentioned before, is the equality $H^1_0(\Omega) = \{u \in H^1(\Omega)\,;\, \operatorname{tr} u = \{0\}\}$ for Ω with continuous boundary and arbitrary $n \in \mathbb{N}$; this generalises Theorem 7.11.

Incidentally, for non-empty bounded open sets $\Omega \subseteq \mathbb{R}^n$ the equality $H^1_0(\Omega) = \{u \in H^1(\Omega)\,;\, 0 \in \operatorname{tr} u\}$ is equivalent to the property that the Dirichlet problem (**??**) has a unique solution u for all $g \in \operatorname{ran}(\mathrm{tr})$; see [AES22; Theorem 5.2]. Thus uniqueness holds if Ω has continuous boundary.

Exercises

12.2 Let V be a $\mathbb{C}$-vector space, H a complex Hilbert space, $0 \neq j\colon V \to H$ linear, $a\colon V \times V \to \mathbb{C}$ a sesquilinear form. Let $\omega \in \mathbb{R}$, $\theta \in [0, \pi/2)$. Show that the following two properties are equivalent:

(i) (a, j) is quasi-sectorial with vertex $-\omega$ and angle θ,

(ii) $\{a(u)\,;\, u \in V,\ \|ju\|_H = 1\} \subseteq -\omega + \overline{\Sigma_\theta}$.

Hint: First show that (ii) implies $\{a(u)\,;\, u \in V,\ ju = 0\} \subseteq \overline{\Sigma_\theta}$.

12.3 Let Ω be a non-empty bounded open subset of $\mathbb{R}^n$, and let a be the classical Dirichlet form in $L_2(\Omega)$ with $\mathrm{dom}(a) := C_c^\infty(\Omega)$. Recall that the m-sectorial operator A associated with a by (12.5) is the negative Dirichlet Laplacian; see Example 12.5.

(a) Show that the operator A_0 from Proposition 12.2 is given by $A_0 = -\Delta$, with $\mathrm{dom}(A_0) = C_c^\infty(\Omega)$.

(b) Show that $\overline{A_0}$ is not m-sectorial. (This implies, in particular, that $\overline{A_0} \neq A$.)

(c) Show that the Friedrichs extension of the operator $-A_0$ (with A_0 from part (a)) is the Dirichlet Laplacian. (Comment: The Neumann Laplacian is another accretive self-adjoint extension of $-A_0$.)

12.4 Let H be a complex Hilbert space, a an embedded sectorial form in H. Show the following criteria for a being closed or closable:

(a) a is closed if and only if for any Cauchy sequence (u_n) in $(\mathrm{dom}(a), \|\cdot\|_a)$ with $u_n \to u$ in H one has $\|u_n - u\|_a \to 0$.

(b) a is closable if and only if for any Cauchy sequence (u_n) in $(\mathrm{dom}(a), \|\cdot\|_a)$ with $u_n \to 0$ in H one has $\|u_n\|_a \to 0$.

12.5 Let $H := L_2(-1,1)$, and let the forms a_1, a_2 in H be defined by $\mathrm{dom}(a_j) = C_c^\infty(-1,1)$,

$$a_1(u,v) := u(0)\overline{v(0)}, \qquad a_2(u,v) := \int_{-1}^{1} u'(x)\overline{v'(x)}\,\mathrm{d}x + u(0)\overline{v(0)}$$

for all $u, v \in C_c^\infty(-1,1)$.

For $j = 1, 2$ determine whether a_j is closable. Find the completion of $(\mathrm{dom}(a_j), (\cdot\,|\,\cdot)_{a_j})$ and the operator associated with a_j.

12.6 This exercise illustrates Remark 12.13(b). Let $u \in C^2(\overline{(-1,1)})$, $u'(-1) = u'(1) = 0$, $u'(0) \neq 0$, and put $\Omega := (-1,0) \cup (0,1)$.

(a) Show that for the weak normal derivative in the sense of Section 12.5 one has $\partial_\nu u = 0$.

(b) Show that one does *not* have '$\partial_\nu u = 0$' – observe the quotes! – in the sense of Section 7.4. (Hint: Choose $v \in H^1(\Omega)$ with $v(0-) \neq v(0+)$ as a 'test function'.)

12.7 Put $\Omega := (-1,0) \cup (0,1)$.

(a) Determine the relation tr of Section 12.5, and show that dom(tr) is not dense.

(b) Find $\partial_\nu u$ for those $u \in H^1(\Omega)$ with $\Delta u \in L_2(\Omega)$ for which $\partial_\nu u \in L_2(\partial\Omega)$.

(c) Determine the Robin Laplacian for $\beta = 1$.

12.8 Put $S := [0,1] \times \{0\} \subseteq \mathbb{R}^2$, and let (x_n) be a bounded sequence in $\mathbb{R}^2 \setminus S$ having the set S as its cluster points. Let (r_n) be a sequence in $(0,\infty)$ such that $\sum_{n=1}^\infty r_n < \infty$, $B[x_n, r_n] \cap S = \varnothing$ $(n \in \mathbb{N})$ and $B[x_n, r_n] \cap B[x_m, r_m] = \varnothing$ $(m, n \in \mathbb{N},\ m \neq n)$. Put $\Omega := \bigcup_{n\in\mathbb{N}} B(x_n, r_n)$.

(a) Determine $\partial\Omega$ and $\mathcal{H}_1(\partial\Omega)$ (1-dimensional Hausdorff measure). (Note that $S \subseteq \partial\Omega$.)

(b) Show that dom(tr) is dense in $H^1(\Omega)$ and that $\mathrm{tr}\,0 = L_2(S)$ (where $L_2(S)$ is considered as a subspace of $L_2(\partial\Omega)$ in the natural way).

(c) Let D_0 be the Dirichlet-to-Neumann operator for Ω. Show that $L_2(S) \subseteq \mathrm{dom}(D_0)$ and that $D_0|_{L_2(S)} = 0$.

12.9 Complete the proof of Theorem 12.15 as suggested in Remark 12.16.

Chapter 13

Approximation of strongly continuous semigroups

In this chapter we present two important classical results on the approximation of semigroups, the Trotter approximation theorem and the Chernoff product formula. In view of the applications in Chapter 15 we treat these topics in the more general context of 'degenerate strongly continuous semigroups'. This concept will be introduced in Section 13.1. The chapter closes with an introduction to the spectral theorem for self-adjoint operators.

13.1 Degenerate strongly continuous semigroups and m-accretive linear relations

Let X be a Banach space, and let T be a one-parameter semigroup on X. Recall from Remark 1.1(b) that then $T(0)$ is a projection. We call T a **degenerate strongly continuous semigroup** if

$$T(0) = \text{s-lim}_{t\to 0+} T(t)$$

(not necessarily $T(0) = I$!). Note that a C_0-semigroup is also a degenerate strongly continuous semigroup. We would have preferred to omit 'degenerate' in the previous definition; however, in the literature the notions 'strongly continuous semigroup' and 'C_0-semigroup' are usually synonymous.

A degenerate strongly continuous semigroup T is the direct sum of a C_0-semigroup and a semigroup that is identically zero. Indeed, define $P := T(0)$, $X_e := \text{ran}(P)$, and observe that X_e is invariant under T. By T_{X_e} we denote the restriction of T to X_e, i.e. $T_{X_e}(t) := T(t)|_{X_e}$ for all $t \geqslant 0$. Then T_{X_e} is a C_0-semigroup on X_e, whereas $T(t) = T(t)P$ vanishes on $\ker(P)$ for all $t \geqslant 0$. In particular it follows that $[0,\infty) \ni t \mapsto T(t)$ is strongly continuous.

There exist $\omega \in \mathbb{R}$, $M \geqslant 1$ such that $\|T(t)\| \leqslant M\mathrm{e}^{\omega t}$ $(t \geqslant 0)$. (This property is known for the C_0-semigroup T_{X_e} and carries over to T because $T(t) = T_{X_e}(t)P$ for all $t \geqslant 0$.) Let A_e denote the generator of T_{X_e}. Then it follows from Theorem 2.7 that

$$R(\lambda, A_e)P = \int_0^\infty \mathrm{e}^{-\lambda t} T_{X_e}(t)\,\mathrm{d}t\, P = \int_0^\infty \mathrm{e}^{-\lambda t} T(t)\,\mathrm{d}t \qquad \text{(strong integral)}$$

for all $\lambda \in \mathbb{K}$ with $\operatorname{Re}\lambda > \omega$. It is easy to see that the mapping $\rho(A_e) \ni \lambda \mapsto R(\lambda, A_e)P$ is a **pseudo-resolvent**, i.e., $R(\cdot, A_e)P$ is continuous and satisfies the resolvent equation (2.1).

We define the **generator** of T as the linear relation

$$A := \big\{(x,y) \in X \times X\,;\, (x, Py) \in A_e\big\}. \tag{13.1}$$

In Exercise 13.1, the reader is asked to show that then the pseudo-resolvent $R(\cdot, A_0)P$ introduced above is the **resolvent** of the generator A, which is defined by

$$R(\lambda, A) := (\lambda I - A)^{-1}$$

for all $\lambda \in \rho(A_0)$, where

$$\lambda I - A := \big\{(x, \lambda x - y)\,;\, (x,y) \in A\big\};$$

recall Subsection 1.3.1 concerning the inverse of a relation. According to the definition (13.1), A is the direct sum $A_0 \oplus \big(\{0\} \times \ker(P)\big)$ in $X \times X$; in particular, $\{0\} \times \ker(P)$ is a subset of A.

We illustrate these notions by an example.

13.1 Example. Let X be a Banach space, $P \in \mathcal{L}(X)$ a projection, $P \neq 0$, $X_e := \operatorname{ran}(P)$. Then $T(t) := P$ $(t \geqslant 0)$ defines a degenerate strongly continuous semigroup, the generator of T is given by

$$A = \big(X_e \times \{0\}\big) \oplus \big(\{0\} \times \ker(P)\big) = X_e \times \ker(P),$$

and the resolvent of A is given by $R(\lambda, A) = \lambda^{-1}P$, for $\lambda \in \rho(A_e) = \rho(0_{\mathcal{L}(X_e)}) = \mathbb{K}\backslash\{0\}$. $\triangle$

In the remainder of this section we deal with the special case in which the underlying space is a Hilbert space H. If T is a contractive degenerate strongly continuous semigroup on H, then $P := T(0)$ is a contractive projection onto $H_e := \operatorname{ran}(P)$; hence P is the orthogonal projection onto H_e. Let A_e be the generator of the C_0-semigroup T_{H_e}. Then the generator A of T is the relation $A := A_e \oplus (\{0\} \times H_u)$ (orthogonal direct sum in $H \oplus H$), where $H_u := H_e^{\perp}$. For $(x,y) \in A$ one obtains

$$\operatorname{Re}(y \,|\, x) = \operatorname{Re}(Py \,|\, x) = \operatorname{Re}(A_e x \,|\, x) \leqslant 0$$

(because A_e is m-accretive), i.e. the linear relation $-A = \{(x, -y)\,;\, (x,y) \in A\}$ is accretive. It is even m-accretive since

$$\operatorname{ran}(I - A) = \operatorname{ran}(I_e - A_e) \oplus \operatorname{ran}(\{0\} \times H_u) = H_e \oplus H_u = H,$$

where I_e is the identity operator in H_e.

Here, a linear relation A in H is called **accretive** if $\operatorname{Re}(y \,|\, x) \geqslant 0$ for all $(x,y) \in A$, and A is called **m-accretive** if additionally $\operatorname{ran}(I + A) = \{x + y\,;\, (x,y) \in A\} = H$.

Next we show that all m-accretive linear relations arise as an orthogonal direct sum of an m-accretive operator and a 'trivial' m-accretive linear relation, as above.

13.2 Theorem. *Let A be a linear relation in H.*

(a) *Let A be m-accretive. Put $H_e := \overline{\mathrm{dom}(A)}$ and $H_u := H_e^\perp$. Then $A_e := A \cap (H_e \times H_e)$ is an m-accretive operator in H_e, $A_u := A \cap (H_u \times H_u) = \{0\} \times H_u$ is an m-accretive linear relation in H_u, and $A = A_e \oplus A_u$.*

(b) *The linear relation $-A$ is the generator of a contractive degenerate strongly continuous semigroup on H if and only if A is m-accretive.*

For the proof we single out a technical detail.

13.3 Lemma. *Let A be an accretive linear relation in H with dense domain. Then A is an operator.*

Proof. Let $(0, z) \in A$. Then for all $(x, y) \in A$, $\lambda \in \mathbb{K}$, one obtains $(x, y + \lambda z) \in A$,

$$0 \leqslant \mathrm{Re}(y + \lambda z \,|\, x) = \mathrm{Re}(y \,|\, x) + \mathrm{Re}(\lambda(z \,|\, x)).$$

This shows that $z \perp \mathrm{dom}(A)$, hence $z = 0$. □

Proof of Theorem 13.2. (a) Obviously, A_e and A_u are accretive linear relations, and $\mathrm{dom}(A_u) = \{0\}$. Let $z \in H_u$. By the m-accretivity of A, there exists $(x, y) \in A$ such that $x + y = z$. Then $\mathrm{Re}(y \,|\, x) \geqslant 0$, and

$$0 = (z \,|\, x) = (x + y \,|\, x) = \|x\|^2 + (y \,|\, x) \geqslant \|x\|^2$$

implies $x = 0$, hence $(0, z) = (0, y) = (x, y) \in A$. This shows that $A_u = \{0\} \times H_u$.

Let $(x, y) \in A$. There exist $y_e \in H_e$, $y_u \in H_u$ such that $y = y_e + y_u$. Then $(0, y_u) \in A_u \subseteq A$, hence $(x, y_e) \in A \cap (H_e \times H_e) = A_e$. It follows that $\mathrm{dom}(A_e) = \mathrm{dom}(A)$, so $\mathrm{dom}(A_e)$ is dense in H_e, and Lemma 13.3 implies that A_e is an operator.

For the proof of the m-accretivity of A_e let $z \in H_e$. There exists $(x, y) \in A$ such that $x + y = z$, hence $y = z - x \in \overline{\mathrm{dom}(A)} = H_e$, which implies that $(x, y) \in A_e$, $z \in \mathrm{ran}(I_e + A_e)$.

Summing up, we have shown that A is the direct sum of the m-accretive operator A_e and the m-accretive linear relation A_u.

(b) The necessity follows from the considerations before Theorem 13.2.

Conversely, if A is m-accretive, then, with the notation of part (a), the Lumer–Phillips theorem (Theorem 3.16) shows that $-A_e$ generates a contractive C_0-semigroup T_e on H_e. Let P denote the orthogonal projection onto H_e. Then $T(t) := T_e(t)P$ $(t \geqslant 0)$ defines the degenerate strongly continuous semigroup T on H with generator $-A$. □

Next, let T be a **quasi-contractive** degenerate strongly continuous semigroup on H, i.e., there exists $\omega \in \mathbb{R}$ such that $t \mapsto \mathrm{e}^{-\omega t} T(t) =: T_\omega(t)$ is contractive, and let $-A$ be the generator of T. Then the generator $-A_\omega$ of T_ω and A are related by $-A_\omega = -A - \omega I$, and hence for $(x, y) \in A$ one obtains $(x, y + \omega x) \in A_\omega$,

$$\mathrm{Re}(y \,|\, x) = \mathrm{Re}(y + \omega x \,|\, x) - \omega \|x\|^2 \geqslant -\omega \|x\|^2.$$

This means that the relation $A = A_\omega - \omega I$ is **quasi-accretive**. In fact, as $A + (\omega + 1)I = A_\omega + I$ has range equal to H, the linear relation A is even **quasi-m-accretive**. A statement analogous to Theorem 13.2 holds also for quasi-m-accretive linear relations.

13.2 A Trotter approximation theorem

In this section X will be a Banach space.

13.4 Theorem (Trotter approximation theorem for degenerate strongly continuous semigroups). *Let T and T_n ($n \in \mathbb{N}$) be contractive degenerate strongly continuous semigroups on X, with generators A and A_n ($n \in \mathbb{N}$), respectively. Assume that there exists $\lambda > 0$ such that*

$$R(\lambda, A_n)x \to R(\lambda, A)x \qquad (n \to \infty) \tag{13.2}$$

for all $x \in \operatorname{ran}(T(0))$. Then

$$T_n(t)x \to T(t)x \qquad (n \to \infty) \tag{13.3}$$

uniformly on compact subsets of $[0, \infty)$, for all $x \in \operatorname{ran}(T(0))$.

13.5 Remark. It is worth noticing that the converse of Theorem 13.4 is also true. Indeed, if T and T_n ($n \in \mathbb{N}$) are contractive degenerate strongly continuous semigroups, and

$$T(t)x = \lim_{n\to\infty} T_n(t)x \tag{13.4}$$

for all $x \in \operatorname{ran}(T(0))$, $t \geqslant 0$, then

$$R(\lambda, A)x = \int_0^\infty \mathrm{e}^{-\lambda t} T(t)x \,\mathrm{d}t = \lim_{n\to\infty} \int_0^\infty \mathrm{e}^{-\lambda t} T_n(t)x \,\mathrm{d}t = \lim_{n\to\infty} R(\lambda, A_n)x$$

for all $x \in \operatorname{ran}(T(0))$, $\lambda > 0$, by the dominated convergence theorem.

We point out that, even though only pointwise convergence is required in (13.4), Theorem 13.4 implies that the convergence in (13.4) is in fact uniform for t in compact subsets of $[0, \infty)$. △

13.6 Example. The following example shows that the convergence (13.2) for all $x \in X$ does not imply the convergence (13.3) for all $x \in X$.

Let $X := \mathbb{C}$, $A_n := \mathrm{i}n$, $A := \{0\} \times \mathbb{C}$. Then $R(\lambda, A_n) = (\lambda - \mathrm{i}n)^{-1} \to 0 = R(\lambda, A)$ for all $\lambda > 0$, but $T_n(t) = \mathrm{e}^{\mathrm{i}nt}$ does not converge to $T(t) = 0$. △

13.7 Lemma. *Let T and S be contractive degenerate strongly continuous semigroups, with generators A and B, respectively. Then, for $\lambda > 0$, $x \in X$, $t > 0$, we have*

$$R(\lambda, B)\big(T(t) - S(t)\big)R(\lambda, A)x = \int_0^t S(t-s)\big(R(\lambda, A) - R(\lambda, B)\big)T(s)x \,\mathrm{d}s. \tag{13.5}$$

Proof. We write $P_T := T(0)$, and we denote the generator of $T_{\operatorname{ran}(P_T)}$ by A_0. In the computation made below we will use the identities $T(t) = P_T T(t) = T(t)P_T$ and $R(\lambda, A) = R(\lambda, A_0)P_T$. The corresponding notation and properties will be used for S.

For $x \in X$, the function $s \mapsto S(t-s)R(\lambda,B)T(s)R(\lambda,A)x$ is differentiable on $[0,t]$, with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}&S(t-s)R(\lambda,B)T(s)R(\lambda,A)x\\ &= S(t-s)\big(-B_0R(\lambda,B)R(\lambda,A) + R(\lambda,B)A_0R(\lambda,A)\big)T(s)x\\ &= S(t-s)\big(P_SR(\lambda,A) - R(\lambda,B)P_T\big)T(s)x\\ &= S(t-s)\big(R(\lambda,A) - R(\lambda,B)\big)T(s)x,\end{aligned}$$

where in the middle equality we have used $-B_0R(\lambda,B) = (\lambda - B_0 - \lambda)R(\lambda,B) = P_S - \lambda R(\lambda,B)$, and similarly $A_0R(\lambda,A) = -P_T + \lambda R(\lambda,A)$. Integrating with respect to s for $0 \leqslant s \leqslant t$ we obtain (13.5). □

Another ingredient in the proof of Theorem 13.4 is the following important fact; it will also be needed in the proof of Theorem 13.13.

13.8 Lemma. *Let A be the generator of a C_0-semigroup on X. Then* $\mathrm{dom}(A^2)$ *is dense in X.*

Proof. Let $B \in \mathcal{L}(X)$ have dense range. Then obviously $\mathrm{ran}(B^2) = B(\mathrm{ran}(B))$ is dense. Applying this to a resolvent of A – and recalling that $\mathrm{dom}(A)$ is dense – one obtains the assertion. □

Proof of Theorem 13.4. As before we write $P := T(0)$, $X_0 := \mathrm{ran}(P)$ and denote by A_0 the generator of the C_0-semigroup T_{X_0}.

Let $y \in X_0$, $t_0 > 0$. We estimate

$$\begin{aligned}\big\|\big(T(t) - T_n(t)\big)R(\lambda,A)y\big\| \leqslant{}& \big\|\big(R(\lambda,A) - R(\lambda,A_n)\big)T(t)y\big\|\\ &+ \big\|R(\lambda,A_n)\big(T(t) - T_n(t)\big)y\big\|\\ &+ \big\|T_n(t)\big(R(\lambda,A_n) - R(\lambda,A)\big)y\big\|.\end{aligned} \tag{13.6}$$

The third term on the right-hand side of (13.6) converges to 0 uniformly for t in $[0,\infty)$. For the first term we note that $\{T(t)y;\, 0 \leqslant t \leqslant t_0\}$ is a compact subset of X_0 – by the strong continuity of T – and that strong convergence of a sequence of operators implies uniform convergence on compact sets; see Exercise 3.2(b). So we conclude uniform convergence to 0 for $0 \leqslant t \leqslant t_0$.

For the treatment of the second term on the right-hand side of (13.6) we confine our attention to the case in which y is of the form $y = R(\lambda,A)z$, for some $z \in X_0$, and we apply Lemma 13.7 to obtain

$$\begin{aligned}\big\|R(\lambda,A_n)\big(T(t) - T_n(t)\big)R(\lambda,A)z\big\| &= \left\|\int_0^t T_n(t-s)\big(R(\lambda,A) - R(\lambda,A_n)\big)T(s)z\,\mathrm{d}s\right\|\\ &\leqslant \int_0^t \big\|\big(R(\lambda,A) - R(\lambda,A_n)\big)T(s)z\big\|\,\mathrm{d}s.\end{aligned}$$

As above one concludes that the integrand converges to 0 uniformly for $0 \leqslant s \leqslant t_0$, and therefore the second term converges to 0 uniformly for $0 \leqslant t \leqslant t_0$.

So far, we have shown that $T_n(t)x \to T(t)x$ as $n \to \infty$, uniformly for t in compact subsets of $[0,\infty)$, for all x of the form $x = R(\lambda, A)^2 z$ with $z \in X_0$, i.e. for all $x \in \mathrm{dom}(A_0^2)$. As $\mathrm{dom}(A_0^2)$ is dense in X_0 – by Lemma 13.8 – and T_n ($n \in \mathbb{N}$) as well as T are contractive, one obtains the asserted convergence for all $x \in X_0$. □

Example 13.6 illustrates that in the setting of Theorem 13.4 there is no information on the convergence of the sequence of semigroups on the complement of $\mathrm{ran}(T(0))$. The objective of the remaining part of this section is to present a setting in which this deficiency can be remedied; see Theorem 13.10.

We will need the following concept of operator convergence that is important in the context of semigroup approximation. If A and A_n, for $n \in \mathbb{N}$, are generators of contractive degenerate strongly continuous semigroups, then we say that (A_n) converges to A in the **strong resolvent sense** if $R(\lambda, A_n) \to R(\lambda, A)$ strongly as $n \to \infty$, for all $\lambda > 0$. It follows from the subsequent Lemma 13.9 that it suffices to require the convergence for one $\lambda > 0$. Actually, we will also use this terminology for generators $-A_n$ and $-A$, as well as for the case when all operators are shifted by a common real multiple of the identity I. We refer to [Kat80; VIII, § 1.1], where convergence in the strong resolvent sense is introduced under the name 'convergence in the generalised sense'.

Note that a property similar to the above definition had already been used in Theorem 13.4, but in this instance the strong convergence of the resolvents was only required on a subspace of the Banach space.

13.9 Lemma. *Let A and A_n ($n \in \mathbb{N}$) be generators of contractive degenerate strongly continuous semigroups on X. Assume that there exists $\lambda > 0$ such that*

$$R(\lambda, A_n) \to R(\lambda, A) \qquad (n \to \infty)$$

strongly. Then

$$R(\mu, A_n)x \to R(\mu, A)x \qquad (n \to \infty),$$

uniformly for μ in compact subsets of $(0,\infty)$, for all $x \in X$.

Proof. A standard computation with applications of the resolvent equation (2.1) yields

$$\begin{aligned} &R(\mu, A) - R(\mu, A_n) \\ &\quad = \big(I + (\lambda - \mu)R(\mu, A_n)\big)R(\mu, A) - R(\mu, A_n)\big(I + (\lambda - \mu)R(\mu, A)\big) \\ &\quad = \big(I + (\lambda - \mu)R(\mu, A_n)\big)\big(R(\lambda, A) - R(\lambda, A_n)\big)\big(I + (\lambda - \mu)R(\mu, A)\big) \end{aligned}$$

for all $\mu > 0$. Let $x \in X$, and let $0 < \mu_0 < \mu_1$. Then the set

$$\big\{\big(I + (\lambda - \mu)R(\mu, A)\big)x\,;\ \mu_0 \leqslant \mu \leqslant \mu_1\big\}$$

is compact, and

$$\sup_{\mu_0 \leqslant \mu \leqslant \mu_1,\, n \in \mathbb{N}} \|I + (\lambda - \mu)R(\mu, A_n)\| < \infty.$$

Since $R(\lambda, A_n)y \to R(\lambda, A)y$ as $n \to \infty$, uniformly for y in compact subsets of X – by Exercise 3.2(b) –, one obtains the assertion. □

In the following supplement to Theorem 13.4 we present a more special setting in which one also obtains information on the convergence of the approximating semigroups on the complement of $\mathrm{ran}(T(0))$, as announced above.

13.10 Theorem. *Let H be a Hilbert space, and let T and T_n ($n \in \mathbb{N}$) be degenerate strongly continuous semigroups of self-adjoint contractions on H, with corresponding generators A and A_n ($n \in \mathbb{N}$). Assume that (A_n) converges to A in the strong resolvent sense. Then*

$$T_n(t)x \to T(t)x \qquad (n \to \infty)$$

uniformly on compact subsets of $[0,\infty)$, for all $x \in \mathrm{ran}(T(0))$, and

$$T_n(t)x \to 0 = T(t)x \qquad (n \to \infty)$$

uniformly for t in compact subsets of $(0,\infty)$, for all $x \in \ker(T(0)) = \mathrm{ran}(T(0))^\perp$. (Note that $T(0)$ is a self-adjoint, therefore orthogonal, projection.)

We point out that, even if all the semigroups T_n in Theorem 13.10 are C_0-semigroups, it is possible that the limiting semigroup T is degenerate, i.e. $\mathrm{ran}(T(0))$ is a proper subspace of H. In this case $T(0)x = 0 \neq x = T_n(0)x$ for all $x \in \mathrm{ran}(T(0))^\perp \setminus \{0\}$. This explains why on $\mathrm{ran}(T(0))^\perp$ the convergence can only be expected to be uniform on compact subsets of $(0,\infty)$.

For the proof of Theorem 13.10 we need another auxiliary result, Lemma 13.12 below, in which it will be convenient to use the following piece of notation.

13.11 Remark. In Section 11.5 we have used the inequality sign between operators for inequalities in the sense of order in function spaces. In the theory of operators in a Hilbert space H it is also common to denote the accretivity of a symmetric operator A by writing '$A \geqslant 0$'. It will be convenient to use '$A \leqslant B$', for self-adjoint operators $A, B \in \mathcal{L}(H)$, to mean that $B - A$ is accretive, spelled out, that

$$(Ax \,|\, x) \leqslant (Bx \,|\, x) \qquad (x \in H).$$

The reader will have to be aware of this double meaning of the inequality sign; usually it will be clear from the context in which sense it should be interpreted. (In particular, in the context of a Hilbert space which is not a priori an L_2-space, the interpretation in the sense of order in function spaces would be meaningless.) △

13.12 Lemma. *Let H be a Hilbert space, and let T be a degenerate strongly continuous semigroup of self-adjoint contractions on H, with generator A. Then*

$$T(t) \leqslant (I - tA)^{-1} \qquad (t \geqslant 0).$$

Proof. As explained in Section 13.1, $P := T(0)$ is an orthogonal projection, and H decomposes as $H_0 \oplus H_1$, where $H_0 = \mathrm{ran}(P)$, $H_1 = \ker(P)$, H_0 is invariant under T and the restriction T_{H_0} of T to H_0 is a C_0-semigroup, whereas $T(t)|_{H_1} = 0$ ($t \geqslant 0$). Then the generator A_0 of T_{H_0} is self-adjoint, and $-A_0$ is accretive.

Now we apply the spectral theorem for self-adjoint operators, Theorem 13.20, presented in Section 13.4 below, which states that A_0 is unitarily equivalent to a maximal multiplication operator M in $L_2(\mu)$, for some measure space $(\Omega, \mathcal{A}, \mu)$. Because of the accretivity of $-A_0$ we may assume that M is induced by a measurable function $\alpha\colon \Omega \to (-\infty, 0]$; see Remark 13.21, in particular property (ii). Let $t \geqslant 0$. Then Exercise 13.3(b) implies that $T_{H_0}(t)$ corresponds to the operator of multiplication by $\mathrm{e}^{t\alpha}$, whereas it is easy to see that $(I - tA_0)^{-1}$ corresponds to multiplication by $(1 - t\alpha)^{-1}$. From $\mathrm{e}^s \geqslant 1 + s$ $(s \geqslant 0)$ one obtains $\mathrm{e}^{t\alpha} \leqslant (1 - t\alpha)^{-1}$, and this inequality implies $T_{H_0}(t) \leqslant (I - tA_0)^{-1}$, by property (iv) of Remark 13.21. From $T(t) = PT_{H_0}(t)P$ and $(I - tA)^{-1} = P(I - tA_0)^{-1}P$ we then obtain the assertion of the lemma. □

Proof of Theorem 13.10. The first assertion is immediate from Theorem 13.4.

For the second convergence we apply Lemma 13.12 to T_n and A_n and obtain

$$\begin{aligned} \|T_n(t)x\|^2 = (T_n(2t)x \,|\, x) &\leqslant \big((I - 2tA_n)^{-1}x \,\big|\, x\big) \\ &\leqslant \big\|(I - 2tA_n)^{-1}x\big\|\|x\| = \tfrac{1}{2t}\big\|\big(\tfrac{1}{2t} - A_n\big)^{-1}x\big\|\|x\| \end{aligned} \tag{13.7}$$

for all $x \in H$, $t > 0$. Now let $x \in \ker(T(0))$. It follows from Lemma 13.9 that

$$(\lambda - A_n)^{-1}x \to (\lambda - A)^{-1}x = (\lambda - A)^{-1}T(0)x = 0$$

as $n \to \infty$, uniformly for λ in compact subsets of $(0, \infty)$. Therefore, (13.7) implies that $\|T_n(t)x\|^2 \to 0$ uniformly for t in compact subsets of $(0, \infty)$. □

13.3 The Chernoff product formula

In the first part of this section X will be a Banach space.

13.13 Theorem (Chernoff product formula for degenerate strongly continuous semigroups). *Let T be a contractive degenerate strongly continuous semigroup on X, with generator A. Let $F\colon [0, \infty) \to \mathcal{L}(X)$ satisfy $F(0)T(0) = T(0)$, $\|F(t)\| \leqslant 1$ for all $t \geqslant 0$. Define*

$$A(s) := \frac{1}{s}(F(s) - I) \qquad (s > 0),$$

and assume that there exists $\lambda > 0$ such that

$$R\big(\lambda, A(s)\big)x \to R(\lambda, A)x \qquad (s \to 0)$$

for all $x \in \mathrm{ran}(T(0))$. Then

$$F(t/n)^n x \to T(t)x, \tag{13.8}$$

uniformly for t in compact subsets of $[0, \infty)$, for all $x \in \mathrm{ran}(T(0))$.

13.14 Remarks. (a) The hypothesis '$F(0)T(0) = T(0)$' in Theorem 13.13 is a short way to express that $F(0)x = x$ for all $x \in \mathrm{ran}(T(0))$.

(b) We mention that the hypotheses of Theorem 13.13 imply that

$$\big\|\mathrm{e}^{tA(s)}\big\| = \big\|\mathrm{e}^{(t/s)F(s)}\mathrm{e}^{-t/s}\big\| \leqslant \mathrm{e}^{-t/s}\sum_{j=0}^{\infty}\frac{1}{j!}\big\|(t/s)F(s)\big\|^j \leqslant 1$$

for all $t \geqslant 0$, $s > 0$. This explains why $\lambda > 0$ belongs to $\rho\big(A(s)\big)$; the above estimate will also be needed in the proof of Theorem 13.13. △

13.15 Lemma. *Let $S \in \mathcal{L}(X)$ satisfy $\|S^m\| \leqslant M$ for some $M \geqslant 1$ and all $m \in \mathbb{N}$. Then*

$$\big\|\mathrm{e}^{n(S-I)}x - S^n x\big\| \leqslant \sqrt{n}\, M\|Sx - x\|$$

for all $n \in \mathbb{N}$, $x \in X$.

Proof. We will prove the asserted estimate in the form

$$\big\|\mathrm{e}^{nS}x - \mathrm{e}^n S^n x\big\| \leqslant \mathrm{e}^n\sqrt{n}\, M\|Sx - x\|.$$

First we show that

$$\big\|S^k x - S^m x\big\| \leqslant |k-m|\, M\|Sx - x\|$$

for all $k, m \in \mathbb{N}_0$, $x \in X$. Without loss of generality $k > m$. Then

$$\big\|\big(S^k - S^m\big)x\big\| = \left\|\sum_{j=m}^{k-1}\big(S^{j+1} - S^j\big)x\right\| \leqslant \sum_{j=m}^{k-1}\big\|S^j\big\|\|Sx - x\| \leqslant |k-m|\, M\|Sx - x\|.$$

We now estimate

$$\begin{aligned}\big\|\mathrm{e}^{nS}x - \mathrm{e}^n S^n x\big\| &= \left\|\sum_{k=0}^{\infty}\frac{n^k}{k!}\big(S^k - S^n\big)x\right\| \\ &\leqslant \left(\sum_{k=0}^{\infty}\left(\frac{n^k}{k!}\right)^{1/2}\left(\frac{n^k}{k!}\right)^{1/2}|k-n|\right) M\|Sx - x\| \\ &\leqslant \left(\sum_{k=0}^{\infty}\frac{n^k}{k!}\right)^{1/2}\left(\sum_{k=0}^{\infty}\frac{n^k}{k!}(k-n)^2\right)^{1/2} M\|Sx - x\|.\end{aligned}$$

Expanding $(k-n)^2$, one computes $\sum\limits_{k=0}^{\infty}\dfrac{n^k}{k!}(k-n)^2 = n\mathrm{e}^n$; hence

$$\big\|\mathrm{e}^{nS}x - \mathrm{e}^n S^n x\big\| \leqslant (\mathrm{e}^n)^{1/2}(n\mathrm{e}^n)^{1/2} M\|Sx - x\| = \mathrm{e}^n\sqrt{n}\, M\|Sx - x\|. \qquad \square$$

Proof of Theorem 13.13. As before we write $X_0 := \operatorname{ran}(T(0))$ and denote by A_0 the generator of the C_0-semigroup T_{X_0}.

Let $t_1 > 0$, $x \in X_0$. The convergence (13.8) for $t = 0$ is trivial, so it remains to show the uniform convergence (13.8) for $t \in (0, t_1]$. Theorem 13.4 implies that $\mathrm{e}^{tA(s)}x \to T(t)x$ $(s \to 0)$ uniformly for t in compact subsets of $[0, \infty)$. From this convergence one easily infers that

$$\mathrm{e}^{tA(t/n)}x \to T(t)x \qquad (n \to \infty), \tag{13.9}$$

uniformly for $t \in (0, t_1]$.

Now let $x \in \mathrm{dom}(A_0)$. For $s > 0$ we put

$$x(s) := \big(\lambda - A(s)\big)^{-1}(\lambda - A_0)x.$$

Then $x(s) \to x$ and $A(s)x(s) = \lambda x(s) - (\lambda - A(s))x(s) \to \lambda x - (\lambda - A_0)x = A_0 x$ as $s \to 0$. Applying Lemma 13.15 with $S = F(t/n)$ and $M = 1$ we obtain

$$\begin{aligned} \big\|\big(\mathrm{e}^{tA(t/n)} - F(t/n)^n\big)x(t/n)\big\| &= \big\|\big(\mathrm{e}^{n(F(t/n)-I)} - F(t/n)^n\big)x(t/n)\big\| \\ &\leqslant \sqrt{n}\,\big\|\big(F(t/n) - I\big)x(t/n)\big\| \\ &= \frac{t}{\sqrt{n}}\|A(t/n)x(t/n)\| \to 0 \qquad (n \to \infty), \end{aligned}$$

uniformly for $t \in (0, t_1]$. Now observe that $\big\|\mathrm{e}^{tA(t/n)} - F(t/n)^n\big\| \leqslant 2$ for all $t > 0$, by Remark 13.14(b). We thus conclude that

$$\begin{aligned} &\big\|\big(\mathrm{e}^{tA(t/n)} - F(t/n)^n\big)x\big\| \\ &\quad \leqslant 2\|x - x(t/n)\| + \big\|\big(\mathrm{e}^{tA(t/n)} - F(t/n)^n\big)x(t/n)\big\| \to 0 \qquad (n \to \infty), \end{aligned} \tag{13.10}$$

uniformly for $t \in (0, t_1]$, and the denseness of $\mathrm{dom}(A_0)$ in X_0 implies that the last property carries over to all $x \in X_0$.

The convergences (13.9) and (13.10) imply the assertion. □

The following examples serve to illustrate Theorems 13.4 and 13.13. In Exercise 13.2 the reader is asked to carry out the details.

13.16 Examples. Let T be a contractive C_0-semigroup on X, with generator A.

(a) Put $F(s) := T(s)$ and $A(s) := \frac{1}{s}(T(s) - I)$, for $s > 0$. Then $R\big(\lambda, A(s)\big) \to R(\lambda, A)$ strongly as $s \to 0$, for all $\lambda > 0$. Hence Theorem 13.4 implies that

$$T(t) = \text{s-lim}_{s \to 0}\, \mathrm{e}^{tA(s)} \qquad (t \geqslant 0),$$

and Theorem 13.13 yields the trivial 'convergence' $T(t) = \text{s-lim}_{n \to \infty}\, T(\frac{t}{n})^n$.

(b) Put $F(s) := (I - sA)^{-1}$ and $A(s) := \frac{1}{s}(F(s) - I) = A(I - sA)^{-1}$, for $s > 0$. Again one shows that $R\big(\lambda, A(s)\big) \to R(\lambda, A)$ strongly as $s \to 0$, for all $\lambda > 0$. Then Theorem 13.4 yields

$$T(t) = \text{s-lim}_{s \to 0}\, \mathrm{e}^{tA(I - sA)^{-1}} \qquad (t \geqslant 0). \tag{13.11}$$

(We point out that $A_n := A(\frac{1}{n})$ are the Yosida approximations of A and that (13.11) is the formula used in the proof of Theorem 2.9 for the definition of the semigroup T.) Remarkably, Theorem 13.13 yields the exponential formula of Theorem 2.12. △

For self-adjoint semigroups on Hilbert spaces one obtains the following stronger version of the Chernoff product formula. This result will be important for the Trotter product formula for forms, the main result of Chapter 15.

13.17 Theorem (Chernoff product formula for self-adjoint degenerate strongly continuous semigroups). *Let T be a degenerate strongly continuous semigroup of self-adjoint contractions on a Hilbert space H, and let $-A$ be its generator. Let $F\colon [0,\infty) \to \mathcal{L}(H)$ be a function taking its values in the accretive self-adjoint contractions, $F(0)T(0) = T(0)$. Define*

$$A(s) := \frac{1}{s}\big(I - F(s)\big) \qquad (s > 0),$$

and assume that $(A(s))_{s>0}$ converges to A in the strong resolvent sense as $s \to 0$.

Then

$$F(t/n)^n x \to T(t)x \qquad (n \to \infty)$$

uniformly for t in compact subsets of $[0,\infty)$, for all $x \in \operatorname{ran}(T(0))$, and

$$F(t/n)^n x \to 0 \ (= T(t)x) \qquad (n \to \infty)$$

uniformly for t in compact subsets of $(0,\infty)$, for all $x \in \ker(T(0)) = \operatorname{ran}(T(0))^\perp$. (Recall that $T(0)$ is a self-adjoint, therefore orthogonal, projection.)

13.18 Remarks. (a) Notice the change of the sign for the relation A as well as for the approximations $A(s)$, with respect to Theorem 13.13. The reason is that in the proof it is preferable to work with accretive self-adjoint operators. Also, the theorem will be applied – in the proof of Theorem 15.6 – to a linear relation A associated with a form, and then $-A$ is the generator of the corresponding semigroup.

(b) In the assertions of Theorem 13.17, the second convergence (to $T(t)x$) holds for all $x \in H$. Indeed, it is obvious that, given the first assertion of the theorem, the second assertion is equivalent to the convergence $F(t/n)^n x \to T(t)x$ $(n \to \infty)$, uniformly for t in compact subsets of $(0,\infty)$, for all $x \in H$. △

Proof of Theorem 13.17. The first statement is an immediate consequence of Theorem 13.13.

In the proof of the second assertion we use the spectral theorem for self-adjoint operators, Theorem 13.20, presented in the subsequent interlude. It implies that, for fixed $t > 0$, the operator $tA(t) = I - F(t)$ is unitarily equivalent to an operator M_α of multiplication by a measurable function α; by Remark 13.21, α takes its values in $[0,1]$. Observe that $(1-\alpha)^{-n} \geqslant (1+\alpha)^n \geqslant 1 + n\alpha$ and thus $0 \leqslant (1-\alpha)^n \leqslant (1+n\alpha)^{-1}$, for all $n \in \mathbb{N}$. Applying Remark 13.21, property (iv), we obtain

$$0 \leqslant F(t)^n = (I - tA(t))^n \leqslant \big(I + ntA(t)\big)^{-1}.$$

Now, let $x \in \operatorname{ran}(T(0))^\perp = \ker(T(0))$. Then

$$\begin{aligned} \|F(\tfrac{t}{n})^n x\|^2 = \big(F(\tfrac{t}{n})^{2n} x \,\big|\, x\big) &\leqslant \Big(\big(I + 2tA(\tfrac{t}{n})\big)^{-1} x \,\Big|\, x\Big) \\ &= \tfrac{1}{2t}\Big(\big(\tfrac{1}{2t} + A(\tfrac{t}{n})\big)^{-1} x \,\Big|\, x\Big), \end{aligned}$$

and this converges to $\frac{1}{2t}\big((\frac{1}{2t} + A)^{-1} x \,\big|\, x\big) = 0$ as $n \to \infty$, uniformly for t in compact subsets of $(0,\infty)$, by virtue of Lemma 13.9. (In fact, the hypothesis on $(A(s))_{s>0}$ together with Lemma 13.9 implies that, given $x \in H$, a compact interval $[a,b] \subseteq (0,\infty)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $\|(r - A(s))^{-1}x - (r - A)^{-1}x\| \leqslant \varepsilon$ for all $r \in [a,b]$, $s \in (0,\delta)$.) □

13.19 Remark. We note that in Theorems 13.13 and 13.17 the assertions remain true if the powers $F(\frac{t}{n})^n$ are replaced by $F(\frac{t}{n+1})^n$.

This is a consequence of the following elementary observation. Let $J = [0,\infty)$ or $J = (0,\infty)$, let (f_n) be a sequence of continuous functions $f_n\colon J \to X$, let $f\colon J \to X$ and $f_n \to f$ locally uniformly, and let (r_n) be a sequence in $(0,\infty)$ with $\lim_{n\to\infty} r_n = 1$. Then $f_n(r_n\cdot) \to f$ locally uniformly. △

13.4 Interlude: the spectral theorem for self-adjoint operators

Multiplication operators, as described in Exercise 1.6, are particularly nice and simple operators. The spectral theorem for self-adjoint operators – in the version we present below – states that a self-adjoint operator is always unitarily equivalent to a maximal multiplication operator, as follows.

13.20 Theorem (Spectral theorem for self-adjoint operators). *Let H be a Hilbert space, and let A be a self-adjoint operator in H. Then there exist a measure space $(\Omega,\mathcal{A},\mu)$, a measurable function $\alpha\colon \Omega \to \mathbb{R}$ and a unitary operator $J\colon H \to L_2(\Omega,\mathcal{A},\mu)$ such that*

$$A = J^{-1} M_\alpha J. \tag{13.12}$$

In this statement, M_α is the maximal multiplication operator described in Exercise 1.6(a), and (13.12) means in particular that $J(\mathrm{dom}(A)) = \mathrm{dom}(M_\alpha)$. We refer to Appendix F for the proof of this version of the spectral theorem.

The representation of A as a multiplication operator is far from unique. In contrast to this non-uniqueness, the subspaces

$$H_{[a,b]} := J^{-1}\big(\{\varphi \in L_2(\mu);\, \varphi = \mathbf{1}_{[a\leqslant\alpha\leqslant b]}\varphi\}\big) \qquad (a,b\in\mathbb{R},\ a\leqslant b) \tag{13.13}$$

of H do not depend on the measure space and the unitary operator J in Theorem 13.20; see Exercise F.4.

The spectral theorem can be used to define functions of self-adjoint operators: for any Borel measurable function $f\colon \mathbb{R}\to\mathbb{R}$ one puts

$$f(A) := J^{-1} M_{f\circ\alpha} J. \tag{13.14}$$

Observe that then $\mathbf{1}(A) = I$, $\mathrm{id}_{\mathbb{R}}(A) = A$. The uniqueness of the spaces $H_{[a,b]}$ in (13.13) implies that $f(A)$ does not depend on the measure space and the isomorphism J. The proof of this property is delegated to Exercise F.5.

13.21 Remark. The measure space $(\Omega,\mathcal{A},\mu)$ constructed in the proof of Theorem 13.20 is **semi-finite**, i.e., every set $A\in\mathcal{A}$ with $\mu(A)=\infty$ contains a set $B\in\mathcal{A}$ with $0<\mu(B)<\infty$; see Remark F.7(b). In our applications of Theorem 13.20 we will suppose that $(\Omega,\mathcal{A},\mu)$ is semi-finite, for convenience. In Exercise 13.4 the reader is asked to show that the semi-finiteness implies the following properties.

(i) A is bounded if and only if $\alpha\in L_\infty(\mu)$; in this case $\|A\| = \|\alpha\|_\infty$.

(ii) A is accretive if and only if $\alpha \geqslant 0$ a.e.

(iii) $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if $\mu([\alpha = \lambda]) > 0$.

(iv) If the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are Borel measurable and $f \circ \alpha, g \circ \alpha \in L_\infty(\mu)$, then $f(A) \leqslant g(A)$ if and only if $f \circ \alpha \leqslant g \circ \alpha$ μ-a.e. (where the former '$\leqslant$' is meant as in Remark 13.11). $\triangle$

If A is an accretive self-adjoint operator, represented as in Theorem 13.20, then – as explained in Remark 13.21, property (ii) – we may assume that $\alpha \geqslant 0$, and therefore there exists a unique accretive self-adjoint square root $A^{1/2} := J^{-1} M_{\alpha^{1/2}} J$; cf. Exercise 13.5.

In part (a) of the next proposition we present an important relation between accretive symmetric forms and square roots of accretive self-adjoint operators, Kato's 'second representation theorem'; see [Kat80; VI, Theorem 2.23]. Part (b) contains a description of the form domain of an accretive self-adjoint operator that will be needed in the proof of the Trotter product formula in Chapter 15; see Lemma 15.9. We recall that the terminology 'form a in H' refers to an 'embedded form', which means that $\mathrm{dom}(a)$ is a subset of H; see Section 12.2.

13.22 Proposition. *Let a be a closed accretive symmetric form in H with dense domain, and let A be the associated self-adjoint operator.*

(a) *Then* $\mathrm{dom}(a) = \mathrm{dom}(A^{1/2})$ *and*

$$a(u, v) = \big(A^{1/2} u \,\big|\, A^{1/2} v\big) \qquad (u, v \in \mathrm{dom}(a)).$$

(b) *Let T denote the C_0-semigroup generated by $-A$. Then the function*

$$(0, \infty) \ni s \mapsto \frac{1}{s}\Big(\big(I - T(s)\big)x \,\Big|\, x\Big) \in [0, \infty)$$

is decreasing for all $x \in H$,

$$\mathrm{dom}(a) = \Big\{x \in H;\, \sup_{s>0} \frac{1}{s}\Big(\big(I - T(s)\big)x \,\Big|\, x\Big) < \infty\Big\},$$

$$A^{1/2} x = \lim_{s \to 0+} \Big(\frac{1}{s}\big(I - T(s)\big)\Big)^{1/2} x \qquad \big(x \in \mathrm{dom}(a) = \mathrm{dom}(A^{1/2})\big),$$

and

$$a(x, y) = \lim_{s \to 0+} \frac{1}{s}\Big(\big(I - T(s)\big)x \,\Big|\, y\Big) \qquad (x, y \in \mathrm{dom}(a)).$$

Proof. Because of Theorem 13.20 and Remark 13.21 we may assume, without loss of generality, that $H = L_2(\mu)$ and $A = M_\alpha$, with $\alpha \geqslant 0$.

(a) It is easy to see that A is associated with the form b given by

$$b(f, g) = \int \alpha f \overline{g} \, \mathrm{d}\mu = \big(A^{1/2} f \,\big|\, A^{1/2} g\big)$$

on $\mathrm{dom}(b) = \big\{f \in L_2(\mu);\, \alpha^{1/2} f \in L_2(\mu)\big\} = \mathrm{dom}(A^{1/2})$; cf. Example 5.9. The uniqueness stated in Corollary 12.8 shows that $a = b$.

(b) It follows from Exercise 13.3(b) that $T(t)$ acts as multiplication by $\mathrm{e}^{-\alpha(\cdot)}$; hence

$$\frac{1}{s}\Big(\big(I-T(s)\big)f\,\Big|\,f\Big)=\frac{1}{s}\int_\Omega\big(1-\mathrm{e}^{-s\alpha(\xi)}\big)|f(\xi)|^2\,\mathrm{d}\mu(\xi)\qquad(f\in L_2(\mu),\ s>0).$$

For all $\xi\in\Omega$, the function $(0,\infty)\ni s\mapsto\frac{1}{s}\big(1-\mathrm{e}^{-s\alpha(\xi)}\big)\in[0,\infty)$ is decreasing – this yields the first assertion –, and $\lim_{s\to0+}\frac{1}{s}\big(1-\mathrm{e}^{-s\alpha(\xi)}\big)=\alpha(\xi)$. Therefore the monotone convergence theorem implies that $\sup_{s>0}\frac{1}{s}\big(\big(I-T(s)\big)f\,\big|\,f\big)<\infty$ if and only if $\alpha|f|^2\in L_1(\mu)$, and the latter is equivalent to $f\in\mathrm{dom}(a)$.

Using the properties of the function $s\mapsto\frac{1}{s}\big(1-\mathrm{e}^{-s\alpha(\cdot)}\big)$ once more, we conclude for $f\in\mathrm{dom}(A^{1/2})$ that

$$\Big(\frac{1}{s}\big(I-T(s)\big)\Big)^{1/2}f=\Big(\frac{1}{s}\big(1-\mathrm{e}^{-s\alpha(\cdot)}\big)\Big)^{1/2}f\to\alpha^{1/2}f=A^{1/2}f\qquad(s\to0+),$$

by the dominated convergence theorem. The indicated formula for $a(x,y)$ is then immediate because

$$\Big(\big(I-T(s)\big)x\,\Big|\,y\Big)=\Big(\big(I-T(s)\big)^{1/2}x\,\Big|\,\big(I-T(s)\big)^{1/2}y\Big).\qquad\square$$

Notes

Besides C_0-semigroups, in the literature there appear mainly two other types of one-parameter semigroups, which are not strongly continuous at zero or which do not satisfy $T(0)=I$, respectively. The first of these types are holomorphic and are used in the theory of parabolic equations in non-reflexive spaces such as L_∞ and spaces of continuous functions; see [Lun95; Chapter 2 and Section 3.1.2], [ABHN11; Chapter 3.7 and Theorem 6.1.9]. (See Exercise 13.6(c) for an example in a simple context.)

In our context we need semigroups of the second type. We were tempted to simply call them 'strongly continuous semigroups' – because that would be the 'correct' name, according to our definition of one-parameter semigroups in Chapter 1. However, this terminology would collide with the use in the literature. This is why we call them 'degenerate strongly continuous semigroups'.

The Trotter approximation theorem, Theorem 13.4, appeared in [Tro58], and the Chernoff product formula was published in [Che68]; see also [Che74]. Both results can now be found in most books on C_0-semigroups. Degenerate strongly continuous semigroups were studied in [Are01]; see [Are01; Theorem 4.2] for a variant of Theorem 13.4. The additional convergence in Theorem 13.10 for the self-adjoint case is generalised to sequences of holomorphic semigroups in [Are01; Theorem 5.2] (where the convergence of $T_n(t)x$ is uniform on compact subsets of $(0,\infty)$). The idea for the proof of Theorem 13.17 is due to Kato [Kat78; Section 3].

The spectral theorem for self-adjoint operators comes in different versions. A common formulation involves a spectral resolution $\mathbb{R}\ni\lambda\mapsto E(\lambda)$, an increasing function of orthogonal projections, which are in fact the orthogonal projections onto the spaces H_λ we have introduced in Section 13.4. The formulation we have presented is also well-established and very useful. For more information we refer to the Notes of Appendix F.

Exercises

13.1 Let X be a Banach space, $X_0 \subseteq X$ a complemented closed subspace, i.e., there exists a bounded projection P from X onto X_0. Let A_0 be a closed operator in X_0, and define $A \subseteq X \times X$ by (13.1).

Show that A_0 is invertible in $\mathcal{L}(X_0)$ if and only if A is invertible in $\mathcal{L}(X)$, and that then $A^{-1} = A_0^{-1}P \in \mathcal{L}(X)$.

13.2 Prove the assertions stated in Examples 13.16.

13.3 (a) Let X_1, X_2 be Banach spaces, and let $J \in \mathcal{L}(X_1, X_2)$ be an isomorphism. For $j \in \{1, 2\}$ let A_j be the generator of a C_0-semigroup T_j on X_j. In addition suppose that $A_2 = JA_1J^{-1}$. Show that $T_2(t) = JT_1(t)J^{-1}$ for all $t \geqslant 0$.

Hints: Determine the generator of the C_0-semigroup $\big(JT_1(t)J^{-1}\big)_{t\geqslant 0}$ on X_2. Recall Theorem 1.13(b).

(b) Let A be an accretive self-adjoint operator on a Hilbert space H, and let $(\Omega, \mathcal{A}, \mu)$, α, J be as in Theorem 13.20, with a semi-finite measure μ. Show that the C_0-semigroup generated by $-A$ on H corresponds to the C_0-semigroup $(M_{\mathrm{e}^{-t\alpha}})_{t\geqslant 0}$ on $L_2(\mu)$.

Hints: Recall Exercise 1.6(b), and use part (a) above.

13.4 Show the properties stated in Remark 13.21. First treat the special case in which $A = M_\alpha$ for some semi-finite measure space $(\Omega, \mathcal{A}, \mu)$ and a measurable function $\alpha\colon \Omega \to \mathbb{R}$. Then convince yourself that the general case follows from this special case.

13.5 Let A and B be accretive self-adjoint operators in H, $B^2 = A$. Show that $B = A^{1/2}$. (Hint: Start with a representation of B as a multiplication operator.)

13.6 We define the operator A in the Banach space ℓ_∞ by $A(x_n) := (-nx_n)$, for $x = (x_n) \in \mathrm{dom}(A) := \{x \in \ell_\infty;\ (nx_n) \in \ell_\infty\}$. For $\mathrm{Re}\, t \geqslant 0$ we define $T(t) \in \mathcal{L}(\ell_\infty)$ by $T(t)(x_n) := (\mathrm{e}^{-tn}x_n)$. Show the following properties.

(a) $[\mathrm{Re} > 0] \subseteq \rho(A)$, and $\|(\lambda - A)^{-1}\| \leqslant \frac{1}{|\lambda|}$ for all $\mathrm{Re}\,\lambda > 0$.

(b) $(T(t))_{t\geqslant 0}$ is a contractive one-parameter semigroup on ℓ_∞, strongly continuous on $(0, \infty)$, $T(t)(\ell_\infty) \subseteq c_0$ $(t > 0)$, and $\{x \in \ell_\infty;\ x = \lim_{t\to 0+} T(t)x\} = c_0$ (the space of null sequences). In particular, the restriction T_0 of T to c_0 is a C_0-semigroup on c_0.

(c) In the complex case, T is a contractive holomorphic semigroup of angle $\pi/2$. The restriction of T to c_0 is a holomorphic C_0-semigroup.

(d) $\overline{\mathrm{dom}(A)} = c_0$, and $A_0 := A \cap (c_0 \times c_0)$ is the generator of T_0.

(e) A does not generate a degenerate strongly continuous semigroup on ℓ_∞.

Note. The operator A in this exercise is a special type of a Hille–Yosida operator; see Exercise 13.7.

13.7 An operator A in a Banach space X is called a **Hille–Yosida operator** if there exist $\omega \in \mathbb{R}$, $M \geqslant 1$ such that $(\omega, \infty) \subseteq \rho(A)$ and

$$\big\|(\lambda - A)^{-n}\big\| \leqslant \frac{M}{(\lambda - \omega)^n} \qquad (\lambda > \omega,\ n \in \mathbb{N}).$$

For such operators it follows that the part $A_0 := A \cap (X_0 \times X_0)$ of A in $X_0 := \overline{\mathrm{dom}(A)}$ generates a C_0-semigroup T_0 on X_0; see [EnNa00; II, Corollary 3.21]. The operator A in Exercise 13.6 is a Hille–Yosida operator (with $\omega = 0$, $M = 1$). In that exercise there exists an extension of T_0 to a one-parameter semigroup T on X, satisfying

(i) $T(t)|_{X_0} = T_0(t) \quad (t \geqslant 0)$,

(ii) $T(t)(\lambda - A)^{-1}x = (\lambda - A)^{-1}T(t)x \quad (x \in X,\ t \geqslant 0)$.

The present exercise shows that this need not be the case for Hille–Yosida operators, in general.

On $X := C[0,1]$ we define the operator A by $\mathrm{dom}(A) := \{f \in C^1[0,1];\ f(0) = 0\}$, $Af := -f'$ $(f \in \mathrm{dom}(A))$. Show the following properties.

(a) $(0,\infty) \subseteq \rho(A)$, $\|(\lambda - A)^{-1}\| \leqslant \frac{1}{\lambda}$ $(\lambda > 0)$ (i.e. A is a Hille–Yosida operator).

(b) $X_0 := \overline{\mathrm{dom}(A)} = \{f \in C[0,1];\ f(0) = 0\} \cong C_0(0,1]$. Note that, by Exercise 1.4(d), the C_0-semigroup T_0 generated by A_0, defined above, is the semigroup of right translations.

(c) There exists no one-parameter semigroup T on X with the above properties (i), (ii). (Hint: These properties imply that $\mathrm{dom}(A)$ is invariant under T_0.)

(d) There exists a bounded projection $P\colon C[0,1] \to C_0(0,1]$, $Pf := f - f(0)\mathbf{1}$ $(f \in C[0,1])$. Then $\widetilde{T}(t) := T_0(t)P$ $(t \geqslant 0)$ defines a degenerate strongly continuous semigroup on $C[0,1]$; determine its generator.

Note. Combining Exercises 2.6 and 2.4 one concludes that a Hille–Yosida operator in a reflexive Banach space X is always the generator of a C_0-semigroup on X; examples as in Exercise 13.6 and in the present exercise are only possible in non-reflexive Banach spaces.

Chapter 14

Form convergence theorems

The objective of this chapter is to show that convergence of a sequence of forms – suitably defined – implies strong resolvent convergence of the sequence of the associated operators. This supplements the central result of the previous chapter, the Trotter approximation theorem (Theorem 13.4), where strong resolvent convergence of a sequence of semigroup generators was shown to imply the convergence of the corresponding semigroups. These results are of considerable interest for differential operators, but in our context they also serve as preparation for the Trotter product formula for forms, presented in Chapter 15.

We begin this chapter by treating non-densely defined forms and their relation to degenerate strongly continuous semigroups. In the last section we explain the Galerkin method of numerical analysis and its close relation to one of our form convergence theorems.

14.1 Non-densely defined sectorial forms

Let H be a complex Hilbert space, and let (a, j) be a (not necessarily densely defined) sectorial form of angle $\theta \in [0, \pi/2)$ in H. Then (a, j) is a densely defined sectorial form in $H_0 := \overline{\mathrm{ran}(j)}$. Let A_0 denote the associated m-sectorial operator in H_0; see (12.5). (We point out that the index '0' in A_0 corresponds to the index in H_0; the present definition of A_0 is not the one in (12.3)!) Theorem 3.20 implies that $-A_0$ generates a contractive holomorphic C_0-semigroup T_0 of angle $\pi/2 - \theta$ on H_0. Let P be the orthogonal projection in H onto H_0. Then

$$T(z) := T_0(z)P \qquad (z \in \Sigma_{\pi/2-\theta,0})$$

defines a **holomorphic degenerate strongly continuous semigroup** on H; we say that T is **associated with** (a, j). Then the generator of T is given by $-A$, where $A = A_0 \oplus \big(\{0\} \times H_0^\perp\big)$; see Section 13.1.

Let (V, q) denote the completion of $(\mathrm{dom}(a), (\cdot \mid \cdot)_{a,j})$, and let $\tilde{\jmath}, \tilde{a}$ be as in Remark 12.3(c). Then one obtains

$$A = \big\{(x, y) \in H \times H\,;\, \exists u \in V\colon \tilde{\jmath}(u) = x,\ \tilde{a}(u, v) = (y \mid \tilde{\jmath}(v))_H\ (v \in V)\big\}, \tag{14.1}$$

which means that A is the **linear relation in H associated with the form** (a, j); the proof is delegated to Exercise 14.1(a). (The formula for the associated operator A is the same as in the case of densely defined forms; see (12.5).) If $\mathrm{dom}(a)$ is a subset of H and

$j\colon \mathrm{dom}(a) \hookrightarrow H$ is the embedding, then a is an embedded form, and we will suppress j in the notation, as before.

We illustrate these notions by a simple example.

14.1 Example. Let H_0 be a closed subspace of H, and define the (embedded) form a by $a(u,v) := 0$ for all $u, v \in \mathrm{dom}(a) := H_0$. Then one obtains

$$A = \big\{(u,v) \in H_0 \times H\,;\, 0 = (y \,|\, v)_H \ (v \in H_0)\big\} = H_0 \times H_0^\perp,$$

and the associated degenerate strongly continuous semigroup is given by $T(t) = P$ $(t \geqslant 0)$, with the orthogonal projection P onto H_0. $\triangle$

14.2 Remark. Let (a,j) be a quasi-sectorial form with vertex $-\omega \in \mathbb{R}$, and let A be the linear relation defined in (14.1). Then the same argument as in Remark 5.10 shows that the linear relation

$$A + \omega I = \{(x, y + \omega x)\,;\, (x,y) \in A\}$$

is associated with the shifted form (a_ω, j). $\triangle$

A linear relation A in H is called **sectorial** if there exists $\theta \in [0, \pi/2)$ such that $(y \,|\, x) \in \overline{\Sigma_\theta}$ for all $(x,y) \in A$. If additionally $\mathrm{ran}(I + A) = H$, then A is called **m-sectorial**. A linear relation A is called **quasi-sectorial** (**quasi-m-sectorial**) if there exists $\omega \in \mathbb{R}$ such that $A + \omega I$ is sectorial (m-sectorial). In the remainder of this section we will formulate everything without 'quasi', for simplicity (and without loss of generality).

14.3 Remarks. (a) If (a,j) is a sectorial form in H, then the linear relation A associated with (a,j) by (14.1) is an m-sectorial linear relation in H; see Exercise 14.1(b).

(b) Conversely, if A is a m-sectorial linear relation in H, then A is associated with a unique closed embedded sectorial form in H. This is because $A_0 := A \cap (H_0 \times H_0)$ is an m-sectorial operator in $H_0 := \overline{\mathrm{dom}(A)}$, and $A = A_0 \oplus \big(\{0\} \times H_0^\perp\big)$; see Exercise 14.2. $\triangle$

We now consider the special case of an accretive symmetric form (a,j). In this case we allow the underlying Hilbert space H to be real or complex; for symmetric forms the above definitions of the associated degenerate strongly continuous semigroup T and of the associated linear relation A also apply in real Hilbert spaces. Since (a,j) is symmetric, the associated operator A_0 in $H_0 = \overline{\mathrm{ran}(j)}$ is self-adjoint, and as a consequence the generator $-A$ of the associated degenerate strongly continuous semigroup will also be self-adjoint, in the sense treated subsequently (cf. Remark 14.4(a)).

The **adjoint** A^* of a linear relation A is defined in the same way as in Section 6.1 as

$$A^* := \big((-A)^\perp\big)^{-1},$$

and a linear relation A is called **self-adjoint** if $A^* = A$. We emphasise that the results on adjoints and on self-adjoint operators presented in Section 6.1 carry over appropriately to linear relations. It is clear from the definition that a linear relation A is self-adjoint if and only if its inverse A^{-1} is self-adjoint.

A self-adjoint relation A is accretive if and only if $(x \,|\, y) \geqslant 0$ for all $(x,y) \in A$. Clearly, A is accretive if and only if A^{-1} is accretive.

In the following remarks we sketch that the decomposition of m-accretive linear relations, obtained in Theorem 13.2(a), also holds for self-adjoint linear relations, but here even without the hypothesis of accretivity.

14.4 Remarks. (a) If H_0 and H_1 are Hilbert spaces and A_j is a self-adjoint linear relation in H_j for $j = 0, 1$, then

$$A_0 \oplus A_1 := \big\{ \big((x_0, x_1), (y_0, y_1)\big) ; \, (x_j, y_j) \in A_j \ (j = 0, 1) \big\}$$

is a self-adjoint linear relation in $H_0 \oplus H_1$.

(b) A self-adjoint linear relation A in H with dense domain $\mathrm{dom}(A) = \{x \in H ; \exists y \in H : (x, y) \in A\}$ is an operator, by Theorem 6.3(a).

(c) The "extreme opposite" to a densely defined linear relation is a linear relation A with $\mathrm{dom}(A) = \{0\}$; the only self-adjoint linear relation with this property is $A := \{0\} \times H$.

(d) Any self-adjoint linear relation A in H is of the form $A = A_0 \oplus A_1$ as in part (a), with $H_0 := \overline{\mathrm{dom}(A)}$ and the densely defined operator $A_0 := A \cap (H_0 \times H_0)$ in H_0, and with $H_1 := H_0^\perp$ and $A_1 := \{0\} \times H_1$; see Exercise 14.3 or [Are61; Theorem 5.3]. △

Analogously to Remarks 14.3, we have the following correspondence between accretive symmetric forms and m-accretive self-adjoint linear relations.

14.5 Proposition. *Let H be a Hilbert space.*

(a) *Let (a, j) be an accretive symmetric form in H. Then the linear relation A associated with (a, j), given by* (14.1)*, is self-adjoint.*

(b) *Conversely, if A is an accretive self-adjoint linear relation in H, then there exists a unique embedded closed accretive symmetric form a in H such that A is associated with a.*

Proof. (a) is clear from the previous development.

(b) The self-adjoint relation A decomposes as $A = A_0 \oplus A_1$ as in Remark 14.4(d). Then A_0 is associated with a densely defined embedded closed accretive symmetric form a in H_0, and interpreting a as a form in H one obtains A as the associated operator.

Concerning uniqueness, if A is associated with a form a, then $\overline{\mathrm{dom}(a)} = \overline{\mathrm{dom}(A_0)}$, where A_0 is as above, and then the uniqueness of a follows from Corollary 12.8. □

14.2 Form convergence for increasing sequences

The main result of this section, Theorem 14.10, deals with convergence of an increasing sequence of symmetric forms. (We refer to the Notes for generalisations to non-symmetric forms.) In order to explain what 'increasing' means, we first define an order relation for accretive symmetric forms, and we characterise it via the inverses of the associated operators.

Let H be a Hilbert space. Given two accretive symmetric forms a and b in H, we say that $a \leqslant b$ if $\mathrm{dom}(a) \supseteq \mathrm{dom}(b)$ and $a(x) \leqslant b(x)$ for all $x \in \mathrm{dom}(b)$. We recall Remark 13.11 concerning inequalities between bounded self-adjoint operators.

14.6 Proposition. *Let a, b be closed accretive symmetric forms in H. Let A and B be the associated self-adjoint linear relations, and assume that the inverses of A and B are operators belonging to $\mathcal{L}(H)$. Then $a \leqslant b$ if and only if $B^{-1} \leqslant A^{-1}$.*

Our proof of this equivalence is quite different from the proofs that can be found in the literature. The key observation of our treatment is Proposition 14.8 below, whose proof uses the following elementary lemma.

14.7 Lemma. *Let X be a normed space, H a Hilbert space, $P \in \mathcal{L}(X,H)$, $\eta \in X'$, $c \geqslant 0$ with the property that*

$$|\eta(x)| \leqslant c\|Px\| \qquad (x \in X). \tag{14.2}$$

Then there exists $z \in H$ such that $\|z\| \leqslant c$ and $\eta(x) = (Px \,|\, z)$ for all $x \in X$.

Proof. Define $\tilde{\eta}\colon \operatorname{ran}(P) \to \mathbb{K}$ by $\tilde{\eta}(Px) := \eta(x)$ for all $x \in X$; note that (14.2) implies that $\tilde{\eta}$ is well-defined and continuous on $\overline{\operatorname{ran}(P)}$, $\|\tilde{\eta}\| \leqslant c$. The Fréchet–Riesz representation theorem implies that there exists $z \in \overline{\operatorname{ran}(P)}$ such that $\|z\| \leqslant c$ and $\eta(x) = \tilde{\eta}(Px) = (Px \,|\, z)$ for all $x \in X$. □

Assuming that G and H are Hilbert spaces and that C and D are linear relations in $G \times H$, we will say that D **dominates** C if for all $(x,y) \in D$ there exists $z \in H$ such that $(x,z) \in C$ and $\|z\| \leqslant \|y\|$. If C and D are operators, this simply means that $\operatorname{dom}(D) \subseteq \operatorname{dom}(C)$ and $\|Cx\| \leqslant \|Dx\|$ for all $x \in \operatorname{dom}(D)$. The following fundamental property concerning this notion is a more elaborate version of [Kat80; VI, Lemma 2.30].

14.8 Proposition. *Let G, H be Hilbert spaces, and let C, D be closed linear relations in $G \times H$. Then D dominates C if and only if $C^\perp$ dominates $D^\perp$.*

Proof. It clearly suffices to show '⇒'. Let $(x,y) \in C^\perp$.

Let $(f,g) \in D$. By hypothesis, there exists $h \in H$ such that $(f,h) \in C$ and $\|h\| \leqslant \|g\|$. Then $(f,h) \perp (x,y)$, hence

$$|(-f \,|\, x)| = |(h \,|\, y)| \leqslant \|h\|\|y\| \leqslant \|g\|\|y\|,$$

and with $P\colon D \to H$, $(f,g) \mapsto g$ and $\eta\colon D \to \mathbb{K}$, $(f,g) \mapsto (-f \,|\, x)$ it follows that $\big|\eta\big((f,g)\big)\big| \leqslant \|y\|\|P(f,g)\|$ (note that h has dropped out of these properties). We can now apply Lemma 14.7 to obtain $z \in H$ such that $\|z\| \leqslant \|y\|$ and $(-f \,|\, x) = (P(f,g) \,|\, z) = (g \,|\, z)$ for all $(f,g) \in D$, i.e., $(x,z) \in D^\perp$ and $\|z\| \leqslant \|y\|$.

Summarising, we have shown that $C^\perp$ dominates $D^\perp$. □

Proof of Proposition 14.6. Let A_0 be the accretive self-adjoint operator in $\overline{\operatorname{dom}(a)}$ associated with a, and denote by P_a the orthogonal projection onto $\overline{\operatorname{dom}(a)}$. The (accretive self-adjoint) square root of A^{-1} will be denoted by $A^{-1/2}$. We point out that then $A_0^{-1/2}$, the square root of A_0^{-1}, is the restriction of $A^{-1/2}$ to $\overline{\operatorname{dom}(a)}$, and we define

$$A^{1/2} := (A^{-1/2})^{-1} = \big\{(x,y) \in H \times H\,;\, (x, P_a y) \in A_0^{1/2}\big\}.$$

The corresponding notation and properties will also be used for b.

It is evident that $a \leqslant b$ if and only if $B_0^{1/2}$ dominates $A_0^{1/2}$. The latter, in turn, holds if and only if $B^{1/2}$ dominates $A^{1/2}$. (Clearly $A_0^{1/2}$ dominates $A^{1/2}$, and the converse is true as well: if $(x,y) \in A^{1/2}$, then $(x, P_a y) \in A_0^{1/2}$ and $\|P_a y\| \leqslant \|y\|$. By the same token,

$B_0^{1/2}$ and $B^{1/2}$ dominate each other. As it is easily seen that domination is transitive, one concludes the last assertion above.)

On the other hand, $A^{-1} \geqslant B^{-1}$ if and only if $A^{-1/2}$ dominates $B^{-1/2}$.

Finally we observe that

$$(A^{-1/2})^\perp = \big((-A^{-1/2})^*\big)^{-1} = -(A^{-1/2})^{-1} = -A^{1/2}$$

by the self-adjointness of $A^{-1/2}$, and similarly $(B^{-1/2})^\perp = -B^{1/2}$. Now, applying Proposition 14.8 we conclude that $B^{1/2}$ dominates $A^{1/2}$ if and only if $A^{-1/2}$ dominates $B^{-1/2}$.

This proves the asserted equivalence. □

Before we turn to the form convergence theorem announced above we show the following lemma on decreasing sequences of bounded accretive self-adjoint operators. We refer to Corollary 14.20 for a form version – incomparably more general – of this lemma.

14.9 Lemma. *Let (B_n) be a decreasing sequence of accretive self-adjoint operators in $\mathcal{L}(H)$. Then $B := \text{s-lim}_{n\to\infty} B_n$ exists, and B is an accretive self-adjoint operator.*

Proof. For $x \in H$, the sequence $\big((B_n x \,|\, x)\big)_n$ is decreasing and bounded below by 0, hence convergent. The polarisation identity – see Remark 5.1 – implies that $\big((B_n x \,|\, y)\big)_n$ is convergent for all $x, y \in H$. It follows from Exercise 5.1 that there exists $B \in \mathcal{L}(H)$ such that $(Bx \,|\, y) = \lim_{n\to\infty}(B_n x \,|\, y)$ for all $x, y \in H$. Then it is straightforward that B is self-adjoint and accretive.

In order to show the strong convergence we assume without loss of generality that $B = 0$. Then, using the accretive self-adjoint square root of accretive self-adjoint operators – see Section 13.4 –, we obtain

$$\|B_n x\|^2 = \big(B_n B_n^{1/2} x \,\big|\, B_n^{1/2} x\big) \leqslant \big(B_1 B_n^{1/2} x \,\big|\, B_n^{1/2} x\big) \leqslant \|B_1\|\,(B_n x \,|\, x) \to 0$$

as $n \to \infty$, for all $x \in H$. □

Now we state the main result of this section.

14.10 Theorem. *Let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence of closed accretive symmetric forms in H. Then*

$$\operatorname{dom}(a) := \Big\{ x \in \bigcap_{n\in\mathbb{N}} \operatorname{dom}(a_n) \,;\, \sup_{n\in\mathbb{N}} a_n(x) < \infty \Big\}$$

is a vector space,

$$a(x, y) := \lim_{n\to\infty} a_n(x, y) \tag{14.3}$$

exists for all $x, y \in \operatorname{dom}(a)$, and a thus defined is a closed accretive symmetric form in H.

Let A be the self-adjoint linear relation associated with a, and let A_n be the self-adjoint linear relation associated with a_n, for $n \in \mathbb{N}$. Then (A_n) converges to A in the strong resolvent sense, i.e. $(\lambda + A_n)^{-1} \to (\lambda + A)^{-1}$ strongly as $n \to \infty$, for all $\lambda > 0$.

14.11 Remarks. (a) The assertion in Theorem 14.10 is precisely the hypothesis for the linear relations $-A_n$ and $-A$ in Theorem 13.10. This has the important consequence that the degenerate strongly continuous semigroups generated by $-A_n$ converge to the semigroup generated by $-A$, in the sense formulated in Theorem 13.10.

(b) In Theorem 14.10, the closedness hypothesis for the forms a_n is essential. We refer to [BaEl14; Example 3.6] for a counterexample. △

Proof of Theorem 14.10. If $x, y \in \mathrm{dom}(a)$, then the Cauchy–Schwarz inequality, Proposition 5.2, implies that $\big(a_n(x+y)\big)_{n\in\mathbb{N}}$ is bounded, hence $x+y \in \mathrm{dom}(a)$. As it is obvious that $\lambda x \in \mathrm{dom}(a)$ for all $x \in \mathrm{dom}(a)$, $\lambda \in \mathbb{K}$, it follows that $\mathrm{dom}(a)$ is a vector space. The polarisation identity (see Remark 5.1) implies that the limit (14.3) exists for all $x, y \in \mathrm{dom}(a)$.

Clearly a is an accretive symmetric form. We show that it is closed. Let $(x_n)_{n\in\mathbb{N}}$ be a $\|\cdot\|_a$-Cauchy sequence. Then $x := \lim x_n$ exists in H. Also, (x_n) is a $\|\cdot\|_{a_m}$-Cauchy sequence for all $m \in \mathbb{N}$, and therefore $x_n \to x$ in $\big(\mathrm{dom}(a_m), \|\cdot\|_{a_m}\big)$ since a_m is closed. Let $n \in \mathbb{N}$. Then

$$\sup_{m\in\mathbb{N}} a_m(x-x_n) = \sup_{m\in\mathbb{N}} \lim_{k\to\infty} a_m(x_k-x_n) \leqslant \sup_{m\in\mathbb{N},\, k\geqslant n} a_m(x_k-x_n) = \sup_{k\geqslant n} a(x_k-x_n) < \infty.$$

This inequality implies that $x \in \mathrm{dom}(a)$, and

$$a(x-x_n) \leqslant \sup_{k\geqslant n} a(x_k-x_n) \to 0 \qquad (n\to\infty)$$

shows that $x_n \to x$ in the $\|\cdot\|_a$-norm. It follows that a is closed.

In view of Lemma 13.9 it is sufficient to show the asserted strong convergence for $\lambda = 1$. Recall from Remark 14.2 that the operator $I + A_n$ is associated with the shifted form $e + a_n$, for $n \in \mathbb{N}$, where e is the 'unit form', $e(x,y) := (x \,|\, y)$ $(x, y \in H)$. Similarly, $I + A$ is associated with $e + a$. Now Proposition 14.6 implies that $\big((I+A_n)^{-1}\big)_n$ is decreasing and that $(I+A_n)^{-1} \geqslant (I+A)^{-1}$ for all $n \in \mathbb{N}$. From Lemma 14.9 we infer that $R := \text{s-}\lim_{n\to\infty}(I+A_n)^{-1}$ exists and is an accretive self-adjoint operator; clearly $(I+A_n)^{-1} \geqslant R \geqslant (I+A)^{-1}$ for all $n \in \mathbb{N}$. Then $B := R^{-1}$ is an accretive self-adjoint linear relation; let b denote the corresponding closed accretive symmetric form, according to Proposition 14.5(b). Using Proposition 14.6 we infer that $e + a_n \leqslant b \leqslant e + a$ for all $n \in \mathbb{N}$. Thus from the definition of a we obtain $b = e + a$. Since B is associated with b and $I + A$ is associated with $e + a$ we conclude that $(I+A)^{-1} = B^{-1} = R = \text{s-}\lim_{n\to\infty}(I+A_n)^{-1}$. □

14.12 Example. Let $C \subseteq \mathbb{R}^n$ be a closed set, and let (Ω_k) be a decreasing sequence of open subsets of $\mathbb{R}^n$ such that $C = \bigcap_{k=1}^{\infty} \Omega_k$.

For $k \in \mathbb{N}$ let a_k be the classical Dirichlet form with domain $\mathrm{dom}(a_k) = H_0^1(\Omega_k)$,

$$a_k(u,v) = \int \nabla u \cdot \overline{\nabla v}\, \mathrm{d}x \qquad (u, v \in \mathrm{dom}(a_k)).$$

We consider all the forms a_k as forms in $L_2(\mathbb{R}^n)$. Then (a_k) is an increasing sequence of closed accretive symmetric forms, hence convergent in the sense of Theorem 14.10. It is

not difficult to see that the limiting form a is given by

$$\operatorname{dom}(a) = H^1_0(C) := \big\{u \in H^1(\mathbb{R}^n);\, u|_{\mathbb{R}^n\setminus C} = 0\big\} = \bigcap_{k\in\mathbb{N}} H^1_0(\Omega_k),$$
$$a(u,v) = \int \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x \qquad (u,v \in \operatorname{dom}(a)).$$

Suppose, more specifically, that $C = \overline{\Omega}$ for some bounded open set $\Omega \subseteq \mathbb{R}^n$. Then clearly $H^1_0(\Omega) \subseteq H^1_0(\overline{\Omega})$. If in addition $\partial\Omega$ is a Lebesgue null set, we conclude that

$$H^1_0(\Omega) \subseteq H^1_0(\overline{\Omega}) \subseteq L_2(\Omega) \ \big(= \big\{u \in L_2(\mathbb{R}^n);\, u|_{\mathbb{R}^n\setminus\Omega} = 0\big\}\big).$$

This implies that a is a densely defined form in $L_2(\Omega)$. Following [ArDa08b], we call the associated self-adjoint operator in $L_2(\Omega)$ the **pseudo-Dirichlet Laplacian**.

It is natural to ask for conditions implying that $H^1_0(\Omega) = H^1_0(\overline{\Omega})$. Proposition 7.9 implies that Ω having continuous boundary implies this equality; note that $\partial\Omega$ is a Lebesgue null set, by Exercise 14.5. The example $n = 1$, $\Omega := (-1,0)\cup(0,1)$ shows that the equality does not hold for all Ω. We refer to the Notes for more information. △

14.3 Form convergence 'from above'

A counterpart to the results on increasing sequences of accretive symmetric forms are decreasing sequences. This topic, treated in Corollary 14.20, turns out to lead to the problem that the limiting form may turn out to be non-closable, even if all the forms in the sequence are closed. This is why we allow non-closed forms for the members of the sequence as well. Corollary 14.20 will be a consequence of Theorem 14.17, a more general convergence theorem for quasi-sectorial forms.

We begin with a warm-up. If a is a symmetric bounded coercive form in a Hilbert space V, then a simple approximation of a 'from above' is given by $a_n := a|_{V_n\times V_n}$, where (V_n) is an increasing sequence of subspaces of V whose union is dense in V. (Note that $a_n \geqslant a$ for all $n \in \mathbb{N}$.) In the subsequent Remark 14.14(a) we will see that this approximation idea works even for non-symmetric forms and non-monotone sequences of subspaces. This will be a consequence of the inequality in the following important approximation result.

14.13 Lemma (Céa's lemma). *Let V be a Hilbert space, and let a be a bounded coercive form on V,*

$$|a(u,v)| \leqslant M\|u\|_V\|v\|_V, \quad \operatorname{Re} a(u) \geqslant \alpha\|u\|_V^2 \qquad (u,v \in V)$$

for some $M \geqslant 0$, $\alpha > 0$. Let $\breve{V}$ be a closed subspace of V.

Let $\eta \in V^*$, and let $u \in V$, $\breve{u} \in \breve{V}$ be the unique elements such that

$$a(u,v) = \langle \eta, v\rangle_{V^*,V} \quad (v \in V), \qquad a(\breve{u},v) = \langle \eta, v\rangle_{V^*,V} \quad (v \in \breve{V}).$$

Then $\|u - \breve{u}\|_V \leqslant \frac{M}{\alpha}\operatorname{dist}(u,\breve{V})$.

Proof. The existence of u and $\check{u}$ is a consequence of Theorem 5.4, the Lax–Milgram lemma, where for $\check{u}$ the Lax–Milgram lemma is applied to the restriction of a to $\check{V}$.

For $v \in \check{V}$ we compute

$$a(u - \check{u}, v) = \langle \eta, v \rangle_{V^*,V} - a(\check{u}, v) = 0. \tag{14.4}$$

Hence, for any $\check{v} \in \check{V}$ we obtain

$$\begin{aligned} \alpha \|u - \check{u}\|_V^2 &\leqslant \operatorname{Re} a(u - \check{u}, (u - \check{v}) + (\check{v} - \check{u})) = \operatorname{Re} a(u - \check{u}, u - \check{v}) \\ &\leqslant M \|u - \check{u}\|_V \|u - \check{v}\|_V \end{aligned}$$

for all $\check{v} \in \check{V}$. □

14.14 Remarks. (a) Let the hypotheses be as in Lemma 14.13, let (V_n) be a sequence of closed subspaces of V such that $\operatorname{dist}(u, V_n) \to 0$, and let $u_n \in V_n$ $(n \in \mathbb{N})$ be the unique elements such that

$$a(u_n, v) = \langle \eta, v \rangle_{V^*,V} \qquad (v \in V_n).$$

Then Lemma 14.13 implies that $u_n \to u$ in V.

(b) The application of Lemma 14.13 sketched in part (a) is an essential ingredient for the Galerkin method in numerical analysis. In this context, equation (14.4) is called "Galerkin orthogonality". An important feature in this application is the circumstance that the sequence of subspaces need not be increasing. The estimate $\|u - u_n\|_V \leqslant \frac{M}{\alpha} \operatorname{dist}(u, V_n)$ from Céa's lemma states that, up to the factor $\frac{M}{\alpha}$, the "Galerkin approximation" $u_n \in V_n$ is as close to the solution u as the best approximation to u in V_n. △

14.15 Proposition. *Let V be a complex Hilbert space, and let a be a bounded coercive form on V,*

$$|a(u, v)| \leqslant M \|u\|_V \|v\|_V, \quad \operatorname{Re} a(u) \geqslant \alpha \|u\|_V^2 \qquad (u, v \in V)$$

for some $M \geqslant 0$, $\alpha > 0$.

Let $\check{V}$ be a complex Hilbert space, $\check{a}$ a bounded coercive form on $\check{V}$, $J \in \mathcal{L}(\check{V}, V)$, and assume that there exists $\theta \in [0, \frac{\pi}{2})$ such that

$$\check{a}(v) - a(Jv) \in \overline{\Sigma_\theta} \qquad (v \in \check{V}).$$

Let $\eta \in V^*$, and let $u \in V$, $\check{u} \in \check{V}$ be the unique elements such that

$$a(u, v) = \langle \eta, v \rangle_{V^*,V} \quad (v \in V), \qquad \check{a}(\check{u}, v) = \langle \eta, Jv \rangle_{V^*,V} \quad (v \in \check{V}).$$

Then

$$\|u - J\check{u}\|_V^2 \leqslant \inf_{v \in \check{V}} \Big(\frac{M^2}{\alpha^2} \|u - Jv\|_V^2 + \frac{c^2}{2\alpha} \big| \check{a}(v) - a(Jv) \big| \Big), \tag{14.5}$$

where $c := 1 + \tan \theta$.

Proof. We define a form b on $\check{V}$ by

$$b(w,v) := \check{a}(w,v) - a(Jw, Jv).$$

The assumptions imply that

$$|b(w,v)| \leqslant c(\operatorname{Re} b(w))^{1/2}(\operatorname{Re} b(v))^{1/2} \qquad (w,v \in \check{V}); \tag{14.6}$$

see (12.2). For $v \in \check{V}$ we compute

$$a(u - J\check{u}, Jv) = \langle \eta, Jv\rangle_{V^*,V} - a(J\check{u}, Jv) = \check{a}(\check{u}, v) - a(J\check{u}, Jv) = b(\check{u}, v).$$

Hence, for $v \in \check{V}$ we obtain

$$\begin{aligned}\alpha\|u - J\check{u}\|_V^2 &\leqslant \operatorname{Re} a\big(u - J\check{u}, (u - Jv) + J(v - \check{u})\big)\\ &= \operatorname{Re} a(u - J\check{u}, u - Jv) + \operatorname{Re} b(\check{u}, v - \check{u}).\end{aligned}$$

Now we use (14.6), the boundedness of a and the Peter–Paul inequality (twice) to estimate

$$\begin{aligned}\alpha\|u - J\check{u}\|_V^2 &\leqslant M\|u - J\check{u}\|_V\|u - Jv\|_V + c(\operatorname{Re} b(\check{u}))^{1/2}(\operatorname{Re} b(v))^{1/2} - \operatorname{Re} b(\check{u})\\ &\leqslant \frac{\alpha}{2}\|u - J\check{u}\|_V^2 + \frac{M^2}{2\alpha}\|u - Jv\|_V^2 + \frac{c^2}{4}\operatorname{Re} b(v).\end{aligned}$$

Reshuffling terms we conclude that

$$\|u - J\check{u}\|_V^2 \leqslant \frac{M^2}{\alpha^2}\|u - Jv\|_V^2 + \frac{c^2}{2\alpha}\operatorname{Re}\big(\check{a}(v) - a(Jv)\big). \tag{14.7}$$

□

14.16 Remarks. (a) If the space $\check{V}$ is a subspace of V, $J\colon \check{V} \hookrightarrow V$ is the embedding and $\check{a} = a|_{\check{V}\times\check{V}}$, then Proposition 14.15 reduces to Céa's lemma.

(b) The following considerations serve to transform (14.5) into an inequality that will be used in the proof of the subsequent theorem. In the setting of Proposition 14.15, let $\mathcal{A} \in \mathcal{L}(V, V^*)$ and $\check{\mathcal{A}} \in \mathcal{L}(\check{V}, \check{V}^*)$ be the Lax–Milgram operators associated with a and $\check{a}$, respectively. We define the dual operator $J' \in \mathcal{L}(V^*, \check{V}^*)$ by

$$J'\check{\eta} := \check{\eta}J \qquad (\check{\eta} \in V^*).$$

Then, for $\eta \in V^*$ as specified in Proposition 14.15, the elements $u \in V$, $\check{u} \in \check{V}$ are given by

$$u = \mathcal{A}^{-1}\eta, \quad \check{u} = \check{\mathcal{A}}^{-1}J'\eta.$$

In this context, (14.5) reads as

$$\big\|\mathcal{A}^{-1}\eta - J\check{\mathcal{A}}^{-1}J'\eta\big\|_V^2 \leqslant \inf_{v\in\check{V}}\Big(\frac{M^2}{\alpha^2}\big\|\mathcal{A}^{-1}\eta - Jv\big\|_V^2 + \frac{c^2}{2\alpha}\big|\check{a}(v) - a(Jv)\big|\Big). \tag{14.8}$$

△

14.17 Theorem. *Let H be a complex Hilbert space, and let (a, j) be a quasi-sectorial form in H. For $n \in \mathbb{N}$ let a_n be a form with $\operatorname{dom}(a_n) \subseteq \operatorname{dom}(a)$. Let $\theta \in [0, \frac{\pi}{2})$, and assume that*

$$a_n(u) - a(u) \in \overline{\Sigma_\theta} \qquad \big(u \in \operatorname{dom}(a_n),\ n \in \mathbb{N}\big). \tag{14.9}$$

(This implies that the forms a_n are quasi-sectorial with the same vertex as a and with a common angle.) Let D be a core for a, and suppose that for all $u \in D$ there exists a sequence (u_n) in $\operatorname{dom}(a)$, $u_n \in \operatorname{dom}(a_n)$ for all $n \in \mathbb{N}$, such that $u_n \to u$ in $\operatorname{dom}(a)$ and $a_n(u_n) - a(u_n) \to 0$ as $n \to \infty$.

Let A be the linear relation associated with (a, j) (recall the definition from Section 14.1), and let A_n be the linear relation associated with $(a_n, j|_{\operatorname{dom}(a_n)})$, for $n \in \mathbb{N}$. Then (A_n) converges to A in the strong resolvent sense, i.e. $(\lambda + A_n)^{-1} \to (\lambda + A)^{-1}$ $(n \to \infty)$ strongly for all $\lambda > \omega$, where $-\omega$ is a vertex of (a, j).

Analogously to Remark 14.11(a), it follows from Theorem 13.4 that in the situation of this theorem the degenerate strongly continuous semigroups generated by the linear relations $-A_n$ converge to the semigroup generated by $-A$, in the sense described in Theorem 13.4.

14.18 Remarks. (a) In the hypotheses of Theorem 14.17, the condition $u_n \to u$ in $\operatorname{dom}(a)$ implies that $a(u_n) \to a(u)$. Therefore '$a_n(u_n) - a(u_n) \to 0$ as $n \to \infty$' is equivalent to requiring '$a_n(u_n) \to a(u)$ as $n \to \infty$'.

(b) In the proof of Theorem 14.17 it is shown that the convergence property required for all $u \in D$ is in fact satisfied for all $u \in \operatorname{dom}(a)$; see the argument connected with the reformulation (14.10) of this property.

(c) The hypotheses of the theorem express a kind of convergence $a_n \to a$. In previous results on form convergence 'from above' (see, e.g., [Kat80; VIII, Theorem 3.6] and [ArEl12b; Theorem 3.7]) a more restrictive assumption is used, namely that the set

$$D := \left\{ u \in \bigcup_{k \in \mathbb{N}} \bigcap_{n \geqslant k} \operatorname{dom}(a_n);\ a_n(u) \to a(u)\ (n \to \infty) \right\}$$

is a core for a. (Then the hypotheses of Theorem 14.17 are satisfied, by part (b) above.) $\triangle$

Proof of Theorem 14.17. Without loss of generality we assume that $-\omega = 1$ is a vertex of (a, j); cf. Remark 14.2. Then $\operatorname{Re} a$ is a semi-inner product on $\operatorname{dom}(a)$ that is equivalent to the semi-inner product $(\cdot \,|\, \cdot)_{a,j}$ defined in (12.4). Then (a, j) is sectorial, and (14.9) implies that (a_n, j) is sectorial for all $n \in \mathbb{N}$.

Let (V, q) denote the completion of $(\operatorname{dom}(a), \operatorname{Re} a)$. Then there exist a unique $\tilde{\jmath} \in \mathcal{L}(V, H)$ and a unique bounded form $\tilde{a} \colon V \times V \to \mathbb{C}$ such that $\tilde{\jmath} \circ q = j$ and $\tilde{a}(q(u), q(v)) = a(u, v)$ for all $u, v \in \operatorname{dom}(a)$ (see Remark 12.3(c)); recall from (14.1) that A is the linear relation associated with $(\tilde{a}, \tilde{\jmath})$. Analogously we define V_n, q_n, $\tilde{\jmath}_n$ and $\tilde{a}_n$ corresponding to a_n, for $n \in \mathbb{N}$; then A_n is the linear relation associated with $(\tilde{a}_n, \tilde{\jmath}_n)$. It follows from (14.9) that the embedding $\operatorname{dom}(a_n) \hookrightarrow \operatorname{dom}(a)$ is continuous for all $n \in \mathbb{N}$, and by Remark 12.3(c) there exists $J_n \in \mathcal{L}(V_n, V)$ such that $J_n \circ q_n = q|_{\operatorname{dom}(a_n)}$. Then $\tilde{\jmath}_n \circ q_n = j|_{\operatorname{dom}(a_n)} = \tilde{\jmath} \circ q|_{\operatorname{dom}(a_n)} =$

$$\begin{array}{ccccc} \operatorname{dom}(a) & \xrightarrow{q} & V & \xrightarrow{\tilde{\jmath}} & \\ {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle J_n}\uparrow & \searrow & H \\ \operatorname{dom}(a_n) & \xrightarrow[q_n]{} & V_n & \nearrow_{\tilde{\jmath}_n} & \end{array}$$

$\tilde{\jmath} \circ J_n \circ q_n$; hence $\tilde{\jmath}_n = \tilde{\jmath} \circ J_n$ on $\operatorname{ran}(q_n)$, and by denseness on all of V_n.

Let $n \in \mathbb{N}$. Then for all $u \in \operatorname{dom}(a_n)$ we have

$$\tilde{a}_n(q_n(u)) - \tilde{a}(J_n q_n(u)) = a_n(u) - a(u) \in \overline{\Sigma_\theta}\,.$$

Since $\operatorname{ran}(q_n)$ is dense in V_n, it follows that $\tilde{a}_n(v) - \tilde{a}(J_n v) \in \overline{\Sigma_\theta}$ for all $v \in V_n$.

For the application of Proposition 14.15 we note that the convergence hypothesis of the theorem is equivalent to requiring that

$$\inf_{v \in \operatorname{dom}(a_n)} \big(\|u - v\|_a^2 + \big| a_n(v) - a(v) \big| \big) \to 0 \qquad (n \to \infty), \tag{14.10}$$

for all $u \in D$. As D is a core for a, the convergence (14.10) carries over to all $u \in \operatorname{dom}(a)$. In view of the properties of the mappings introduced above this implies that

$$\inf_{v \in \operatorname{ran}(q_n)} \big(\|u - v\|_V^2 + \big| \tilde{a}_n(v) - \tilde{a}(J_n v) \big| \big) \to 0 \qquad (n \to \infty),$$

for all $u \in \operatorname{ran}(q)$. Then, using the denseness of $\operatorname{ran}(q_n)$ in V_n as well as the denseness of $\operatorname{ran}(q)$ in V, one also obtains

$$\inf_{v \in V_n} \big(\|u - J_n v\|_V^2 + \big| \tilde{a}_n(v) - \tilde{a}(J_n v) \big| \big) \to 0 \qquad (n \to \infty), \tag{14.11}$$

for all $u \in V$.

Let $\tilde{\mathcal{A}} \in \mathcal{L}(V, V^*)$ and $\tilde{\mathcal{A}}_n \in \mathcal{L}(V_n, V_n^*)$ $(n \in \mathbb{N})$ be the Lax–Milgram operators associated with $\tilde{a}$ and $\tilde{a}_n$ $(n \in \mathbb{N})$, respectively. By Proposition 15.4 we have $A^{-1} = \tilde{\jmath} \tilde{\mathcal{A}}^{-1} \tilde{k}$ and $A_n^{-1} = \tilde{\jmath}_n \tilde{\mathcal{A}}_n^{-1} \tilde{k}_n$ for all $n \in \mathbb{N}$, where $\tilde{k} y = (y \,|\, \tilde{\jmath}(\cdot))_H$ and $\tilde{k}_n y = (y \,|\, \tilde{\jmath}_n(\cdot))_H$ for all $y \in H$. Combining (14.11) with the inequality (14.8) we conclude that $J_n \mathcal{A}_n^{-1} J_n' \to \mathcal{A}^{-1}$ strongly in $\mathcal{L}(V^*, V)$, as $n \to \infty$.

Recall that $\tilde{\jmath}_n = \tilde{\jmath} \circ J_n$ and $\tilde{k}_n = J_n' \circ \tilde{k}$, for all $n \in \mathbb{N}$. Therefore

$$A_n^{-1} = \tilde{\jmath}(J_n \mathcal{A}_n^{-1} J_n') \tilde{k} \to \tilde{\jmath} \mathcal{A}^{-1} \tilde{k} = A^{-1} \qquad (n \to \infty)$$

strongly in $\mathcal{L}(H)$. Applying Lemma 13.9 we conclude that $(\lambda + A)^{-1} = \text{s-lim}_{n \to \infty} (\lambda + A_n)^{-1}$ for all $\lambda > 0$. □

14.19 Remark. Theorem 14.17, with the same proof, also holds for real Hilbert spaces if all the forms a, a_n are symmetric and accretive, and where (14.9) should be interpreted as $a_n(u) - a(u) \geqslant 0$. This is why we do not have to suppose a complex Hilbert space in the following Corollary 14.20. △

14.20 Corollary. *Let H be a Hilbert space, and let (a_n) be a decreasing sequence of (embedded) accretive symmetric forms in H. Define* $\operatorname{dom}(a) := \bigcap_{n \in \mathbb{N}} \operatorname{dom}(a_n)$.

Then

$$a(x, y) := \lim_{n \to \infty} a_n(x, y)$$

exists for all $x, y \in \operatorname{dom}(a)$, and a is an accretive symmetric form in H. Let A be the accretive self-adjoint linear relation associated with a, and similarly, let A_n be associated with a_n, for $n \in \mathbb{N}$. Then $A_n \to A$ in the strong resolvent sense, i.e. $(\lambda + A_n)^{-1} \to (\lambda + A)^{-1}$ strongly as $n \to \infty$, for all $\lambda > 0$.

Proof. The existence of $\lim_{n\to\infty} a_n(x,y)$ is obtained as in the proof of Lemma 14.9, and the asserted properties of a are then obvious. We can now apply Theorem 14.17 to obtain the remaining assertions. (Choose $u_n = u$ for large n in the assumptions of the theorem.) □

14.21 Example. Let Ω be an open subset of $\mathbb{R}^n$, and let (Ω_k) be an increasing sequence of open subsets of Ω satisfying $\Omega = \bigcup_{k\in\mathbb{N}} \Omega_k$. For all $k \in \mathbb{N}$ let a_k be the classical Dirichlet form on $C_c^\infty(\Omega_k)$, and let a be the classical Dirichlet form on $C_c^\infty(\Omega)$ (cf. Example 12.5).

Then, for all $k \in \mathbb{N}$, the self-adjoint linear relation in $L_2(\Omega)$ associated with a_k is the negative Dirichlet Laplacian on $L_2(\Omega_k)$, supplemented by $\{0\} \times L_2(\Omega \setminus \Omega_k)$. Corollary 14.20 implies that (A_k) converges in the strong resolvent sense to the negative Dirichlet Laplacian on $L_2(\Omega)$.

We refer to Exercise 14.4 for a more sophisticated version of this example. △

14.22 Remark. In contrast to Theorem 14.10, where increasing sequences of forms are treated, in Corollary 14.20 it is not supposed that the forms a_n are closed. Even if the forms a_n are closed, the limiting form a in Corollary 14.20 need not be closable, as is demonstrated by the example in Exercise 14.6. The example also shows how to avoid this problem if one can guess a closed limiting form: then one can apply Theorem 14.17 instead of Corollary 14.20. △

Notes

One of the important consequences of the form convergence theorems, Theorems 14.10 and 14.17 and Corollary 14.20, is the convergence of the corresponding generated semigroups. For an outline of the early history of convergence theorems for sequences of forms – mainly in the 70's – we refer to Reed–Simon [ReSi80; Supplementary Material, Notes to Supplement VIII.7].

The use of non-densely defined forms was advocated in [Sim78; Section 4], for symmetric forms, and further developed, for sectorial forms, in [ArBa93; Section 7] [MVV05], [BaEl14], [VoVo20a]. We refer to [AEKS14] for an application of non-densely defined forms in the context of compact coercivity (or 'compact ellipticity' in the terminology of [AEKS14]).

The form convergence theorem for increasing sequences of forms, Theorem 14.10, is contained in [Sim78; Theorem 4.1]; the full proof can be found in[VoVo20a]. See also Simon [Sim78; Theorem 3.1], Kato [Kat80; VIII, Theorem 3.13a] for the special case in which the limiting form is densely defined. The proof of Proposition 14.6 is taken from [VoVo20a].

As there is no natural notion of order between sectorial forms, a generalisation of Theorem 14.10 to sequences of sectorial forms is not straightforward. We mention [Ouh95; Theorem 6], [BaEl14; Theorem 1.2] and [VoVo20a] for results concerning this topic. In the latter two references the input is a sequence (a_n) of sectorial forms for which $a_{n+1} - a_n$, for all $n \in \mathbb{N}$, is sectorial of a fixed angle $\theta \in [0, \frac{\pi}{2})$.

In Example 14.12 the question was raised under what conditions the equality

$$H_0^1(\Omega) = H_0^1(\overline{\Omega}) = \left\{ u|_{\overline{\Omega}};\ u \in H^1(\mathbb{R}^n),\ u|_{\mathbb{R}^n \setminus \overline{\Omega}} = 0 \right\}$$

holds for a bounded open set $\Omega \subseteq \mathbb{R}^n$. It turns out that this property is equivalent to an approximation property for harmonic functions. (Recall that $u\colon \Omega \to \mathbb{K}$ is called harmonic if $u \in C^2(\Omega)$ and $\Delta u = 0$.)

The following result is Theorem 11.8 in the survey of Hedberg [Hed93], where methods of abstract potential theory are used for the proof. Versions of the result go back to Keldysh [Kel41], with more direct proofs.

Theorem (Hedberg-Keldysh) *Let $n \geqslant 2$, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with $\overset{\circ}{\overline{\Omega}} = \Omega$. Then the following properties are equivalent.*

(i) $H_0^1(\Omega) = H_0^1(\overline{\Omega})$;

(ii) *for each function $u \in C(\overline{\Omega})$, $u|_\Omega$ harmonic, there exist functions u_k that are defined and harmonic on an open neighbourhood Ω_k of $\overline{\Omega}$ such that $u_k(x) \to u(x)$ as $k \to \infty$, uniformly on $\overline{\Omega}$.*

We point out that in property (ii) the function u need not belong to $H^1(\Omega)$, even if Ω is a very 'nice' set such as the unit disc $B_{\mathbb{R}^2}(0,1)$; see the discussion of Hadamard's example in the Notes of Chapter 7. Moreover, property (ii) is not related to the solvability of the Dirichlet problem: there is no implication between (ii) and the Dirichlet regularity of Ω. In particular, the set of traces $u|_{\partial\Omega}$ of the functions u occurring in property (ii) may be a proper subset of $C(\partial\Omega)$.

Theorem 14.17 generalises [Kat80; VIII, Theorem 3.6], [ArEl12b; Theorem 3.7]. The noteworthy new features of our result, when compared with the quoted sources, are that we allow more general domains of the forms a_n – which may in fact have pairwise trivial intersection – and that the limiting form a need not be densely defined. An essential ingredient of the proof is Proposition 14.15, which is an extended version of Céa's lemma, Lemma 14.13. The origin of explicit estimates of the type stated in Lemma 14.13 seems to be [Cea64; Proposition 3.1 on p. 365], where a stronger estimate is stated for the case of symmetric forms. The lemma in the general non-symmetric case is not stated explicitly in [Cea64].

It was mentioned in Remarks 14.14 that Céa's lemma is fundamental for the Galerkin method, which in turn is the basis of the finite-element method for the solution of partial differential equations. We will not expand on this topic here, but we mention the recent paper [ACE22] for the disussion and analysis of various aspects concerning the Galerkin approximation.

Corollary 14.20 generalises [Sim78; Theorems 3.2 and 4.1], where it is assumed that the forms a_n are closed. (Note that, by Remark 12.12(c), the operator A is associated with the form $\overline{a_r}$, where a_r is Simon's 'regular part' of a.)

Exercises

14.1 Let H be a complex Hilbert space, (a, j) a sectorial form in H.

(a) Let T be the holomorphic degenerate strongly continuous semigroup associated with (a, j), and let A be the linear relation associated with (a, j); see (14.1). Show that $-A$ is the generator of T.

(b) Show that the linear relation A is m-sectorial, and show that the corresponding properties hold for the quasi-sectorial case.

14.2 Let H be a complex Hilbert space, and let A be an m-sectorial linear relation in H. Show the properties stated in Remark 14.3(b). (Hint: Note that A is an m-accretive linear relation; apply Theorem 13.2(a) and Corollary 12.8.)

14.3 Verify the assertions stated in Remarks 14.4.

Hint concerning part (d): Let P be the orthogonal projection in H onto H_0. Show first that $(x, y) \in A$ if and only if $(x, Py) \in A$, then that $A \cap (H_0 \times H_0)$ is an operator A_0 with $\operatorname{dom}(A_0) = \operatorname{dom}(A)$, and finally that A_0 is self-adjoint in H_0. Conclude that $A \cap (H_1 \times H_1) = \{0\} \times H_1$ and that $A = A_0 \oplus (\{0\} \times H_1)$ in $H \times H$.

14.4 (a) Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $(\Omega_k)_{k\in\mathbb{N}}$ be a sequence of open subsets of Ω with the property that for each compact set $K \subseteq \Omega$ there exists $k_K \in \mathbb{N}$ such that $K \subseteq \Omega_k$ for all $k \geqslant k_K$.

Let a be the classical Dirichlet form on $C_c^\infty(\Omega)$,

$$a(f, g) = \int \nabla f \cdot \overline{\nabla g}\, \mathrm{d}x \qquad (f, g \in C_c^\infty(\Omega)),$$

let a_k be the restriction of a to $\operatorname{dom}(a_k) := C_c^\infty(\Omega_k)$, and let A and A_k $(k \in \mathbb{N})$ be the self-adjoint linear relation associated with a and a_k $(k \in \mathbb{N})$, respectively. (In other words, the operator A is the negative Dirichlet Laplacian in $L_2(\Omega)$, whereas for $k \in \mathbb{N}$, A_k is the negative Dirichlet Laplacian in $L_2(\Omega_k)$ supplemented by $\{0\} \times L_2(\Omega \setminus \Omega_k)$ to a self-adjoint linear relation in $L_2(\Omega)$.)

Show that $A_k \to A$ in the strong resolvent sense.

(b) Show that the sets $\Omega := \mathbb{R}^n$, $\Omega_k := B(0, k) \cup (\mathbb{R}^n \setminus B[0, k + 1])$ $(k \in \mathbb{N})$ satisfy the conditions posed in (a) above. Show that $k, l \in \mathbb{N}$, $\Omega_k \subseteq \Omega_l$ implies that $k = l$.

This example shows that, in general, in the context of part (a) one cannot expect to find a decreasing subsequence $(a_{k_j})_j$ of $(a_k)_k$.

14.5 Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with continuous boundary. Show that $\partial\Omega$ has Lebesgue measure 0 and that $\overset{\circ}{\overline{\Omega}} = \Omega$.

14.6 (From [Kat80; VIII, Example 3.10]) Let $H := L_2(0, 1; \mathbb{R})$, and for $n \in \mathbb{N}$ define the form a_n in H by $\operatorname{dom}(a_n) := H^1(0, 1)$,

$$a_n(u, v) := \frac{1}{n} \int_0^1 u'(\xi)v'(\xi)\, \mathrm{d}\xi + u(0)v(0) + u(1)v(1).$$

(a) Show that (a_n) is a decreasing sequence of (embedded) closed accretive symmetric forms. Determine the form a according to Corollary 14.20 and show that a is not closable.

(b) Choose $a_0 := 0$ with domain $\operatorname{dom}(a_0) := H$, and use Theorem 14.17 with $D := H_0^1(0, 1)$ to show that the sequence (A_n) converges to $A = 0$ in the strong resolvent sense, where A_n is associated with a_n.

(c) Conclude from (b) that the operator $A = 0$ is associated with the form a from part (a).

Chapter 15

The Trotter product formula for forms

The ultimate objective of this chapter is the Trotter product formula for sectorial forms, which will be achieved in Section 15.3. The proof relies on the material presented in Chapters 13 and 14; in addition we will need Vitali's convergence theorem for sequences of holomorphic functions (Section 15.1, an interlude) and holomorphic dependence of forms and operators (Section 15.2).

15.1 Interlude: The Vitali convergence theorem

Convergence of a sequence of holomorphic functions on a domain $\Omega \subseteq \mathbb{C}$ propagates from very small subsets to the entire domain. This surprising phenomenon has been discovered by Vitali. In the functional analytic setting used in our proof we obtain Vitali's theorem as a consequence of the unique extension theorem.

15.1 Theorem (Vitali). *Let X be a complex Banach space. Let $\Omega \subseteq \mathbb{C}$ be open and connected. Let (f_n) be a sequence of holomorphic functions $f_n \colon \Omega \to X$, and assume that (f_n) is locally uniformly bounded. Assume further that the set*

$$\Omega_0 := \Big\{ z \in \Omega ;\ \big(f_n(z)\big)_{n\in\mathbb{N}} \text{ convergent} \Big\}$$

has a cluster point in Ω.

Then there exists a holomorphic function $f \colon \Omega \to X$ such that $f_n \to f$ locally uniformly.

Proof. Denote by $\ell_\infty(\mathbb{N}; X)$ the space of bounded sequences in X, provided with the supremum norm, and by $c(\mathbb{N}; X)$ the closed subspace of convergent sequences.

We define the function $g \colon \Omega \to \ell_\infty(\mathbb{N}; X)$ by $g(z) := \big(f_n(z)\big)_{n\in\mathbb{N}}$ $(z \in \Omega)$. Then g is holomorphic, by the equivalence of (i) and (iv) in Theorem 3.2; take $c_{\mathrm{c}}(\mathbb{N}; X') := \big\{(x'_n) \in (X')^{\mathbb{N}};\ \{n \in \mathbb{N};\ x'_n \neq 0\}$ finite$\big\}$ as a norming subset of the dual of $\ell_\infty(\mathbb{N}; X)$. On the set Ω_0, the function g takes its values in $c(\mathbb{N}; X)$. Applying the identity theorem for holomorphic functions together with the Hahn–Banach theorem we conclude that $g(z) \in c(\mathbb{N}; X)$ for all $z \in \Omega$, i.e., (f_n) converges pointwise.

Now the assertion is a consequence of Theorem 3.5. □

15.2 Holomorphic families of operators defined by forms

The following result shows that the holomorphic dependence – on a complex variable – of a family of sectorial forms implies holomorphic dependence of the associated sectorial operators. A rather striking application, due to B. Simon, will be given in Corollary 15.10.

15.2 Theorem. *Let H be a complex Hilbert space, $V \subseteq H$ a subspace, and let $\Omega \subseteq \mathbb{C}$ be open. For each $z \in \Omega$ let a_z be a closed sectorial form in H with domain $\mathrm{dom}(a_z) = V$, and let A_z denote the linear relation associated with a_z (according to Section 14.1). Assume that for all $x, y \in V$ the mapping $\Omega \ni z \mapsto a_z(x, y)$ is holomorphic.*

Then the function $\Omega \ni z \mapsto (I + A_z)^{-1} \in \mathcal{L}(H)$ is holomorphic.

15.3 Lemma. *Let X, Y be complex Banach spaces. Let $r > 0$, and let $A\colon B_{\mathbb{C}}(0, r) \to \mathcal{L}(X, Y)$ be holomorphic. Assume that $A(0)$ is invertible with $A(0)^{-1} \in \mathcal{L}(Y, X)$.*

Then there exists $\delta \in (0, r)$ such that $A(z)$ has an inverse in $\mathcal{L}(Y, X)$ for all $z \in B(0, \delta)$, and the function $B(0, \delta) \ni z \mapsto A(z)^{-1} \in \mathcal{L}(Y, X)$ is holomorphic.

Proof. Replacing $A(z)$ by $A(0)^{-1}A(z) \in \mathcal{L}(X)$ $(z \in B(0, r))$ we see that without loss of generality we may assume that $Y = X$.

It is an easy consequence of Remark 2.3(a) (Neumann series) that

$$\mathcal{I}(X) := \big\{A \in \mathcal{L}(X);\ A \text{ invertible in } \mathcal{L}(X)\big\}$$

is open in $\mathcal{L}(X)$ and that the mapping $\mathcal{I}(X) \ni A \mapsto A^{-1} \in \mathcal{I}(X)$ is continuous. This implies that there exists $\delta \in (0, r)$ such that $A(z) \in \mathcal{I}(X)$ for all $z \in B(0, \delta)$, and $\sup_{|z|<\delta} \big\|A(z)^{-1}\big\| < \infty$. Dividing the equality

$$A(w)^{-1} - A(z)^{-1} = A(w)^{-1}\big(A(z) - A(w)\big)A(z)^{-1} \qquad (w, z \in B(0, \delta))$$

by $w - z$ and taking $w \to z$, one concludes that $z \mapsto A(z)^{-1}$ is holomorphic and that

$$\frac{\mathrm{d}}{\mathrm{d}z}A(z)^{-1} = -A(z)^{-1}A'(z)A(z)^{-1}. \qquad \square$$

In the proof of Theorem 15.2 we will also use the following version of Proposition 5.7 for non-densely defined coercive forms.

15.4 Proposition. *Let V and H be Hilbert spaces, $j \in \mathcal{L}(V, H)$, $a\colon V \times V \to \mathbb{K}$ a bounded coercive form, and let A be the strictly m-accretive linear relation associated with (a, j). Let $\mathcal{A}\colon V \to V^*$ be the Lax–Milgram operator associated with a, and let $k\colon H \to V^*$, $y \mapsto (y \,|\, j(\cdot))$ be as in Proposition 5.7. Then*

$$A^{-1} = j\mathcal{A}^{-1}k.$$

(Recall from Theorem 5.4 that $\mathcal{A}$ is invertible with inverse in $\mathcal{L}(V^*, V)$.)

Proof. For $x, y \in H$ one has $(x, y) \in A$ if and only if there exists $u \in V$ such that $ju = x$ and $a(u, v) = (y \,|\, jv)$ for all $v \in V$. The latter property is equivalent to $\mathcal{A}u = ky$, i.e. $u = \mathcal{A}^{-1}ky$. Therefore $(x, y) \in A$ if and only if $x = ju = (j\mathcal{A}^{-1}k)y$. $\square$

Proof of Theorem 15.2. The closedness of a_z means that the space $(V, \|\cdot\|_{a_z})$, with the norm

$$\|u\|_{a_z} = \big(\operatorname{Re} a_z(u) + \|u\|_H^2\big)^{1/2} \qquad (u \in V),$$

is complete. Using the continuity of the embedding $(V, \|\cdot\|_{a_z}) \hookrightarrow (H, \|\cdot\|_H)$ and applying the closed graph theorem we conclude that the norms $\|\cdot\|_{a_z}$ are pairwise equivalent. For notational convenience we can therefore assume that $(V, (\cdot \,|\, \cdot)_V)$ is a Hilbert space with a norm equivalent to all the norms $\|\cdot\|_{a_z}$. We point out that the equivalence of the norms implies that the forms $a_z + e$ are coercive (where e denotes the 'unit form', $e(u,v) := (u \,|\, v)_H$ $(u, v \in V)$). Recall that the linear relation $I + A_z$ is associated with the form $a_z + e$; see Remark 14.2.

Now we use the description of the inverse of $I + A_z$ presented in Proposition 15.4: if we define $\mathcal{B}_z \in \mathcal{L}(V, V^*)$ by

$$\langle \mathcal{B}_z u, v\rangle := a_z(u,v) + (u \,|\, v)_H \qquad (u, v \in V), \tag{15.1}$$

then $\mathcal{B}_z^{-1} \in \mathcal{L}(V^*, V)$ and

$$(I + A_z)^{-1} = j\mathcal{B}_z^{-1}k \in \mathcal{L}(H), \tag{15.2}$$

where $j\colon V \hookrightarrow H$ is the embedding, and $k\colon H \hookrightarrow V^*$ is a in Proposition 15.4.

The definition (15.1) together with the hypotheses and Theorem 3.2 yield the holomorphy of the function $\Omega \ni z \mapsto \mathcal{B}(z) \in \mathcal{L}(V, V^*)$. Then Lemma 15.3 implies that $\Omega \ni z \mapsto \mathcal{B}_z^{-1} \in \mathcal{L}(V^*, V)$ is holomorphic. By (15.2), this implies the holomorphy of $\Omega \ni z \mapsto (I + A_z)^{-1} \in \mathcal{L}(H)$. □

Next we show that the holomorphic depencence on z of a family of generators implies that the corresponding degenerate strongly continuous semigroups also depend holomorphically on z.

15.5 Theorem. *Let X be a complex Banach space. Let $\Omega \subseteq \mathbb{C}$ be open, $(A_z)_{z\in\Omega}$ a family of generators of degenerate strongly continuous semigroups T_z on X. Assume that $M := \sup\big\{\|T_z(t)\|;\ t \geqslant 0,\ z \in \Omega\big\} < \infty$ and that $\Omega \ni z \mapsto (I - A_z)^{-1} \in \mathcal{L}(X)$ is holomorphic. Then*

(a) *the function $\Omega \ni z \mapsto T_z(\cdot)x \in C([0,t_0]; X)$ is holomorphic for all $t_0 > 0$, $x \in X$,*

(b) *the function $\Omega \ni z \mapsto T_z(t) \in \mathcal{L}(X)$ is holomorphic for all $t \geqslant 0$.*

Proof. (a) The boundedness hypothesis implies that

$$\big\|(\lambda - A_z)^{-n}\big\| \leqslant \frac{M}{\lambda^n} \qquad (n \in \mathbb{N},\ \lambda > 0); \tag{15.3}$$

see Theorem 2.7. The holomorphy hypothesis implies that $\Omega \ni z \mapsto (\lambda - A_z)^{-1}$ is holomorphic for all $\lambda > 0$. For the verification of this assertion put

$$U := \big\{\lambda > 0;\, \Omega \ni z \mapsto (\lambda - A_z)^{-1} \text{ holomorphic}\big\}.$$

It follows from (15.3) and Corollary 3.6 that U is a closed subset of $(0, \infty)$. Also, $U \neq \varnothing$, because $1 \in U$. If $\mu \in U$, then, using the resolvent equation, one concludes that

$$(\lambda - A_z)^{-1} = (\mu - A_z)^{-1}\big(I - (\lambda - \mu)(\mu - A_z)^{-1}\big)^{-1} \tag{15.4}$$

for all $z \in \Omega$ and all $\lambda > 0$ such that $|\lambda - \mu|\frac{M}{\mu} < 1$, and Lemma 15.3 implies that the right-hand side of (15.4) is holomorphic in z. This shows that U is open, and as $(0, \infty)$ is connected one concludes that $U = (0, \infty)$.

For $z \in \Omega$ let $P_z := T_z(0)$ denote the projection onto the 'active' subspace of T_z; then $\|P_z\| \leqslant M$. From the exponential formula, Theorem 2.12, we know that

$$T_z(t)x = \lim_{n\to\infty} P_z\Big(I - \frac{t}{n}A_z\Big)^{-n} x \qquad (x \in X,\ t \geqslant 0), \tag{15.5}$$

with uniform convergence for t in compact subsets of $[0, \infty)$. (In fact, Theorem 2.12 was only shown for C_0-semigroups; this implies that the convergence holds for $x \in \operatorname{ran}(P_z)$. However, on $\ker(P_z)$, both sides of the previous formula vanish, and therefore the formula also holds for all $x \in X$. The inclusion of the factor P_z entails that the function $t \mapsto P_z\big(I - \frac{t}{n}A_z\big)^{-n}$ is strongly continuous at $t = 0$.) Inequality (15.3) implies

$$\Big\|P_z\Big(I - \frac{t}{n}A_z\Big)^{-n}\Big\| \leqslant M \qquad (t \geqslant 0,\ z \in \Omega). \tag{15.6}$$

(For $t > 0$, the factor $\big(I - \frac{t}{n}A_z\big)^{-n}$ has already its range in $\operatorname{ran}(P_z)$, so that P_z can be omitted, and therefore (15.3) takes over. For $t = 0$ the estimate is clear.)

Let $x \in X$, $t_0 > 0$ and define $F_n \colon \Omega \to C([0, t_0]; X)$ by

$$F_n(z) := \Big[t \mapsto P_z\Big(I - \frac{t}{n}A_z\Big)^{-n} x\Big].$$

For all $t \in (0, t_0]$ the mapping $\Omega \ni z \mapsto F_n(z)(t) \in X$ is holomorphic. This property together with the boundedness (15.6) implies that $\Omega \ni z \mapsto F_n(z) \in C([0, t_0]; X)$ is holomorphic; see Exercise 15.1. The convergence (15.5) shows that $F_n(z) \to T_z(\cdot)x|_{[0,t_0]}$ in $C([0, t_0]; X)$ for all $z \in \Omega$, and Theorem 3.5 implies the assertion.

(b) In view of part (a), the assertion is an immediate consequence of the implication '(ii) $\Rightarrow$ (i)' of Theorem 3.4. □

15.3 The Trotter product formula for forms

Now we come to the main result of this chapter. Not only the results of the present chapter treated so far, but also the topics presented in the previous two chapters will play a decisive role in the proof of this result. We first treat the case of symmetric forms; the more general case of sectorial forms will then be obtained as a corollary.

Let H be a Hilbert space.

15.6 Theorem (Trotter product formula for symmetric forms). *Let a and b be closed accretive symmetric forms in H, let $c := a + b$, with domain $\operatorname{dom}(c) := \operatorname{dom}(a) \cap \operatorname{dom}(b)$, and let T_a, T_b, T_c be the degenerate strongly continuous semigroups associated with a, b, c, respectively. Then*

$$\big(T_a(\tfrac{t}{n})T_b(\tfrac{t}{n})\big)^n x \to T_c(t)x \qquad (n \to \infty),$$

uniformly for t in compact subsets of $[0,\infty)$, for all $x \in \overline{\mathrm{dom}(c)}$, and

$$\big(T_a(\tfrac{t}{n})T_b(\tfrac{t}{n})\big)^n x \to 0 \ (= T_c(t)x) \qquad (n \to \infty),$$

uniformly for t in compact subsets of $(0,\infty)$, for all $x \in \mathrm{dom}(c)^\perp$.

In the proof of the above theorem we need some auxiliary results; the first lemma is of a general nature.

15.7 Lemma. *Let $A, B \in \mathcal{L}(H)$, A self-adjoint and accretive, $(Ax \,|\, x) \leqslant \mathrm{Re}(Bx \,|\, x)$ for all $x \in H$. Then $\|(I+B)^{-1}A^{1/2}\| \leqslant 1$.*

Proof. The hypotheses imply that B and B^* are accretive. We have $\|(I+B)^{-1}A^{1/2}\| = \big\|(A^{1/2})^*\big((I+B)^{-1}\big)^*\big\| = \|A^{1/2}(I+B^*)^{-1}\|$ and for the last norm we estimate

$$\begin{aligned}\big\|A^{1/2}(I+B^*)^{-1}x\big\|^2 &= \big(A(I+B^*)^{-1}x \,\big|\, (I+B^*)^{-1}x\big)\\ &\leqslant \mathrm{Re}\big((I+B)(I+B^*)^{-1}x \,\big|\, (I+B^*)^{-1}x\big)\\ &\leqslant \big|\big((I+B^*)^{-1}x \,\big|\, x\big)\big| \leqslant \|x\|^2\end{aligned}$$

for all $x \in H$. This shows the asserted inequality. □

For the proof of Theorem 15.6 as well as for two further auxiliary lemmas we fix the following additional notation. Let A be the linear relation associated with a. We denote the orthogonal projection associated with the degenerate strongly continuous semigroup T_a by $P_a := T_a(0)$, and by A_0 we denote the accretive self-adjoint operator associated with a as a form in $\overline{\mathrm{dom}(a)} = \mathrm{ran}(P_a)$. For the forms b and c we will use the corresponding notation.

For $t \geqslant 0$, $s > 0$ we define

$$F(t) := T_a(t)T_b(t), \quad R(s) := \tfrac{1}{s}\big(I - F(s)\big)$$

and

$$A(s) := \tfrac{1}{s}\big(I - T_a(s)\big), \quad B(s) := \tfrac{1}{s}\big(I - T_b(s)\big), \quad C(s) := A(s) + B(s).$$

It is easy to check that with these definitions one gets

$$R(s) = C(s) - sA(s)B(s). \tag{15.7}$$

The overall plan for the proof of Theorem 15.6 is to show that

$$\text{s-lim}_{s\to 0}\big(I + R(s)\big)^{-1} = \text{s-lim}_{s\to 0}\big(I + C(s)\big)^{-1} = (I+C)^{-1},$$

and then to apply Theorems 13.13 and 13.17. For the application of the latter we will also need a symmetrisation procedure.

15.8 Lemma. *For all $s > 0$ one has*

$$\mathrm{Re}\big(R(s)x \,\big|\, x\big) \geqslant \tfrac{1}{2}\big(C(s)x \,\big|\, x\big) \geqslant \tfrac{1}{2}\big(A(s)x \,\big|\, x\big) \geqslant 0 \qquad (x \in H);$$

in particular, $R(s)$ is accretive.

Proof. For $x \in H$ we estimate

$$\begin{aligned}\big|\big(sA(s)B(s)x \,\big|\, x\big)\big| &= \big|\big(sA(s)^{1/2}B(s)^{1/2}B(s)^{1/2}x \,\big|\, A(s)^{1/2}x\big)\big| \\ &\leqslant \big\|sA(s)^{1/2}B(s)^{1/2}\big\|\,\big\|B(s)^{1/2}x\big\|\,\big\|A(s)^{1/2}x\big\|\end{aligned}$$

(Because of $0 \leqslant T_a(s) \leqslant I$ we have $\|A(s)\| = \frac{1}{s}\|I - T_a(s)\| \leqslant \frac{1}{s}$, hence $\big\|\big(sA(s)\big)^{1/2}\big\| \leqslant 1$; similarly for $B(s)$, and therefore $\big\|sA(s)^{1/2}B(s)^{1/2}\big\| \leqslant 1$.)

$$\leqslant \frac{1}{2}\Big(\big\|B(s)^{1/2}x\big\|^2 + \big\|A(s)^{1/2}x\big\|^2\Big) = \tfrac{1}{2}\big(C(s)x \,\big|\, x\big).$$

This implies

$$\mathrm{Re}\big(R(s)x \,\big|\, x\big) = \big(C(s)x \,\big|\, x\big) - \mathrm{Re}\big(sA(s)B(s)x \,\big|\, x\big) \geqslant \tfrac{1}{2}\big(C(s)x \,\big|\, x\big).$$

Combining this estimate with the equality three lines above, we obtain the assertion. □

15.9 Lemma. *One has*

(a) $\text{s-lim}_{s\to 0}\big(I + C(s)\big)^{-1} = (I + C)^{-1}$,

(b) $\text{s-lim}_{s\to 0}\, s^{1/2}B(s)\big(I + C(s)\big)^{-1} = 0$.

Proof. (a) Let $s > 0$. As $T_a(s)$ is self-adjoint and $\|T_a(s)\| \leqslant 1$, the form $H \times H \ni (x, y) \mapsto \frac{1}{s}\big((I - T_a(s))x \,\big|\, y\big)$ is bounded, symmetric and accretive, and similarly for T_b. The form associated with the bounded operator $C(s)$ is given by the sum of these two forms,

$$c_s(x, y) := (C(s)x \,|\, y) = \frac{1}{s}\big((I - T_a(s))x \,\big|\, y\big) + \frac{1}{s}\big((I - T_b(s))x \,\big|\, y\big).$$

We observe that $(0, \infty) \ni s \mapsto c_s(x) \in [0, \infty)$ is decreasing for all $x \in H$. Indeed, if $x \in \overline{\mathrm{dom}(a)}$, then Proposition 13.22(b) implies that $s \mapsto \frac{1}{s}\big((I - T_a(s))x \,\big|\, x\big)$ is decreasing, and therefore

$$s \mapsto \frac{1}{s}\big((I - T_a(s))(x + y) \,\big|\, x + y\big) = \frac{1}{s}\big((I - T_a(s))x \,\big|\, x\big) + \frac{1}{s}\|y\|^2$$

is decreasing for all $x \in \overline{\mathrm{dom}(a)}$, $y \in \mathrm{dom}(a)^\perp$. The analogous property holds for b.

If $0 \neq x \in \mathrm{dom}(a)^\perp$, then $T_a(s)x = 0$, therefore $c_s(x) \to \infty$ as $s \to 0$, and similarly for $0 \neq x \in \mathrm{dom}(b)^\perp$. For $x \in \overline{\mathrm{dom}(a)} \cap \overline{\mathrm{dom}(b)}$, Proposition 13.22(b) shows that $\sup_{s>0} c_s(x) < \infty$ if and only if $x \in \mathrm{dom}(a) \cap \mathrm{dom}(b)$, and that

$$\begin{aligned}\lim_{s\to 0} c_s(x, y) &= \lim_{s\to 0} \frac{1}{s}\big((I - T_a(s))x \,\big|\, y\big) + \lim_{s\to 0} \frac{1}{s}\big((I - T_b(s))x \,\big|\, y\big) \\ &= a(x, y) + b(x, y)\end{aligned}$$

for all $x, y \in \mathrm{dom}(a) \cap \mathrm{dom}(b)$. Using Theorem 14.10 we conclude that $\big(I + C(s)\big)^{-1} \to (I + C)^{-1}$ strongly as $s \to 0$.

(b) The main part of the proof consists in showing that

$$\begin{aligned}\text{s-lim}_{s\to 0}\, A(s)^{1/2}\big(I + C(s)\big)^{-1} &= A_0^{1/2}(I + C)^{-1}, \\ \text{s-lim}_{s\to 0}\, B(s)^{1/2}\big(I + C(s)\big)^{-1} &= B_0^{1/2}(I + C)^{-1}.\end{aligned} \tag{15.8}$$

We fix $x \in H$ and put

$$y(s) := \big(I + C(s)\big)^{-1} x, \quad y_a(s) := A(s)^{1/2} y(s), \quad y_b(s) := B(s)^{1/2} y(s).$$

From part (a) we know that $y := \lim_{s\to 0} y(s) = (I + C)^{-1} x$.

We show that $P_a y_a(s) \to A_0^{1/2} y$ and $P_b y_b(s) \to B_0^{1/2} y$ weakly as $s \to 0$. We have $\|y(s)\| \leqslant \|x\|$, because $C(s)$ is accretive. It follows that

$$\begin{aligned} \|y(s)\|^2 + \|y_a(s)\|^2 + \|y_b(s)\|^2 &= \big(y(s) + A(s)y(s) + B(s)y(s) \,\big|\, y(s)\big) \\ &= (x \,|\, y(s)) \leqslant \|x\| \, \|y(s)\| \leqslant \|x\|^2. \end{aligned} \tag{15.9}$$

This inequality implies that the set $\{y_a(s);\ s > 0\}$ is bounded. Let $z \in \mathrm{dom}(a) = \mathrm{dom}(A_0^{1/2})$. Then Proposition 13.22(b) shows that $A(s)^{1/2} z \to A_0^{1/2} z$, and we obtain

$$\begin{aligned} (P_a y_a(s) \,|\, z) &= \big(A(s)^{1/2} y(s) \,\big|\, z\big) = \big(y(s) \,\big|\, A(s)^{1/2} z\big) \\ &\to \big(y \,\big|\, A_0^{1/2} z\big) = \big(A_0^{1/2} y \,\big|\, z\big) \qquad (s \to 0). \end{aligned}$$

Taking into account $\mathrm{ran}(P_a) = \overline{\mathrm{dom}(a)}$ and $A_0^{1/2} y \in \mathrm{ran}(P_a)$ as well as the boundedness shown above, we obtain the weak convergence $P_a y_a(s) \to A_0^{1/2} y$ asserted above. The corresponding weak convergence $P_b y_b(s) \to B_0^{1/2} y$ follows by the symmetry of the hypotheses with respect to a and b.

For the last part of the proof of (15.8) we note that the equation $y = (I + C)^{-1} x$ ($\in \mathrm{dom}(C_0) \subseteq \mathrm{dom}(c)$) can be reformulated as $(I + C_0)y = P_c x$. From (15.9) we infer, recalling $y(s) \to y = P_c y$, that

$$\begin{aligned} &\|y_a(s)\|^2 + \|y_b(s)\|^2 = (x \,|\, y(s)) - \|y(s)\|^2 \\ &\to (x \,|\, y) - \|y\|^2 = (P_c x - y \,|\, y) = (C_0 y \,|\, y) = a(y) + b(y) = \big\|A_0^{1/2} y\big\|^2 + \big\|B_0^{1/2} y\big\|^2 \end{aligned}$$

as $s \to 0$. Since $P_a y_a(s) \to A_0^{1/2} y$ and $P_b y_b(s) \to B_0^{1/2} y$ weakly, we conclude that

$$\begin{aligned} \big\|y_a(s) - A_0^{1/2} y\big\|^2 + \big\|y_b(s) - B_0^{1/2} y\big\|^2 &= \|y_a(s)\|^2 - 2\,\mathrm{Re}\big(y_a(s) \,\big|\, P_a A_0^{1/2} y\big) + \big\|A_0^{1/2} y\big\|^2 \\ &\quad + \|y_b(s)\|^2 - 2\,\mathrm{Re}\big(y_b(s) \,\big|\, P_b B_0^{1/2} y\big) + \big\|B_0^{1/2} y\big\|^2 \end{aligned}$$

converges to 0 as $s \to 0$. This completes the proof of (15.8).

Now for the asserted convergence we note – with the notation from above – that

$$\big\|s^{1/2} B(s)\big(I + C(s)\big)^{-1} x\big\|^2 = \big\|s^{1/2} B(s)^{1/2} y_b(s)\big\|^2 = \big(sB(s) y_b(s) \,\big|\, y_b(s)\big),$$

and this converges to $\big((I - P_b) B_0^{1/2} y \,\big|\, B_0^{1/2} y\big)$ as $s \to 0$, which in fact equals 0 because $\mathrm{ran}(B_0^{1/2}) \subseteq \overline{\mathrm{dom}(b)} = \mathrm{ran}(P_b)$. □

Proof of Theorem 15.6. (i) In the first step we show that $\big(I + R(s)\big)^{-1} - \big(I + C(s)\big)^{-1} \to 0$ strongly as $s \to 0$. In view of Lemma 15.9(a), this will imply that $\text{s-}\lim_{s\to 0} \big(I + R(s)\big)^{-1} = (I + C)^{-1}$, and applying Theorem 13.13 we then obtain the first assertion of the theorem. (Note that $F(0)x = P_a P_b x = x$ for all $x \in \mathrm{ran}(T_c(0)) = \mathrm{ran}(P_c)$.)

Using (15.7) we obtain

$$\begin{aligned}\big(I+R(s)\big)^{-1}-\big(I+C(s)\big)^{-1}&=\big(I+R(s)\big)^{-1}sA(s)B(s)\big(I+C(s)\big)^{-1}\\&=\Big(\big(I+R(s)\big)^{-1}A(s)^{1/2}\Big)\big(sA(s)\big)^{1/2}\Big(s^{1/2}B(s)\big(I+C(s)\big)^{-1}\Big).\end{aligned}$$

For the first factor, the application of Lemmas 15.7 and 15.8 yields

$$\big\|\big(I+R(s)\big)^{-1}A(s)^{1/2}\big\|\leqslant\sqrt{2}\qquad(s>0),$$

and $\big\|\big(sA(s)\big)^{1/2}\big\|\leqslant 1$ has been observed already in the proof of Lemma 15.8. Therefore Lemma 15.9(b) implies the assertion.

(ii) In order to obtain the second assertion of the theorem we have to symmetrise the operators $F(t)$ and $R(s)$. For $t\geqslant 0$, $s>0$ we define

$$G(t):=T_a(\tfrac{t}{2})T_b(t)T_a(\tfrac{t}{2}),\quad S(s):=\tfrac{1}{s}\big(I-G(s)\big).$$

From

$$sI=s\big(I+S(s)\big)\big(I+S(s)\big)^{-1}=\big((1+s)I-G(s)\big)\big(I+S(s)\big)^{-1}$$

we obtain

$$\begin{aligned}(1+s)\big(I+S(s)\big)^{-1}&=sI+T_a(\tfrac{s}{2})T_b(s)T_a(\tfrac{s}{2})\big(I+S(s)\big)^{-1}\\&=sI+T_a(\tfrac{s}{2})T_b(s)\big(I+R(s)\big)^{-1}T_a(\tfrac{s}{2}),\end{aligned}$$

where the last equality is a consequence of

$$\big(I+R(s)\big)T_a(\tfrac{s}{2})=T_a(\tfrac{s}{2})\big(I+S(s)\big).$$

This shows that

$$\begin{aligned}\text{s-lim}_{s\to 0}\big(I+S(s)\big)^{-1}&=\text{s-lim}_{s\to 0}\tfrac{1}{1+s}\big(sI+T_a(\tfrac{s}{2})T_b(s)\big(I+R(s)\big)^{-1}T_a(\tfrac{s}{2})\big)\\&=0+P_aP_b(I+C)^{-1}P_a=(I+C)^{-1},\end{aligned}$$

where in the last equality we have used the identities $(I+C)^{-1}=P_c(I+C)^{-1}P_c$ and $P_a(P_bP_c)=P_c=P_cP_a$ (the latter being true because $\operatorname{ran}(P_c)\subseteq\operatorname{ran}(P_a)\cap\operatorname{ran}(P_b)$). Now Theorem 13.17 implies that $G(\frac{t}{n})^nx\to T_c(t)x$ $(n\to\infty)$, uniformly for t in compact subsets of $(0,\infty)$, for all $x\in H$; recall also Remark 13.18(b). Remark 13.19 shows that the same holds for the convergence $G(\frac{t}{n})^{n-1}x\to T_c(t)x$.

As the result we conclude that

$$F(\tfrac{t}{n})^nx=T_a(\tfrac{t}{2n})G(\tfrac{t}{n})^{n-1}T_a(\tfrac{t}{2n})T_b(\tfrac{t}{n})x\to P_aT_c(t)P_aP_bx=T_c(t)x\qquad(n\to\infty),$$

uniformly for t in compact subsets of $(0,\infty)$, for all $x\in H$. □

For the remainder of this section let H be a complex Hilbert space. We now come to the striking application of Theorem 15.2 announced above.

15.10 Corollary (Trotter product formula for sectorial forms). *Let a and b be closed sectorial forms in H, $c := a + b$, and let T_a, T_b and T_c be the associated degenerate strongly continuous semigroups on H.*

Then we obtain the same conclusions as in Theorem 15.6.

Proof. There exists a constant $C > 0$ such that $|\operatorname{Im} a| \leqslant C \operatorname{Re} a$ and $|\operatorname{Im} b| \leqslant C \operatorname{Re} b$; see the last paragraph of Section 5.1. Then for $z \in \Omega := \{z \in \mathbb{C};\ |\operatorname{Re} z| < \frac{1}{C}\}$ the forms

$$a_z := \operatorname{Re} a + z \operatorname{Im} a, \quad b_z := \operatorname{Re} b + z \operatorname{Im} b, \quad c_z := \operatorname{Re} c + z \operatorname{Im} c$$

are closed sectorial forms in H. Indeed, for $z \in \Omega$ we have

$$\operatorname{Re} a_z \geqslant \operatorname{Re} a - |\operatorname{Re} z \operatorname{Im} a| \geqslant (1 - |\operatorname{Re} z| C) \operatorname{Re} a \geqslant 0,$$
$$|\operatorname{Im} a_z| = |\operatorname{Im} z \operatorname{Im} a| \leqslant |\operatorname{Im} z|\, C \operatorname{Re} a \leqslant \frac{|\operatorname{Im} z| C}{1 - |\operatorname{Re} z| C} \operatorname{Re} a_z,$$

and the same for b and c. Clearly, the mappings $z \mapsto a_z$, $z \mapsto b_z$, $z \mapsto c_z$ are holomorphic in the sense formulated in Theorem 15.2.

Applying first Theorem 15.2 and then Theorem 15.5 we obtain the holomorphic dependence $\Omega \ni z \mapsto T_{a_z}(t) \in \mathcal{L}(H)$ for all $t > 0$, and the same for T_{b_z} and T_{c_z}. (Note that the boundedness condition of Theorem 15.5 is satisfied because of $\|T_{a_z}(t)\| \leqslant 1$ for all $t \geqslant 0$, and similarly for b_z and c_z.)

For $n \in \mathbb{N}$, $z \in \Omega$ we define the function $F_{n,z}\colon [0, \infty) \to \mathcal{L}(H)$,

$$F_{n,z}(t) := \big(T_{a_z}(\tfrac{t}{n}) T_{b_z}(\tfrac{t}{n})\big)^n \quad (t \geqslant 0).$$

Then $F_{n,z}$ is strongly continuous for all $n \in \mathbb{N},\ z \in \Omega$, and $\|F_{n,z}(t)\| \leqslant 1$ for all $n \in \mathbb{N},\ z \in \Omega,\ t \geqslant 0$.

Let $t_0 > 0$, $x \in \overline{\operatorname{dom}(c)}$. Then $F_{n,z}(\cdot)x|_{[0,t_0]} \in C\big([0, t_0]; H\big)$ and $\|F_{n,z}(\cdot)x\| \leqslant \|x\|$ for all $n \in \mathbb{N},\ z \in \Omega$. Furthermore, for all $n \in \mathbb{N},\ 0 \leqslant t \leqslant t_0$ the function $\Omega \ni z \mapsto F_{n,z}(t)x$ is holomorphic. Observe that the set of functionals

$$E := \{\eta_{t,y};\ y \in H,\ \|y\| \leqslant 1,\ 0 \leqslant t \leqslant t_0\} \subseteq C\big([0, t_0]; H\big)',$$

where

$$\eta_{t,y}(f) := (f(t) \,|\, y) \qquad \big(f \in C\big([0, t_0]; H\big)\big),$$

is norming for $C\big([0, t_0]; H\big)$. Thus we conclude from Theorem 3.2 that $\Omega \ni z \mapsto F_{n,z}(\cdot)x|_{[0,t_0]}$ is holomorphic. Finally, for $z \in \Omega \cap \mathbb{R}$ we already know from Theorem 15.6 that

$$F_{n,z}(\cdot)x|_{[0,t_0]} \to T_{c_z}(\cdot)x|_{[0,t_0]} \qquad (n \to \infty).$$

Now Vitali's theorem, Theorem 15.1, implies that $F_{n,z}(\cdot)x|_{[0,t_0]} \to T_{c_z}(\cdot)x|_{[0,t_0]}$ in $C\big([0, t_0]; H\big)$ $(n \to \infty)$, uniformly for z in compact subsets of Ω. Observing that $a = a_{\mathrm{i}}$, $b = b_{\mathrm{i}}$ and $c = c_{\mathrm{i}}$, we finally obtain the first convergence asserted in the theorem.

The second assertion is proved analogously. □

15.4 Applications of the Trotter product formula

The following applications of the Trotter product formula, Theorem 15.3, illustrate the interaction between diffusion semigroups and domain restrictions or multiplication operators.

15.11 Example (Dirichlet boundary conditions via absorption). Let

$$a(u,v) := \int_{\mathbb{R}^n} \nabla u(x) \cdot \overline{\nabla v(x)}\, \mathrm{d}x$$

be the classical Dirichlet form, defined on $\mathrm{dom}(a) := H^1(\mathbb{R}^n)$. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and define b by $\mathrm{dom}(b) := L_2(\Omega) = \{f \in L_2(\mathbb{R}^n);\ f = 0 \text{ a.e. on } \mathbb{R}^n \setminus \Omega\}$,

$$b(u,v) := 0 \qquad (u, v \in \mathrm{dom}(b)).$$

Then $\mathrm{dom}(a)$ is dense in $L_2(\mathbb{R}^n)$, and the C_0-semigroup T_a associated with a is governed by the heat equation. The degenerate strongly continuous semigroup T_b associated with b is given by $T_b(t) = P_\Omega$ $(t \geqslant 0)$, where P_Ω is the orthogonal projection onto $L_2(\Omega)$, $P_\Omega f = \mathbf{1}_\Omega f$ $(f \in L_2(\mathbb{R}^n))$.

The form $c = a + b$ is then given by

$$\mathrm{dom}(c) = H^1(\mathbb{R}^n) \cap L_2(\Omega), \qquad c(u,v) = a(u,v) \quad (u, v \in \mathrm{dom}(c)),$$

and Theorem 15.6 implies that T_c is given by the interesting formula

$$T_c(t)f = \lim_{k\to\infty} \big(T_a(\tfrac{t}{k})P_\Omega\big)^k f,$$

where the convergence is uniform for t in compact subsets of $[0,\infty)$, if $f \in L_2(\Omega)$, and uniform for t in compact subsets of $(0,\infty)$, if $f \in L_2(\mathbb{R}^n)$.

Assuming additionally that Ω is bounded and has continuous boundary, one obtains a particularly interesting result. In this case the resticted semigroup $T_c|_{L_2(\Omega)}$ is the C_0-semigroup generated by the Dirichlet Laplacian in $L_2(\Omega)$. Indeed, it was shown in Proposition 7.9 that under these hypotheses one has $\mathrm{dom}(c) = H^1(\mathbb{R}^n) \cap L_2(\Omega) = H^1_0(\Omega)$.

We warn the reader that the previous statement does not hold without the hypothesis of continuous boundary, as the example $n = 1$, $\Omega = (-1,0) \cup (0,1)$ shows. Indeed, in this situation one has $\mathrm{dom}(c) = H^1_0(-1,1)$, but $H^1_0(-1,1)$ is not a subset of $H^1_0(\Omega)$. Clearly, one can construct similar examples in any dimension.

We will see in Example 15.12(b) that there is another way to approximate the semigroup T_c. Placing ourselves into the context of Example 15.12 and defining $q\colon \mathbb{R} \to [0,\infty]$ by $q := \infty \mathbf{1}_{\mathbb{R}^n\setminus\Omega}$, we find that the semigroup T_c described above is the same as the one obtained in Example 15.12. △

15.12 Example. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let

$$a(u,v) := \int_{\Omega} \nabla u(x) \cdot \overline{\nabla v(x)}\, \mathrm{d}x$$

be the classical Dirichlet form, defined on $\mathrm{dom}(a) = V_0 := H_0^1(\Omega)$ (corresponding to the Dirichlet Laplacian) or on $\mathrm{dom}(a) = V_1 := H^1(\Omega)$ (corresponding to the Neumann Laplacian).

Let $q\colon \Omega \to [0,\infty]$ be measurable, and define

$$b(u,v) := \int_\Omega q(x)\, u(x)\, \overline{v(x)}\, \mathrm{d}x$$

for $u, v \in \mathrm{dom}(b) := \{u \in L_2(\Omega);\, q|u|^2 \in L_1(\Omega)\}$.

Then $c := a + b$ is a closed symmetric form.

(a) One way to approximate T_c is by applying Theorem 15.6. Note that the degenerate strongly continuous semigroup T_b associated with the form b is given by $T_b(t)f = \mathrm{e}^{-tq}f$ (with $\mathrm{e}^{-t\infty} := 0$). The application of Theorem 15.6 yields

$$\big(T_a(\tfrac{t}{k})\mathrm{e}^{-\frac{t}{k}q}\big)^k f \to T_c(t)f, \tag{15.10}$$

with the specifications concerning the sets of convergence according to the assertion in Theorem 15.6. It follows from (15.10) that all the operators $T_c(t)$, for $t > 0$, are sub-Markovian on $L_2(\Omega)$. We now show that there exists a measurable set $\Omega_0 \subseteq \Omega$ such that

$$H_0 := \overline{\mathrm{dom}(c)} = L_2(\Omega_0) = \{f \in L_2(\Omega);\, f = 0 \text{ a.e. on } \Omega \setminus \Omega_0\}; \tag{15.11}$$

then the restriction of the degenerate strongly continuous semigroup T_c associated with c to H_0 is a sub-Markovian C_0-semigroup.

Indeed, the strong continuity of T_c implies that $T_c(0) = \text{s-}\lim_{t\to 0+} T_c(t)$ is also sub-Markovian; in fact it is a sub-Markovian projection. More strongly, the convergence (15.10) implies that $0 \leqslant T_c(t) \leqslant T_a(t)$ for all $t > 0$. (Here and in what follows, the inequality $A \leqslant B$ between two operators $A, B \in \mathcal{L}(L_2(\mu))$ will denote inequality in the order sense, i.e., $Af \leqslant Bf$ for all $f \in L_2(\Omega)_+$.) This implies that

$$T_c(0) = \text{s-}\lim_{t\to 0} T_c(t) \leqslant \text{s-}\lim_{t\to 0} T_a(t) = T_a(0) = I;$$

hence $T_c(0)$ is a projection satisfying $0 \leqslant T_c(0) \leqslant I$. In Proposition 15.13 we will show that this implies that

$$H_0 := \overline{\mathrm{dom}(c)} = \mathrm{ran}(T_c(0))$$

is a closed **order ideal** in $L_2(\Omega)$, i.e., H_0 is a closed subspace, and $f \in H_0$, $g \in L_2(\mu)$, $|g| \leqslant |f|$ implies $g \in H_0$; therefore Exercise 15.2(c) shows that there exists a measurable subset $\Omega_0 \subseteq \Omega$ such that (15.11) holds.

(b) Another way to approximate the degenerate strongly continuous semigroup T_c is by using the forms b_k, defined by $\mathrm{dom}(b_k) := L_2(\Omega)$,

$$b_k(u,v) := \int_\Omega (q \wedge k) u\bar{v}\, \mathrm{d}x.$$

Then (b_k) is an increasing sequence of closed symmetric forms, and $(a + b_k)_k$ converges to $c = a + b$ in the sense of Theorem 14.10. Let T_k be the C_0-semigroup associated with $a + b_k$. Then, using Theorem 14.10 together with Theorem 13.10 one concludes that $T_k(t)f \to T_c(t)f$ for all $f \in L_2(\Omega_0)$, uniformly for t in compact subsets of $[0,\infty)$, and that $T_k(t)f \to 0$ for all $f \in L_2(\Omega \setminus \Omega_0)$, uniformly for t in compact subsets of $(0,\infty)$. $\triangle$

15.13 Proposition. *Let (Ω, μ) be a measure space. Let $P \in \mathcal{L}(L_2(\mu))$ be a projection satisfying $0 \leqslant P \leqslant I$. Then* $\operatorname{ran}(P)$ *is a closed order ideal in $L_2(\mu)$.*

Proof. First we treat the case $\mathbb{K} = \mathbb{R}$. It is easy to see that the assertion will follow if we show the equivalence

$$f \in \operatorname{ran}(P) \quad \Longleftrightarrow \quad |f| \wedge |g| = 0 \quad (g \in \operatorname{ran}(I - P)). \tag{15.12}$$

First we show '$\Rightarrow$'. Let $0 \leqslant f, g \in L_2(\mu)$, and put $h := Pf \wedge (I - P)g$. Then $0 \leqslant Ph \leqslant P(I - P)g = 0$, $0 \leqslant (I - P)h \leqslant (I - P)Pf = 0$, hence $Pf \wedge (I - P)g = 0$. This shows the asserted implication of (15.12) for $f, g \geqslant 0$. For the general case decompose f and g into their positive and negative parts.

In order to show '$\Leftarrow$' we take $f \in L_2(\mu)_+$ such that $f \wedge (I - P)g = 0$ for all $g \in L_2(\mu)_+$. Then $Pf \wedge (I - P)f = 0$ as well; hence $0 = (f - Pf) \wedge (I - P)f$, $f = Pf \in \operatorname{ran}(P)$. For the general case decompose $f = f^+ - f^-$.

Concerning the complex case, we just observe that $f \in \operatorname{ran}(P)$ if and only if $\operatorname{Re} f, \operatorname{Im} f \in \operatorname{ran}(P)$. □

Notes

The elegant proof of Vitali's theorem, Theorem 15.1, is due to Arendt and Nikolski; see [ArNi00; Theorem 2.1]. Note that even for the case of a sequence of $\mathbb{C}$-valued functions one needs the concept of Banach space valued holomorphic functions.

The proof of Theorem 15.2 is taken from [VoVo18]; the new feature introduced here, in comparison with the proof given in Kato [Kat80; VII, Theorem 4.2], is the use of the Lax–Milgram lemma. Theorem 15.5 is taken from [VoVo18] as well. The Trotter product formula has first been proved by Trotter [Tro59] for the case of contraction semigroups, with suitable hypotheses on the domains of the generators. The Trotter product formula for forms, Theorem 15.6, is due to T. Kato. Our proof follows [Kat78] but is somewhat simpler because we do not prove the theorem in the full generality stated there. Corollary 15.10, due to B. Simon, is contained in the Addendum in [Kat78].

There has been considerable interest in investigating variants of the Trotter product formula. One natural question is: when do the Trotter products converge in the operator norm or even stronger norms? This is discussed in [CaZa99], [Tam00], [CaZa01a], [CaZa01b], [NSZ18], [Zag19; Chap. 5]. Result on numerical methods in connection with Trotter products can be found in [JaLu00], [Tha08].

Another issue is the "Lapidus problem", which arises in the context of Theorem 15.6. If $\mathbb{K} = \mathbb{C}$, then the semigroups T_a, T_b, T_c are bounded holomorphic semigroups of angle $\pi/2$, and as such they have strongly continuous extensions to the closed right half-plane (cf. Exercise 3.7(c)). In Exercise 15.3 the reader is asked to show that the Trotter approximation also holds on the open right half-plane. The Lapidus problem is the question of whether the Trotter approximation, for $x \in \overline{\operatorname{dom}(c)}$, holds on the imaginary axis as well. (It follows from [MaSh03; Subsection 3.1] that one cannot expect convergence for $x \notin \overline{\operatorname{dom}(c)}$.) We refer to [JoLa00; Problem 11.3.9 and Section 11.7], [Cac05], [ENZ11] for more information and partial results concerning this (open!) problem.

Exercises

15.1 Let M be a compact metric space, X a complex Banach space, $\Omega \subseteq \mathbb{C}$ an open set. Let $F\colon \Omega \to C(M;X)$ be a bounded function, and assume that there exists a dense subset $D \subseteq M$ such that $\Omega \ni z \mapsto F(z)(t) \in X$ is holomorphic for all $t \in D$.

Show that $F\colon \Omega \to C(M;X)$ is holomorphic.

Hint: Show that $E := \{C(M;X) \ni f \mapsto \langle f(t), x'\rangle\,;\, t \in D,\ x' \in X'\} \subseteq C(M;X)'$ is a norming subset and apply Theorem 3.2.

15.2 Let (Ω, μ) be a measure space, and let $H_0 \subseteq L_2(\mu)$ be a closed order ideal in $L_2(\mu)$; see Example 15.12 for the definition. Let P denote the orthogonal projection onto H_0.

(a) Let $g \in L_2(\mu)$. Show that $g \perp H_0$ if and only if $|g| \wedge |f| = 0$ for all $f \in H_0$. Show that $|Pf| \wedge |(I-P)f| = 0$ for all $f \in L_2(\mu)$. Hint for '$\Rightarrow$' in the first part: For $f \in H_0$ one has $f \cdot \overline{\operatorname{sgn} f} \cdot \operatorname{sgn} g \in H_0$.

(b) Assume additionally that $\mu(\Omega) < \infty$. Show that there exists a measurable set $\Omega_0 \subseteq \Omega$ such that $P\mathbf{1}_\Omega = \mathbf{1}_{\Omega_0}$, and that $Pf = \mathbf{1}_{\Omega_0} f$ for all $f \in L_2(\Omega)$. Hint: Show the latter property first for $0 \leqslant f \leqslant \mathbf{1}_\Omega$.

(c) Assume that μ is σ-finite. Show that there exists a measurable set $\Omega_0 \subseteq \Omega$ such that $Pf = \mathbf{1}_{\Omega_0} f$ for all $f \in L_2(\Omega)$.

15.3 Let H be a complex Hilbert space, let a, b be accretive symmetric forms in H, and put $c := a + b$. Let T_a, T_b, T_c be the degenerate holomorphic C_0-semigroups of angle $\pi/2$ associated with a, b, c, respectively.

(a) Show that $T_c(z) = \text{s-lim}\big(T_a(z/n)T_b(z/n)\big)^n$ uniformly for z in compact subset of the right half-plane $[\mathrm{Re} > 0] \subseteq \mathbb{C}$. (Hint: Use Vitali's theorem, Theorem 15.1.)

(b) Let $\theta \in (-\pi/2, \pi/2)$. Show that $T_c(z)x = \lim\big(T_a(z/n)T_b(z/n)\big)^n x$ uniformly for z in compact subsets of the half-line $\mathrm{e}^{\mathrm{i}\theta}[0,\infty)$, for all $x \in \overline{\mathrm{dom}(c)}$. (Hint: Use Corollary 15.10.)

(c) Let $R > 0$, $\theta \in (0, \pi/2)$, $D := \big\{r\mathrm{e}^{\mathrm{i}\alpha}\,;\, 0 \leqslant r \leqslant R,\ |\alpha| \leqslant \theta\big\}$. Show that $T_c(z)x = \lim\big(T_a(z/n)T_b(z/n)\big)^n x$ uniformly for $z \in D$, for all $x \in \overline{\mathrm{dom}(c)}$. (Hint: Combine (a) and (b), and apply the maximum principle for holomorphic functions.)

Chapter 16

The Stokes operator

The Stokes operator arises in the context of the (nonlinear!) Navier–Stokes equation and acts in a subspace of a $\mathbb{K}^n$-valued L_2-space. In our context we define it using a variant of the classical Dirichlet form. One of the features appearing in the description of the Stokes operator is the use of a Sobolev space of negative order, which is introduced at the beginning. Another important feature is the appearance of divergence free vector fields. This extra condition of vanishing divergence has interesting implications for the theory of the related Sobolev spaces, and a large part of the chapter is devoted to the investigation of these properties.

16.1 Interlude: the Sobolev space $H^{-1}(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$ be open. The space $H^1_0(\Omega)$ is a Hilbert space, and by the Fréchet–Riesz theorem, each continuous antilinear functional on $H^1_0(\Omega)$ is represented by an element of $H^1_0(\Omega)$. For some purposes, however, it is more convenient to use a different representation of the antidual space of $H^1_0(\Omega)$ that contains $L_2(\Omega)$ als a dense subspace.

The basic observation is that each element $f \in L_2(\Omega)$ acts in a natural way as a continuous antilinear functional on $H^1_0(\Omega)$, by

$$H^1_0(\Omega) \ni u \mapsto (f \,|\, u)_{L_2(\Omega)} =: \langle f, u\rangle_{H^{-1},H^1_0}\,.$$

The operator $L_2(\Omega) \ni f \mapsto \langle f, \cdot\rangle_{H^{-1},H^1_0} \in H^1_0(\Omega)^*$ is injective, and it has dense range because $u \in H^1_0(\Omega)$, $\langle f, u\rangle_{H^{-1},H^1_0} = 0$ for all $f \in L_2(\Omega)$ implies $u = 0$; see Exercise 17.1. In this context the antidual $H^1_0(\Omega)^*$ is denoted by $H^{-1}(\Omega)$.

16.1 Remark. We note that in the situation described above one has

$$H^1_0(\Omega) \hookrightarrow L_2(\Omega) \hookrightarrow H^{-1}(\Omega) = H^1_0(\Omega)^*,$$

with dense embeddings, and the embeddings are dual to each other. In such a situation one calls $\big(H^1_0(\Omega), L_2(\Omega), H^{-1}(\Omega)\big)$ a **Gelfand triple**; this will be treated in greater generality in Section 17.1. △

If $f \in L_2(\Omega)$ and $j \in \{1, \dots, n\}$, then the mapping

$$H^1_0(\Omega) \ni u \mapsto \langle \partial_j f, u\rangle_{H^{-1},H^1_0} := -(f \,|\, \partial_j u)_{L_2(\Omega)}$$

belongs to $H_0^1(\Omega)^*$. This definition of $\partial_j f$ is consistent with the definition of the distributional derivative in Section 4.1 because $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, and the mapping $\partial_j \colon L_2(\Omega) \to H^{-1}(\Omega)$ is linear and continuous. As a consequence, the differential operator $\Delta = \sum_{j=1}^n \partial_j \partial_j$ acts as a bounded operator $\Delta \colon H_0^1(\Omega) \to H^{-1}(\Omega)$. With this interpretation of the Laplace operator we obtain

$$\langle \Delta u, \varphi \rangle_{H^{-1}, H_0^1} = (u \,|\, \Delta\varphi)_{L_2(\Omega)}$$

for all $u \in H_0^1(\Omega),\ \varphi \in C_c^\infty(\Omega)$.

It turns out that the mapping $I - \Delta \colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isometric isomorphism. If Ω is bounded and one provides $H_0^1(\Omega)$ with the scalar product $(u, v) \mapsto \int \nabla u \cdot \overline{\nabla v}\, \mathrm{d}x$, then $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isometric isomorphism. See Exercise 16.1 for these properties. (Strictly speaking, the notation '$I - \Delta$' is not quite correct: the identity I is meant to be the embedding $H_0^1(\Omega) \hookrightarrow H^{-1}(\Omega)$.)

16.2 The Stokes operator

Let $\Omega \subseteq \mathbb{R}^n$ be open. The Stokes operator is an operator in a subspace H of $L_2(\Omega; \mathbb{K}^n)$. We will define it as the operator associated with a variant of the classical Dirichlet form on

$$V := H_{0,\sigma}^1(\Omega; \mathbb{K}^n) := \big\{ u \in H_0^1(\Omega; \mathbb{K}^n) = H_0^1(\Omega)^n ;\ \operatorname{div} u = 0 \big\}.$$

The index 'σ' should be remindful of 'solenoidal', which is the classical term for divergence free vectors fields. Note that V is a closed subspace of $H_0^1(\Omega; \mathbb{K}^n)$. The Hilbert space H is defined as the closure of V in $L_2(\Omega; \mathbb{K}^n)$,

$$H := L_{2,\sigma,0}(\Omega; \mathbb{K}^n) := \overline{V}^{L_2(\Omega;\mathbb{K}^n)}.$$

16.2 Remarks. (a) The space $L_{2,\sigma}(\Omega; \mathbb{K}^n) := \{ f \in L_2(\Omega; \mathbb{K}^n) ;\ \operatorname{div} f = 0 \}$ is a closed subspace of $L_2(\Omega; \mathbb{K}^n)$; this is because $\operatorname{div} \colon L_2(\Omega; \mathbb{K}^n) \to H^{-1}(\Omega)$ is continuous. Therefore H is a (closed) subspace of $L_{2,\sigma}(\Omega; \mathbb{K}^n)$.

We point out that $C_c^\infty(\Omega; \mathbb{K}^n) \cap H$ need not be dense in H, so that the notation $L_{2,\sigma,0}$ is not entirely consistent with previous notation. (There is no such reservation for sets Ω that are bounded and have Lipschitz boundary; cf. Theorem 16.14.)

(b) One could consider the form a defined below as a non-densely defined form in the Hilbert space $L_2(\Omega, \mathbb{K}^n)$; cf. Chapter 14. In the present chapter we are not interested in this point of view, instead we work with the Hilbert space H adapted to the form domain V. △

We define the form $a \colon V \times V \to \mathbb{R}$ by

$$a(u, v) := \sum_{j=1}^n \int_\Omega \nabla u_j \cdot \overline{\nabla v_j}\, \mathrm{d}x.$$

In each component of $H_0^1(\Omega)^n$, the form a is the classical Dirichlet form. Therefore we conclude from Section 5.4 that a is symmetric, accretive and H-coercive; for bounded Ω it is coercive. It follows that the operator

$$A := \Big\{(u,f) \in V \times H\,;\, \sum_{j=1}^n \int_\Omega \nabla u_j \cdot \overline{\nabla v_j}\,\mathrm{d}x = (f \,|\, v)_H \ (v \in V)\Big\} \tag{16.1}$$

associated with a is self-adjoint and accretive. In view of Section 16.1, the equality appearing in the description of A can be rewritten as

$$0 = -\sum_{j=1}^n \int_\Omega \nabla u_j \cdot \overline{\nabla v_j}\,\mathrm{d}x + (f \,|\, v)_H = \sum_{j=1}^n \langle \Delta u_j + f_j, v_j\rangle_{H^{-1},H_0^1} \qquad (v \in V),$$

where f_j is considered as an element of $H^{-1}(\Omega)$ via the injection $L_2(\Omega) \hookrightarrow H^{-1}(\Omega)$, and Δ is the operator $\Delta\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ from Section 16.1.

Note that the antidual of $H_0^1(\Omega)^n$ is given by $H^{-1}(\Omega)^n$, with the dual pairing

$$\langle \eta, u\rangle_{H^{-1},H_0^1} := \sum_{j=1}^n \langle \eta_j, u_j\rangle_{H^{-1},H_0^1} \qquad \big(\eta \in H^{-1}(\Omega)^n,\ u \in H_0^1(\Omega)^n\big).$$

Then the condition in (16.1) becomes

$$\langle \Delta u + f, v\rangle_{H^{-1},H_0^1} = 0 \qquad (v \in V),$$

with the abbreviating notation $\Delta u = (\Delta u_1, \dots, \Delta u_n)$. Defining the polar of $V \subseteq H_0^1(\Omega)^n$,

$$V^\circ := \big\{\eta \in H^{-1}(\Omega)^n\,;\, \langle \eta, v\rangle_{H^{-1},H_0^1} = 0 \ (v \in V)\big\},$$

as the 'orthogonal complement' of V in $H^{-1}(\Omega)^n$ we can write the condition in (16.1) in the concise form

$$\eta := \Delta u + f \in V^\circ.$$

Summarising we obtain the following description of the **Stokes operator**; as mentioned above, it is self-adjoint and accretive.

16.3 Theorem. *The operator A in H associated with the form a is given by*

$$A = \big\{(u,f) \in V \times H\,;\, \exists \eta \in V^\circ\colon -\Delta u + \eta = f\big\}$$

(where the equality '$-\Delta u + \eta = f$' is an equality in $H^{-1}(\Omega)^n$, with $f \in H \hookrightarrow H^{-1}(\Omega)^n$). Written differently,

$$\begin{aligned} \mathrm{dom}(A) &= \big\{u \in V\,;\, \exists \eta \in V^\circ\colon -\Delta u + \eta \in H\big\},\\ Au &= -\Delta u + \eta \quad \textit{(with } \eta \textit{ as in } \mathrm{dom}(A)\textit{)}. \end{aligned}$$

Note that for $f \in L_2(\Omega)$ one has $\langle \nabla f, v\rangle_{H^{-1},H^1_0} = (f \,|\, \operatorname{div} v)_H = 0$ for all $v \in V$, so the gradient $\nabla f \in H^{-1}(\Omega)^n$ belongs to V°. For the physical interpretation of the Stokes operator one requires that *all* elements $\eta \in V^\circ$ appearing in the description of A can be expressed in this way:

$$\text{for all } \eta \in V^\circ \text{ there exists } p \in L_2(\Omega) \text{ with } \eta = \nabla p. \tag{H_0}$$

(The index 0 in the label (H_0) refers to the index 0 in $V = H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$.)

We will comment on hypothesis (H_0) in Remark 16.6. In Section 16.4 we show that it holds if Ω is bounded and has Lipschitz boundary. With this hypothesis we get another description of the Stokes operator A.

16.4 Theorem. *Assume that Ω satisfies hypothesis* (H_0). *Then*

$$A = \big\{(u,f) \in V \times H\,;\, \exists p \in L_2(\Omega)\colon -\Delta u + \nabla p = f\big\}$$

is the operator associated with the form a. Expressed differently,

$$\operatorname{dom}(A) = \big\{u \in V\,;\, \exists p \in L_2(\Omega)\colon -\Delta u + \nabla p \in H\big\},$$
$$Au = -\Delta u + \nabla p \quad \textit{(with } p \textit{ as in } \operatorname{dom}(A)\textit{)}.$$

In fluid dynamics one interprets p in the statement of Theorem 16.4 as a pressure, which is the reason for the notation. However, we will not endeavour to enter the physical interpretation of the Stokes operator.

In the following remarks we describe some conditions and properties related to (H_0) in the 'classical' context.

16.5 Remarks. Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $u\colon \Omega \to \mathbb{K}^n$ be a continuous vector field.

(a) Assume that u satisfies the condition

$$\int u \cdot \varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{K}^n) \text{ with } \operatorname{div}\varphi = 0. \tag{16.2}$$

(This condition should be considered as a weakened version of the hypothesis '$\eta \in V^\circ$' in (H_0); note that $\{\varphi \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{K}^n)\,;\, \operatorname{div}\varphi = 0\}$ need not be dense in V.) Then one can show that there exists a potential $p \in C^1(\Omega)$ for u, i.e. $\nabla p = u$. Clearly it is sufficient to treat the case when Ω is connected. Then, fixing an 'initial point' $x^0 \in \Omega$, one defines $p(x) := \int_0^1 u(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t$, where $\gamma\colon [0,1] \to \Omega$ is a piecewise continuously differentiable path connecting x^0 with x. Using (16.2) one can show that p is well-defined – this is the main issue – and that then $u = \nabla p$. Conversely, the existence of a potential for the vector field implies the validity of (16.2).

The proof of these statements is delegated to Exercise 16.2(a).

(b) If u is continuously differentiable and satisfies (16.2), then it satisfies the 'compatibility condition' $\partial_j u_k = \partial_k u_j$ for all $j,k \in \{1,\dots,n\}$. Indeed, let $\psi \in C^\infty_{\mathrm{c}}(\Omega)$, and put $\varphi_j := \partial_k\psi$, $\varphi_k := -\partial_j\psi$, $\varphi_\ell := 0$ for all other components (where without loss of generality $j \neq k$). Then $\operatorname{div}\varphi = 0$, and therefore

$$\int (\partial_j u_k - \partial_k u_j)\psi \,\mathrm{d}x = \int (u_j \partial_k\psi - u_k \partial_j\psi)\,\mathrm{d}x = \int u \cdot \varphi \,\mathrm{d}x = 0.$$

If one requires from u merely the compatibility condition (and not the validity of (16.2)), then one can show that every point of Ω possesses a neighbourhood in which a potential exists, given by the explicit formula (16.4) in Section 16.3 below. An example of a vector field u without a global potential can be found in Exercise 16.2(b). However, if one adds the hypothesis that Ω is simply connected, then one can show that a global potential for u exists. This is a version of the 'Poincaré lemma'.

We will not prove these statements, because in this generality they will not be of importance in our context. However, it is noteworthy that the smoothed version (16.5) of the explicit formula mentioned above will be basic for the further development.

(c) We mention that both of the phenomena described in parts (a) and (b) can also be treated in the more general context of distributions. If u is a vector of distributions on Ω satisfying a condition analogous to (16.2), then one can show that there exists a distribution p such that $\nabla p = u$. (This is a special case of 'de Rham's theorem'.) Likewise, if Ω is simply connected, and a vector $u = (u_1, \dots, u_n)$ of distributions satisfies the compatibility condition $\partial_j u_k = \partial_k u_j$ for all $j, k \in \{1, \dots, n\}$, then one can find a distribution p on Ω with $\nabla p = u$. (This is a generalised version of Poincaré's lemma.) For a recent treatment of these topics we refer to [Voi22]. △

The topics described in the preceding remark do not explicitly enter our further treatment, but serve to motivate the investigation of corresponding properties in the context of Sobolev spaces, as sketched in the following remark.

16.6 Remark. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $f \in L_2(\Omega)$. Then clearly $\eta := \nabla f \in H^{-1}(\Omega)^n$ has the following property corresponding to (16.2):

$$\langle \eta, \varphi \rangle = 0 \text{ for all } \varphi \in C_c^\infty(\Omega)^n \text{ satisfying } \operatorname{div} \varphi = 0. \tag{16.3}$$

In view of Remarks 16.5(a) and (c) it is natural to ask whether for any $\eta \in H^{-1}(\Omega)^n$ satisfying (16.3) there exists $f \in L_2(\Omega)$ such that $\nabla f = \eta$. (Note that the condition on η in (16.3) is weaker than '$\eta \in V^\circ$'; so the requirement that η satisfying (16.3) can be represented as $\eta = \nabla f$ – later this will be defined as property $(\mathrm{H_c})$ – is stronger than $(\mathrm{H_0})$.) A positive answer will be given in Theorem 16.14 for the case that Ω has Lipschitz boundary.

The remaining sections of the present chapter are mainly motivated by this topic. △

16.3 Interlude: the Bogovskiĭ formula

This section could also run under the heading "some functions are divergences". For $u \in C_c^\infty(\mathbb{R}^n; \mathbb{K}^n)$ one has $\int \operatorname{div} u \, dx = 0$, and in fact the converse holds as well: for each $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\int \varphi \, dx = 0$ there exists $u \in C_c^\infty(\mathbb{R}^n; \mathbb{K}^n)$ with $\operatorname{div} u = \varphi$. Here we show that for suitable bounded open $\Omega \subseteq \mathbb{R}^n$ one has a Sobolev space version of this statement: for any $f \in L_2(\Omega)$ with $\int f \, dx = 0$ there exists $u \in H_0^1(\Omega)^n$ such that $\operatorname{div} u = f$; see Theorem 16.11.

Let $u \in C^\infty(\mathbb{R}^n; \mathbb{K}^n)$ be a vector field satisfying the compatibility condition $\partial_j u_k = \partial_k u_j$ for all $j, k = 1, \dots, n$. Then it is well-known and easy to show that, for any $y \in \mathbb{R}^n$, a

potential for u is given by

$$p(x) := \int_0^1 u(y + t(x-y)) \cdot (x-y)\,\mathrm{d}t, \tag{16.4}$$

i.e. $u = \nabla p$. We smooth this formula out with the help of a function $\rho \in C_c^\infty(\mathbb{R}^n)_+$ satisfying $\int \rho(x)\,\mathrm{d}x = 1$: we define

$$Au(x) := \int \rho(y) \int_0^1 u(sy + (1-s)x) \cdot (x-y)\,\mathrm{d}s\,\mathrm{d}y \tag{16.5}$$

and obtain $Au \in C^\infty(\mathbb{R}^n)$, $\nabla(Au) = u$. In order to write A as an integral operator, we substitute $z = sy + (1-s)x$ and $r = \frac{1}{s}$ to obtain

$$\begin{aligned} Au(x) &= \int_0^1 \int \rho\big(\tfrac{1}{s}(z - (1-s)x)\big) u(z) \cdot \frac{x-z}{s}\, s^{-n}\,\mathrm{d}z\,\mathrm{d}s \\ &= \int_1^\infty \int \rho(x + r(z-x)) u(z) \cdot (x-z)\,\mathrm{d}z\, r^{n-1}\,\mathrm{d}r. \end{aligned}$$

This means that one can write $Au(x) = \int k(x,y) \cdot u(y)\,\mathrm{d}y$, with

$$k(x,y) = \int_1^\infty \rho(x + r(y-x)) r^{n-1}\,\mathrm{d}r\,(x-y).$$

Let ℓ be the negative transposed kernel of k, i.e.,

$$\ell(x,y) := -k(y,x) = (x-y) \int_1^\infty \rho(y + r(x-y)) r^{n-1}\,\mathrm{d}r \qquad (x, y \in \mathbb{R}^n).$$

It will be shown in the next theorem that then the definition

$$\begin{aligned} Bf(x) &:= \int \ell(x,y) f(y)\,\mathrm{d}y \\ &= \int f(y)(x-y) \int_1^\infty \rho(y + r(x-y)) r^{n-1}\,\mathrm{d}r\,\mathrm{d}y, \end{aligned} \tag{16.6}$$

for $x \in \mathbb{R}^n$ and $f \in C_c^\infty(\mathbb{R}^n)$, yields a mapping $B\colon C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n; \mathbb{K}^n)$. This definition is such that for all $u \in C_c^\infty(\mathbb{R}^n; \mathbb{K}^n)$, $f \in C_c^\infty(\mathbb{R}^n)$ one has $\int (Au) f\,\mathrm{d}x = -\int u \cdot Bf\,\mathrm{d}x$.

16.7 Theorem. *For all $f \in C_c^\infty(\mathbb{R}^n)$ one has $Bf \in C_c^\infty(\mathbb{R}^n; \mathbb{K}^n)$,*

$$\operatorname{spt}(Bf) \subseteq \big\{\lambda z_1 + (1-\lambda) z_2;\ z_1 \in \operatorname{spt} f,\ z_2 \in \operatorname{spt}\rho,\ 0 \leqslant \lambda \leqslant 1\big\} =: E. \tag{16.7}$$

If $\int f(x)\,\mathrm{d}x = 0$, then $\operatorname{div} Bf = f$.

Proof. Let $f \in C_c^\infty(\mathbb{R}^n)$.

(i) First we show that $Bf = 0$ on $\mathbb{R}^n \setminus E$. Note that E is a compact set (since $\operatorname{spt} f$, $\operatorname{spt}\rho$ and $[0,1]$ are compact). Let $x \in \mathbb{R}^n \setminus E$. If $y \in \operatorname{spt} f$ and $r \geqslant 1$ then $y + r(x-y) \notin \operatorname{spt}\rho$ (because $z = y + r(x-y) \in \operatorname{spt}\rho$ would lead to $x = \frac{1}{r} z + (1 - \frac{1}{r}) y \in E$ – a contradiction), and therefore $\rho(y + r(x-y)) = 0$. Hence (16.6) implies $Bf(x) = 0$.

(ii) By the variable transformation $z = x - y$ and then $r = 1 + \frac{s}{|z|}$ in the inner integral we can write Bf in the form

$$Bf(x) = \int f(x-z)\frac{z}{|z|^n}\int_0^\infty \rho\Big(x + s\frac{z}{|z|}\Big)(s+|z|)^{n-1}\,\mathrm{d}s\,\mathrm{d}z.$$

In this form one can differentiate under the integral to obtain $Bf \in C^\infty(\mathbb{R}^n;\mathbb{K}^n)$. (If $R > 0$ is such that ρ, f have their supports in $B(0,R)$, then we know from step (i) that $Bf = 0$ on $\mathbb{R}^n \setminus B(0,R)$, and for $x \in B(0,R)$ we can use a multiple of $(s,z) \mapsto \mathbf{1}_{(0,2R)}(s)\mathbf{1}_{B(0,2R)}(z)|z|^{1-n}$ as a dominating function.)

(iii) Let $f, \varphi \in C_\mathrm{c}^\infty(\mathbb{R}^n)$, $\int f\,\mathrm{d}x = 0$. Note that $\nabla\varphi$ satisfies the compatibility condition, therefore $\nabla A\nabla\varphi = \nabla\varphi$, $\nabla(A\nabla\varphi - \varphi) = 0$, i.e. $A\nabla\varphi - \varphi$ is constant. It follows that $\int f(A\nabla\varphi - \varphi)\,\mathrm{d}x = 0$, and we obtain

$$\int (\operatorname{div} Bf)\varphi\,\mathrm{d}x = -\int Bf\cdot\nabla\varphi\,\mathrm{d}x = \int fA\nabla\varphi\,\mathrm{d}x = \int f\varphi\,\mathrm{d}x.$$

As this holds for all $\varphi \in C_\mathrm{c}^\infty(\mathbb{R}^n)$ we conclude that $\operatorname{div} Bf = f$. □

16.8 Remarks. (a) The formula (16.6) is the **Bogovskiĭ formula**. The linear mapping B in (16.6) as well as its extension obtained in Theorem 16.10 below is the **Bogovskiĭ operator**. (The reader should keep in mind the fact that the Bogovskiĭ operator depends on the function ρ; so the use of the definite article 'the' might be somewhat misleading.)

(b) If $n \geqslant 2$, then there exist vector fields $0 \neq u \in C_\mathrm{c}^\infty(\mathbb{R}^n;\mathbb{K}^n)$ satisfying $\operatorname{div} u = 0$. Therefore the solution of $\operatorname{div} u = f$ obtained by the Bogovskiĭ formula is not unique. For the Bogovskiĭ formula in dimension $n = 1$ we refer to Exercise 16.3.

(c) Let $\Omega \subseteq \mathbb{R}^n$ be an open set containing a ball $B(x^0,r)$ with the property that Ω is star-shaped with respect to each point of $B(x^0,r)$. Let $\rho \in C_\mathrm{c}^\infty(\mathbb{R}^n)_+$ satisfy $\operatorname{spt}\rho \subseteq B(x^0,r)$ and $\int\rho(x)\,\mathrm{d}x = 1$. Then (16.7) implies that $B(C_\mathrm{c}^\infty(\Omega)) \subseteq C_\mathrm{c}^\infty(\Omega;\mathbb{K}^n)$, with the identification $C_\mathrm{c}^\infty(\Omega) = \{\varphi \in C_\mathrm{c}^\infty(\mathbb{R}^n)\,;\,\operatorname{spt}\varphi \subseteq \Omega\}$. In this sense, B is a linear mapping from $C_\mathrm{c}^\infty(\Omega)$ to $C_\mathrm{c}^\infty(\Omega;\mathbb{K}^n)$. Thus, an important consequence of Theorem 16.7 is that for each $\varphi \in C_\mathrm{c}^\infty(\Omega)$ with $\int\varphi(x)\,\mathrm{d}x = 0$ there exists a vector field $\Phi\ (= B\varphi) \in C_\mathrm{c}^\infty(\Omega;\mathbb{K}^n)$ such that $\operatorname{div}\Phi = \varphi$. △

We now show that the last property mentioned in Remark 16.8(c) carries over to more general open sets.

16.9 Proposition. *Let $\Omega \subseteq \mathbb{R}^n$ be a connected open set, and let $\varphi \in C_\mathrm{c}^\infty(\Omega)$ satisfy $\int\varphi(x)\,\mathrm{d}x = 0$. Then there exists a vector field $\Phi \in C_\mathrm{c}^\infty(\Omega;\mathbb{K}^n)$ such that* $\operatorname{div}\Phi = \varphi$.

Proof. (i) In the first step we show the assertion for functions of the special form $\varphi = \hat\varphi - \check\varphi$, with $\check\varphi, \hat\varphi \in C_\mathrm{c}^\infty(\Omega)$, $\int\check\varphi(x)\,\mathrm{d}x = \int\hat\varphi(x)\,\mathrm{d}x$, and $\operatorname{spt}\check\varphi \subseteq \check B$, $\operatorname{spt}\hat\varphi \subseteq \hat B$ for open balls $\check B, \hat B \subseteq \Omega$.

There exist open balls $\check B = B_0, B_1, \ldots, B_m = \hat B \subseteq \Omega$ such that $B_{j-1} \cap B_j \neq \varnothing$ for $j = 1,\ldots,m$. Choose $\varphi_1,\ldots,\varphi_m \in C_\mathrm{c}^\infty(\Omega)$ such that $\int\varphi_j(x)\,\mathrm{d}x = \int\check\varphi(x)\,\mathrm{d}x$ and $\operatorname{spt}\varphi_j \subseteq B_{j-1}\cap B_j$ for $j = 1,\ldots,m$, and put $\varphi_0 := \check\varphi$, $\varphi_{m+1} := \hat\varphi$. By Remark 16.8(c) there exist $\Phi_0,\ldots,\Phi_m \in C_\mathrm{c}^\infty(\Omega;\mathbb{K}^n)$ satisfying $\operatorname{spt}\Phi_j \subseteq B_j$ and $\operatorname{div}\Phi_j = \varphi_{j+1} - \varphi_j$ for $j = 0,\ldots,m$. Then the vector field $\Phi := \sum_{j=0}^m \Phi_j$ is as asserted.

(ii) For the general case we find a covering $(B_j)_{j=1,\dots,m}$ of $\operatorname{spt}\varphi$ by open balls $B_j \subseteq \Omega$. There exists a subordinate partition of unity $(\chi_j)_{j=1,\dots,m}$ in $C_c^\infty(\Omega)_+$ on $\operatorname{spt}\varphi$, i.e. a family of functions with $\operatorname{spt}\chi_j \subseteq B_j$ for all $j \in \{1,\dots,m\}$ and $\sum_{j=1}^m \chi_j(x) = 1$ for all $x \in \operatorname{spt}\varphi$; see Exercise 4.3(b). For $j = 1,\dots,m$ put $\varphi_j := \chi_j\varphi$. Let $B_0 \subseteq \Omega$ be an open ball, and choose $\varphi_0 \in C_c^\infty(\Omega)$ with $\operatorname{spt}\varphi_0 \subseteq B_0$ and $\int \varphi_0(x)\,\mathrm{d}x = 1$. For $1 \leqslant j \leqslant m$ we apply step (i) with $\check\varphi := (\int \varphi_j)\varphi_0$ and $\hat\varphi := \varphi_j$ and obtain $\Phi_j \in C_c^\infty(\Omega;\mathbb{K}^n)$ satisfying $\operatorname{div}\Phi_j = \varphi_j - (\int\varphi_j)\varphi_0$. Then $\Phi := \sum_{j=1}^m \Phi_j$ satisfies $\operatorname{div}\Phi = \big(\sum_{j=1}^m \chi_j\big)\varphi - (\int\varphi)\varphi_0 = \varphi$. □

The next issue is extending the Bogovskiĭ operator to a bounded linear operator $B\colon L_2(\Omega) \to H_0^1(\Omega;\mathbb{K}^n)$, for suitable bounded open Ω. Then the equality $\operatorname{div} Bf = f$ carries over to all $f \in L_2^0(\Omega) := \big\{f \in L_2(\Omega)\,;\, \int f(x)\,\mathrm{d}x = 0\big\}$ because $C_c^\infty \cap L_2^0(\Omega)$ is dense in $L_2^0(\Omega)$; see Exercise 16.4(a). We confine ourselves to a special form of the 'smoothing function' ρ.

16.10 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set containing a ball $B(x^0, r_0)$ with the property that Ω is star-shaped with respect to every point of $B(x^0, r_0)$. Let $\rho_0, \tilde\rho \in C_c^\infty(\mathbb{R}^n)$ be such that $\operatorname{spt}\rho_0 \subseteq B(0, r_0/2)$, $\operatorname{spt}\tilde\rho \subseteq B(x^0, r_0/2)$, $\int \rho_0(x)\,\mathrm{d}x = \int\tilde\rho(x)\,\mathrm{d}x = 1$, and let $\rho := \rho_0 * \tilde\rho$. (Note that $\rho \in C_c^\infty(R^n)$, $\operatorname{spt}\rho \subseteq B(x^0, r_0)$ and $\int\rho(x)\,\mathrm{d}x = 1$.)*

Then $B\colon C_c^\infty(\Omega) \to C_c^\infty(\Omega;\mathbb{K}^n)$, defined in Remark 16.8(c), has a continuous (linear) extension $B\colon L_2(\Omega) \to H_0^1(\Omega;\mathbb{K}^n)$. For all $f \in L_2^0(\Omega)$ one has $\operatorname{div} Bf = f$.

The proof of Theorem 16.10 is delegated to Appendix H. Showing that there exists $c > 0$ such that $\|Bf\|_2 \leqslant c\|f\|_2$ is not the problem; see Exercise 16.5. The hard part is the corresponding estimate for the derivatives of Bf. In order to stay with our philosophy to provide complete information on the treated topics we present the proof of the 'hard part' mentioned above in Appendix H, Section H.3.

Traditionally, for proving Theorem 16.10 one uses the Calderón–Zygmund theory of singular integral operators; see Section H.4 and the Notes of Appendix H. Our hypothesis of the special form of $\rho = \rho_0 * \tilde\rho$ helps avoiding harder parts of the Calderón–Zygmund theory and thereby facilitates a much easier access. The restriction to this special form of the Bogovskiĭ operator has no restricting effect on the remaining results because we really need only *one* operator $B\colon C_c^\infty(\Omega) \to C_c^\infty(\Omega;\mathbb{R}^n)$ with the continuity property and the property that $\operatorname{div} Bf = f$ for all $f \in C_c^\infty(\Omega)$ with $\int f(x)\,\mathrm{d}x = 0$.

An important consequence of Theorem 16.10 is the surjectivity of the mapping $\operatorname{div}\colon H_0^1(\Omega) \to L_2^0(\Omega)$. (Note that $\operatorname{div}(C_c^\infty(\Omega;\mathbb{K}^n)) \subseteq L_2^0(\Omega)$ and thus $\operatorname{div}(H_0^1(\Omega)) \subseteq L_2^0(\Omega)$ for *any* bounded open $\Omega \subseteq \mathbb{R}^n$.) We now show that Theorem 16.10 implies the surjectivity of $\operatorname{div}\colon H_0^1(\Omega) \to L_2^0(\Omega)$ for more general Ω.

16.11 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be a connected bounded open set with Lipschitz boundary. Let $f \in L_2^0(\Omega)$. Then there exists $u \in H_0^1(\Omega;\mathbb{K}^n)$ such that $\operatorname{div} u = f$.*

Proof. (i) It is not difficult to see that for all $x \in \overline\Omega$ there exists an open neighbourhood U_x such that $U_x \cap \Omega$ is star-shaped with respect to the points of a ball in $U_x \cap \Omega$. This is obvious for $x \in \Omega$, and for $x \in \partial\Omega$ it results from the Lipschitz property of $\partial\Omega$. Compactness of $\overline\Omega$ implies that there exists a finite open covering $(\Omega_j)_{j=1,\dots,m}$ of Ω by sets to which Thereom 16.10 can be applied.

(ii) It is not too difficult to show that there exist functions $f_1, \dots, f_m \in L_2^0(\Omega)$ such that $[f_j \neq 0] \subseteq \Omega_j$ for all $j \in \{1, \dots, m\}$ and $f = \sum_{j=1}^m f_j$. (Let us illustrate this for $m = 2$. In that case $\Omega_1 \cap \Omega_2$ is non-empty, so we can choose $g \in L_2(\Omega)$ with $[g \neq 0] \subseteq \Omega_1 \cap \Omega_2$, $\int g \, dx = \int \mathbf{1}_{\Omega_1} f \, dx$, and put $f_1 := \mathbf{1}_{\Omega_1} f - g$, $f_2 := \mathbf{1}_{\Omega_2 \setminus \Omega_1} f + g$. The general case is delegated to Exercise 16.4(b).) Then for all $j \in \{1, \dots, m\}$ there exists $u^j \in H_0^1(\Omega_j; \mathbb{K}^n)$ with $\operatorname{div} u^j = f_j$, and $u := \sum_{j=1}^m u^j$ has the required properties. □

16.4 The hypothesis (H_0)

In this section we show that the hypothesis (H_0) is satisfied for bounded open sets $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary.

16.12 Theorem. *Let Ω be as stated above. Then for any $\eta \in H^1_{0,\sigma}(\Omega; \mathbb{K}^n)^\circ$, i.e. $\eta \in H^{-1}(\Omega)^n$ with $\langle \eta, u \rangle_{H^{-1}, H_0^1} = 0$ for all $u \in H^1_{0,\sigma}(\Omega; \mathbb{K}^n)$, there exists $p \in L_2(\Omega)$ such that $\eta = \nabla p$. In other words, Ω satisfies hypothesis* (H_0).

For the proof we need the following special case of the 'closed range theorem'.

16.13 Proposition. *Let G, H be Hilbert spaces, and let $A \in \mathcal{L}(G, H)$ have closed range. Then* $\operatorname{ran}(A^*)$ *is closed,* $\operatorname{ran}(A^*) = \ker(A)^\perp$.

Proof. The bounded inverse theorem implies that the operator $A_1 := A|_{\ker(A)^\perp} : \ker(A)^\perp \to \operatorname{ran}(A)$ is boundedly invertible. It follows that $A_1^* : \operatorname{ran}(A) \to \ker(A)^\perp$ is boundedly invertible (with $(A_1^*)^{-1} = (A_1^{-1})^*$).

Let $J : \ker(A)^\perp \hookrightarrow G$ be the embedding, and let P be the orthogonal projection from H onto $\operatorname{ran}(A) = \ker(A^*)^\perp$. Then J^* is the orthogonal projection from G onto $\ker(A)^\perp = \overline{\operatorname{ran}(A^*)}$, and $P^* : \ker(A^*)^\perp \hookrightarrow H$ is the embedding; see Exercise 16.6. Since $A_1^* = (PAJ)^* = J^* A^* P^*$, we conclude that $\operatorname{ran}(A^*) = \operatorname{ran}(A_1^*) = \ker(A)^\perp$, and the latter set is closed. □

Proof of Theorem 16.12. Without loss of generality we assume that Ω is connected.

The fundamental observation for the proof is that the bounded linear operators $\operatorname{div} : H_0^1(\Omega; \mathbb{K}^n) \to L_2(\Omega)$ and $\nabla : L_2(\Omega) \to H^{-1}(\Omega)^n$ are negative adjoints of each other. Indeed, for $f \in L_2(\Omega)$, $u \in H_0^1(\Omega; \mathbb{K}^n)$ one has

$$(f \mid \operatorname{div} u) = \sum_{j=1}^n \int f \overline{\partial_j u_j} \, dx = -\sum_{j=1}^n \langle \partial_j f, u_j \rangle_{H^{-1}, H_0^1} = -\langle \nabla f, u \rangle_{H^{-1}, H_0^1} .$$

We know from Lemma 6.8 that $\operatorname{ran}(\nabla)^\circ = \ker(\operatorname{div})$, and this implies $\overline{\operatorname{ran}(\nabla)} = \ker(\operatorname{div})^\circ$. (Note that for this argument no special properties of Ω are required.) It is shown in Theorem 16.11 that $\operatorname{ran}(\operatorname{div}) = L_2^0(\Omega)$, which is a closed subspace of $L_2(\Omega)$. Therefore Proposition 16.13 implies that $\operatorname{ran}(\nabla) = \ker(\operatorname{div})^\circ$. As $\ker(\operatorname{div}) = H^1_{0,\sigma}(\Omega; \mathbb{K}^n)$ by definition, we obtain the assertion of the theorem. □

16.5 Supplement: the space $H^1_{\sigma,0}(\Omega;\mathbb{K}^n)$ and the hypothesis $(\mathrm{H_c})$

The final issue of the chapter will be to investigate a slightly stronger version of hypothesis (H_0). Using the space $H^1_{\sigma,0}(\Omega;\mathbb{K}^n) := \overline{C^\infty_{\mathrm{c},\sigma}(\Omega;\mathbb{K}^n)}^{H^1_0}$, where

$$C^\infty_{\mathrm{c},\sigma}(\Omega;\mathbb{K}^n) := \{u \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{K}^n);\ \operatorname{div} u = 0\} = C^\infty_{\mathrm{c}}(\Omega;\mathbb{K}^n) \cap H^1_{0,\sigma}(\Omega;\mathbb{K}^n),$$

we can formulate it as follows:

$$\text{for all } \eta \in H^1_{\sigma,0}(\Omega;\mathbb{K}^n)^\circ \text{ there exists } p \in L_2(\Omega) \text{ with } \eta = \nabla p. \tag{$\mathrm{H_c}$}$$

The space $H^1_{\sigma,0}(\Omega;\mathbb{K}^n)$ is a closed subspace of $H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$, and $(\mathrm{H_c})$ can be rephrased as the property that $H^1_{\sigma,0}(\Omega;\mathbb{K}^n)^\circ = \operatorname{ran}(\nabla)$, where $\nabla\colon L_2(\Omega) \to H^{-1}(\Omega)^n,\ f \mapsto \nabla f$.

We insert the important observation that for any open set $\Omega \subseteq \mathbb{R}^n$ one has the inclusions $\operatorname{ran}(\nabla) \subseteq H^1_{0,\sigma}(\Omega;\mathbb{K}^n)^\circ \subseteq H^1_{\sigma,0}(\Omega;\mathbb{K}^n)^\circ$ $(= C^\infty_{\mathrm{c},\sigma}(\Omega;\mathbb{K}^n)^\circ)$. Recall that (H_0) is equivalent to $H^1_{0,\sigma}(\Omega;\mathbb{K}^n)^\circ = \operatorname{ran}(\nabla)$. Summarising, property $(\mathrm{H_c})$ is equivalent to (H_0) together with the equality $H^1_{\sigma,0}(\Omega;\mathbb{K}^n) = H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$, and this equality holds if and only if $C^\infty_{\mathrm{c},\sigma}(\Omega;\mathbb{K}^n)$ is dense in $H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$.

The following theorem strengthens Theorem 16.12.

16.14 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then $(\mathrm{H_c})$ is satisfied, and $H^1_{\sigma,0}(\Omega;\mathbb{K}^n) = H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$, i.e. $C^\infty_{\mathrm{c},\sigma}(\Omega;\mathbb{K}^n)$ is dense in $H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$.*

Proof. (i) Assume additionally that Ω is star-shaped with respect to some ball $B(x^0,r) \subseteq \Omega$. Without loss of generality assume that $x^0 = 0$. Then one can easily see that $\lambda\overline{\Omega} \subseteq \Omega$ for all $\lambda \in (0,1)$. Let $u \in H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$. Extend u to $\mathbb{R}^n$ by 0; then it follows from Exercise 4.6 that $\operatorname{div} u = 0$ for the extended function as well. For $\lambda \in (0,1)$ put $u_\lambda := u(\lambda^{-1}\cdot)$; then $\operatorname{spt} u_\lambda \subseteq \lambda\overline{\Omega}$ is a compact subset of Ω. If $(\rho_k)_{k\in\mathbb{N}}$ is a δ-sequence in $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$, then one concludes that $\operatorname{div}(\rho_k * u_\lambda) = 0$ for all $k \in \mathbb{N}$, $\rho_k * u_\lambda \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{K}^n)$ for large k, and $\rho_k * u_\lambda \to u_\lambda$ in $H^1_0(\Omega;\mathbb{K}^n)$ as $k \to \infty$. As a consequence, $u_\lambda \in H^1_{\sigma,0}(\Omega;\mathbb{K}^n)$ for all $\lambda \in (0,1)$. Now $u_\lambda \to u$ in $H^1_0(\Omega;\mathbb{K}^n)$ as $\lambda \to 1$; hence $u \in H^1_{\sigma,0}(\Omega;\mathbb{K}^n)$.

So we have shown that $H^1_{\sigma,0}(\Omega;\mathbb{K}^n) = H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$. This implies $(\mathrm{H_c})$ since (H_0) is satisfied by Theorem 16.12.

(ii) Without loss of generality we assume that Ω is connected. From the proof of Theorem 16.11 we recall that there exists an open covering $(\Omega_j)_{j=1,\dots,m}$ of Ω for which each Ω_j is star-shaped with respect to a ball.

Let $\eta \in H^1_{\sigma,0}(\Omega;\mathbb{K}^n)^\circ$. Then clearly $\eta^j := \eta|_{H^1_0(\Omega_j;\mathbb{K}^n)} \in H^1_{\sigma,0}(\Omega_j;\mathbb{K}^n)^\circ$, and from step (i) we conclude that there exists $f_j \in L_2(\Omega_j)$ (unique up to a constant) such that $\nabla f_j = \eta^j$, for all $j = 1,\dots,m$. We are going to show that one can choose versions of the f_j such that the family $(f_j)_{j=1,\dots,m}$ is consistent, in the sense that $f_j = f_k$ on $\Omega_j \cap \Omega_k$ for all $j,k = 1,\dots,m$.

For all $j = 1,\dots,m$ we choose a function $\varphi_j \in C^\infty_{\mathrm{c}}(\Omega_j)_+$ with $\int_{\Omega_j} \varphi_j(x)\,\mathrm{d}x = 1$. For each $j \in \{1,\dots,m\}$ there exists a function $\Phi_j \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{K}^n)$ such that $\operatorname{div}\Phi_j = \varphi_1 - \varphi_j$; see Proposition 16.9. With these functions we choose the version of f_j with $(f_j \,|\, \varphi_j) = \langle \eta, \Phi_j\rangle$,

for all $j = 1, \dots, m$. Given $j, k \in \{1, \dots, m\}$ we now show that $f_j = f_k$ on $\Omega_j \cap \Omega_k$. Let $\varphi \in C_c^\infty(\Omega_j \cap \Omega_k)$, $\int \varphi(x)\,\mathrm{d}x = 1$; then by Proposition 16.9 we can find $\Phi_{\varphi,j} \in C_c^\infty(\Omega_j; \mathbb{K}^n)$ such that $\operatorname{div} \Phi_{\varphi,j} = \varphi - \varphi_j$, and similarly $\Phi_{\varphi,k} \in C_c^\infty(\Omega_k; \mathbb{K}^n)$ with $\operatorname{div} \Phi_{\varphi,k} = \varphi - \varphi_k$. Then

$$(f_j \,|\, \varphi) = (f_j \,|\, \operatorname{div} \Phi_{\varphi,j}) + (f_j \,|\, \varphi_j) = -\langle \nabla f_j, \Phi_{\varphi,j} \rangle + \langle \eta, \Phi_j \rangle = \langle \eta, \Phi_j - \Phi_{\varphi,j} \rangle.$$

Note that η belonging to $H^1_{\sigma,0}(\Omega; \mathbb{K}^n)^\circ$ means that $\langle \eta, \Phi \rangle$ depends only on the divergence of $\Phi \in C_c^\infty(\Omega; \mathbb{K}^n)$. Because of $\operatorname{div}(\Phi_j - \Phi_{\varphi,j}) = (\varphi_1 - \varphi_j) - (\varphi - \varphi_j) = \varphi_1 - \varphi = \operatorname{div}(\Phi_k - \Phi_{\varphi,k})$ we thus obtain $(f_j \,|\, \varphi) = (f_k \,|\, \varphi)$. It follows that $(f_j \,|\, \varphi) = (f_k \,|\, \varphi)$ for all $\varphi \in C_c^\infty(\Omega_j \cap \Omega_k)$, the consistency of the family $(f_j)_{j=1,\dots,n}$ is established, and the function f, $f|_{\Omega_j} := f_j$ for $j = 1, \dots, m$, is a well-defined function in $L_2(\Omega)$ satisfying $\nabla f = \eta$. □

The fundamental problem in the proof of Theorem 16.14 is that one cannot simply approximate elements of $H^1_{0,\sigma}(\Omega; \mathbb{K}^n)$ by convolution with a δ-sequence. We refer to [ACM15; proof of Theorem 4.1] for another approach to dealing with this problem.

16.15 Remarks. In these remarks we sketch how Theorems 16.12 and 16.14 are obtained in [Tem77; Chap. I, § 1].

(a) The basis is Nečas' inequality, stating that there exists $c > 0$ such that

$$\|f\|_{L_2} \leqslant c\Big(\sum_{j=1}^n \|\partial_j f\|_{H^{-1}} + \|f\|_{H^{-1}}\Big) \qquad (f \in L_2(\Omega)),$$

if Ω has Lipschitz boundary, asserted in [Neč12; Lemma 7.1, p. 186]. Using the compactness of the embedding $H^1_0(\Omega) \hookrightarrow L_2(\Omega)$ one can show that this inequality implies

$$\|f\|_2 \leqslant c \sum_{j=1}^n \|\partial_j f\|_{H^{-1}} \qquad \Big(f \in L_2(\Omega) \text{ with } \int_\Omega f(x)\,\mathrm{d}x = 0\Big) \tag{16.8}$$

if Ω is bounded and has Lipschitz boundary. From (16.8) one concludes that $\operatorname{ran}(\nabla)$ is closed (where $\nabla\colon L_2(\Omega) \to H^{-1}(\Omega)^n$); see Exercise 16.7. As $\operatorname{ran}(\nabla)$ is also dense in $H^1_{0,\sigma}(\Omega; \mathbb{K}^n)^\circ$, Theorem 16.12 is proved.

(b) Theorem 16.14 is also derived in [Tem77; Chap. I, Remark 1.4]. The argument there is based on a stronger version of (16.8), in which f is not a priori in $L_2(\Omega)$. The source [Neč66] for this version, cited in [Tem77], was not available to us. △

Notes

Our introduction and presentation of the Stokes operator follows [Mon06] and [ArEl12a]. As in several of the previous chapters, properties of Sobolev spaces play an important role – for the Stokes operator we need Sobolev spaces of divergence free vector fields. These spaces are also of importance for the treatment of the Navier–Stokes equation.

The treatment given for the space $H^1_{0,\sigma}(\Omega; \mathbb{K}^n)$ in Section 16.3 can be found to a large part in [Gal11; Section III.3]. The *derivation* of the Bogovskiĭ formula presented at

the beginning of Section 16.3 was found by the authors, and the same holds for the proof of Theorem 16.14. We mention that the Bogovskiĭ operator is also treated in the L_p-context in [Gal11] and in Sobolev spaces of negative order in [GHH06]. The treatment in Section 16.4 uses ideas contained in [Tem77; Chap. I, § 1].

In [ACM15; Theorem 4.1] one can find an account of various important relations between spaces arising in connection with the Stokes operator and the Navier–Stokes equation.

Exercises

16.1 Let $\Omega \subseteq \mathbb{R}^n$ be open.

(a) Show that the mapping $I - \Delta\colon H^1_0(\Omega) \to H^{-1}(\Omega)$ is an isometric isomorphism.

(b) Assume additionally that Ω is bounded, and provide $H^1_0(\Omega)$ with the scalar product $(u, v) \mapsto \int \nabla u \cdot \overline{\nabla v}\, \mathrm{d}x$. Show that $-\Delta\colon H^1_0(\Omega) \to H^{-1}(\Omega)$ is an isometric isomorphism.

16.2 (a) Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $u\colon \Omega \to \mathbb{K}^n$ be a continuous vector field. Show that the following properties are equivalent:

(i) u is a gradient field, $u = \nabla p$ for some $p \in C^1(\Omega)$.

(ii) u satisfies (16.2), i.e., $\int u \cdot \varphi\, \mathrm{d}x = 0$ for all $\varphi \in C^1_{\mathrm{c}}(\Omega; \mathbb{K}^n)$ with $\operatorname{div} \varphi = 0$.

(iii) u is conservative, i.e., $\int_\gamma u = \int_0^1 u(\gamma(t)) \cdot \gamma'(t)\, \mathrm{d}t = 0$ for all piecewise continuously differentiable closed paths $\gamma\colon [0, 1] \to \Omega$.

Hint concerning '(ii) ⇒ (iii)': Let (ρ_k) be a δ-sequence in $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$. Show that $\varphi_k(y) := \int_0^1 \rho_k(\gamma(t) - y)\gamma'(t)\, \mathrm{d}t$ defines a divergence free element of $C^\infty_{\mathrm{c}}(\Omega; \mathbb{K}^n)$, for k large enough, and that $\int u \cdot \varphi_k\, \mathrm{d}x \to \int_\gamma u$ for all $u \in C(\Omega; \mathbb{K}^n)$.

Hint concerning '(iii) ⇒ (i)': Define p as in Remark 16.5(a).

Supplemental comment: The sequence (φ_k) converges in a suitable sense to a divergence free distributional vector field supported on $\operatorname{im} \gamma = \{\gamma(t);\ 0 \leqslant t \leqslant 1\}$. The limiting distributional vector field is given by $C^\infty_{\mathrm{c}}(\Omega; \mathbb{K}^n) \ni \psi \mapsto \int_0^1 \psi(\gamma(t)) \cdot \gamma'(t)\, \mathrm{d}t \in \mathbb{K}$.

(b) Show that the vector field $u\colon \mathbb{R}^2 \times \{0\} \to \mathbb{R}^2$, $u(x_1, x_2) := \left(\dfrac{-x_2}{x_1^2 + x_2^2}, \dfrac{x_1}{x_1^2 + x_2^2} \right)$, satisfies the compatibility condition $\partial_1 u_2 = \partial_2 u_1$. Show that u has a potential on $(0, \infty) \times \mathbb{R}$, given by $p(x_1, x_2) := \arctan \frac{x_2}{x_1}$. Extend this potential to $\mathbb{R}^2 \setminus (-\infty, 0] \times \{0\}$, and show that it cannot be extended to $\mathbb{R}^2 \setminus \{0\}$.

16.3 Find a simple form of the Bogovskiĭ formula in space dimension $n = 1$. In this form it should be immediately visible that Bf does not depend on ρ for f with $\int f(x)\, \mathrm{d}x = 0$, whereas Bf depends on ρ if $\int f(x)\, \mathrm{d}x \neq 0$. (Hint: Use the expression for the Bogovskiĭ operator indicated in the proof of Theorem 16.7, step (ii).)

16.4 Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set.

(a) Show that $L^0_2(\Omega)$ is a closed subspace of $L_2(\Omega)$ and that $C^\infty_{\mathrm{c}} \cap L^0_2(\Omega)$ is dense in $L^0_2(\Omega)$. (Hint: For the denseness show first that $L^0_{2,\mathrm{c}}(\Omega) := \{f \in L^0_2(\Omega);\ \operatorname{spt} f \text{ compact}\}$ is dense in $L^0_2(\Omega)$.)

(b) Assume additionally that Ω is connected. Let $(\Omega_j)_{j=1,\dots,m}$ be a finite open covering of Ω. Let $f \in L^0_2(\Omega)$. Show that there exist functions $f_1, \dots, f_m \in L^0_2(\Omega)$, $[f_j \neq 0] \subseteq \Omega_j$ for all $j = 1, \dots, m$, such that $f = \sum_{j=1}^m f_j$.

16.5 Let Ω be as in Theorem 16.10. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be such that $\operatorname{spt}\rho \subseteq B(x^0, r_0)$, $\int \rho(x)\,\mathrm{d}x = 1$, and let the Bogovskiĭ operator B be as in (16.6). Show that there exists $c > 0$ such that $\|Bf\|_2 \leqslant c\|f\|_2$ for all $f \in C_c^\infty \cap L_2^0(\Omega)$.

Hint: Recall the dominating function indicated in the proof of Theorem 16.7. Then either use Proposition 4.3(b), or else show an L_1-L_1 bound and an L_∞-L_∞ bound for B, and then use Riesz–Thorin.

16.6 Let H be a Hilbert space. Let $H_0 \subseteq H$ be a closed subspace, $J\colon H_0 \hookrightarrow H$ the embedding. Show that J^* is the orthogonal projection from H onto H_0.

16.7 Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Show that the inequality (16.8) implies that $\operatorname{ran}(\nabla)$ is closed, for the mapping $\nabla\colon L_2(\Omega) \to H^{-1}(\Omega)^n$.

Chapter 17

Non-autonomous equations

So far we have considered forms that do not depend on time. The aim of this chapter is to establish well-posedness of the non-autonomous problem

$$u'(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0\,, \tag{17.1}$$

where each operator $\mathcal{A}(t)$ is associated with a form $a(t,\cdot,\cdot)$, defined on a t-independent Hilbert space $V \overset{\mathrm{d}}{\hookrightarrow} H$. In contrast to previous results, the existence of solutions will be investigated in the antidual V^*, which will be considered as a superspace of H. In fact, the situation is slightly more delicate and interesting: the solution will take its values in V, but the equation will be solved only in V^*.

We begin this chapter by describing the setup for V^*. Sections 17.2 and 17.3 are devoted to the Bochner integral for Hilbert space valued functions and to Hilbert space valued Sobolev spaces. Of special interest are the 'maximal regularity spaces', certain mixed Sobolev spaces, and their properties. In Section 17.4 we present the important representation theorem of J.-L. Lions, which extends the Lax–Milgram lemma. Lions' representation theorem is the key for the elegant treatment of (17.1), in the last section.

17.1 Gelfand triples

Let V, H be Hilbert spaces, $V \overset{\mathrm{d}}{\hookrightarrow} H$. We have frequently encountered this situation in previous chapters; typical examples are $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$ or $H^1(\Omega) \hookrightarrow L_2(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is open. In Section 16.1 we have introduced the Gelfand triple corresponding to the embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$; here we treat the general setup.

For $x \in H$ we define $\langle x, \cdot\rangle \in V^*$ by

$$\langle x, u\rangle = \langle x, u\rangle_{V^*,V} := (x \,|\, u)_H \qquad (u \in V).$$

It is not difficult to see that the mapping $H \ni x \mapsto \langle x, \cdot\rangle \in V^*$ belongs to $\mathcal{L}(H, V^*)$ and is injective. It also has dense range because $u \in V$, $\langle x, u\rangle = 0$ for all $x \in H$ implies $u = 0$; see Exercise 17.1. Incidentally, the mapping $x \mapsto \langle x, \cdot\rangle$ coincides with the mapping $k \in \mathcal{L}(H, V^*)$ used in Proposition 5.7. We will identify $\langle x, \cdot\rangle$ and x and thereby consider H as a subspace of V^*. Thus we now have a triple of injected spaces

$$V \overset{\mathrm{d}}{\hookrightarrow} H \overset{\mathrm{d}}{\hookrightarrow} V^*,$$

called a **Gelfand triple**. We illustrate this situation in the following remark.

17.1 Remarks. (a) Let (Ω, μ) be a measure space, $m\colon \Omega \to [\delta, \infty)$ a measurable function, where $\delta > 0$, and let $H := L_2(\Omega, \mu)$, $V := L_2(\Omega, m\mu)$. Then $V \overset{\mathrm{d}}{\hookrightarrow} H$; for the denseness of the embedding we refer to Example 5.9. We can identify V^* with $L_2(\Omega, \frac{1}{m}\mu)$, where the duality is given by

$$\langle f, u\rangle = \int_\Omega f\overline{u}\,\mathrm{d}\mu \qquad \big(f \in L_2(\Omega, \tfrac{1}{m}\mu),\ u \in V\big).$$

One can easily see this by using the fact that $\Phi u := mu$ defines an isometric isomorphism $\Phi\colon L_2(\Omega; m\mu) \to L_2(\Omega; \frac{1}{m}\mu)$.

(b) In fact, the situation described in part (a) is generic. This is a consequence of the spectral theorem for self-adjoint operators, Theorem 13.20; we refer to Exercise 17.2(a) for more information. △

Now let $a\colon V \times V \to \mathbb{K}$ be a bounded form. Recall from (5.2) the corresponding Lax–Milgram operator $\mathcal{A} \in \mathcal{L}(V, V^*)$, given by $\langle \mathcal{A}u, v\rangle = a(u, v)$. Let A be the operator in H associated with the form a; see Proposition 5.5. From the definition it is immediate that A is the part of $\mathcal{A}$ in H, i.e.

$$A = \mathcal{A} \cap (V \times H),$$

with $V \subseteq H \subseteq V^*$ considered as subspaces. If a is H-coercive, then we know that $-A$ generates a holomorphic C_0-semigroup on H. One can show that $-\mathcal{A}$, considered as an operator in V^*, generates a holomorphic C_0-semigroup on V^*. We will not pursue this issue, but we recall from Remark 3.11(b) that as a consequence one can solve the initial value problem

$$u' + \mathcal{A}u = 0, \quad u(0) = u_0 \in V^*$$

in V^*. Our aim in the present chapter is to study the *inhomogeneous* Cauchy problem

$$u' + \mathcal{A}u = f, \quad u(0) = u_0\,,$$

where $\mathcal{A}$ *depends on time*; see Section 17.5.

17.2 Interlude: The Bochner integral for Hilbert space valued functions

There is a general theory extending the Lebesgue integral to Banach space valued functions (the Bochner integral, see [ABHN11; Section 1.1]). On separable Hilbert spaces one may use a more elementary approach, which we will present here (cf. [ArUr23; Section 8.5]).

Let H be a separable Hilbert space, and let $-\infty \leqslant a < b \leqslant \infty$. A function $f\colon (a, b) \to H$ is called **measurable** if $(f(\cdot)\,|\,x)$ is measurable for all $x \in H$. If (x_n) is a dense sequence in the unit ball $B_H(0, 1)$, then $\|f(t)\| = \sup_{n\in\mathbb{N}} |(f(t)\,|\,x_n)|$ for all $t \in (a, b)$, and this implies that $\|f(\cdot)\|\colon (a, b) \to \mathbb{R}$ is measurable. We define

$$L_1(a, b; H) := \Big\{f\colon (a, b) \to H;\ f \text{ measurable},\ \int_a^b \|f(t)\|\,\mathrm{d}t < \infty\Big\},$$

where the elements of $L_1(a, b; H)$ are to be understood as equivalence classes of a.e. equal functions.

17.2 Lemma. *Let $f \in L_1(a,b;H)$. Then there exists a unique $x \in H$ such that*

$$\int_a^b (f(t)\,|\,y)_H \,\mathrm{d}t = (x\,|\,y)_H \qquad (y \in H),$$

and we define $\int_a^b f(t)\,\mathrm{d}t := x$. The mapping $L_1(a,b;H) \ni f \mapsto \int_a^b f(t)\,\mathrm{d}t \in H$ is linear and continuous, $\big\|\int_a^b f(t)\,\mathrm{d}t\big\| \leqslant \int_a^b \|f(t)\|\,\mathrm{d}t$ for all $f \in L_1(a,b;H)$.

Lemma 17.2 is an easy consequence of the theorem of Fréchet–Riesz. The above definition of the integral is consistent with the definition given in Subsection 1.3.2, by Theorem 1.8(a). As in that theorem, bounded linear operators commute with integration; see Exercise 17.3.

We also introduce the space

$$L_2(a,b;H) := \Big\{ f\colon (a,b) \to H\,;\ f \text{ measurable},\ \int_a^b \|f(t)\|^2\,\mathrm{d}t < \infty \Big\},$$

again identifying a.e. equal functions. Note that $L_2(a,b;H) \subseteq L_1(a,b;H)$ if (a,b) is a bounded interval.

17.3 Proposition. *The space $L_2(a,b;H)$ is a Hilbert space for the scalar product*

$$(f\,|\,g)_{L_2(a,b;H)} := \int_a^b (f(t)\,|\,g(t))_H \,\mathrm{d}t.$$

In order to see that the function $t \mapsto (f(t)\,|\,g(t))_H$ in Proposition 17.3 is measurable, we first show the following denseness property.

17.4 Lemma. *For each $f \in L_2(a,b;H)$ there exists a sequence (f_n) in*

$$L_2(a,b) \otimes H := \operatorname{lin}\big\{\varphi(\cdot)x\,;\ \varphi \in L_2(a,b),\ x \in H\big\}$$

such that $\|f_n(\cdot)\| \leqslant \|f(\cdot)\|$ for all $n \in \mathbb{N}$ and $f_n \to f$ a.e.

In particular, $L_2(a,b) \otimes H$ is dense in $L_2(a,b;H)$, and $L_2(a,b;H)$ is separable.

Proof. The assertions are obvious if $\dim H < \infty$, so we assume that H is infinite-dimensional. Let $f \in L_2(a,b;H)$. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of H. Then, with $f_n := \sum_{k=1}^n (f(\cdot)\,|\,e_k)e_k$, the sequence (f_n) has the required properties. The dominated convergence theorem implies that $f_n \to f$ in $L_2(a,b;H)$. □

Proof of Proposition 17.3. We first show that $t \mapsto (f(t)\,|\,g(t))_H$ is measurable. If $h \in L_2(a,b) \otimes H$, $h = \sum_{j=1}^n \varphi_j(\cdot)x_j$, then $(h(\cdot)\,|\,g(\cdot))_H = \sum_{j=1}^n \varphi_j(\cdot)(x_j\,|\,g(\cdot))_H$ is measurable. Every $f \in L_2(a,b;H)$ can be approximated pointwise a.e. by a sequence in $L_2(a,b) \otimes H$, by Lemma 17.4, and the measurability of $(f(\cdot)\,|\,g(\cdot))_H$ follows.

The completeness is proved in the same way as the completeness of the scalar-valued $L_2(a,b)$. (Show that every absolutely convergent series is convergent.) □

A nice and important application of Lemma 17.4 is the following fact which will be needed in the next section.

17.5 Lemma. *Let $u \in L_2(\mathbb{R}; H)$. Then the function $\mathbb{R} \ni \tau \mapsto u(\cdot - \tau) \in L_2(\mathbb{R}; H)$ is continuous.*

Proof. The assertion is obvious if $u \in C_c(\mathbb{R}) \otimes H = \operatorname{lin}\{\varphi(\cdot)x;\ \varphi \in C_c(\mathbb{R}),\ x \in H\}$. By Lemma 17.4 and the denseness of $C_c(\mathbb{R})$ in $L_2(\mathbb{R})$ there exists a sequence (u_k) in $C_c(\mathbb{R}) \otimes H$ such that $u_k \to u$ in $L_2(\mathbb{R}; H)$ as $k \to \infty$. Then $u_k(\cdot - \tau) \to u(\cdot - \tau)$ in $L_2(\mathbb{R}; H)$ as $k \to \infty$, uniformly for $\tau \in \mathbb{R}$. This implies the assertion. □

17.3 Vector-valued Sobolev spaces

We now define Hilbert space valued Sobolev spaces. As before, let H be a separable Hilbert space.

Let $-\infty \leqslant a < b \leqslant \infty$. As in the scalar case, given $u \in L_2(a, b; H)$, a function $u' \in L_2(a, b; H)$ is called a **weak derivative** of u if

$$-\int_a^b u(t)\varphi'(t)\,\mathrm{d}t = \int_a^b u'(t)\varphi(t)\,\mathrm{d}t \qquad (\varphi \in C_c^\infty(a, b)).$$

Such a weak derivative is unique whenever it exists; see Exercise 17.4(a). We define the **Sobolev space**

$$H^1(a, b; H) := \{u \in L_2(a, b; H);\ u \text{ has a weak derivative } u' \text{ in } L_2(a, b; H)\}.$$

It is easy to see that $H^1(a, b; H)$ is a separable Hilbert space for the scalar product

$$(u \mid v)_{H^1} = \int_a^b \big((u(t) \mid v(t))_H + (u'(t) \mid v'(t))_H\big)\,\mathrm{d}t$$

(cf. Theorem 4.9 and Proposition 17.3).

For the remainder of this section we will assume throughout that $-\infty < a < b < \infty$. We state the following basic result concerning $H^1(a, b; H)$ without proof. Parts (a) and (b) can be shown essentially as Proposition 4.7, and part (c) is analogous to Theorem 4.11.

17.6 Proposition. (a) *Let $v \in L_2(a, b; H)$, $u_0 \in H$, and put $u(t) := u_0 + \int_a^t v(s)\,\mathrm{d}s$ $(t \in (a, b))$. Then $u \in H^1(a, b; H)$ and $u' = v$.*

(b) *Conversely, let $u \in H^1(a, b; H)$. Then there exists $u_0 \in H$ such that*

$$u(t) = u_0 + \int_a^t u'(s)\,\mathrm{d}s \qquad (\text{a.e. } t \in (a, b)).$$

In particular, each function $u \in H^1(a, b; H)$ has a representative in $C([a, b]; H)$, and with this representative one has $\int_a^b u'(s)\,\mathrm{d}s = u(b) - u(a)$.

(c) *The embedding $H^1(a, b; H) \hookrightarrow C([a, b]; H)$ thus defined is continuous.*

We will always use the continuous representative for functions $u \in H^1(a, b; H)$.

17.7 Remark. We note the following product rule for differentiation: if $u \in H^1(a,b;H)$ and $\varphi \in C^\infty[a,b]$, then $\varphi u \in H^1(a,b;H)$, and $(\varphi u)' = \varphi' u + \varphi u'$.

Indeed, for $\psi \in C_c^\infty(a,b)$ one has

$$\begin{aligned}\int_a^b (\varphi u)\psi' \,\mathrm{d}t &= \int_a^b (\varphi\psi)' u \,\mathrm{d}t - \int_a^b \varphi'\psi u \,\mathrm{d}t \\ &= -\int_a^b \varphi\psi u' \,\mathrm{d}t - \int_a^b \varphi' u\psi \,\mathrm{d}t = -\int_a^b (\varphi u' + \varphi' u)\psi \,\mathrm{d}t. \qquad \triangle\end{aligned}$$

Next we suppose that V is a separable Hilbert space, $V \overset{\mathrm{d}}{\hookrightarrow} H$, and we identify H with a dense subspace of V^* as in Section 17.1. The mixed Sobolev space

$$MR(a,b;V,V^*) := L_2(a,b;V) \cap H^1(a,b;V^*),$$

with $-\infty < a < b < \infty$ as before, plays an important role for evolutionary problems. Here the symbol "*MR*" stands for "maximal regularity". It is easy to see that $MR(a,b;V,V^*)$ is a separable Hilbert space for the norm

$$\|u\|_{MR(a,b;V,V^*)} := \big(\|u\|^2_{L_2(a,b;V)} + \|u'\|^2_{L_2(a,b;V^*)}\big)^{1/2}.$$

By Proposition 17.6(b) each $u \in MR(a,b;V,V^*)$ has a representative $u \in C([a,b];V^*)$. We will see in Theorem 17.9 below that in fact the representative is even continuous with values in the smaller space H. In the statement of this result we will use the scalar **Sobolev space**

$$W_1^1(a,b) := \big\{u \in L_1(a,b);\ u' = \partial u \in L_1(a,b)\big\},$$

with norm

$$\|u\|_{W_1^1(a,b)} := \|u\|_{L_1(a,b)} + \|u'\|_{L_1(a,b)} \qquad (u \in W_1^1(a,b)).$$

17.8 Proposition. *Every $u \in W_1^1(a,b)$ has a representative in $C[a,b]$, and the embedding $W_1^1(a,b) \hookrightarrow C[a,b]$ thus defined is continuous; explicitly,*

$$\|u\|_{C[a,b]} \leqslant \frac{1}{b-a}\|u\|_{L_1(a,b)} + \|u'\|_{L_1(a,b)}\,.$$

The proof is the same as for Theorem 4.11; the inequality is on the first line of (4.3). Proposition 4.7 implies that, choosing the continuous representative of $u \in W_1^1(a,b)$, one has the formula

$$u(t) = u(a) + \int_a^t u'(s)\,\mathrm{d}s \qquad (t \in [a,b]). \tag{17.2}$$

For the following theorem, concerning the regularity of elements in $MR(a,b;V,V^*)$, recall the embeddings $V \hookrightarrow H \hookrightarrow V^*$ and the dual pairing $\langle\cdot,\cdot\rangle$ from Section 17.1.

17.9 Theorem. (a) *If $u \in MR(a,b;V,V^*)$, then the function $\|u(\cdot)\|_H^2$ belongs to $W_1^1(a,b)$, and*

$$\big(\|u(\cdot)\|_H^2\big)' = 2\,\mathrm{Re}\langle u'(\cdot), u(\cdot)\rangle. \tag{17.3}$$

(b) *One has $MR(a,b;V,V^*) \hookrightarrow C([a,b];H)$.*

Note that the function $t \mapsto f(t) := \langle u'(t), u(t)\rangle$ in part (b) is measurable; this is shown as in the proof of Proposition 17.3. Since $|f(t)| \leqslant \|u'(t)\|_{V^*}\|u(t)\|_V$ for all $t \in (a,b)$, the Cauchy–Schwarz inequality then implies that $f \in L_1(a,b)$.

For the proof of Theorem 17.9 we need the following denseness property. The result is analogous to Theorem 7.7.

17.10 Proposition. *The space $C^\infty([a,b];V)$ is dense in $MR(a,b;V,V^*)$.*

Proof. (i) Let $\alpha \in C^\infty(\mathbb{R})$ be such that $\alpha = 1$ in a neighbourhood of $(-\infty, a]$, $\alpha = 0$ in a neighbourhood of $[b,\infty)$. For $\tau > 0$ put $u_\tau := (\alpha u)(\cdot + \tau) + \big((1-\alpha)u\big)(\cdot - \tau)$. Then $u_\tau \to u$ in $MR(a,b;V,V^*)$ as $\tau \to 0$, by Lemma 17.5. (For the derivative of u_τ, we note that $(u_\tau)' = (\alpha u)'(\cdot+\tau) + \big((1-\alpha)u\big)'(\cdot - \tau)$, by the choice of α.) For fixed $\tau > 0$ we choose $\psi \in C_c^\infty(\mathbb{R})$ with $\operatorname{spt}\psi \subseteq (a-\tau, b+\tau)$ and $\psi = 1$ on (a,b); then $\psi u_\tau \in MR(\mathbb{R};V,V^*)$. (For convenience we use the notation $MR(\mathbb{R};V,V^*) := L_2(\mathbb{R};V) \cap H^1(\mathbb{R};V^*)$, although MR was only defined for bounded intervals (a,b).)

(ii) Now it suffices to approximate $v \in MR(\mathbb{R};V,V^*)$ by a sequence in $C^\infty(\mathbb{R};V) \cap MR(\mathbb{R};V,V^*)$. Clearly $MR(\mathbb{R};V,V^*)$ is invariant under translations, and Lemma 17.5 implies that $\mathbb{R} \ni t \mapsto v(\cdot - t) \in MR(\mathbb{R};V,V^*)$ is continuous. Let ρ_k be a δ-sequence in $C_c^\infty(\mathbb{R})$. For $k \in \mathbb{N}$ we define $v_k := \int_{\mathbb{R}} \rho_k(s) v(\cdot - s)\,\mathrm{d}s \in MR(\mathbb{R};V,V^*)$. Then

$$\begin{aligned}\|v_k - v\|_{MR(\mathbb{R};V,V^*)} &\leqslant \int_{\mathbb{R}} \rho_k(s)\|v(\cdot - s) - v\|_{MR(\mathbb{R};V,V^*)}\,\mathrm{d}s \\ &\leqslant \sup\big\{\|v(\cdot - s) - v\|_{MR(\mathbb{R};V,V^*)};\ |s| \leqslant 1/k\big\} \to 0 \qquad (k \to \infty).\end{aligned}$$

For $t \in \mathbb{R}$, Proposition 17.6(c) implies that point evaluation $MR(\mathbb{R};V,V^*) \ni u \mapsto u(t) \in V^*$ is a bounded linear operator. Hence, using Exercise 17.3(b), we obtain

$$v_k(t) = \int_{\mathbb{R}} \rho_k(s) v(t-s)\,\mathrm{d}s =: \rho_k * v(t),$$

where the integral is taken in V^*. Applying Exercise 17.3(b) to the embedding $V \hookrightarrow V^*$ we see that the integral can also be taken in V; then with a proof as for Lemma 4.1 one shows that $\rho_k * v \in C^\infty(\mathbb{R};V)$. □

Proof of Theorem 17.9. (a) If $u \in C^1([a,b];V)$, then it is immediate that $\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_H^2 = (u'(t)\,|\,u(t))_H + (u(t)\,|\,u'(t))_H = 2\operatorname{Re}\langle u'(t), u(t)\rangle$. Now let $u \in MR(a,b;V,V^*)$. By Proposition 17.10 there exists a sequence (u_n) in $C^1([a,b];V)$ converging to u in $MR(a,b;V,V^*)$. Then

$$\big(\|u_n(\cdot)\|_H^2\big)' = 2\operatorname{Re}\langle u_n'(\cdot), u_n(\cdot)\rangle \to 2\operatorname{Re}\langle u'(\cdot), u(\cdot)\rangle$$

in $L_1(a,b)$. Moreover $u_n \to u$ in $L_2(a,b;H)$ since $V \hookrightarrow H$, and therefore $\|u_n(\cdot)\|_H^2 \to \|u(\cdot)\|_H^2$ in $L_1(a,b)$. Hence, $2\operatorname{Re}\langle u'(\cdot), u(\cdot)\rangle$ is the distributional derivative of $\|u(\cdot)\|_H^2$, by Lemma 4.10.

(b) For $u \in C^1([a,b];V)$, applying Proposition 17.8 to the function $t \mapsto \|u(t)\|_H^2$ we obtain

$$\begin{aligned}\|u\|_{C([a,b];H)}^2 &\leqslant \frac{1}{b-a}\int_a^b \|u(t)\|_H^2 \,\mathrm{d}t + \int_a^b \left|\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_H^2\right| \mathrm{d}t \\ &\leqslant \frac{1}{b-a}\int_a^b \|u(t)\|_H^2 \,\mathrm{d}t + \int_a^b 2\|u'(t)\|_{V^*}\|u(t)\|_V \,\mathrm{d}t \\ &\leqslant \frac{c^2}{b-a}\|u\|_{L_2(a,b;V)}^2 + 2\|u'\|_{L_2(a,b;V^*)}\|u\|_{L_2(a,b;V)}\,,\end{aligned}$$

with the embedding constant $c > 0$ of $V \hookrightarrow H$. As $C^1([a,b];V)$ is dense in $MR(a,b;V,V^*)$, by Proposition 17.10, this inequality shows that $MR(a,b;V,V^*) \hookrightarrow C([a,b];H)$. □

17.4 Lions' representation theorem

Of great importance in the theory of Hilbert spaces is the representation theorem of Fréchet–Riesz, which often gives us weak solutions of partial differential equations. One of the consequences of this theorem is the reflexivity of Hilbert spaces. Reflexivity is a key ingredient in the proof of the following much more general representation theorem.

17.11 Theorem (Lions' representation theorem). *Let $\mathcal{V}$ be a Hilbert space, $\mathcal{W}$ a normed space, $\mathcal{W} \hookrightarrow \mathcal{V}$. Let $E\colon \mathcal{V} \times \mathcal{W} \to \mathbb{K}$ be sesquilinear, and assume that*

(i) *$E(\cdot, w) \in \mathcal{V}'$ for all $w \in \mathcal{W}$,*

(ii) *$|E(w,w)| \geqslant \alpha\|w\|_{\mathcal{W}}^2$ for all $w \in \mathcal{W}$, with some $\alpha > 0$.*

Let $L \in \mathcal{W}^*$. Then there exists $u \in \mathcal{V}$ such that $L(w) = E(u,w)$ for all $w \in \mathcal{W}$ and $\|u\|_{\mathcal{V}} \leqslant \frac{c}{\alpha}\|L\|_{\mathcal{W}^*}$, where $c > 0$ is an embedding constant of the embedding $\mathcal{W} \hookrightarrow \mathcal{V}$.

The circumstance that $\mathcal{W}$ need not be complete makes Lions' theorem more widely applicable than the Lax–Milgram lemma. The larger the norm on $\mathcal{W}$, the less restrictive is the assumption on L to be continuous with respect to the norm of $\mathcal{W}$. On the other hand, the 'coercivity hypothesis' (ii) becomes more restrictive if we take larger norms. Even if we choose as norm on $\mathcal{W}$ the norm of $\mathcal{V}$, it is an advantage that E need not be defined on all of $\mathcal{V}$ in the second variable; note that there is no continuity requirement on E with respect to the second variable.

In Exercise 17.7(a) we present a slightly more general version of Theorem 17.11.

Proof of Theorem 17.11. We define an antilinear operator $T\colon \mathcal{W} \to \mathcal{V}'$ by $Tw := E(\cdot, w)$ for all $w \in \mathcal{W}$. It follows from property (ii) that

$$\alpha\|w\|_{\mathcal{W}}^2 \leqslant |E(w,w)| \leqslant \|E(\cdot,w)\|_{\mathcal{V}'}\|w\|_{\mathcal{V}} \leqslant c\|Tw\|_{\mathcal{V}'}\|w\|_{\mathcal{W}}\,;$$

thus $\frac{\alpha}{c}\|w\|_{\mathcal{W}} \leqslant \|Tw\|_{\mathcal{V}'}$ for all $w \in \mathcal{W}$. This implies that T is injective and that $T^{-1}\colon T(\mathcal{W}) \to \mathcal{W}$ is antilinear and bounded, where $T(\mathcal{W})$ is provided with the norm of $\mathcal{V}'$. Since $L \in \mathcal{W}^*$, the functional $L \circ T^{-1}$ extends to a bounded linear functional $\ell \in (\mathcal{V}')'$, with $\|\ell\| \leqslant \frac{c}{\alpha}\|L\|_{\mathcal{W}^*}$. By the reflexivity of $\mathcal{V}$ there exists $u \in \mathcal{V}$ such that $\ell(v') = v'(u)$ for all $v' \in \mathcal{V}'$, and $\|u\| = \|\ell\|$. In particular, for $w \in \mathcal{W}$ one has $L(w) = L(T^{-1}Tw) = \ell(Tw) = Tw(u) = E(u,w)$. □

17.12 Remark (Uniqueness in Theorem 17.11). The vector $u \in \mathcal{V}$ is unique if and only if

$$v \in \mathcal{V},\ E(v,w) = 0 \text{ for all } w \in \mathcal{W} \ \text{ implies } \ v = 0.$$

This is the same as saying that $T(\mathcal{W})$ is dense in $\mathcal{V}'$, with the operator T from the proof. △

17.5 The non-autonomous equation

Now we come to the main result of this chapter. We study the non-autonomous inhomogeneous evolution equation

$$u'(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0\,. \tag{17.4}$$

Our assumptions are as follows. Let V, H be separable Hilbert spaces, $V \overset{\mathrm{d}}{\hookrightarrow} H$. With these spaces we form the Gelfand triple $V \overset{\mathrm{d}}{\hookrightarrow} H \overset{\mathrm{d}}{\hookrightarrow} V^*$ and use the notation introduced in Section 17.1.

Let $\tau \in (0,\infty)$, and let a be a **non-autonomous form** on V, i.e., $a\colon [0,\tau]\times V\times V \to \mathbb{K}$ is a mapping such that

(Li1) $a(t,\cdot,\cdot)\colon V\times V\to\mathbb{K}$ is sesquilinear for all $t\in[0,\tau]$;
(Li2) $a(\cdot,x,y)$ is measurable for all $x,y\in V$.

We also suppose that there exist $M \geqslant \alpha > 0$ such that

(Li3) $|a(t,x,y)| \leqslant M\|x\|_V\|y\|_V$ for all $t\in[0,\tau]$, $x,y\in V$ (**boundedness**);
(Li4) $\operatorname{Re} a(t,x,x) \geqslant \alpha\|x\|_V^2$ for all $t\in[0,\tau]$, $x\in V$ (**coercivity**).

We point out that α and M are independent of t. It follows from Lemma 17.4 and the conditions (Li2) and (Li3) that $t\mapsto a(t,u(t),v(t))$ is integrable for all $u,v\in L_2(0,\tau;V)$. This property will be used below without further notice.

For each $t\in[0,\tau]$ we denote by $\mathcal{A}(t)\in\mathcal{L}(V,V^*)$ the operator given by

$$\langle \mathcal{A}(t)x, y\rangle = a(t,x,y) \qquad (x,y\in V).$$

First we want to interpret $\mathcal{A}$ as a 'multiplication operator' in the following way.

17.13 Proposition. *Defining*

$$(\mathcal{A}u)(t) := \mathcal{A}(t)u(t) \qquad (t\in(0,\tau))$$

for $u\in L_2(0,\tau;V)$, one obtains a bounded linear operator $\mathcal{A}\colon L_2(0,\tau;V)\to L_2(0,\tau;V^*)$.

Proof. Let $u\in L_2(0,\tau;V)$. Recalling that the function $t\mapsto\langle\mathcal{A}u(t),y\rangle = a(t,u(t),y)$ is measurable for all $y\in V$, one deduces that $\mathcal{A}u$ is measurable. The estimate

$$\|\mathcal{A}(t)u(t)\|_{V^*} \leqslant M\|u(t)\|_V \qquad (t\in(0,\tau)) \tag{17.5}$$

shows that $\mathcal{A}u = \mathcal{A}(\cdot)u(\cdot)\in L_2(0,\tau;V^*)$.

Obviously the operator $\mathcal{A}$ is linear, and the previous inequality shows that $\mathcal{A}$ is bounded. □

In the proof of Theorem 17.15 below we will apply Theorem 17.11 with a suitable sesquilinear map E; the definition of E is motivated by the following proposition. We will use the notation $C_c^\infty[0,\tau) := \{\varphi \in C^\infty[0,\tau)\,;\, \operatorname{spt}\varphi \text{ compact}\}$.

17.14 Proposition. *Let $u_0 \in H$, $f \in L_2(0,\tau;V^*)$, $u \in L_2(0,\tau;V)$. Then the following properties are equivalent.*

(i) *$u \in MR(0,\tau;V,V^*)$ and $u' + \mathcal{A}u = f$, $u(0) = u_0$.*

(ii) *For all $\varphi \in C_c^\infty[0,\tau)$, $x \in V$ one has*

$$-\int_0^\tau (u(t)\,|\,\varphi'(t)x)_H\,\mathrm{d}t + \int_0^\tau a(t,u(t),\varphi(t)x)\,\mathrm{d}t = \int_0^\tau \langle f(t),\varphi(t)x\rangle\,\mathrm{d}t + (u_0\,|\,\varphi(0)x)_H\,.$$

Proof. Note that property (ii) is equivalent to

$$-\int_0^\tau \varphi'(t)u(t)\,\mathrm{d}t + \int_0^\tau \varphi(t)(\mathcal{A}u)(t)\,\mathrm{d}t = \int_0^\tau \varphi(t)f(t)\,\mathrm{d}t + \varphi(0)u_0 \tag{17.6}$$

for all $\varphi \in C_c^\infty[0,\tau)$. This is an equation in V^*; recall that $\mathcal{A}u = \mathcal{A}(\cdot)u(\cdot) \in L_2(0,\tau;V^*)$.

(i) $\Rightarrow$ (ii). Let $\varphi \in C_c^\infty[0,\tau)$. Using $\mathcal{A}u = f - u'$ and Remark 17.7 we obtain

$$-\int_0^\tau \varphi'(t)u(t)\,\mathrm{d}t + \int_0^\tau \varphi(t)(\mathcal{A}u)(t)\,\mathrm{d}t = -\int_0^\tau (\varphi u)'(t)\,\mathrm{d}t + \int_0^\tau \varphi(t)f(t)\,\mathrm{d}t.$$

Since $\varphi(\tau) = 0$ and $u(0) = u_0$, it follows from Proposition 17.6(b) that (17.6) holds.

(ii) $\Rightarrow$ (i). Using (17.6) with $\varphi \in C_c^\infty(0,\tau)$ we obtain

$$-\int_0^\tau \varphi'(t)u(t)\,\mathrm{d}t = \int_0^\tau \varphi(t)\big(f(t) - (\mathcal{A}u)(t)\big)\,\mathrm{d}t.$$

This implies that $u \in H^1(0,\tau;V^*)$ and $u' = f - \mathcal{A}u$.

From (i) $\Rightarrow$ (ii) it follows that (17.6) also holds with $u(0)$ in place of u_0. Choosing $\varphi \in C_c^\infty[0,\tau)$ with $\varphi(0) = 1$ we conclude that $u(0) = u_0$. □

We can now formulate and prove the main result of this chapter.

17.15 Theorem (Lions). *Let $u_0 \in H$, $f \in L_2(0,\tau;V^*)$. Then there exists a unique $u \in MR(0,\tau;V,V^*)$ such that*

$$u' + \mathcal{A}u = f, \quad u(0) = u_0\,. \tag{17.7}$$

For the solution u one has the estimate

$$\|u\|_{L_2(0,\tau;V)} \leqslant \tfrac{1}{\alpha}\|f\|_{L_2(0,\tau;V^*)} + \sqrt{\tfrac{2}{\alpha}}\|u_0\|_H\,. \tag{17.8}$$

Note that both terms u', $\mathcal{A}u$ belong to the same space $L_2(0,\tau;V^*)$ as f. For this reason we say that the problem has **maximal regularity** in V^*. Note also that, in view of Theorem 17.9(b), a solution $u \in MR(0,\tau;V,V^*)$ of (17.7) can only exist if the initial value u_0 belongs to H. In Exercise 17.7(b) we indicate how to obtain a slightly sharper estimate than (17.8) for the solution u.

17.16 Remark. Theorem 17.15 implies that the linear operator

$$H \times L_2(0,\tau;V^*) \ni (u_0, f) \mapsto u \in MR(0,\tau;V,V^*)$$

(mapping the data u_0, f to the solution u) is bounded. Concerning the norm of u in $L_2(0,\tau;V)$ we refer to (17.8). For the derivative of u we apply (17.5) to estimate

$$\|u'\|_{L_2(0,\tau;V^*)} = \|f - \mathcal{A}u\|_{L_2(0,\tau;V^*)} \leqslant \|f\|_{L_2(0,\tau;V^*)} + M\|u\|_{L_2(0,\tau;V)}$$

and then use (17.8) once more. △

Proof of Theorem 17.15. To show existence we apply Lions' representation theorem, Theorem 17.11, with $\mathcal{V} := L_2(0,\tau;V)$ and the pre-Hilbert space

$$\mathcal{W} := C_c^\infty[0,\tau) \otimes V = \operatorname{lin}\bigl\{\varphi(\cdot)x;\ \varphi \in C_c^\infty[0,\tau),\ x \in V\bigr\}$$

with norm $\|w\|_{\mathcal{W}} := \bigl(\alpha\|w\|_{\mathcal{V}}^2 + \frac{1}{2}\|w(0)\|_H^2\bigr)^{1/2}$. We define $E\colon \mathcal{V} \times \mathcal{W} \to \mathbb{K}$ by

$$E(v,w) = -\int_0^\tau (v(t) \,|\, w'(t))_H \,\mathrm{d}t + \int_0^\tau a(t, v(t), w(t))\,\mathrm{d}t.$$

For $w \in \mathcal{W}$ one has

$$|E(v,w)| \leqslant \|v\|_{\mathcal{V}}\|w'\|_{L_2(0,\tau;V^*)} + M\|v\|_{\mathcal{V}}\|w\|_{\mathcal{V}} \qquad (v \in \mathcal{V}).$$

Thus condition (i) of Theorem 17.11 is satisfied.

In order to show (ii) let $w \in \mathcal{W}$. Then $\|w(\cdot)\|_H^2 \in C_c^\infty[0,\tau)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_H^2 = 2\operatorname{Re}(w'(t) \,|\, w(t))_H$; thus

$$-\int_0^\tau \operatorname{Re}(w'(t) \,|\, w(t))_H \,\mathrm{d}t = \frac{1}{2}\|w(0)\|_H^2\,.$$

By the coercivity condition (Li4) it follows that

$$\operatorname{Re} E(w,w) \geqslant \frac{1}{2}\|w(0)\|_H^2 + \alpha\|w\|_{\mathcal{V}}^2 = \|w\|_{\mathcal{W}}^2\,; \tag{17.9}$$

thus condition (ii) of Theorem 17.11 is satisfied.

Define $L \in \mathcal{W}^*$ by

$$L(w) := \int_0^\tau \langle f(t), w(t)\rangle \,\mathrm{d}t + (u_0 \,|\, w(0))_H \qquad (w \in \mathcal{W}).$$

By Lions' representation theorem, there exists $u \in \mathcal{V}$ such that $E(u,w) = L(w)$ for all $w \in \mathcal{W}$, which means that u satisfies property (ii) of Proposition 17.14. It follows that $u \in MR(0,\tau;V,V^*)$ and that u is a solution of (17.7).

In order to show uniqueness, let $u \in MR(0,\tau;V,V^*)$ be such that $u' + \mathcal{A}u = 0$ and $u(0) = 0$. Then by Theorem 17.9(a) we obtain

$$\bigl(\|u(\cdot)\|_H^2\bigr)'(t) = 2\operatorname{Re}\langle u'(t), u(t)\rangle = -2\operatorname{Re}\langle(\mathcal{A}u)(t), u(t)\rangle = -2\operatorname{Re} a(t, u(t), u(t)) \leqslant 0$$

for a.e. $t \in [0,\tau]$. Using formula (17.2) we conclude that $\|u(\cdot)\|_H$ is decreasing, so $u(t) = 0$ for all $t \in [0,\tau]$.

Obviously $\|w\|_{\mathcal{V}} \leqslant \frac{1}{\sqrt{\alpha}}\|w\|_{\mathcal{W}}$ for all $w \in \mathcal{W}$, and by the Cauchy–Schwarz inequality one obtains $\|L\|_{\mathcal{W}^*} \leqslant \bigl(\frac{1}{\alpha}\|f\|_{L_2(0,\tau;V^*)}^2 + 2\|u_0\|_H^2\bigr)^{1/2} \leqslant \frac{1}{\sqrt{\alpha}}\|f\|_{L_2(0,\tau;V^*)} + \sqrt{2}\,\|u_0\|_H$. In view of these inequalities and (17.9), the estimate in Theorem 17.11 implies (17.8). □

17.17 Remark. The notion of solution we use here is rather weak. This is because our space of test functions $\mathcal{W}$ is small and consists of very regular functions. Of course, if we have a weak notion of solution, existence of solutions might become easy to prove (an extreme version would be to call every function a solution). But uniqueness may become hard to prove. Here for existence we need the key equality (17.3) only for functions in $\mathcal{W}$, for which it is in fact trivial. With the help of Lions' representation theorem the proof of existence is rather elegant and easy. In contrast, uniqueness is more technical, since (17.3) is needed for a rather general class of functions. △

Notes

The results of Sections 17.4 and 17.5 go back to J.-L. Lions and his school in the 1950s and 1960s. Lions' representation theorem, Theorem 17.11, was first proved in [Lio57]. The existence and uniqueness result Theorem 17.15 is contained in Lions' book ([Lio61; IV, Théorème 1.1]), with a slightly different formulation. Lions cites [Vis56] and [Lio59] for the first proofs of existence. Uniqueness is shown in [Lio59]; this had been an open problem for a while. It is also possible to prove Theorem 17.15 by the Galerkin method; see [DaLi92]. This is of great interest for the numerical treatment but less elegant than via the representation theorem.

A revival of the subject is due to the interest in maximal regularity. We refer to the thesis [Die14] and to [ADLO14] for further information, in particular for the discussion of maximal regularity in H.

Exercises

17.1 Let V be a Hilbert space and $M \subseteq V^*$ a subset with the property that

$$u \in V,\ \langle \eta, u\rangle = 0 \text{ for all } \eta \in M \ \text{ implies } \ u = 0.$$

Show that M is a dense subset of V^*.

Hint: Let $V^* \ni \eta \mapsto \hat{\eta} \in V$ be the inverse of the Fréchet–Riesz isomorphism, i.e. $(\hat{\eta}\,|\,\cdot)_V = \eta$ for all $\eta \in V^*$. Show that $\widehat{M} = \{\hat{\eta};\ \eta \in M\}$ is dense in V. (Note that, in view of the reflexivity of V, the assertion can also be obtained from the Hahn–Banach theorem.)

17.2 Let V, H be Hilbert spaces, $V \overset{\mathrm{d}}{\hookrightarrow} H$.

(a) Show that there exist a measure space (Ω, μ), a unitary operator $J\colon H \to L_2(\Omega, \mu)$ and a measurable function $m\colon \Omega \to [\delta, \infty)$ (with $\delta > 0$) such that $J|_V$ is a unitary operator from V onto $L_2(\Omega, m\mu)$; see Remarks 17.1. (Hint: Show that $a(u, v) := (u\,|\,v)_V$ $(u, v \in V)$ defines a symmetric bounded coercive form, and let A be the associated strictly accretive self-adjoint operator. Use the spectral theorem for self-adjoint operators, Theorem 13.20.)

(b) In the context of part (a), let the form $a\colon V \times V \to \mathbb{K}$ be given by

$$a(u,v) = \int_\Omega mu\bar{v}\,\mathrm{d}\mu.$$

Show that the operator $\mathcal{A}$ in V^*, described in Section 17.1, is given by

$$\mathcal{A}u = mu \qquad (u \in \mathrm{dom}(\mathcal{A}) = V).$$

(c) Let $\mathbb{K} = \mathbb{C}$. Show that the operator $-\mathcal{A}$ from part (b) generates a holomorphic C_0-semigroup T on V^*. Show that $T(t)(V^*) \subseteq V$ for all $t > 0$, and that $(0,\infty) \ni t \mapsto T(t)f \in V$ is infinitely differentiable, for all $f \in V^*$.

17.3 (a) Prove Lemma 17.2.

(b) Let G be a Hilbert space, $A \in \mathcal{L}(H,G)$. Show that $A\int_a^b f(t)\,\mathrm{d}t = \int_a^b Af(t)\,\mathrm{d}t$ for all $f \in L_1(a,b;H)$.

17.4 Let $-\infty < a < b < \infty$.

(a) Let H be a separable Hilbert space, $u \in L_1(a,b;H)$. Assume that

$$\int_a^b \varphi(t)u(t)\,\mathrm{d}t = 0$$

for all $\varphi \in C_c^\infty(a,b)$. Show that $u = 0$ a.e.

Deduce the uniqueness of the weak derivative defined in Section 17.3.

(b) Find a (non-separable) Hilbert space H and a function $u\colon (a,b) \to H$ with the properties that $\|u(t)\| = 1$ for all $t \in (a,b)$ and $(u(\cdot)\,|\,v) = 0$ a.e. for all $v \in H$. (Note that then $\int_a^b \varphi(t)u(t)\,\mathrm{d}t = 0$ for all $\varphi \in C_c^\infty(a,b)$, if the integral were defined as in Lemma 17.2.)

17.5 Let $V \overset{\mathrm{d}}{\hookrightarrow} H$ be Hilbert spaces, let a be a bounded coercive form on V, with associated operator A, and let S be the C_0-semigroup generated by $-A$. Let $\tau > 0$, and interpret a as a non-autonomous form (not depending on t).

Let $u_0 \in H$. Show that $u := S(\cdot)u_0$ is the solution of

$$u' + \mathcal{A}u = 0, \quad u(0) = u_0$$

obtained from Theorem 17.15. (Hint: First treat the case $u_0 \in \mathrm{dom}(A)$. For the general case use (17.8).)

17.6 Prove the existence and uniqueness parts of Theorem 17.15 when the form a is not necessarily coercive but H-coercive, i.e.,

$$\mathrm{Re}\,a(t,x,x) + \omega\|x\|_H^2 \geqslant \alpha\|x\|_V^2 \qquad (t \in [0,\tau],\ x \in V)$$

holds for some $\omega \geqslant 0$, $\alpha > 0$. Show that the solution satisfies the estimate

$$\|u\|_{L_2(0,\tau;V)} \leqslant \mathrm{e}^{\omega\tau}\big(\tfrac{1}{\alpha}\|f\|_{L_2(0,\tau;V^*)} + \sqrt{\tfrac{2}{\alpha}}\|u_0\|_H\big).$$

Hint: Solve $v' + (\mathcal{A} + \omega)v = \mathrm{e}^{-\omega\cdot}f$.

17.7 (a) Prove a generalised version of Lions' representation theorem, Theorem 17.11, in which hypothesis (ii) is replaced by the weaker condition

(ii′) $|E(w,w)| \geqslant \alpha \|w\|_{\mathcal{W}} \|w\|_{\mathcal{V}}$ for all $w \in \mathcal{W}$, with some $\alpha > 0$.

Show that then one obtains the estimate $\|u\|_{\mathcal{V}} \leqslant \frac{1}{\alpha}\|L\|_{\mathcal{W}^*}$ for the solution u. (Hint: Inspect the proof of Theorem 17.11.)

(b) Using part (a), give a proof of Theorem 17.15, with the improved estimate

$$\|u\|_{L_2(0,\tau;V)} \leqslant \tfrac{1}{\alpha}\|f\|_{L_2(0,\tau;V^*)} + \tfrac{1}{\sqrt{2\alpha}}\|u_0\|_H$$

for the solution u.

Hint: $\mathcal{V}$ and $\mathcal{W}$ as before, but $\|w\|_{\mathcal{W}} := \max\bigl\{\|w\|_{\mathcal{V}}, \sqrt{\tfrac{2}{\alpha}}\|w(0)\|_H\bigr\}$; then

$$\|w\|_{\mathcal{W}}\|w\|_{\mathcal{V}} \leqslant \|w\|_{\mathcal{V}}^2 + \tfrac{1}{2\alpha}\|w(0)\|_H^2 \leqslant \tfrac{1}{\alpha}\operatorname{Re} E(w,w) \qquad (w \in \mathcal{W})$$

(with the aid of Peter–Paul in the first inequality), and $\|L\|_{\mathcal{W}^*} \leqslant \|f\|_{L_2(0,\tau;V^*)} + \sqrt{\tfrac{\alpha}{2}}\,\|u_0\|_H$.

(c) Consider $V = H = \mathbb{R}$ and $\mathcal{A} = \alpha$ in order to show that the coefficient $\frac{1}{\sqrt{2\alpha}}$ at $\|u_0\|_H$, in the estimate in (b), is optimal (if one wants a τ-independent estimate).

17.8 (a) Let $x, a, b \in [0,\infty)$ satisfy the 'quadratic inequality' $x^2 \leqslant ax + b$. Show that this implies $x \leqslant a + \sqrt{b}$.

(b) Under the hypotheses of Theorem 17.15, let $u \in MR(0,\tau;V,V^*)$ be a solution of (17.7).

Insert $u' = -\mathcal{A}u + f$ into (17.3), integrate the resulting equation, and apply the coercivity of a, to obtain

$$\alpha\|u\|_{L_2(0,\tau;V)}^2 \leqslant \|f\|_{L_2(0,\tau;V^*)}\|u\|_{L_2(0,\tau;V)} + \tfrac{1}{2}\|u_0\|_H^2\,.$$

Apply part (a) to reproduce the estimate from Exercise 17.7(b).

Conclude the uniqueness for solutions of (17.7).

Chapter 18

Maximal regularity for non-autonomous equations

The last chapter was devoted to the non-autonomous Cauchy problem

$$u'(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0\,,$$

where $\mathcal{A}$ is associated with a non-autonomous form on V. The problem was treated in the setup of a Gelfand triple $V \overset{\mathrm{d}}{\hookrightarrow} H \overset{\mathrm{d}}{\hookrightarrow} V^*$, and in Theorem 17.15, well-posedness was obtained in V^*. In the present chapter we will establish well-posedness and maximal regularity in H. The price to pay is that the non-autonomous form has to be Lipschitz continuous in time. The importance of maximal regularity in H will be demonstrated in Chapter 19, by applications to nonlinear problems. We refer to the Notes for more information on what is true and not true concerning maximal regularity in H.

18.1 Well-posedness in H

As in Chapter 17 let V, H be separable Hilbert spaces, $V \overset{\mathrm{d}}{\hookrightarrow} H$, giving rise to the Gelfand triple $V \overset{\mathrm{d}}{\hookrightarrow} H \overset{\mathrm{d}}{\hookrightarrow} V^*$. Let $\tau \in (0,\infty)$, and assume that a is a bounded non-autonomous form on V, i.e., a mapping $a\colon [0,\tau] \times V \times V \to \mathbb{K}$ satisfying the properties (Li1), (Li2) and (Li3) stated in Section 17.5. For $t \in [0,\tau]$ we recall the definition of $\mathcal{A}(t) \in \mathcal{L}(V,V^*)$,

$$\langle \mathcal{A}(t)x, y\rangle = a(t,x,y) \qquad (x,y \in V),$$

and we define the operator $A(t)$ as the part of $\mathcal{A}(t)$ in H, $A(t) = \mathcal{A}(t) \cap (V \times H)$; see the discussion following Remarks 17.1. In analogy to the notation $MR(a,b;V,V^*)$, introduced in Section 17.3, we define

$$MR(0,\tau;V,H) := L_2(0,\tau;V) \cap H^1(0,\tau;H),$$

with norm

$$\|u\|_{MR(0,\tau;V,H)} := \big(\|u\|^2_{L_2(0,\tau;V)} + \|u'\|^2_{L_2(0,\tau;H)}\big)^{1/2} \qquad (u \in MR(0,\tau;V,H)).$$

We say that the form a satisfies **maximal regularity** in H if, given $u_0 \in V$, $f \in L_2(0,\tau;H)$, there exists a unique function $u \in MR(0,\tau;V,H)$ such that

$$u' + \mathcal{A}u = f, \quad u(0) = u_0\,. \tag{18.1}$$

(Concerning the notation $\mathcal{A}u$ we refer to Proposition 17.13.) Note that it then follows that $u(t) \in \operatorname{dom}(A(t))$ and

$$u'(t) + A(t)u(t) = f(t) \quad \text{for a.e. } t.$$

Thus both functions u', $A(\cdot)u(\cdot)$ belong to the same space $L_2(0,\tau;H)$ as f, which justifies the term "maximal regularity in H". Maximal regularity in H is not always valid for bounded non-autonomous forms; additional assumptions are needed. Here we will make the following assumptions.

We suppose that the form $a\colon [0,\tau]\times V\times V\to\mathbb{K}$ can be written as $a = a_1 + b$, where a_1 and b are bounded non-autonomous forms on V, with the following additional requirements.

The form a_1 is **symmetric**, i.e.

$$a_1(t,x,y) = \overline{a_1(t,y,x)} \qquad (t\in[0,\tau],\ x,y\in V),$$

coercive, i.e., there exists $\alpha > 0$ such that

$$a_1(t,x,x) \geqslant \alpha\|x\|_V^2 \qquad (t\in[0,\tau],\ x\in V)$$

(see condition (Li4) in Section 17.5 for the non-symmetric case), and **Lipschitz continuous**, i.e., there exists non-autonomous form!Lipschitz continuous$M_1' \geqslant 0$ such that

$$|a_1(t,x,y) - a_1(s,x,y)| \leqslant M_1'|t-s|\,\|x\|_V\|y\|_V \qquad (s,t\in[0,\tau],\ x,y\in V).$$

In this context we will call M_1' a **Lipschitz bound** of a_1.

For the form b we do not require any regularity in time (besides measurability) but impose a stronger boundedness property, namely

$$|b(t,x,y)| \leqslant M_b\|x\|_V\|y\|_H \qquad (t\in[0,\tau],\ x,y\in V), \tag{18.2}$$

for some $M_b \geqslant 0$.

18.1 Remarks. (a) The above hypotheses imply that the form a is ***H*-coercive**, i.e.

$$\operatorname{Re} a(t,x,x) + \omega\|x\|_H^2 \geqslant \alpha\|x\|_V^2 \qquad (t\in[0,\tau],\ x\in V),$$

for some $\alpha > 0$, $\omega\in\mathbb{R}$.

Indeed, the Peter–Paul inequality implies that

$$|b(t,x,x)| \leqslant M_b\|x\|_V\|x\|_H \leqslant \frac{\alpha}{2}\|x\|_V^2 + \frac{M_b^2}{2\alpha}\|x\|_H^2\,;$$

hence by the coercivity of a_1 we obtain

$$\operatorname{Re} a(t,x,x) = a_1(t,x,x) + \operatorname{Re} b(t,x,x) \geqslant \frac{\alpha}{2}\|x\|_V^2 - \frac{M_b^2}{2\alpha}\|x\|_H^2\,,$$

for all $t\in[0,\tau]$, $x\in V$.

(b) Let a be a Lipschitz continuous symmetric bounded non-autonomous form on V which is H-coercive. Then a satisfies the above requirements, with

$$a_1(t,x,y) := a(t,x,y) + \omega(x \,|\, y)_H\,, \qquad b(t,x,y) := -\omega(x \,|\, y)_H\,.$$

(c) Let $a = a_1 + b$ be a non-autonomous form satisfying the hypotheses formulated above. Then, as shown in (a), the form a is H-coercive. Let $u_0 \in V$, $f \in L_2(0,\tau;H)$. Then, by Theorem 17.15 and Exercise 17.6, there exists a unique solution $u \in MR(0,\tau;V,V^*)$ of (18.1). On the other hand, a solution $u \in MR(0,\tau;V,H)$ of (18.1) will belong to $MR(0,\tau;V,V^*)$; hence, the uniqueness statement in Theorem 18.2 below is a consequence of the uniqueness recalled above. △

Now we formulate the main result of this chapter. We assume that $a = a_1 + b$ is a non-autonomous form satisfying the hypotheses formulated above. The operators $\mathcal{A}(t) \in \mathcal{L}(V,V^*)$ are given by $\langle \mathcal{A}(t)x, y\rangle = a(t,x,y)$, as before.

18.2 Theorem. *Let $u_0 \in V$, $f \in L_2(0,\tau;H)$. Then there exists a unique solution $u \in MR(0,\tau;V,H)$ of*

$$u' + \mathcal{A}u = f, \quad u(0) = u_0\,.$$

In short, the form a satisfies maximal regularity in H.

For the solution u one has the estimate

$$\|u\|_{MR(0,\tau;V,H)} \leqslant 2\mathrm{e}^{\gamma\tau}\big(\|f\|_{L_2(0,\tau;H)} + M_{1,0}^{1/2}\|u_0\|_V\big), \tag{18.3}$$

where $\gamma := \frac{1+M_1'+M_b^2}{\alpha}$, and $M_{1,0}$ is a bound for the form $a_1(0,\cdot,\cdot)$.

The proof of this theorem will be given in Section 18.5. First we will need to study Lipschitz continuous functions and product rules; see Sections 18.3 and 18.4.

The solution u in Theorem 18.2 depends continuously on the part b of the form; this is the topic of Exercise 18.1.

18.3 Remarks. (a) With the form $b(t,\cdot,\cdot)$ we associate the Lax–Milgram operator $\mathcal{B}(t) \in \mathcal{L}(V,V^*)$, for all $t \in [0,\tau]$. Because of the H-norm at the element y in the estimate (18.2), the element $\mathcal{B}(t)x \in V^*$ belongs to H, for all $x \in V$, $t \in [0,\tau]$. This means that in fact the range of $\mathcal{B}(t)$ is contained in H, and that $\mathcal{B}(t)$ also acts as a bounded operator from V to H, with norm $\leqslant M_b$.

The properties of $\mathcal{B}$ imply that $\mathcal{B}u \in L_2(0,\tau;H)$ and $\|\mathcal{B}u\|_{L_2(0,\tau;H)} \leqslant M_b\|u\|_{L_2(0,\tau;V)}$ for all $u \in L_2(0,\tau;V)$. (The measurability of $\mathcal{B}u$ is proved in the same way as in the proof of Proposition 17.13.)

(b) With the notation from part (a) and with $\mathcal{A}_1(t)$ denoting the Lax–Milgram operator associated with $a_1(t,\cdot,\cdot)$, the equation in Theorem 18.2 can be written as

$$u' + \mathcal{A}_1 u + \mathcal{B}u = f, \quad u(0) = u_0\,.$$

The solution $u \in MR(0,\tau;V,H)$ also has maximal regularity in H in the sense that both the terms $\mathcal{A}_1 u$ and $\mathcal{B}u$ are in $L_2(0,\tau;H)$. Indeed, $\mathcal{A}u \in L_2(0,\tau;H)$ together with

$\mathcal{B}u \in L_2(0,\tau;H)$ implies that $\mathcal{A}_1 u = \mathcal{A}u - \mathcal{B}u \in L_2(0,\tau;H)$. From Theorem 19.7 in the next chapter one can conclude a further regularity property: the solution u even belongs to $C([0,\tau];V)$.

(c) Assume that, in Theorem 18.2, the form a_1 does not depend on t, and let A_1 be the self-adjoint operator in H associated with a_1. Then the solution u of $u' + \mathcal{A}u = f$ has the stronger regularity property $u \in MR(0,\tau;\mathrm{dom}_{A_1},H)$, where dom_{A_1} denotes $\mathrm{dom}(A_1)$, provided with the 'graph norm' $\|\cdot\|_{A_1}$,

$$\|x\|_{A_1} = \big(\|x\|_H^2 + \|A_1 x\|_H^2\big)^{1/2} \qquad (x \in \mathrm{dom}(A_1)).$$

Indeed, A_1 is the part of $\mathcal{A}_1$ in H, and $A_1^{-1} = \mathcal{A}_1^{-1}|_H$ maps H continuously to dom_{A_1}. This shows that $u = A_1^{-1}(\mathcal{A}_1 u) \in L_2(0,\tau;\mathrm{dom}_{A_1})$. △

18.2 Examples for non-autonomous problems

In this section we give examples illustrating the existence theorems for non-autonomous problems presented so far. The first issue is an application of Theorem 17.15.

18.4 Example. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Let $\tau \in (0,\infty)$, and for $j,k = 1,\dots,n$ let $a_{jk}\colon [0,\tau] \to L_\infty(\Omega)$ be bounded and measurable as an $L_2(\Omega)$-valued function. Assume that there exists $\alpha > 0$ such that

$$\mathrm{Re} \sum_{j,k=1}^n a_{jk}(t,x)\xi_k\overline{\xi_j} \geqslant \alpha|\xi|^2$$

for all $t \in [0,\tau]$, $x \in \Omega$, $\xi \in \mathbb{K}^n$.

Let $V := H_0^1(\Omega)$ and $H := L_2(\Omega)$; then $V^* = H^{-1}(\Omega)$ (cf. Section 16.1). Define $a\colon [0,\tau] \times V \times V \to \mathbb{K}$ by

$$a(t,u,v) := \int_\Omega \sum_{j,k=1}^n a_{jk}(t,x)\partial_k u(x)\overline{\partial_j v(x)}\,\mathrm{d}x.$$

Obviously, a satisfies the properties (Li1), (Li2) and (Li4) in Section 17.5. In order to see the measurability property (Li3), note that, by hypothesis,

$$t \mapsto \int_\Omega a_{jk}(t,x)f(x)\,\mathrm{d}x$$

is measurable for all $f \in L_2(\Omega)$, and hence, by approximation, for $f = \partial_k u \cdot \overline{\partial_j v} \in L_1(\Omega)$ as well.

Let $u_0 \in L_2(\Omega)$, $f \in L_2(0,\tau;H^{-1}(\Omega))$. Then Theorem 17.15 implies that there exists a unique solution $u \in MR(0,\tau;H_0^1(\Omega),H^{-1}(\Omega))$ of the initial value problem

$$u' + \mathcal{A}u = f, \quad u(0) = u_0$$

(where $\mathcal{A}$ is defined as in Section 17.5). △

Our second example illustrates the application of Theorem 18.2.

18.5 Example. Let the hypotheses be as in Example 18.4, but rename the non-autonomous form a to a_1. Additionally to these hypotheses assume that, for $j,k = 1,\dots,n$, the functions $a_{jk}\colon [0,\tau] \to L_\infty(\Omega)$ are Lipschitz continuous, and that $a_{jk}(t) = \overline{a_{kj}(t)}$ for all $t \in [0,\tau]$, $j,k = 1,\dots,n$.

In addition, for $j = 1,\dots,n$ let $b_j\colon [0,\tau] \to L_\infty(\Omega)$ be bounded and measurable as an $L_2(\Omega)$-valued function, and define $b\colon [0,\tau] \times V \times V \to \mathbb{K}$ by

$$b(t,u,v) := \int_\Omega \Big(\sum_{j=1}^n b_j(t,x)\partial_j u(x)\Big)\overline{v(x)}\,\mathrm{d}x.$$

One easily sees that a_1 and b satisfy the requirements stated in Section 18.1. Define $\mathcal{A}$ as in Section 18.1.

Let $u_0 \in H^1_0(\Omega)$, $f \in L_2(0,\tau;L_2(\Omega))$. Then by Theorem 18.2 there exists a unique $u \in MR(0,\tau;H^1_0(\Omega),L_2(\Omega))$ such that

$$u' + \mathcal{A}u = f, \quad u(0) = u_0\,.$$

Spelled out differently, the equation the function u satisfies reads as

$$\partial_t u(t,x) = \sum_{j,k=1}^n \partial_j(a_{jk}(t,x)\partial_k u)(t,x) - \sum_{j=1}^n b_j(t,x)\partial_j u(t,x) + f(t,x);$$

this is a non-autonomous diffusion equation with a drift term. △

18.3 Interlude: Lipschitz continuous functions, scalar product rule

Let $-\infty \leqslant a < b \leqslant \infty$. It follows from Proposition 4.7 that each function $u \in C(a,b)$ with distributional derivative $u' \in L_\infty(a,b)$ is Lipschitz continuous. This property has a converse, as follows.

18.6 Proposition. *Let $u\colon (a,b) \to \mathbb{K}$ be Lipschitz continuous, i.e.*

$$|u(t) - u(s)| \leqslant L|t-s| \qquad (s,t \in (a,b)),$$

for some $L \geqslant 0$. Then $u' = \partial u \in L_\infty(a,b)$, with $\|u'\|_\infty \leqslant L$.

Proof. It is easy to see that one can extend u to a Lipschitz continuous function $u\colon \mathbb{R} \to \mathbb{K}$ with the same Lipschitz constant L. Therefore, without loss of generality, we may assume that $(a,b) = \mathbb{R}$.

Let $(\rho_k)_k$ be a δ-sequence in $C^\infty_{\mathrm{c}}(\mathbb{R})$. Then $\rho_k * u \in C^1(\mathbb{R})$ for all $k \in \mathbb{N}$, by Lemma 4.1. For $k \in \mathbb{N}$, $s,t \in \mathbb{R}$ we estimate

$$|\rho_k * u(t) - \rho_k * u(s)| \leqslant \int_{\mathbb{R}} \rho_k(r)\big|u(t-r) - u(s-r)\big|\,\mathrm{d}r \leqslant L|t-s|.$$

It follows that $\|(\rho_k * u)'\|_\infty \leqslant L$, for all $k \in \mathbb{N}$.

Let $R > 0$. Then $\big((\rho_k * u)'|_{(-R,R)}\big)_k$ is a bounded sequence in $L_2(-R,R)$ and therefore contains a weakly convergent subsequence $\big((\rho_{k_j} * u)'|_{(-R,R)}\big)_j$, with weak limit $v_R \in L_2(-R,R)$. Proposition 4.3(a) shows that $\rho_k * u \to u$ uniformly on $(-R,R)$. Arguing as in the proof of Lemma 4.10, one concludes from these convergences that $\partial\big(u|_{(-R,R)}\big) = v_R$. Moreover, for all $w \in L_2(-R,R)$ one obtains

$$\left|\int_{-R}^{R} v_R(t)w(t)\,\mathrm{d}t\right| = \lim_{j\to\infty}\left|\int_{-R}^{R}(\rho_{k_j} * u)'(t)w(t)\,\mathrm{d}t\right| \leqslant L\|w\|_1\,,$$

and this implies that $\|v_R\|_\infty \leqslant L$.

Using a standard diagonal procedure one can choose the subsequence $\big((\rho_{k_j} * u)'\big)_j$ such that $\big((\rho_{k_j} * u)'|_{(-R,R)}\big)_j$ is weakly convergent simultaneously for all $R > 0$. Then the definition $v|_{(-R,R)} := v_R$, for all $R > 0$, yields the distributional derivative v of u, with $\|v\|_\infty \leqslant L$. □

The second topic of this section is the following product rule for scalar functions with distributional derivatives in $L_{1,\mathrm{loc}}$; it serves as a warm-up for the next interlude.

18.7 Proposition. *Let $-\infty \leqslant a < b \leqslant \infty$, and let $u, v \in C(a,b)$, $u', v' \in L_{1,\mathrm{loc}}(a,b)$. Then $(uv)' = u'v + uv' \in L_{1,\mathrm{loc}}(a,b)$.*

Proof. We only need to show the formula on (c,b), for $c \in (a,b)$. For $c \leqslant t < b$ we compute, applying Fubini's theorem in the last step,

$$\begin{aligned} u(t)v(t) - u(c)v(c) &= \int_c^t u'(s)\,\mathrm{d}s\,v(t) + u(c)\int_c^t v'(r)\,\mathrm{d}r \\ &= \int_c^t u'(s)\left(v(s) + \int_s^t v'(r)\,\mathrm{d}r\right)\mathrm{d}s + \int_c^t \Big(u(r) - \int_c^r u'(s)\,\mathrm{d}s\Big)v'(r)\,\mathrm{d}r \\ &= \int_c^t \big(u'(s)v(s) + u(s)v'(s)\big)\,\mathrm{d}s + 0. \end{aligned}$$

This shows the asserted equality, by Proposition 4.7. □

18.4 Interlude: the product rule – vector-valued

In this section let G and H be separable Hilbert spaces of infinite dimension, and let $\tau \in (0,\infty)$ be fixed. In the arguments below we will need the space

$$L_\infty(0,\tau;H) := \big\{u\colon (0,\tau) \to H\,;\, u \text{ measurable, } \|u(\cdot)\|_H \text{ essentially bounded}\big\},$$

where a.e. equal functions are identified, and with norm

$$\|u\|_\infty := \big\|\|u(\cdot)\|_H\big\|_{L_\infty(0,\tau)}\,.$$

18.8 Proposition. *Let $T \in \mathcal{L}(G, L_\infty(0,\tau;H))$. Then there exists a bounded function $k\colon (0,\tau) \to \mathcal{L}(G,H)$, $\|k(t)\| \leqslant \|T\|$ for all $t \in (0,\tau)$, such that $Tx = k(\cdot)x$ for all $x \in G$.*

Proof. We choose an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of G and define $G_0 := \mathrm{lin}_{\mathbb{Q}_\mathbb{K}}\{e_j\,;\, j\in\mathbb{N}\}$ as the set of rational linear combinations of $\{e_j\,;\, j\in\mathbb{N}\}$, where $\mathbb{Q}_\mathbb{R} := \mathbb{Q}$ and $\mathbb{Q}_\mathbb{C} := \mathbb{Q} + \mathrm{i}\mathbb{Q}$. For each $x\in G_0$ we choose a representative $k_x\colon (0,\tau)\to H$ of Tx. Then there exists a null set $N\subseteq(0,\tau)$ such that for all $t\in(0,\tau)\setminus N$, $x,y\in G_0$, $\lambda\in\mathbb{Q}_\mathbb{K}$ one has

$$\begin{aligned} \|k_x(t)\|_H &\leqslant \|T\|\|x\|_G\,, \\ k_{\lambda x+y}(t) &= \lambda k_x(t) + k_y(t). \end{aligned} \tag{18.4}$$

(This is because of the linearity of T and the countability of $\mathbb{Q}_\mathbb{K}$ and G_0.)

For $t\in(0,\tau)\setminus N$, the continuous $\mathbb{Q}_\mathbb{K}$-linear mapping $G_0\ni x\mapsto k_x(t)\in H$ has a continuous $\mathbb{K}$-linear extension to G, denoted by $k(t)$. Putting $k(t) := 0$ $(t\in N)$, we thus obtain a function $k\colon(0,\tau)\to\mathcal{L}(G,H)$, and (18.4) implies $\|k(t)\|_{\mathcal{L}(G,H)}\leqslant\|T\|$ for all $t\in(0,\tau)$.

It remains to show that $k(\cdot)x = k_x(\cdot)$, which by definition holds for all $x\in G_0$, carries over to all $x\in G$. From the measurability of $k(\cdot)x$ for all $x\in G_0$ we deduce by (18.4) that $k(\cdot)x$ is measurable for all $x\in G$. We define the operator $T_0\in\mathcal{L}(G,L_\infty(0,\tau;H))$ by $T_0x := k(\cdot)x$ $(x\in G)$. Since T and T_0 coincide on G_0, the denseness of G_0 implies that $T = T_0$, i.e. $Tx = k(\cdot)x$ for all $x\in G$. □

18.9 Remark. The following fact concerning weak derivatives will be useful. If $u,v\in L_2(0,\tau;H)$, then $u\in H^1(0,\tau;H)$ and $u' = v$ if and only if $(u(\cdot)\,|\,y)' = (v(\cdot)\,|\,y)$ for all $y\in H$.

This equivalence is an easy consequence of the definitions. △

18.10 Proposition. (a) *Let $f\colon[0,\tau]\to H$ be Lipschitz continuous, with Lipschitz constant L. Then $f\in H^1(0,\tau;H)$, $\|f'(t)\|\leqslant L$ for a.e. $t\in(0,\tau)$.*

(b) *Let $\mathcal{D}\colon[0,\tau]\to\mathcal{L}(G,H)$ be Lipschitz continuous, with Lipschitz constant L. Then there exists a bounded function $\mathcal{D}'\colon(0,\tau)\to\mathcal{L}(G,H)$, $\|\mathcal{D}'(t)\|_{\mathcal{L}(G,H)}\leqslant L$ for all $t\in(0,\tau)$, such that $(\mathcal{D}(\cdot)x)' = \mathcal{D}'(\cdot)x$ for all $x\in G$.*

Proof. (a) Using Proposition 18.6 we can define $T\in\mathcal{L}(H,L_\infty(0,\tau))$ by $Tx := (x\,|\,f(\cdot))'$ $(x\in H)$; then $\|T\|\leqslant L$. We now apply Proposition 18.8 and use the Fréchet–Riesz representation theorem to identify H with $\mathcal{L}(H;\mathbb{K})$ by the anti-isomorphism $y\mapsto(\cdot\,|\,y)$. Thus we obtain a bounded function $k\colon(0,\tau)\to H$, $\|k(t)\|_H\leqslant L$ for all $t\in(0,\tau)$, such that

$$(x\,|\,f(\cdot))' = Tx = (x\,|\,k(\cdot)) \qquad (x\in H).$$

This property yields the measurability of k, and Remark 18.9 implies that $f\in H^1(0,\tau;H)$, $f' = k$.

(b) For each $x\in G$, part (a) above shows that $Tx := (\mathcal{D}(\cdot)x)'\in L_\infty(0,\tau;H)$, $\|Tx\|_\infty\leqslant L\|x\|$. Applying Proposition 18.8 to the operator $T\in\mathcal{L}(G,L_\infty(0,\tau;H))$ we obtain a function $\mathcal{D}'\colon(0,\tau)\to\mathcal{L}(G,H)$ as asserted. □

For the proof of the following 'product rule' we refer to Exercise 18.2.

18.11 Lemma. *Let $u,v\in H^1(0,\tau;H)$. Then $(u(\cdot)\,|\,v(\cdot))\in H^1(0,\tau)$,*

$$(u(\cdot)\,|\,v(\cdot))' = (u'(\cdot)\,|\,v(\cdot)) + (u(\cdot)\,|\,v'(\cdot)).$$

18.12 Proposition. *Let $\mathcal{D}\colon [0,\tau] \to \mathcal{L}(G,H)$ be Lipschitz continuous, and let $\mathcal{D}'$ be as in Proposition 18.10(b). Let $u \in H^1(0,\tau;G)$. Then $\mathcal{D}u \in H^1(0,\tau;H)$,*

$$(\mathcal{D}u)' = \mathcal{D}'u + \mathcal{D}u'.$$

(Here we have used the notation $\mathcal{D}v := \mathcal{D}(\cdot)v(\cdot)$, for $v \in L_2(0,\tau;G)$, and similarly for $\mathcal{D}'$.)

The mapping $u \mapsto \mathcal{D}u$ belongs to $\mathcal{L}(H^1(0,\tau;G), H^1(0,\tau;H))$.

Proof. Let $y \in H$; then the function $\mathcal{D}^*(\cdot)y\colon [0,\tau] \to G$ is Lipschitz continuous. For all $x \in G$ we get – observing Remark 18.9 –

$$(x \,|\, \mathcal{D}(\cdot)^*y)'_G = (\mathcal{D}(\cdot)x \,|\, y)'_H = (\mathcal{D}'(\cdot)x \,|\, y)_H = (x \,|\, \mathcal{D}'(\cdot)^*y)_G\,.$$

Applying Remark 18.9 again we conclude that $(\mathcal{D}(\cdot)^*y)' = \mathcal{D}'(\cdot)^*y$. Thus, by Lemma 18.11 we obtain

$$\begin{aligned}\big(\mathcal{D}(\cdot)u(\cdot)\,\big|\,y\big)'_H &= \big(u(\cdot)\,\big|\,\mathcal{D}^*(\cdot)y\big)'_G = \big(u'(\cdot)\,\big|\,\mathcal{D}^*(\cdot)y\big)_G + \big(u(\cdot)\,\big|\,(\mathcal{D}^*(\cdot)y)'\big)_G \\ &= \big(\mathcal{D}(\cdot)u'(\cdot) + \mathcal{D}'(\cdot)u(\cdot)\,\big|\,y\big)_H\,.\end{aligned}$$

As this holds for all $y \in H$, we can use Remark 18.9 once more to conclude that $\mathcal{D}u \in H^1(0,\tau;H)$ and $(\mathcal{D}u)' = \mathcal{D}'u + \mathcal{D}u'$.

In view of the formula for $(\mathcal{D}u)'$, the continuity of the mapping $H^1(0,\tau;G) \ni u \mapsto \mathcal{D}u \in H^1(0,\tau;H)$ is obvious. □

We employ Proposition 18.12 for the proof of a product rule for non-autonomous forms, as follows. Note that in this result there is no coercivity requirement on the non-autonomous form a.

18.13 Corollary. *Let V be a separable Hilbert space. Let $a\colon [0,\tau] \times V \times V \to \mathbb{K}$ be a Lipschitz continuous bounded non-autonomous form with Lipschitz bound M'. Then $a(\cdot,u(\cdot),v(\cdot)) \in H^1(0,\tau)$ for all $u,v \in H^1(0,\tau;V)$. There exists a bounded non-autonomous form $a'\colon (0,\tau)\times V\times V \to \mathbb{K}$, $|a'(t,x,y)| \leqslant M'\|x\|_V\|y\|_V$ $(x,y \in V,\ t \in (0,\tau))$, such that*

$$a(\cdot,u(\cdot),v(\cdot))' = a'(\cdot,u(\cdot),v(\cdot)) + a(\cdot,u'(\cdot),v(\cdot)) + a(\cdot,u(\cdot),v'(\cdot)),$$

for all $u,v \in H^1(0,\tau;V)$.

Proof. We start by recalling that with the form a we can associate the Lax–Milgram operaters $\mathcal{A}(t) \in \mathcal{L}(V,V^*)$, defined by

$$\mathcal{A}(t)x = a(t,x,\cdot) \qquad (x \in V,\ t \in (0,\tau)).$$

It is easy to see that the Lipschitz continuity of the form a is equivalent to the Lipschitz continuity of $\mathcal{A}\colon [0,\tau] \to \mathcal{L}(V,V^*)$ with Lipschitz bound M'. Now Proposition 18.12 and Lemma 18.11 imply that $a(\cdot,u(\cdot),v(\cdot)) \in H^1(0,\tau)$,

$$\begin{aligned}a(\cdot,u(\cdot),v(\cdot))' &= \langle \mathcal{A}u, v\rangle'_{V^*,V} = \langle \mathcal{A}'u, v\rangle + \langle \mathcal{A}u', v\rangle + \langle \mathcal{A}u, v'\rangle \\ &= \langle \mathcal{A}'u, v\rangle + a(\cdot,u'(\cdot),v(\cdot)) + a(\cdot,u(\cdot),v'(\cdot)),\end{aligned} \tag{18.5}$$

where we have used the dual pairing $\langle \cdot, \cdot \rangle_{V^*,V}$ instead of the scalar product in V. It remains to find the form a' as in the assertion. From Proposition 18.10(b) we know that $\mathcal{A}' \colon (0,\tau) \to \mathcal{L}(V,V^*)$ is a bounded function, $\|\mathcal{A}'(t)\| \leqslant M'$ $(t \in (0,\tau))$. Putting

$$a'(t,x,y) := \langle \mathcal{A}'(t)x, y \rangle \qquad (x,y \in V,\ t \in (0,\tau))$$

we obtain a bounded non-autonomous form $a' \colon (0,\tau) \times V \times V \to \mathbb{K}$ with bound M'. (The measurability property (Li2) from Section 17.5 follows from the measurability of $\mathcal{A}'(\cdot)x = (\mathcal{A}(\cdot)x)'$ $(x \in V)$, in Proposition 18.10(b).) □

18.5 Proof of Theorem 18.2

Throughout this section we assume that $a = a_1 + b$ is a non-autonomous form as in Section 18.1.

Given $u_0 \in V \subseteq H$ and $f \in L_2(0,\tau;H) \subseteq L_2(0,\tau;V^*)$, we already know the existence of a solution $u \in MR(0,\tau;V,V^*)$ of (18.1), from Theorem 17.15 and Exercise 17.6. Our method of proof will *not* be to show that this solution, in the present case, has better qualities than stated in Theorem 17.15, but we give a new existence proof. The idea of the new proof is to again use Lions' representation theorem, but with a different sesquilinear form E that results from a characterisation of solutions which is different from Proposition 17.14 – see the third and fourth paragraphs of the proof given below.

Proof of Theorem 18.2. As observed in Remark 18.1(c), the uniqueness of the solution follows from Theorem 17.15 and Exercise 17.6.

To show the existence, let

$$\mathcal{V} := \big\{ u \in MR(0,\tau;V,H)\,;\, u(0) \in V \big\}.$$

(Recall that $MR(0,\tau;H,V) \hookrightarrow H^1(0,\tau;H) \hookrightarrow C([0,\tau];H)$; hence the requirement that $u(0)$ should belong to V makes sense.) Then $\mathcal{V}$ is a Hilbert space for the norm $\|\cdot\|_{\mathcal{V}}$,

$$\|u\|_{\mathcal{V}}^2 := \int_0^\tau \|u'(t)\|_H^2 \,\mathrm{d}t + \int_0^\tau \|u(t)\|_V^2 \,\mathrm{d}t + a_1(0,u(0),u(0)).$$

Let $\mathcal{W} := H^1(0,\tau;V)$ with the same norm; note that $\mathcal{W}$ is not complete.

If $u \in \mathcal{V}$ is a solution of (18.1) (with the given initial value $u_0 \in V$ and the given inhomogeneity f), then one easily sees that $E(u,w) = L(w)$ for all $w \in \mathcal{W}$, where the sesquilinear form $E \colon \mathcal{V} \times \mathcal{W} \to \mathbb{K}$ and $L \in \mathcal{W}^*$ are defined by

$$E(v,w) := \int_0^\tau (v'(t) \,|\, w'(t))_H \,\mathrm{e}^{-\gamma t} \,\mathrm{d}t + \int_0^\tau a(t,v(t),w'(t)) \,\mathrm{e}^{-\gamma t} \,\mathrm{d}t + a_1(0,v(0),w(0)),$$

$$L(w) := \int_0^\tau (f(t) \,|\, w'(t))_H \,\mathrm{e}^{-\gamma t} \,\mathrm{d}t + a_1(0,u_0,w(0)) \qquad (v \in \mathcal{V},\ w \in \mathcal{W});$$

here, the constant $\gamma \in \mathbb{R}$ is arbitrary – for the moment – and will be fixed later.

Conversely, assume that $u \in \mathcal{V}$ is such that

$$E(u,w) = L(w) \qquad (w \in \mathcal{W}). \tag{18.6}$$

We show that then u is a solution of (18.1). Let $\psi \in C_c^\infty(0,\tau)$ and $x \in V$. Then $w(t) := \int_0^t \psi(s)\,\mathrm{d}s\, x$ defines an element $w \in \mathcal{W}$; hence by (18.6) we obtain

$$\int_0^\tau (u'(t)\,|\,x)_H \psi(t) \mathrm{e}^{-\gamma t}\,\mathrm{d}t + \int_0^\tau a(t,u(t),x)\,\psi(t)\mathrm{e}^{-\gamma t}\,\mathrm{d}t = \int_0^\tau (f(t)\,|\,x)_H \psi(t)\mathrm{e}^{-\gamma t}\,\mathrm{d}t.$$

Since $\psi \in C_c^\infty(0,\tau)$ is arbitrary, it follows that

$$(u'(t)\,|\,x)_H + a(t,u(t),x) = (f(t)\,|\,x)_H$$

for a.e. $t \in (0,\tau)$. This being valid for all $x \in V$ implies that $u'(t) + \mathcal{A}(t)u(t) = f(t)$ for a.e. $t \in (0,\tau)$, i.e. u solves the differential equation. Applying (18.6) with the constant function $w(t) = x$ we obtain $a_1(0,u(0),x) = a_1(0,u_0,x)$ for all $x \in V$, which implies $u(0) = u_0$. Thus, u solves (18.1).

Clearly $E(\cdot,w) \in \mathcal{V}'$ for all $w \in \mathcal{W}$, i.e., E satisfies property (i) of Lions' representation theorem, Theorem 17.11. We show that, with γ chosen appropriately, E is coercive, i.e. E satisfies property (ii) as well. This will achieve the proof of existence: then by Theorem 17.11 there exists $u \in \mathcal{V}$ satisfying (18.6), which means that u is a solution of (18.1), as shown above.

Let $w \in \mathcal{W} = H^1(0,\tau;V)$. By Corollary 18.13 there exists a non-autonomous form $a_1' \colon (0,\tau) \times V \times V \to \mathbb{K}$ which is bounded by M_1' such that $a_1(\cdot,w(\cdot),w(\cdot)) \in H^1(0,\tau)$ and

$$a_1(\cdot,w(\cdot),w(\cdot))' = a_1'(\cdot,w(\cdot),w(\cdot)) + 2\,\mathrm{Re}\,a_1(\cdot,w(\cdot),w'(\cdot)); \tag{18.7}$$

for the expression on the right-hand side we have applied the symmetry of a_1. We multiply by $\mathrm{e}^{-\gamma\cdot}$; then by the product rule and the accretivity of a_1 it follows that

$$\begin{aligned}
&\big(a_1(\cdot,w(\cdot),w(\cdot))\mathrm{e}^{-\gamma\cdot}\big)' \\
&\quad = -\gamma a_1(\cdot,w(\cdot),w(\cdot))\mathrm{e}^{-\gamma\cdot} + a_1'(\cdot,w(\cdot),w(\cdot))\mathrm{e}^{-\gamma\cdot} + 2\,\mathrm{Re}\,a_1(\cdot,w(\cdot),w'(\cdot))\mathrm{e}^{-\gamma\cdot} \\
&\quad \leqslant -\gamma\alpha\|w(\cdot)\|_V^2\,\mathrm{e}^{-\gamma\cdot} + M_1'\|w(\cdot)\|_V^2\,\mathrm{e}^{-\gamma\cdot} + 2\,\mathrm{Re}\,a_1(\cdot,w(\cdot),w'(\cdot))\mathrm{e}^{-\gamma\cdot},
\end{aligned}$$

which we rewrite as

$$\mathrm{Re}\,a_1(\cdot,w(\cdot),w'(\cdot))\mathrm{e}^{-\gamma\cdot} \geqslant \frac{1}{2}\big(a_1(\cdot,w(\cdot),w(\cdot))\mathrm{e}^{-\gamma\cdot}\big)' + \frac{1}{2}(\gamma\alpha - M_1')\|w(\cdot)\|_V^2\,\mathrm{e}^{-\gamma\cdot}.$$

By the Peter–Paul inequality we have

$$|b(t,w(t),w'(t))| \leqslant M_b\|w(t)\|_V\|w'(t)\|_H \leqslant \frac{1}{2}\|w'(t)\|_H^2 + \frac{M_b^2}{2}\|w(t)\|_V^2$$

for all $t \in [0,\tau]$. Altogether, recalling $a = a_1 + b$, we can now estimate

$$\begin{aligned}
\mathrm{Re}\,E(w,w) &= \int_0^\tau \|w'(t)\|_H^2\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t + \mathrm{Re}\int_0^\tau a(t,w(t),w'(t))\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t + a_1(0,w(0),w(0)) \\
&\geqslant \frac{1}{2}\int_0^\tau \|w'(t)\|_H^2\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t + \frac{1}{2}\int_0^\tau \big(a_1(\cdot,w(\cdot),w(\cdot))\mathrm{e}^{-\gamma\cdot}\big)'(t)\,\mathrm{d}t \\
&\quad + \frac{\gamma\alpha - M_1' - M_b^2}{2}\int_0^\tau \|w(t)\|_V^2\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t + a_1(0,w(0),w(0)).
\end{aligned}$$

Choosing $\gamma > 0$ such that $\gamma\alpha - M_1' - M_b^2 = 1$ and applying the fundamental theorem of calculus in W_1^1 (see (17.2)), we conclude that

$$\begin{aligned}\operatorname{Re} E(w,w) &\geqslant \frac{1}{2}\int_0^\tau \|w'(t)\|_H^2 \,\mathrm{e}^{-\gamma t}\,\mathrm{d}t + \frac{1}{2}a_1(\tau, w(\tau), w(\tau))\,\mathrm{e}^{-\gamma\tau} - \frac{1}{2}a_1(0, w(0), w(0)) \\ &\quad + \frac{1}{2}\int_0^\tau \|w(t)\|_V^2 \,\mathrm{e}^{-\gamma t}\,\mathrm{d}t + a_1(0, w(0), w(0)) \qquad (18.8) \\ &\geqslant \frac{1}{2}\mathrm{e}^{-\gamma\tau}\|w\|_{\mathcal{V}}^2 .\end{aligned}$$

This proves the coercivity of E.

In order to obtain the estimate (18.3) we first note that the embedding constant of $\mathcal{W} \hookrightarrow \mathcal{V}$ is $c = 1$. Because of $\gamma > 0$, the norm of L can be estimated by

$$\|L\|_{\mathcal{W}^*} \leqslant \big(\|f\|_{L_2(0,\tau;H)}^2 + a_1(0,u_0,u_0)\big)^{1/2} \leqslant \|f\|_{L_2(0,\tau;H)} + M_{1,0}^{1/2}\|u_0\|_V ,$$

and (18.8) together with the estimate in Theorem 17.11 yields (18.3). (Note that the $\mathcal{V}$-norm of u on the left-hand side of the resulting inequality dominates the $MR(0,\tau;H,V)$-norm of u.) □

Notes

Let us consider a non-autonomous form as described in Section 18.1. In the main result of this chapter, Theorem 18.2, it is proved that the form satisfies maximal regularity in H if it is symmetric and Lipschitz continuous (up to some perturbation). This result was proved in [ADLO14; Theorem 4.2] in greater generality and with a different approach. The special case in which the form a is symmetric and C^1 (instead of Lipschitz continuous) is due to J.-L. Lions [Lio61; IV, Théorème 6.1]. Lions [Lio61; p. 68] asked whether maximal regularity in H still holds if the form is merely continuous (in time) or even measurable. Today this question is settled and we want to tell first what is known if we restrict ourselves to the initial value $u_0 = 0$.

We say that the problem

$$u' + \mathcal{A}u = f, \quad u(0) = 0 \qquad (18.9)$$

satifies maximal regularity in H if for all $f \in L_2(0,\tau;H)$ there exists a unique solution $u \in MR(0,\tau;V,H)$ of (18.9). It was proved by Ouhabaz and Spina [OuSp10] that (18.9) has maximal regularity in H if a is Hölder continuous with exponent $\beta > 1/2$, i.e.

$$|a(t,u,v) - a(s,u,v)| \leqslant M|t-s|^\beta \|u\|_V\|v\|_V$$

for all $s,t \in [0,\tau]$, $u,v \in V$. Fackler [Fac17] showed that the exponent $\beta > 1/2$ is optimal even in the symmetric case. More precisely, he constructed a symmetric non-autonomous form a satisfying

$$|a(t,u,v) - a(s,u,v)| \leqslant M|t-s|^{1/2} \|u\|_V\|v\|_V$$

for all $s, t \in [0,\tau]$, $u, v \in V$ such that (18.9) does not have maximal regularity in H. We refer to [ADF17] for more information. What remains open still today, is whether one has maximal regularity in H for Hölder exponents $\beta \leqslant 1/2$ in the special case of a non-autonomous form associated with a family of differential operators as in Example 18.4.

Another question is for which initial values the solution has maximal regularity in H. This is even a problem in the autonomous case. Let $V \overset{\mathrm{d}}{\hookrightarrow} H$, and let $a\colon V \times V \to \mathbb{K}$ be a bounded coercive form. Denote by A the operator in H associated with a and by S the C_0-semigroup generated by $-A$. Let $u_0 \in H$. Then $u := S(\cdot)u_0$ is the unique solution $u \in MR(0,\tau;V,V^*)$ of $u'(t)+\mathcal{A}u(t) = 0$, $u(0) = u_0$ (where $\mathcal{A}$ is given by $\langle \mathcal{A}u, v\rangle = a(u,v)$); see Exercise 17.5. It is known from semigroup theory that $u \in H^1(0,\tau;H)$ if and only if $u_0 \in \mathrm{dom}(A^{1/2})$, the domain of the square root of A (which we do not want to define here). If a is symmetric, then $\mathrm{dom}(A^{1/2}) = V$ (see Proposition 13.22), but in general it is not easy to determine $\mathrm{dom}(A^{1/2})$. McIntosh gave an example of a form a for which $\mathrm{dom}(A^{1/2}) \neq V$ (see [Are04; Section 5.5] for references and more details). The following was a big problem, open for many years.

Kato's square root problem. *Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $a\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{K}$ be the form defined in* (11.3)*. Does it then follow that* $\mathrm{dom}(A^{1/2}) = H^1(\Omega)$*?*

The problem was finally solved by Auscher et al. [AHL&02], who showed that the answer is positive for $\Omega = \mathbb{R}^n$ (see [EHT16] for bounded Lipschitz domains Ω).

Exercises

18.1 Continuous dependence of the solution u on the non-autonomous form b, in Theorem 18.2.

For a bounded non-autonomous form $b\colon [0,\tau] \times V \times H \to \mathbb{K}$, as in Section 18.1, we denote by $B(t) \in \mathcal{L}(V,H)$ the 'refined' Lax–Milgram operator defined by

$$(B(t)x \,|\, y)_H = b(t,x,y) \qquad (x \in V,\ y \in H),$$

for $t \in [0,\tau]$. Note that b is bounded with bound M_b if and only if $\sup_{0\leqslant t\leqslant\tau} \|B(t)\|_{\mathcal{L}(V,H)} \leqslant M_b$.

(a) Let (B_n) be a sequence of operator functions as above, associated with a sequence of forms (b_n) satisfying $\sup_n M_{b_n} < \infty$. For $n \in \mathbb{N}$ and $f \in L_2(0,\tau;H)$ let v_n be the solution of

$$v_n' + \mathcal{A}_1 v_n + B_n v_n = f, \quad v_n(0) = 0.$$

Show that there exists a constant $c \geqslant 0$, not depending on n and f, such that

$$\|v_n\|_{MR(0,\tau;V,H)} \leqslant c\|f\|_{L_2(0,\tau;H)}\,.$$

(Hint: Evaluate the constant in (18.3).)

(b) Let (B_n) be a sequence as in (a), $B_n(t) \to B(t)$ strongly in $\mathcal{L}(V,H)$ for a.e. $t \in [0,\tau]$, with B as above. Let $u_0 \in V$, $f \in L_2(0,\tau;H)$, and let $u_n, u \in MR(0,\tau;V,H)$ be the

unique solutions of

$$u_n' + \mathcal{A}_1 u_n + B_n u_n = f, \quad u_n(0) = u_0\,,$$
$$u' + \mathcal{A}_1 u + Bu = f, \quad u(0) = u_0\,,$$

respectively. Show that $u_n \to u$ in $MR(0,\tau;V,H)$. (Hints: For $v_n := u_n - u$ one obtains, subtracting the two equations,

$$v_n' + \mathcal{A}_1 v_n + B_n v_n = (B - B_n)u, \quad v_n(0) = 0.$$

Show that $(B - B_n)u \to 0$ in $L_2(0,\tau;H)$, using the dominated convergence theorem. Then apply part (a).)

18.2 Prove Lemma 18.11. (Hint: Imitate the proof of Proposition 18.7.)

18.3 Let $H := L_2(0,\pi)$, and let $a\colon V \times V \to \mathbb{K}$ be the classical Dirichlet form,

$$a(u,v) := \int_0^\pi u'\overline{v'}\,\mathrm{d}x$$

on V as indicated below. Denote by $\mathcal{A} \in \mathcal{L}(V,V^*)$ the associated Lax–Milgram operator in the Gelfand triple $V \overset{\mathrm{d}}{\hookrightarrow} H \overset{\mathrm{d}}{\hookrightarrow} V^*$. Let $u := \sin$.

(a) Show that $\mathcal{A}u = u$ if $V = H_0^1(0,\pi)$.

(b) Show that $\mathcal{A}u = u - \delta_0 - \delta_\pi$ if $V = H^1(0,\pi)$, where δ_0, δ_π denote the Dirac measures at $0,\pi$. Conclude that u does not belong to the domain of the operator associated with a on $H^1(0,\pi)$.

18.4 Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. In parts (a) and (b) let $a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x$ be the classical Dirichlet form on the indicated space V. Also, denote by $\mathcal{A} \in \mathcal{L}(V,V^*)$ the associated Lax–Milgram operator in the Gelfand triple $V \overset{\mathrm{d}}{\hookrightarrow} L_2(\Omega) \overset{\mathrm{d}}{\hookrightarrow} V^*$.

(a) Let $V := H_0^1(\Omega)$, $u \in H_0^1(\Omega)$. Show that $\mathcal{A}u \in L_2(\Omega)$ if and only if $\Delta u \in L_2(\Omega)$ (in the distributional sense), and that then

$$\langle \mathcal{A}u, v\rangle = -\int \Delta u \overline{v}\,\mathrm{d}x \qquad (v \in H_0^1(\Omega)).$$

(b) Assume that Ω has C^1-boundary, and let $V := H^1(\Omega)$, $u \in C^2(\overline{\Omega})$. Show that

$$\langle \mathcal{A}u, v\rangle = -\int_\Omega \Delta u \overline{v}\,\mathrm{d}x + \int_{\partial\Omega} \overline{v}\nabla u \cdot \nu\,\mathrm{d}\sigma \qquad (v \in H^1(\Omega)).$$

Determine those $u \in C^2(\overline{\Omega})$ for which $\mathcal{A}u \in L_2(\Omega)$. (Hint: Show the formula first for all $v \in C^1(\overline{\Omega})$; then use Theorem 7.10.)

(c) As in (b), assume that Ω has C^1-boundary, and let $V := H^1(\Omega)$. Let $\beta \in L_\infty(\partial\Omega)$, and define $a\colon V \times V \to \mathbb{K}$ by

$$a(u,v) := \int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x + \int_{\partial\Omega} \beta u \overline{v}\,\mathrm{d}\sigma \qquad (u,v \in V).$$

(Recall from Section 7.5 that a is bounded and $L_2(\Omega)$-coercive.)

Let $u \in C^2(\overline{\Omega})$. Show that

$$\langle \mathcal{A}u, v\rangle = -\int_\Omega \Delta u \overline{v}\,\mathrm{d}x + \int_{\partial\Omega} \overline{v}(\beta u + \nabla u \cdot \nu)\,\mathrm{d}\sigma \qquad (v \in H^1(\Omega)).$$

Determine those $u \in C^2(\overline{\Omega})$ for which $\mathcal{A}u \in L_2(\Omega)$.

18.5 Under the hypotheses of Theorem 18.2, show that there exists a constant $c > 0$, not depending on τ, such that

$$\|u\|_{MR(0,\tau;V,H)} \leqslant c\mathrm{e}^{\tilde{\gamma}\tau}\big(\|f\|_{L_2(0,\tau;H)} + \|u_0\|_V\big)$$

for all solutions $u \in MR(0,\tau;V,H)$ of (18.1), where $\tilde{\gamma} := \frac{M_b^2}{2\alpha}$. (The aim of this exercise is improving the constant γ in (18.3). In particular, if $b = 0$, then the estimate does not depend on τ.)

Hints: Let $u \in MR(0,\tau;V,H)$ be a solution of (18.1).

1. Using Exercise 17.6 and Remark 18.1(a), show that

$$\|u\|_{L_2(0,\tau;V)} \leqslant c_0\mathrm{e}^{\tilde{\gamma}\tau}\big(\tfrac{2}{\alpha}\|f\|_{L_2(0,\tau;H)} + \tfrac{2}{\sqrt{\alpha}}\|u_0\|_V\big),$$

where c_0 is an embedding constant of $V \hookrightarrow H$.

2. Using

$$\|u'\|^2_{L_2(0,\tau;H)} = (f - \mathcal{B}u \,|\, u') - (\mathcal{A}_1 u \,|\, u') = (f - \mathcal{B}u \,|\, u') - a_1(\cdot, u(\cdot), u'(\cdot))$$

and (18.7), show

$$\|u'\|^2_{L_2(0,\tau;H)} \leqslant \|f - \mathcal{B}u\|_{L_2(0,\tau;H)}\|u'\|_{L_2(0,\tau;H)} + \tfrac{1}{2}\big(M_1'\|u\|^2_{L_2(0,\tau;V)} + a_1(0,u_0,u_0)\big).$$

From Exercise 17.8(a) conclude that

$$\|u'\|_{L_2(0,\tau;H)} \leqslant \|f\|_{L_2(0,\tau;H)} + \Big(M_b + \sqrt{\tfrac{M_1'}{2}}\Big)\|u\|_{L_2(0,\tau;V)} + \sqrt{\tfrac{M_{1,0}}{2}}\|u_0\|_V\,.$$

Chapter 19

Nonlinear non-autonomous equations

The aim of this chapter is to treat nonlinear non-autonomous Cauchy problems of the form

$$u'(t) + \mathcal{A}(t)u(t) = Fu(t), \quad u(0) = u_0\,, \tag{19.1}$$

where F is a suitable nonlinear mapping between certain function spaces. A sketch of the method for obtaining a solution is as follows. Given w in the function space on which F is defined, find a solution u of the linear problem

$$u'(t) + \mathcal{A}(t)u(t) = Fw(t), \quad u(0) = u_0\,, \tag{19.2}$$

thus obtaining a (nonlinear) mapping $T\colon w \mapsto u$. Then a fixed point of T will be a solution of (19.1).

The linear problem (19.2) has been treated in the previous two chapters, and it will become clear that the regularity of solutions obtained in Chapter 18 is essential for the method sketched above. Another essential issue is the existence of a fixed point of the mapping T. The corresponding fixed point theorems are the subject of Section 19.1; a compactness property that is needed for their application will be treated in Section 19.2.

19.1 Schauder's and Schaefer's fixed point theorems

Fixed point theorems belong to the principal tools for treating nonlinear problems. In the well-known Banach fixed point theorem the mapping is assumed to be a strict contraction; then there exists a unique fixed point. In contrast, in Schauder's fixed point theorem – which is what we want to apply – the essential hypothesis is compactness, and the fixed point cannot be expected to be unique.

19.1 Theorem (Schauder). *Let X be a Banach space, $C \subseteq X$ a closed convex subset, and let T be a continuous self-map of C, with the property that $T(C)$ is relatively compact in X. Then T has a fixed point, i.e. there exists $x \in C$ such that $Tx = x$.*

We refer to Appendix I for the proof; see Theorem I.8.

As a first application, we show how one can apply Schauder's fixed point theorem to a nonlinear elliptic problem.

19.2 Example (A semilinear elliptic equation). Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $f\colon \mathbb{K} \to \mathbb{K}$ be bounded and continuous. Then there exists $u \in H_0^1(\Omega)$ such that

$$-\Delta u = f \circ u.$$

Proof. Note that for all $u \in L_2(\Omega)$, $f \circ u$ belongs to $L_\infty(\Omega) \subseteq L_2(\Omega)$. We show that the mapping $F\colon L_2(\Omega) \to L_2(\Omega)$, given by $F(u) := f \circ u$, is continuous. Indeed, let (u_k) be a sequence in $L_2(\Omega)$, $u_k \to u$ in $L_2(\Omega)$. There exists a subsequence (u_{k_m}) converging to u almost everywhere. The dominated convergence theorem implies that $f \circ u_{k_m} \to f \circ u$ in $L_2(\Omega)$. Now the standard sub-subsequence argument shows that $f \circ u_k \to f \circ u$ as $k \to \infty$.

Let $R := (-\Delta_D)^{-1} \in \mathcal{L}(L_2(\Omega))$ be the inverse of the negative Dirichlet Laplacian; see Example 6.20. Then $T := R \circ F\colon L_2(\Omega) \to L_2(\Omega)$ is continuous. From Example 6.20 we know that R is a compact operator. The range of F is a bounded subset of $L_2(\Omega)$, and therefore the range of $R \circ F$ is relatively compact in $L_2(\Omega)$. Now Schauder's fixed point theorem, Theorem 19.1, implies the existence of a fixed point $u = Tu = RF(u)$; then $u \in \operatorname{dom}(\Delta_{\mathrm{D}}) \subseteq H_0^1(\Omega)$ and $-\Delta u = f \circ u$. □

In the previous example, the function f in the nonlinearity is assumed to be bounded, which is a rather strong hypothesis. The following fixed point theorem, due to Schaefer, will enable us to treat more general nonlinearities. In this theorem, the relative compactness of the range of the mapping is no longer required; instead, an a priori estimate is needed, which is in fact a very natural assumption for problems in partial differential equations.

A (nonlinear) mapping $T\colon X \to Y$, where X, Y are Banach spaces, is called **compact** if $T(B_X(0,r))$ is relatively compact for all $r > 0$.

19.3 Theorem (Schaefer). *Let X be a Banach space, and let $T\colon X \to X$ be continuous and compact. Assume that the* ***Schaefer set***

$$S := \big\{x \in X\,;\, \exists \lambda \in (0,1)\colon \lambda Tx = x\big\}$$

is bounded. Then T has a fixed point.

Proof. Let $r > 0$ be such that $S \subseteq B(0,r)$, and let $P\colon X \to B[0,r]$ be the projection given by $Px := r\frac{x}{\|x\|}$ if $\|x\| > r$, $P|_{B[0,r]} = \mathrm{id}_{B[0,r]}$. Then $T \circ P$ is continuous and has relatively compact range; therefore Schauder's fixed point theorem, Theorem 19.1, implies that $T \circ P$ has a fixed point $x \in X$, $TPx = x$.

We show that $x \in B[0,r]$; then it follows that $x = Px$ is a fixed point of T. If $\|x\|$ were greater than r one would obtain $Px = \lambda x = \lambda TPx$, with $\lambda = \frac{r}{\|x\|} \in (0,1)$. This would imply that $Px \in S$, contradicting the assumption on r, because $\|Px\| = r$. □

As an application of Schaefer's fixed point theorem we show how the nonlinearity in Example 19.2 can be generalised.

19.4 Example (A semilinear elliptic equation, revisited). Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Denote by $\lambda_1 = \min \sigma(-\Delta_D) > 0$ the first Dirichlet eigenvalue, and let $0 \leqslant \alpha < \lambda_1$, $\beta \geqslant 0$. Let $f\colon \mathbb{K} \to \mathbb{K}$ be continuous, satisfying

$$|f(z)| \leqslant \alpha|z| + \beta \qquad (z \in \mathbb{K}).$$

Let $g \in L_2(\Omega)$. Then there exists $u \in H_0^1(\Omega)$ such that

$$-\Delta u = f \circ u + g.$$

Proof. We show that the mapping $F\colon L_2(\Omega) \to L_2(\Omega)$, defined by $F(u) := f \circ u + g$, is continuous. Let (u_k) be a sequence in $L_2(\Omega)$, $u_k \to u$ in $L_2(\Omega)$. There exist a subsequence (u_{k_m}) and a function $h \in L_2(\Omega)$ such that $u_{k_m} \to u$ a.e. and $|u_{k_m}| \leqslant h$ a.e. for all $m \in \mathbb{N}$. Then $|f \circ u_{k_m}| \leqslant \alpha h + \beta$ a.e. for all $m \in \mathbb{N}$, and the dominated convergence theorem implies that $F(u_{k_m}) \to F(u)$ in $L_2(\Omega)$. Now the standard sub-subsequence argument (Exercise 9.7) shows the asserted convergence $F(u_k) \to F(u)$ as $k \to \infty$.

As in Example 19.2 we define $R := (-\Delta_D)^{-1} \in \mathcal{L}(L_2(\Omega))$ and recall that R is a compact operator. Then $T := R \circ F\colon L_2(\Omega) \to L_2(\Omega)$ is continuous. In order to show that T is compact, let $r > 0$, $u \in B_{L_2(\Omega)}(0, r)$. Since $|f \circ u| \leqslant \alpha|u| + \beta$, it follows that $\|F(u)\|_{L_2(\Omega)} \leqslant \alpha r + \beta\|\mathbf{1}_\Omega\|_{L_2(\Omega)} + \|g\|_{L_2(\Omega)}$. Hence, $F(B(0,r))$ is bounded in $L_2(\Omega)$, and $T(B(0,r)) = R\big(F(B(0,r))\big)$ is relatively compact in $L_2(\Omega)$.

We have shown that T is a compact continuous mapping. In order to apply Schaefer's fixed point theorem, Theorem 19.3, we have to show that the Schaefer set

$$S := \big\{v \in L_2(\Omega);\, \exists \lambda \in (0,1)\colon \lambda T v = v\big\}$$

is bounded. If this is achieved, the theorem implies that there exists $u \in L_2(\Omega)$ with $Tu = u$; then $u \in H_0^1(\Omega)$ and $-\Delta u = f \circ u + g$.

Let $v \in S$, i.e., $v \in H_0^1(\Omega)$ and there exists $\lambda \in (0,1)$ such that

$$-\Delta v = \lambda(f \circ v + g).$$

Then

$$\int_\Omega |\nabla v|^2 \,\mathrm{d}x = \lambda \int_\Omega (f \circ v + g)\,\overline{v}\,\mathrm{d}x \leqslant \int_\Omega \alpha|v|^2\,\mathrm{d}x + \int_\Omega \beta|v|\,\mathrm{d}x + \int_\Omega |g||v|\,\mathrm{d}x.$$

Using the Peter–Paul inequality we obtain

$$\int_\Omega |\nabla v|^2 \,\mathrm{d}x \leqslant \alpha \int_\Omega |v|^2\,\mathrm{d}x + \varepsilon \int_\Omega |v|^2\,\mathrm{d}x + \frac{1}{4\varepsilon}\int_\Omega (\beta + |g|)^2\,\mathrm{d}x,$$

for all $\varepsilon > 0$. By Poincaré's inequality (see Example 6.20) one has $\lambda_1 \int_\Omega |v|^2 \leqslant \int_\Omega |\nabla v|^2$; hence it follows that

$$(\lambda_1 - \alpha - \varepsilon) \int_\Omega |v|^2\,\mathrm{d}x \leqslant \frac{1}{4\varepsilon}\int_\Omega (\beta + |g|)^2\,\mathrm{d}x.$$

Choosing $0 < \varepsilon < \lambda_1 - \alpha$, we conclude that the Schaefer set is bounded, and the proof is complete. □

19.5 Remark. The assertion of Example 19.4 no longer holds for $\alpha = \lambda_1$. Indeed, choose $f(z) := \lambda_1 z$ and $g := \varphi_1$, where $\varphi_1 \in H_0^1(\Omega)$, $-\Delta\varphi_1 = \lambda_1\varphi_1$, $\|\varphi_1\|_{L_2(\Omega)} = 1$. Suppose that $u \in H_0^1(\Omega)$ satisfies $-\Delta u = \lambda_1 u + \varphi_1$. Then $\int \nabla u \cdot \overline{\nabla v} = \lambda_1 \int u\,\overline{v} + \int \varphi_1\,\overline{v}$ for all $v \in H_0^1(\Omega)$. In particular, for $v = \varphi_1$ one obtains

$$\lambda_1 \int u\,\overline{\varphi_1}\,\mathrm{d}x = \int \nabla u \cdot \overline{\nabla\varphi_1}\,\mathrm{d}x = \lambda_1 \int u\,\overline{\varphi_1}\,\mathrm{d}x + \int |\varphi_1|^2\,\mathrm{d}x,$$

which is absurd. △

19.2 Interlude: Compact embeddings of mixed spaces

Let $\tau \in (0,\infty)$. We have seen that the embedding $H^1(0,\tau) \hookrightarrow L_2(0,\tau)$ is compact; see Theorem 7.12. Now, if H is an infinite dimensional separable Hilbert space, then the embedding $H^1(0,\tau;H) \hookrightarrow L_2(0,\tau;H)$ is no longer compact. (Indeed, let (e_n) be an orthonormal sequence in H, and consider the functions $u_n(t) := e_n$ $(0 \leqslant t \leqslant \tau)$ in $H^1(0,\tau;H)$. The sequence $(u_n)_{n\in\mathbb{N}}$ has no convergent subsequence in $L_2(0,\tau;H)$.) However, things change if we consider the vector-valued spaces taking values in two different Hilbert spaces.

19.6 Theorem (Aubin–Lions). *Let V, H be separable Hilbert spaces, with compact embedding $V \hookrightarrow H$. Then the embedding*

$$MR(0,\tau;V,H) \hookrightarrow L_2(0,\tau;H)$$

is compact.

Proof. As $MR(0,\tau;V,H)$ is a Hilbert space, hence reflexive, it is sufficient to show that every weakly convergent sequence (u_n) in $MR(0,\tau;V,H)$ is convergent in $L_2(0,\tau;H)$; see Exercise 6.2(b). Without loss of generality $u_n \to 0$ weakly in $MR(0,\tau;V,H)$, i.e., $u_n \to 0$ weakly in $L_2(0,\tau;V)$ and $u_n' \to 0$ weakly in $L_2(0,\tau;H)$, and also without loss of generality $\|u_n'\|_{L_2(0,\tau;H)} \leqslant 1$ for all $n \in \mathbb{N}$. We have to show that $u_n \to 0$ in $L_2(0,\tau;H)$.

First we fix $t \in (0,\tau)$ and show that $u_n(t) \to 0$ in H. Let $\varepsilon > 0$. For the following computations recall Proposition 17.6(b) (as well as Remark 17.7, for the product rule). For $0 < s_0 < \tau - t$ we have

$$\begin{aligned} u_n(t) &= \frac{1}{s_0}\int_0^{s_0} \frac{\mathrm{d}}{\mathrm{d}s}\big((s-s_0)u_n(t+s)\big)\,\mathrm{d}s \\ &= \frac{1}{s_0}\int_0^{s_0} u_n(t+s)\,\mathrm{d}s + \frac{1}{s_0}\int_0^{s_0}(s-s_0)u_n'(t+s)\,\mathrm{d}s, \end{aligned}$$

where the integrals should be taken in H. Applying the Cauchy–Schwarz inequality one can see that the H-norm of the second term in the last expression is $\leqslant\sqrt{s_0}$; hence, choosing $s_0 \leqslant \varepsilon^2$ makes this norm $\leqslant\varepsilon$. Concerning the first term, we note that this integral can also be considered as an integral in V and that the mapping $L_2(0,\tau;V) \ni u \mapsto \frac{1}{s_0}\int_0^{s_0} u(t+s)\,\mathrm{d}s \in V$ is linear and continuous, hence continuous with respect to the weak topologies. Therefore the sequence $\big(\frac{1}{s_0}\int_0^{s_0} u_n(t+s)\,\mathrm{d}s\big)_n$ converges weakly to 0 in V, hence in norm to 0 in H, by the compactness of the embedding $V \hookrightarrow H$. Thus we have shown that $\limsup_{n\to\infty}\|u_n(t)\|_H \leqslant \varepsilon$.

From Proposition 17.6(c) we recall that $H^1(0,\tau;H)$ is continuously embedded into $C([0,\tau];H)$; hence there exists a constant $c \geqslant 0$ such that $\|u_n(t)\|_H \leqslant c$ for all $t \in (0,\tau)$, $n \in \mathbb{N}$. Now the dominated convergence theorem implies that $u_n \to 0$ in $L_2(0,\tau;H)$ as $n \to \infty$. □

The compactness property in Theorem 19.6 can be reinforced to compactness of the embedding $MR(0,\tau;V,H) \hookrightarrow C([0,\tau];H)$; see Exercise 19.3.

19.3 More on maximal regularity

In this section we return to the non-autonomous problem (18.1) and prove a further regularity property of the solutions obtained in Theorem 18.2. As in Section 18.1, let V, H be separable Hilbert spaces, $V \overset{\mathrm{d}}{\hookrightarrow} H$. Let $\tau \in (0,\infty)$. We assume that $a\colon [0,\tau]\times V\times V \to \mathbb{K}$ is a Lipschitz continuous symmetric bounded coercive non-autonomous form. The bound of a will be denoted by M, the Lipschitz bound by M' and the coercivity constant by α. (This corresponds to the hypotheses on a_1 in Section 18.1; the form b from that section will not be used here.)

For the discussion below as well as for later use we define the space

$$MR_a(0,\tau;V,H) := \{u \in MR(0,\tau;V,H)\,;\, \mathcal{A}u \in L_2(0,\tau;H)\},$$

with norm

$$\|u\|_{MR_a(0,\tau;V,H)} := \big(\|u\|^2_{MR(0,\tau;V,H)} + \|\mathcal{A}u\|^2_{L_2(0,\tau;H)}\big)^{1/2}.$$

We point out that a solution $u \in MR(0,\tau;V,H)$ of (18.1) automatically lies in $MR_a(0,\tau;V,H)$; using $\mathcal{A}u = f - u'$ and the estimate (18.3) one obtains

$$\begin{aligned}\|u\|_{MR_a(0,\tau;V,H)} &\leqslant \|u\|_{MR(0,\tau;V,H)} + \|\mathcal{A}u\|_{L_2(0,\tau;H)} \leqslant 2\|u\|_{MR(0,\tau;V,H)} + \|f\|_{L_2(0,\tau;H)} \\ &\leqslant 5\mathrm{e}^{\gamma\tau}\big(\|f\|_{L_2(0,\tau;H)} + M_0^{1/2}\|u_0\|_V\big),\end{aligned} \tag{19.3}$$

where $\gamma = \frac{1+M'}{\alpha}$, and M_0 is a bound for the form $a(0,\cdot,\cdot)$.

19.7 Theorem. *With the previous notation one has*

$$MR_a(0,\tau;V,H) \hookrightarrow C([0,\tau];V).$$

More explicitly, for each $u \in MR_a(0,\tau;V,H)$ one has $u \in C([0,\tau];V)$ and

$$\|u\|_{C([0,\tau];V)} \leqslant c_0\|u\|_{MR_a(0,\tau;V,H)}\,, \tag{19.4}$$

with a constant $c_0 \geqslant 0$ depending only on τ, α, M, M'. One also has

$$\|u\|_{C([0,\tau];V)} \leqslant c_1\big(\|u\|_{MR_a(0,\tau;V,H)} + \|u(0)\|_V\big), \tag{19.5}$$

with a constant $c_1 \geqslant 0$ depending only on α, M, M', but not on τ.

The proof requires a few preliminary results. The first of these deals with forms without t-dependence.

19.8 Proposition. *Let a be a bounded coercive form on V, $\operatorname{Re} a(u) \geqslant \alpha\|u\|_V^2$ for all $u \in V$, where $\alpha > 0$. Let $\mathcal{A}$ be the associated Lax–Milgram operator, considered as an operator in V^* with domain $\operatorname{dom}(\mathcal{A}) = V$.*

(a) *Then $\|(I+\mathcal{A})^{-1}\|_{\mathcal{L}(V^*,H)} \leqslant 1/(2\sqrt{\alpha})$ and $\|(I+\mathcal{A})^{-1}\|_{\mathcal{L}(H,V)} \leqslant 1/(2\sqrt{\alpha})$.*

(b) *Assume additionally that a is symmetric, and let $M \geqslant 0$ be a bound of a (i.e. $a(u) \leqslant M\|u\|_V^2$ for all $u \in V$). Then $\|(I+\mathcal{A})^{-1}\|_{\mathcal{L}(V)} \leqslant \sqrt{M/\alpha}$.*

Proof. (a) For $u \in V$ we estimate

$$2\|u\|_H\sqrt{\alpha}\|u\|_V \leqslant \|u\|_H^2 + \alpha\|u\|_V^2 \leqslant \mathrm{Re}\langle (I+\mathcal{A})u, u\rangle \leqslant \|(I+\mathcal{A})u\|_{V^*}\|u\|_V\,.$$

This implies $\|(I+\mathcal{A})^{-1}\eta\|_H \leqslant 1/(2\sqrt{\alpha})\|\eta\|_{V^*}$ for all $\eta \in V^*$.

Similarly, for $u \in \mathrm{dom}(A)$ one has

$$2\sqrt{\alpha}\|u\|_V\|u\|_H \leqslant \|u\|_H^2 + \alpha\|u\|_V^2 \leqslant \mathrm{Re}((I+\mathcal{A})u \,|\, u)_H \leqslant \|(I+\mathcal{A})u\|_H\|u\|_H\,,$$

and it follows that $\|(I+\mathcal{A})^{-1}x\|_V \leqslant 1/(2\sqrt{\alpha})\|x\|_H$ for all $x \in H$.

(b) As a is symmetric, the associated operator A in H is an accretive self-adjoint operator. Let $u \in V$, and put $v := (I+A)^{-1}u$. Then $Av = u - v \in V$,

$$\begin{aligned} a(u) = a(v + Av) &= a(v) + 2a(v, Av) + a(Av) \\ &= a(v) + 2\|Av\|_H^2 + a(Av) \geqslant a(v) = a((I+A)^{-1}u), \end{aligned}$$

hence

$$\alpha\|(I+A)^{-1}u\|_V^2 \leqslant a((I+A)^{-1}u) \leqslant a(u) \leqslant M\|u\|_V^2\,.$$

Because of $(I+\mathcal{A})^{-1}|_V = (I+A)^{-1}|_V$ we obtain the assertion. □

The following proposition deals with approximation of the identity on $MR_a(0,\tau;V,H)$ and of the embedding $MR_a(0,\tau;V,H) \hookrightarrow L_2(0,\tau;V)$; it is the final technical tool for the proof of Theorem 19.7, and we will use it to show the compactness property stated in Proposition 19.10 below.

19.9 Proposition. *Under the hypotheses as in Theorem 19.7, for $n \in \mathbb{N}$ we define*

$$\begin{aligned} \mathcal{D}_n(t) &:= (I + \tfrac{1}{n}\mathcal{A}(t))^{-1} \in \mathcal{L}(V^*, V) \qquad (t \in [0,\tau]), \\ \mathcal{D}_n u(t) &:= \mathcal{D}_n(t)u(t) \qquad (t \in (0,\tau),\ H^1(0,\tau;V^*)) \end{aligned}$$

(where we consider $\mathcal{A}(t)$ as an unbounded operator in V^*, with $\mathrm{dom}(\mathcal{A}(t)) = V$). Then

(a) *$\mathcal{D}_n \in \mathcal{L}(H^1(0,\tau;V^*), H^1(0,\tau;V))$, and $\mathcal{D}_n \in \mathcal{L}(MR_a(0,\tau;V,H))$;*

(b) *$\mathcal{D}_n \to I$ $(n \to \infty)$ strongly in $\mathcal{L}(MR_a(0,\tau;V,H))$;*

(c) *$\mathcal{D}_n \to j$ in $\mathcal{L}(MR_a(0,\tau;V,H), L_2(0,\tau;V))$ as $n \to \infty$, where j denotes the embedding $MR_a(0,\tau;V,H) \hookrightarrow L_2(0,\tau;V)$.*

Proof. (a) Note that $I + \frac{1}{n}\mathcal{A}(t)$ is the Lax–Milgram operator associated with the accretive form $\frac{1}{n}a(t,u,v) + (u \,|\, v)_H$ on V. Thus the Lax–Milgram lemma, Theorem 5.4, implies that $\|\mathcal{D}_n(t)\|_{\mathcal{L}(V^*,V)} \leqslant n/\alpha$ $(t \in [0,\tau])$. For all $s,t \in [0,\tau]$ we have

$$\mathcal{D}_n(t) - \mathcal{D}_n(s) = \tfrac{1}{n}\mathcal{D}_n(t)(\mathcal{A}(s) - \mathcal{A}(t))\mathcal{D}_n(s), \tag{19.6}$$

which shows that $\mathcal{D}_n\colon [0,\tau] \to \mathcal{L}(V^*,V)$ is Lipschitz continuous with Lipschitz constant $\frac{n}{\alpha^2}M'$ (where M' is the Lipschitz constant of a). By Proposition 18.12 it follows that $\mathcal{D}_n \in \mathcal{L}(H^1(0,\tau;V^*), H^1(0,\tau;V))$.

Using the continuous embeddings $MR(0,\tau;V,H) \hookrightarrow H^1(0,\tau;V^*)$ and $H^1(0,\tau;V) \hookrightarrow MR(0,\tau;V,H)$ we conclude that $\mathcal{D}_n \in \mathcal{L}(MR(0,\tau;V,H))$. Now the accretivity of $\frac{1}{n}a(t)$ implies that

$$\|\mathcal{D}_n(t)\|_{\mathcal{L}(H)} \leqslant 1 \qquad (t \in [0,\tau],\ n \in \mathbb{N}). \tag{19.7}$$

Since $\mathcal{A}\mathcal{D}_n = \mathcal{A}(I + \frac{1}{n}\mathcal{A})^{-1} = (I + \frac{1}{n}\mathcal{A})^{-1}\mathcal{A} = \mathcal{D}_n\mathcal{A}$ in $\mathcal{L}(V, V^*)$, we obtain

$$\|\mathcal{A}\mathcal{D}_n u\|_{L_2(0,\tau;H)} = \|\mathcal{D}_n\mathcal{A}u\|_{L_2(0,\tau;H)} \leqslant \|\mathcal{A}u\|_{L_2(0,\tau;H)}$$

for all $u \in MR_a(0,\tau;V,H)$. Combining this estimate with the property obtained before we conclude that $\mathcal{D}_n \in \mathcal{L}(MR_a(0,\tau;V,H))$.

(b) Let $u \in MR_a(0,\tau;V,H)$, and put $u_n := \mathcal{D}_n u$ $(n \in \mathbb{N})$. The fundamental observation for the asserted convergence is that $\mathcal{D}_n(t) \to I$ $(n \to \infty)$ strongly in $\mathcal{L}(H)$ for all $t \in [0,\tau]$, by Lemma 2.10. Using (19.7) and the dominated convergence theorem, we conclude that $u_n \to u$ and $\mathcal{A}u_n = \mathcal{A}\mathcal{D}_n u = \mathcal{D}_n\mathcal{A}u \to \mathcal{A}u$ in $L_2(0,\tau;H)$ as $n \to \infty$.

The convergence $\mathcal{A}u_n \to \mathcal{A}u$ in $L_2(0,\tau;H)$ implies $\mathcal{A}u_n \to \mathcal{A}u$ in $L_2(0,\tau;V^*)$. Because of the accretivity of a we have $\|\mathcal{A}(t)^{-1}\|_{\mathcal{L}(V^*,V)} \leqslant 1/\alpha$ for all $t \in [0,\tau]$; hence $u_n = \mathcal{A}^{-1}\mathcal{A}u_n \to \mathcal{A}^{-1}\mathcal{A}u = u$ in $L_2(0,\tau;V)$.

It remains to prove the convergence of the derivatives $u_n' = (\mathcal{D}_n u)'$ to u' in $L_2(0,\tau;H)$. The Lipschitz continuity of $\mathcal{D}_n\colon [0,\tau] \to \mathcal{L}(V^*,V)$, shown in part (a), implies that $\mathcal{D}_n$ is Lipschitz continuous considered as a function from $[0,\tau]$ to $\mathcal{L}(H)$. Hence by Propositions 18.10(b) and 18.12, $\mathcal{D}_n'(\cdot)$ exists in $\mathcal{L}(H)$ and $u_n' = \mathcal{D}_n' u + \mathcal{D}_n u'$. As we know from the previous considerations that $\mathcal{D}_n u' \to u'$ $(n \to \infty)$ in $L_2(0,\tau;H)$, we still have to show that $\mathcal{D}_n' u \to 0$ $(n \to \infty)$ in $L_2(0,\tau;H)$.

First, Proposition 19.8(a) implies that

$$\|\mathcal{D}_n(t)\|_{\mathcal{L}(V^*,H)} \leqslant \sqrt{n/(4\alpha)}, \quad \|\mathcal{D}_n(t)\|_{\mathcal{L}(H,V)} \leqslant \sqrt{n/(4\alpha)} \qquad (t \in [0,\tau],\ n \in \mathbb{N}). \quad (19.8)$$

From these inequalities and (19.6) we conclude that the functions $\mathcal{D}_n\colon [0,\tau] \to \mathcal{L}(H)$ are Lipschitz continuous, with a Lipschitz constant $M'/(4\alpha)$ independent of $n \in \mathbb{N}$. Thus by Proposition 18.10(b) the derivatives of $\mathcal{D}_n\colon [0,\tau] \to \mathcal{L}(H)$ can in fact be chosen such that

$$\|\mathcal{D}_n'(t)\|_{\mathcal{L}(H)} \leqslant M'/(4\alpha) \qquad (t \in (0,\tau),\ n \in \mathbb{N}).$$

Interpreting $\mathcal{D}_n'$ as the 'multiplication operator' $\mathcal{D}_n'\colon L_2(0,\tau;H) \to L_2(0,\tau;H),\ v \mapsto \mathcal{D}_n'(\cdot)v(\cdot)$ we conclude that $(\mathcal{D}_n')$ is a bounded sequence in $\mathcal{L}(L_2(0,\tau;H))$.

From Proposition 19.8(b) we obtain the estimate $\|\mathcal{D}_n(t)\|_{\mathcal{L}(V)} \leqslant \sqrt{M/\alpha}$ for all $t \in [0,\tau]$, $n \in \mathbb{N}$. Combining this estimate with (19.6) and (19.8) we conclude that, for $x \in V$, the function $\mathcal{D}_n(\cdot)x\colon (0,\tau) \to H$ is Lipschitz continuous with Lipschitz constant $c\|x\|_V/\sqrt{n}$, where $c := \sqrt{M}M'/(2\alpha)$. Hence, Proposition 18.10(a) implies that

$$\|(\mathcal{D}_n(\cdot)x)'\|_{L_\infty(0,\tau;H)} \leqslant c\|x\|_V/\sqrt{n} \to 0 \qquad (n \to \infty).$$

It follows that $\mathcal{D}_n'(f(\cdot)x) = f(\mathcal{D}_n(\cdot)x)' \to 0$ in $L_2(0,\tau;H)$ for all $x \in V$, $f \in L_2(0,\tau)$. Since $\operatorname{lin}\{f(\cdot)x\,;\ x \in V,\ f \in L_2(0,\tau)\}$ is dense in $L_2(0,\tau;H)$ and the sequence $(\mathcal{D}_n')$ is bounded in $\mathcal{L}(L_2(0,\tau;H))$, we conclude that $\mathcal{D}_n' \to 0$ strongly in $\mathcal{L}(L_2(0,\tau;H))$; in particular, $\mathcal{D}_n' u \to 0$ in $L_2(0,\tau;H)$.

(c) It is easy to see that $j - \mathcal{D}_n = \frac{1}{n}\mathcal{D}_n\mathcal{A}$. Thus from $\|\mathcal{A}\|_{\mathcal{L}(MR_a(0,\tau;V,H),L_2(0,\tau;H))} \leqslant 1$ and the second estimate in (19.8) we obtain

$$\|j - \mathcal{D}_n\|_{\mathcal{L}(MR_a(0,\tau;V,H),L_2(0,\tau;V))} \leqslant \tfrac{1}{n}\|\mathcal{D}_n\|_{\mathcal{L}(L_2(0,\tau;H),L_2(0,\tau;V))} \leqslant \tfrac{1}{2\sqrt{n\alpha}} \to 0 \qquad (n \to \infty).\ \square$$

Proof of Theorem 19.7. Let $u \in MR_a(0,\tau;V,H)$.

(i) For the proof of the asserted inequalities we first assume in addition that $u \in H^1(0,\tau;V)$; then $u \in C([0,\tau];V)$ by Proposition 17.6(b). By Corollary 18.13 and the symmetry of a we obtain $a(\cdot,u(\cdot),u(\cdot)) \in H^1(0,\tau) \subseteq W_1^1(0,\tau)$ and

$$a(\cdot,u(\cdot),u(\cdot))' = a'(\cdot,u(\cdot),u(\cdot)) + 2\,\mathrm{Re}(\mathcal{A}u(\cdot)\,|\,u'(\cdot))_H\,.$$

Applying Proposition 17.8 we obtain

$$\begin{aligned}\alpha\|u\|^2_{C([0,\tau];V)} &\leqslant \|a(\cdot,u(\cdot),u(\cdot))\|_{C[0,\tau]} \\ &\leqslant \frac{1}{\tau}\|a(\cdot,u(\cdot),u(\cdot))\|_{L_1(0,\tau)} + \|a(\cdot,u(\cdot),u(\cdot))'\|_{L_1(0,\tau)} \\ &\leqslant \frac{1}{\tau}\int_0^\tau M\|u(t)\|_V^2\,\mathrm{d}t + \int_0^\tau \big(|a'(t,u(t),u(t))| + 2\|\mathcal{A}u(t)\|_H\|u'(t)\|_H\big)\,\mathrm{d}t \\ &\leqslant \Big(\frac{M}{\tau}+M'\Big)\|u\|^2_{L_2(0,\tau;V)} + \|\mathcal{A}u\|^2_{L_2(0,\tau;H)} + \|u'\|^2_{L_2(0,\tau;H)}\,.\end{aligned} \tag{19.9}$$

In view of the 'fundamental theorem of calculus formula' (17.2), the estimate (19.9) can also be continued as

$$\begin{aligned}\alpha\|u\|^2_{C([0,\tau];V)} &\leqslant a(0,u(0),u(0)) + \int_0^\tau \big|a(\cdot,u(\cdot),u(\cdot))'(t)\big|\,\mathrm{d}t \\ &\leqslant M\|u(0)\|_V^2 + M'\|u\|^2_{L_2(0,\tau;V)} + \|\mathcal{A}u\|^2_{L_2(0,\tau;H)} + \|u'\|^2_{L_2(0,\tau;H)}\,.\end{aligned}$$

The above two estimates for $\|u\|_{C([0,\tau];V)}$ show (19.4) and (19.5) under the additional assumption $u \in H^1(0,\tau;H)$. The constants in the estimates have the dependencies stated in the theorem.

(ii) For general u we use the mappings $\mathcal{D}_n$ defined in Proposition 19.9 and put $u_n := \mathcal{D}_n u$ ($n \in \mathbb{N}$). Then Proposition 19.9(a) shows that (u_n) is a sequence in $MR_a(0,\tau;V,H) \cap H^1(0,\tau;V)$, and $u_n \to u$ in $MR_a(0,\tau;V,H)$ by Proposition 19.9(b). The inequality (19.4) implies that (u_n) is a Cauchy sequence in $C([0,\tau];V)$, and since $u_n \to u$ in $L_2(0,\tau;V)$, it follows that u has a representative in $C([0,\tau];V)$ satisfying (19.4). Once the convergence $u_n \to u$ in $C([0,\tau];V)$ is established, it is clear that (19.5) also carries over from u_n to u. □

The following compactness property will be needed below in an application of Schaefer's fixed point theorem.

19.10 Proposition. *Suppose that the embedding $V \hookrightarrow H$ is compact. Then the space $MR_a(0,\tau;V,H)$ is compactly embedded into $L_2(0,\tau;V)$.*

Proof. Let $n \in \mathbb{N}$, and let $\mathcal{D}_n$ be defined as in Proposition 19.9. We show that $\mathcal{D}_n\colon MR_a(0,\tau;V,H) \to L_2(0,\tau;V)$ is compact. The Aubin–Lions lemma, Theorem 19.6, says that the embedding $MR(0,\tau;V,H) \hookrightarrow L_2(0,\tau;H)$ is compact, and the second estimate in (19.8) implies that $\mathcal{D}_n\colon L_2(0,\tau;H) \to L_2(0,\tau;V)$ is bounded. This shows, more strongly, that $\mathcal{D}_n\colon MR(0,\tau;V,H) \to L_2(0,\tau;V)$ is compact.

Then Proposition 19.9(c) implies that the embedding $MR_a(0,\tau;H,V) \hookrightarrow L_2(0,\tau;V)$ is approximated in norm by compact mappings, hence is compact. □

19.4 An abstract semilinear problem

Let V and H be Hilbert spaces, with compact embedding $V \overset{\mathrm{d}}{\hookrightarrow} H$. Let $\tau \in (0,\infty)$, and let $a\colon [0,\tau] \times V \times V \to \mathbb{K}$ be a Lipschitz continuous symmetric bounded non-autonomous form which is H-coercive, i.e., there exist $\omega \in \mathbb{R}$, $\alpha > 0$ such that

$$a(t,u,u) + \omega \|u\|_H^2 \geqslant \alpha \|u\|_V^2 \qquad (t \in [0,\tau],\ u \in V).$$

Denote by $\mathcal{A}(t)\colon V \to V^*$ the Lax–Milgram operator associated with $a(t,\cdot,\cdot)$.

Next we introduce the nonlinear term. Let $F\colon L_2(0,\tau;V) \to L_2(0,\tau;H)$ be continuous, and suppose that there exists a constant $c_F \geqslant 0$ such that

$$\|Fw\|_{L_2(0,t;H)} \leqslant c_F(1 + \|w\|_{L_2(0,t;V)}) \qquad (t \in (0,\tau],\ w \in L_2(0,\tau;V)). \tag{19.10}$$

The inequality in (19.10) expresses that the values of Fw on the interval $(0,t)$ should not depend too much on the 'future' values of w (on the interval (t,τ)). In Section 19.6 we will describe concrete examples of mappings F with these properties.

Given these hypotheses, we formulate the main result of the present chapter, concerning the existence of solutions of non-autonomous semilinear parabolic equations.

19.11 Theorem. *Let $u_0 \in V$. Then there exists $u \in MR(0,\tau;V,H)$ such that $u(0) = u_0$ and*

$$u'(t) + \mathcal{A}(t)u(t) = Fu(t) \tag{19.11}$$

for a.e. $t \in [0,\tau]$.

Note that there is no statement concerning uniqueness; we refer to Exercise 19.5 for an example with non-uniqueness.

Proof of Theorem 19.11. By the H-coercivity of a there exists $\omega \geqslant 0$ such that the form a_ω defined by

$$a_\omega(t,x,y) := a(t,x,y) + \omega(x \,|\, y)_H \qquad (x,y \in V,\ t \in [0,\tau])$$

is coercive. Note that a function $u \in MR(0,\tau;V,H))$ satisfies the properties stated in Theorem 19.11 if and only if u is a solution of

$$u'(t) + (\mathcal{A}(t) + \omega)u(t) = F_\omega u(t), \quad u(0) = u_0\,,$$

where

$$F_\omega w := Fw + \omega w \qquad (w \in L_2(0,\tau;V)).$$

This reduces the proof to the case in which a is coercive.

Applying Theorem 18.2, we obtain for each $w \in L_2(0,\tau;V)$ a unique solution $u =: Tw \in MR(0,\tau;V,H)$ of

$$u' + \mathcal{A}u = Fw, \quad u(0) = u_0\,.$$

We will show that $T\colon L_2(0,\tau;V) \to L_2(0,\tau;V)$ satisfies the assumptions of Schaefer's fixed point theorem.

Since F is continuous, Theorem 18.2 implies that T is continuous from $L_2(0,\tau;V)$ to $MR(0,\tau;V,H)$ and hence to $L_2(0,\tau;V)$. Combining (19.3) and (19.10) we estimate

$$\begin{aligned}\|Tw\|_{MR_a(0,t;V,H)} &\leqslant 5\mathrm{e}^{\gamma t}\big(\|Fw\|_{L_2(0,t;H)} + M_0^{1/2}\|u_0\|_V\big)\\ &\leqslant 5\mathrm{e}^{\gamma\tau}\big(c_F(\|w\|_{L_2(0,t;V)}+1) + M_0^{1/2}\|u_0\|_V\big)\end{aligned} \tag{19.12}$$

for all $w \in L_2(0,\tau;V)$, $0 < t \leqslant \tau$. It follows that T maps bounded sets in $L_2(0,\tau;V)$ to bounded sets in $MR_a(0,\tau;V,H)$. Since the embedding $MR_a(0,\tau;V,H) \hookrightarrow L_2(0,\tau;V)$ is compact, by Proposition 19.10, we conclude that $T\colon L_2(0,\tau;V) \to L_2(0,\tau;V)$ is compact.

We want to show that the Schaefer set

$$S = \{w \in L_2(0,\tau;V)\,;\, \exists\lambda \in (0,1)\colon \lambda Tw = w\}$$

is bounded in $L_2(0,\tau;V)$. Let $w \in S$, so $w = \lambda Tw$ with $\lambda \in (0,1)$. Then (19.5) yields

$$\|w\|_{C([0,t];V)} = \lambda\|Tw\|_{C([0,t];V)} \leqslant c_1\big(\|Tw\|_{MR_a(0,t;V,H)} + \|Tw(0)\|_V\big)$$

for all $t \in (0,\tau]$. Since $Tw(0) = u_0$, inequality (19.12) implies that there exists a constant $c > 0$ (not depending on w) such that

$$\|w\|_{C([0,t];V)} \leqslant c\|w\|_{L_2(0,t;V)} + c$$

and hence

$$\|w(t)\|_V^2 \leqslant 2c^2\|w\|_{L_2(0,t;V)}^2 + 2c^2,$$

for all $t \in (0,\tau]$. Then Gronwall's inequality, proved below, shows that $\|w(t)\|_V^2 \leqslant 2c^2\mathrm{e}^{2c^2t}$ for all $t \in [0,\tau]$, and the boundedness of S in $L_2(0,\tau;V)$ follows.

Now we can apply Schaefer's fixed point theorem, Theorem 19.3, and find $u \in L_2(0,\tau;V)$ such that $Tu = u$. By the definition of T, any function u with this property is a solution as asserted in the theorem. □

In the proof above we have used Gronwall's inequality in its simplest form, as follows.

19.12 Lemma (Gronwall's inequality). *Let $f\colon [0,\tau] \to [0,\infty)$ be continuous, and assume that*

$$f(t) \leqslant \alpha + \beta\int_0^t f(s)\,\mathrm{d}s \qquad (0 < t < \tau),$$

with $\alpha, \beta \geqslant 0$. Then $f(t) \leqslant \alpha\mathrm{e}^{\beta t}$ for all $t \in [0,\tau]$.

Proof. Let $g(t) := \alpha + \beta\int_0^t f(s)\,\mathrm{d}s$ for all $t \in [0,\tau]$. Then $g(0) = \alpha$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{-\beta t}g(t)\big) = \mathrm{e}^{-\beta t}(g'(t) - \beta g(t)) = \beta\mathrm{e}^{-\beta t}(f(t) - g(t)) \leqslant 0$$

for all $t \in (0,\tau)$. This shows that $\mathrm{e}^{-\beta t}g(t) \leqslant \alpha$ for all $t \in [0,\tau]$, and the assertion follows. □

19.13 Remark. Recall that in the general context introduced in Section 18.1, the operator $\mathcal{A}$ consisted of two terms, the second coming from a non-autonomous form b that satisfies (18.2). Here we explain how such a term can be included in Theorem 19.11 as well.

Indeed, let b be as in Section 18.1. Then, defining $\mathcal{B}(t) \in \mathcal{L}(V,H)$ $(0 < t < \tau)$ as in Remark 18.3(a), one obtains a mapping $\mathcal{B} \in \mathcal{L}(L_2(0,\tau;V), L_2(0,\tau;H))$ satisfying (19.10) with $c_{\mathcal{B}} = M_b$. The equation

$$u'(t) + \big(\mathcal{A}(t) + \mathcal{B}(t)\big)u(t) = Fu(t),$$

where $\mathcal{A}$ and F are as in Theorem 19.11, can be rewritten as

$$u'(t) + \mathcal{A}(t)u(t) = Fu(t) - \mathcal{B}(t)u(t),$$

and the new nonlinear term, given by $Fw - \mathcal{B}w$ $(w \in L_2(0,\tau;V))$, satisfies (19.10) with constant $c_F + M_b$. △

19.5 Interlude: $L_2((0,\tau)\times\Omega) = L_2(0,\tau;L_2(\Omega))$

With the following proposition we prepare the application of Theorem 19.11 to situations in which $H = L_2(\Omega)$ and F is a Nemytskii type operator.

19.14 Proposition. *Let $\tau > 0$, and let $\Omega \subseteq \mathbb{R}^n$ be open. Then for each $w \in L_2((0,\tau)\times\Omega)$ there exists a unique $\widetilde{w} \in L_2(0,\tau;L_2(\Omega))$ such that $w(t,\cdot) = \widetilde{w}(t)$ in $L_2(\Omega)$ for a.e. $t \in (0,\tau)$. The mapping $w \mapsto \widetilde{w}$ thus defined is an isometric isomorphism.*

The property that $w(t,\cdot) = \widetilde{w}(t)$ for a.e. $t \in (0,\tau)$ should be read with the understanding that it holds for all representatives of the elements w and $\widetilde{w}$.

Proof of Proposition 19.14. We will work with the version of L_2-spaces in which all representatives of the elements are Borel measurable. We consider w as a representative of the given element in $L_2((0,\tau)\times\Omega)$; then by Fubini's theorem

$$\|w\|^2_{L_2((0,\tau)\times\Omega)} = \int_0^\tau \int_\Omega |w(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Thus, the set $N := \big\{t \in (0,\tau);\ \int_\Omega |w(t,x)|^2\,\mathrm{d}x = \infty\big\}$ is a Borel null set. Put

$$w_1(t,\cdot) := \begin{cases} w(t,\cdot) & \text{if } t \in (0,\tau)\setminus N, \\ 0 & \text{if } t \in N. \end{cases}$$

Then $w_1 = w$ a.e. on $(0,\tau)\times\Omega$, and $w_1(t,\cdot) \in L_2(\Omega)$ for all $t \in (0,\tau)$. The function $(0,\tau) \ni t \mapsto w_1(t,\cdot) \in L_2(\Omega)$ is measurable since for $g \in L_2(\Omega)$, the function $(0,\tau)\times\Omega \ni (t,x) \mapsto w_1(t,x)\overline{g}(x)$ is integrable (as product of the L_2-functions w_1 and $(t,x)\mapsto\overline{g}(x)$), and thus

$$t \mapsto \int_\Omega w_1(t,x)\overline{g}(x)\,\mathrm{d}x = (w_1(t,\cdot)\,|\,g)$$

is measurable. Hence, $\widetilde{w} := [t \mapsto w_1(t,\cdot)] \in L_2(0,\tau;L_2(\Omega))$, and $\|\widetilde{w}\|_{L_2(0,\tau;L_2(\Omega))} = \|w_1\|_{L_2((0,\tau)\times\Omega)}$. For the uniqueness of $\widetilde{w}$ it suffices to note that, if w_2 is another representative of w, then

$$\int_0^\tau \int_\Omega |w_2(t,x) - w(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t = 0,$$

hence $w_2(t,\cdot) = w(t,\cdot) = w_1(t,\cdot)$ for a.e. $t \in (0,\tau)$.

The above considerations show that

$$J\colon L_2((0,\tau)\times\Omega) \to L_2(0,\tau;L_2(\Omega)),\ w \mapsto \widetilde{w}$$

is an isometric linear mapping. It remains to show that J has dense range. For $\varphi \in L_2(0,\tau)$ and $g \in L_2(\Omega)$ one easily sees that $w := [(t,x) \mapsto \varphi(t)g(x)] \in L_2((0,\tau)\times\Omega)$ and $Jw = \varphi(\cdot)g$. Thus, $\mathrm{ran}(J)$ contains the set $L_2(0,\tau)\otimes L_2(\Omega)$, which by Lemma 17.4 is dense in $L_2(0,\tau;L_2(\Omega))$. □

19.6 Non-autonomous semilinear parabolic equations

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $H := L_2(\Omega)$, and let V be a closed subspace of $H^1(\Omega)$ containing $H_0^1(\Omega)$. We suppose that the embedding $V \hookrightarrow H$ is compact. This is the case if $V = H_0^1(\Omega)$, by Theorem 6.22, but it also holds under the previous more general hypothesis if Ω has continuous boundary, by Theorem 7.12. Let $\tau \in (0,\infty)$, and let the non-autonomous form a and the notation be as in Section 19.4. In particular, we ask the reader to think of a non-autononous form a as in Example 18.4; this will justify the heading of the present section.

Let $f\colon (0,\tau)\times\Omega\times\mathbb{K}\times\mathbb{K}^n \to \mathbb{K}$ be a Borel measurable function with the property that $f(t,x,\cdot,\cdot)\colon \mathbb{K}\times\mathbb{K}^n \to \mathbb{K}$ is continuous for all $t \in (0,\tau)$, $x \in \Omega$. Assume that there exist $L \geqslant 0$ and $0 \leqslant g \in L_2(0,\tau;L_2(\Omega))$ such that

$$|f(t,x,q,p)| \leqslant g(t)(x) + L(|q|+|p|) \qquad (t \in (0,\tau),\ x \in \Omega,\ q \in \mathbb{K},\ p \in \mathbb{K}^n). \tag{19.13}$$

Given these hypotheses, we can formulate the following consequence of Theorem 19.11; the proof will consist in showing that the right-hand side of (19.14) corresponds to a mapping F as in Theorem 19.11.

19.15 Theorem. *Let $u_0 \in V$. Then there exists $u \in MR(0,\tau;V,H)$ such that $u(0) = u_0$ and*

$$u'(t) + \mathcal{A}(t)u(t) = f(t,\cdot,u(t,\cdot),\nabla u(t,\cdot)) \tag{19.14}$$

for a.e. $t \in [0,\tau]$.

Note that, as in Theorem 19.11, there is no statement concerning uniqueness; the example of Exercise 19.5 applies also for the kind of equation treated here. The right-hand side of (19.14) is quite general; in particular it can include nonlinear first order terms. The following example serves as an application of Theorem 19.15; it is a generalisation of Example 18.5, with nonlinear drift terms.

19.16 Example. Let $b\colon [0,\tau]\times\Omega\times\mathbb{K}\to\mathbb{K}^n$ be a bounded Borel measurable function, $b(t,x,\cdot)$ continuous for all $t\in[0,\tau]$, $x\in\Omega$. Then $b(t,\cdot,v(\cdot))\in L_\infty(\Omega;\mathbb{K}^n)$ for all $v\in L_2(\Omega)$, $t\in[0,\tau]$.

Let $g\in L_2(0,\tau;L_2(\Omega))$ and $u_0\in V$. Applying Corollary 19.15 with $f(t,x,q,p):=g(t)(x)-b(t,x,q)\cdot p$ we obtain: there exists $u\in MR(0,\tau;V,L_2(\Omega))$ such that $u(0)=u_0$ and

$$u'(t)+\mathcal{A}(t)u(t)+b(t,\cdot,u(t,\cdot))\cdot\nabla u(t,\cdot)=g(t) \tag{19.15}$$

for a.e. $t\in[0,\tau]$.

We mention that for this kind of nonlinearity the solution can already be obtained from Schauder's fixed point theorem (whereas in the proof of Theorem 19.11 we have applied Schaefer's fixed point theorem). The method is to map each function $w\in L_2(0,\tau;L_2(\Omega))$ to the solution $u=:Tw\in MR(0,\tau;V,L_2(\Omega))$ of

$$u'(t)+\mathcal{A}(t)u(t)+b(t,\cdot,w(t,\cdot))\cdot\nabla u(t,\cdot)=g(t),\quad u(0)=u_0\,,$$

which exists by Theorem 18.2. Then Exercise 18.1 can be used to show that $T\colon L_2(0,\tau;L_2(\Omega))\to MR(0,\tau;V,L_2(\Omega))$ is continuous. (In fact, as $L_2(0,\tau;L_2(\Omega))$ is a metric space, it is sufficient to prove sequential continuity.) Moreover, it follows from the estimate (18.3) that the range of T is a bounded subset of $MR(0,\tau;V,L_2(\Omega))$; hence by the Aubin–Lions lemma, Theorem 19.6, the range of T is relatively compact in $L_2(0,\tau;H)$. Applying Schauder's fixed point theorem, Theorem 19.3, one obtains $u\in MR(0,\tau;V,L_2(\Omega))\subseteq L_2(0,\tau;L_2(\Omega))$ satisfying $Tu=u$, i.e. u solves (19.15). In Exercise 19.2 the reader is asked to provide the details of these arguments. △

The next lemma is needed for the application of Theorem 19.11 in the proof of Theorem 19.15.

19.17 Lemma. *The mapping $F\colon L_2(0,\tau;V)\to L_2(0,\tau;H)$ given by*

$$Fw(t):=f(t,\cdot,w(t,\cdot),\nabla w(t,\cdot))\qquad\big(w\in L_2(0,\tau;V),\ t\in(0,\tau)\big) \tag{19.16}$$

is defined and continuous, and satisfies (19.10) *with $c_F:=2L+\|g\|_{L_2(0,\tau;H)}$.*

Proof. Using Proposition 19.14 and the Borel measurability of f one easily shows that $Fw\in L_2(0,\tau;H)$ for all $w\in L_2(0,\tau;V)$. Moreover, the assumed estimate (19.13) implies

$$\|Fw\|_{L_2(0,t;H)}\leqslant\|g\|_{L_2(0,\tau;H)}+2L\|w\|_{L_2(0,t;V)}\qquad(t\in(0,\tau),\ w\in L_2(0,\tau;V)). \tag{19.17}$$

This shows the last assertion of the lemma.

Let $w_k\to w$ in $L_2(0,\tau;V)$. Then $w_k\to w$ and $\partial_j w_k\to\partial_j w$ $(j=1,\dots,n)$ in $L_2(0,\tau;H)$. Using Proposition 19.14 again, we find a subsequence (w_{k_m}) and a function $h\in L_2((0,\tau)\times\Omega)$ such that $w_{k_m}\to w$, $\nabla w_{k_m}\to\nabla w$ a.e. on $(0,\tau)\times\Omega$ and $|w_{k_m}|\leqslant h$, $|\nabla w_{k_m}|\leqslant h$ a.e. for all $m\in\mathbb{N}$. Then $v_m(t,x):=f(t,x,w_{k_m}(t,x),\nabla w_{k_m}(t,x))\to f(t,x,w(t,x),\nabla w(t,x))$ for a.e. $(t,x)\in(0,\tau)\times\Omega$, and $|v_m|\leqslant g+2Lh$ a.e. Now the dominated convergence theorem together with Proposition 19.14 and the standard sub-subsequence argument yields the asserted continuity of F. □

Proof of Theorem 19.15. From Lemma 19.17 we know that the mapping F defined in (19.16) can be used in Theorem 19.11, and with this choice of F, equation (19.14) becomes a particular version of (19.11).

Hence Theorem 19.15 follows immediately from Theorem 19.11. □

Finally we give an example of a mapping F exhibiting actions that are 'non-local' in time.

19.18 Example. Let $0 < t_0 < \tau$, and define

$$Fw(t) := (|w| \wedge 1)(t + t_0) + |\nabla w|(t - t_0) \qquad (0 < t < \tau,\ w \in L_2(0, \tau; V)), \tag{19.18}$$

where we suppose w to be extended to $\mathbb{R}$ by zero on $\mathbb{R} \setminus (0, \tau)$.

In Exercise 19.4 the reader is asked to show that F is continuous and satisfies (19.10). The first term in this example takes values of w 'from the future' – appropriately damped – whereas the second is a delay term. △

Notes

Mathematicians working in partial differential equations believe in the following principle: if there is an a priori estimate for solutions, then a solution should exist. Schaefer's fixed point theorem gives us at hand a precise framework in which this statement is correct. This fixed point theorem is part of the Leray–Schauder theory (which started by a paper of Leray and Schauder [LeSc34] and is presented in many textbooks, e.g., in [Sch69], [Ber77], [Dei85]). It was Schaefer [Sch55] who found the short proof we present here and which works even in the framework of locally convex spaces. See also [Eva10; Section 9.2, Theorem 4] for Schaefer's fixed point theorem.

The idea of using maximal regularity in order to gain compactness via the Aubin–Lions lemma, as we present it in Section 19.4, stems from [ArCh10]. (In that paper, the locally convex version of Schaefer's fixed point theorem allows the authors to treat unbounded domains.) What is new in our approach is that the linear part is chosen non-autonomous throughout. Moreover, we first treat the problem in a more abstract context with a rather general class of nonlinear terms, which includes delay terms. An important tool in our proof is the embedding result Theorem 19.7, which is based on the approximation obtained in Proposition 19.9. For related results in a more general context we refer to [ADLO14; Sections 3 and 4].

Exercises

19.1 Show that under the hypotheses of Theorem 19.3 the operator λT has a fixed point, for all $\lambda \in (0, 1)$; in particular, the Schaefer set of T is non-empty. (Hint: Look at the Schaefer set of λT.)

19.2 Carry out the following details of the procedure sketched in Example 19.16.

(a) Show that for all $w \in L_2(0,\tau;L_2(\Omega))$ an operator function $B_w\colon (0,\tau) \to \mathcal{L}(V, L_2(\Omega))$ is defined by

$$B_w v(t) := b(t,\cdot,w(t,\cdot)) \cdot \nabla v(\cdot) \qquad (v \in V,\ t \in [0,\tau]).$$

(b) Let (w_j) be a sequence in $L_2(0,\tau;L_2(\Omega))$, $w_j \to w$ in $L_2(0,\tau;L_2(\Omega))$. Show that there exists a subsequence (w_{j_k}) such that $B_{w_{j_k}}(t) \to B_w(t)$ strongly in $\mathcal{L}(V, L_2(\Omega))$ for a.e. $t \in [0,\tau]$, and that then $Tw_{j_k} \to Tw$ in $MR(0,\tau;V,L_2(\Omega))$. Apply the sub-subsequence argument to show that $Tw_j \to Tw$ in $MR(0,\tau;V,L_2(\Omega))$.

(c) Show that $T\big(L_2(0,\tau;L_2(\Omega))\big)$ is bounded in $MR(0,\tau;V,L_2(\Omega))$.

19.3 Let $\tau > 0$, and let H be a Hilbert space.

(a) Show that the unit ball of $H^1(0,\tau;H)$ is a uniformly equicontinuous subset of $C([0,\tau];H)$. (Hint: Recall Proposition 17.6.)

(b) Additionally let V be a Hilbert space, with compact embedding $V \hookrightarrow H$. Show that the embedding $MR(0,\tau;V,H) \hookrightarrow C([0,\tau];H)$ is compact. (Hint: Use the proof of Theorem 19.6 and part (a) above to show that every weakly convergent sequence in $MR(0,\tau;V,H)$ is convergent in $C([0,\tau];H)$.)

19.4 Show that the mapping F defined in Example 19.18 is continuous and satisfies (19.10).

19.5 (a) Find infinitely many solutions of the initial value problem for the ordinary differential equation $y' = \sqrt{|y|}$, $y(0) = 0$ on $[0,1]$.

(b) As in Section 19.6 let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and $H := L_2(\Omega)$. Let $V := H^1(\Omega)$, $\tau := 1$, and define the form a by $a(u,v) := \int \nabla u \cdot \nabla v \,\mathrm{d}x$ $(u,v \in V)$. Define $f\colon (0,1) \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by

$$f(t,x,q,p) := \sqrt{|q|}\,.$$

Show that f satisfies (19.13), and find infinitely many solutions $u \in MR(0,1;V,H)$ of the initial value problem

$$u'(t) + \mathcal{A}u(t) = f(t,\cdot,u(t,\cdot),\nabla u(t,\cdot)), \quad u(0) = 0.$$

Appendices

Appendix A

The divergence theorem

In this appendix we provide a proof of Gauss' theorem, also known as the divergence theorem. In the first section we will sketch, without proofs, basic facts concerning embedded manifolds in $\mathbb{R}^n$ and integration on manifolds. On the basis of these facts we will give a complete proof of the divergence theorem in the second section.

A.1 Integration on submanifolds of $\mathbb{R}^n$

In this section we sketch the definition of the integral for functions on submanifolds of $\mathbb{R}^n$. We start with some facts concerning submanifolds, by which we always mean C^1-submanifolds.

Let $k, n \in \mathbb{N}_0$, $k < n$. A set $M \subseteq \mathbb{R}^n$ is a ***k*-dimensional submanifold** if for all $z \in M$ there exist an open neighbourhood U of z in $\mathbb{R}^n$, an open set $V \subseteq \mathbb{R}^n$ and a diffeomorphism $h\colon U \to V$ such that $h(M \cap U) = \big(\mathbb{R}^k \times \{0_{n-k}\}\big) \cap V$ (where 0_{n-k} denotes the zero element in $\mathbb{R}^{n-k}$). There are several other equivalent descriptions of submanifolds. The following is the one we will need below.

A set $M \subseteq \mathbb{R}^n$ is a k-dimensional submanifold if and only if M has local parametrisations, i.e., for all $z \in M$ there exist an open neighbourhood $W \subseteq M$, an open set $T \subseteq \mathbb{R}^k$ (the parameter domain) and a regular mapping $\Phi\colon T \to \mathbb{R}^n$ (the local parametrisation) such that $\Phi(T) = W$. Here, 'regular' means: Φ is continuously differentiable, $\operatorname{rnk} \Phi'(t) = k$ for all $t \in T$ (where 'rnk' means rank), Φ is injective, and $\Phi^{-1}\colon W \to T$ is continuous.

In particular, it follows that the boundary $\partial\Omega$ of a bounded open set Ω with C^1-boundary is an $(n-1)$-dimensional manifold: looking at the definition of a C^1-graph W one easily finds a local parametrisation $\Phi\colon T \to \partial\Omega$ with $\Phi(T) = W$.

The definition of the integral on submanifolds is based on the local parametrisations and includes a weight factor that involves the quantity $\gamma(A) := \sqrt{\det(A^\top A)}$, for matrices $A \in \mathbb{R}^{n\times k}$. (Note that $A^\top A \in \mathbb{R}^{k\times k}$ is positive definite, hence $\det(A^\top A) \geqslant 0$. As a motivation for this weight factor we mention that the k-dimensional volume of $A([0,1]^k)$ – the image of the cube $[0,1]^k \subseteq \mathbb{R}^k$ under A – is given by $\gamma(A)$.)

A.1 Remarks (The surface measure $\boldsymbol{\sigma}$ on $\boldsymbol{M}$). Let $M \subseteq \mathbb{R}^n$ be a k-dimensional submanifold.

(a) Let $\Phi\colon T \to M$ be a local parametrisation, $W := \Phi(T)$. Then $\Phi\colon T \to W$ is a homeomorphism, and we define the surface measure σ_W on the Borel sets $A \subseteq W$ by

$$\sigma_W(A) := \int_T \mathbf{1}_A(\Phi(t))\,\gamma(\Phi'(t))\,\mathrm{d}t \in [0,\infty].$$

Note that the integral exists because the function $\mathbf{1}_A \circ \Phi$ is Borel measurable by Exercise A.1(b). Moreover $\mathbf{1}_A \circ \Phi$ is the indicator function on T of the Borel set $\Phi^{-1}(A)$, and thus σ_W is the image on W of the weighted Lebesgue-Borel measure on T, with weight $\gamma(\Phi'(\cdot))$. This definition has the consequence that a Borel measurable function $f\colon W \to \mathbb{K}$ is integrable with respect to σ_W if and only if $f(\Phi(\cdot))\gamma(\Phi'(\cdot))$ is integrable on T, and then

$$\int_W f(z)\,\mathrm{d}\sigma_W(z) = \int_T f(\Phi(t))\gamma(\Phi'(t))\,\mathrm{d}t. \tag{A.1}$$

(b) Now let $\Phi_j\colon T_j \to M$ be local parametrisations, $W_j := \Phi_j(T_j)$ for $j = 1, 2$. It is a major step in the theory to show that the measures σ_{W_1} and σ_{W_2} are consistent, i.e. they coincide on the Borel subsets of $W_1 \cap W_2$. For the proof one establishes that $\int_{W_1} f(z)\,\mathrm{d}\sigma_{W_1}(z) = \int_{W_2} f(z)\,\mathrm{d}\sigma_{W_2}(z)$ for all functions $f \in C_\mathrm{c}(M)$ with $\operatorname{spt} f \subseteq W_1 \cap W_2$, using the theorem of local invertibility and the change of variable formula for k-dimensional integrals.

(c) In order to define the surface measure σ on the Borel subsets of M we assume for simplicity that M is compact. (This is sufficient for our purposes because our aim is to define σ on $\partial\Omega$ for bounded open sets Ω with C^1-boundary.)

The compactness of M implies that there exists a finite set of local parametrisations $\Phi_j\colon T_j \to M$, $j = 1, \dots, m$, such that $M = \bigcup_{j=1}^m W_j$, with $W_j := \Phi_j(T_j)$. By part (b) above, the Borel measures σ_{W_j} on the Borel subsets of W_j are mutually consistent. It follows that there exists a unique Borel measure σ on M such that the restriction of σ to the Borel subsets of W_j equals σ_{W_j} for all $j = 1, \dots, m$; see Exercise A.2.

As a consequence one obtains: if $f \in L_1(M, \sigma)$ is written as a sum $f = \sum_{j=1}^m f_j$, where $f_j \in L_1(M, \sigma)$ and $f_j = 0$ on $M \setminus W_j$ for all $j = 1, \dots, m$, then

$$\int_M f(z)\,\mathrm{d}\sigma(z) = \sum_{j=1}^m \int_{W_j} f_j(z)\,\mathrm{d}\sigma_{W_j}(z) = \sum_{j=1}^m \int_{T_j} f_j(\Phi_j(t))\,\gamma(\Phi_j'(t))\,\mathrm{d}t.$$

(d) An important property of the surface measure σ is 'orthogonal invariance'. More precisely, if $B \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then

$$\int_{B(M)} f(z)\,\mathrm{d}\sigma(z) = \int_M f(Bz)\,\mathrm{d}\sigma(z) \tag{A.2}$$

for all $f \in L_1(B(M), \sigma)$; in particular, inserting $f = \mathbf{1}_{B(A)}$ for a Borel set $A \subseteq M$ one obtains $\sigma(B(A)) = \sigma(A)$. Here we suppose as in part (c) that M is compact; then one easily sees that $B(M)$ is a compact k-dimensional submanifold of $\mathbb{R}^n$ as well. We point out that on the left-hand side of (A.2) σ denotes the surface measure on $B(M)$, and on the right-hand side the surface measure on M. The basic observation for the proof of (A.2) is the identity

$$\gamma(BA) = \det\bigl((BA)^\top BA\bigr) = \det\bigl(A^\top B^\top BA\bigr) = \gamma(A) \qquad (A \in \mathbb{R}^{n\times k}).$$

If $\Phi\colon T \to W \subseteq M$ is a local parametrisation of M, then obviously $B \circ \Phi\colon T \to B(W) \subseteq B(M)$ is a local parametrisation of $B(M)$, and the above identity implies $\gamma\bigl((B\circ\Phi)'(t)\bigr) = \gamma(\Phi'(t))$. Then it is straightforward to derive (A.2). △

In the proof of Gauss' theorem we will apply the formula (A.1) to the special case when $k = n-1$ and $W \subseteq M$ is a standard C^1-graph. With the notation introduced in Section 7.1, a parametrisation is given by $\Phi\colon W' \to W,\ y \mapsto \begin{pmatrix} y \\ g(y) \end{pmatrix}$, where $W' \subseteq \mathbb{R}^{n-1}$ is an open set and $g \in C^1(W')$. In order to evaluate (A.1) we have to determine $\gamma(\Phi'(y))$. Now $\Phi'(y) = \begin{pmatrix} E_{n-1} \\ g'(y) \end{pmatrix}$, hence

$$\Phi'(y)^\top \Phi'(y) = \begin{pmatrix} E_{n-1} & \nabla g(y) \end{pmatrix} \begin{pmatrix} E_{n-1} \\ g'(y) \end{pmatrix} = E_{n-1} + \nabla g(y) g'(y),$$

and the subsequent Lemma A.2 implies that $\gamma(\Phi'(y))^2 = 1 + |\nabla g(y)|^2$. Thus from (A.1) one obtains

$$\int_W f(z)\,\mathrm{d}\sigma(z) = \int_{W'} f(\Phi(y)) \sqrt{1 + |\nabla g(y)|^2}\,\mathrm{d}y. \tag{A.3}$$

A.2 Lemma. *Let $k \in \mathbb{N}$, $v \in \mathbb{R}^k$, and let $E_k \in \mathbb{R}^{k\times k}$ denote the unit matrix. Then* $\det(E_k + vv^\top) = 1 + |v|^2$.

Proof. Without loss of generality $v \neq 0$. Then v is an eigenvector of the matrix $E_k + vv^\top$ with eigenvalue $1 + |v|^2$, whereas the $(k-1)$-dimensional orthogonal complement $v^\perp$ is the eigenspace of $E_k + vv^\top$ associated with the eigenvalue 1. This implies the assertion. □

A.2 Proof of the divergence theorem

Throughout this section we suppose that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with C^1-boundary. We will prove Gauss' theorem, Theorem 7.3, in the equivalent form of the divergence theorem. (Concerning the equivalence we observe that the choice $u(x) = v(x)e_j$ in (A.4) yields (7.1) for the scalar function v; conversely, applying (7.1) to the components of the vector field one obtains (A.4).)

A.3 Theorem (Divergence theorem). *Let $u \in C^1(\overline{\Omega}; \mathbb{K}^n)$. Then*

$$\int_\Omega \operatorname{div} u(x)\,\mathrm{d}x = \int_{\partial\Omega} u(z) \cdot \nu(z)\,\mathrm{d}\sigma(z), \tag{A.4}$$

where $\nu(z)$ is the outer unit normal at $z \in \partial\Omega$.

The following local part of the divergence theorem contains the main technical difficulty of the proof.

A.4 Proposition. *Let $W \subseteq \partial\Omega$ be a C^1-graph. Then there exists an open set $U \subseteq \mathbb{R}^n$ with the following properties: $W = U \cap \partial\Omega$, and if $u \in C^1(\overline{\Omega}; \mathbb{K}^n)$ is such that* $\operatorname{spt} u$ *(formed in $\overline{\Omega}$) is a compact subset of U, then*

$$\int_\Omega \operatorname{div} u(x)\,\mathrm{d}x = \int_{\partial\Omega} u(z) \cdot \nu(z)\,\mathrm{d}\sigma(z), \tag{A.5}$$

where $\nu(z)$ is the outer unit normal at $z \in \partial\Omega$.

Proof. (i) In this step we suppose that W is a standard C^1-graph. Let $W' \subseteq \mathbb{R}^{n-1}$, (a,b) and $g\colon W' \to (a,b)$ be as in Section 7.1. Put $U := W' \times (a,b)$, and let $u \in C^1(\overline{\Omega};\mathbb{K}^n)$, $\operatorname{spt} u \subseteq U$.

Recall from Remark 7.2 that the outer unit normal is given by

$$\nu(z) = \frac{1}{\sqrt{|\nabla g(y)|^2+1}} \begin{pmatrix} -\nabla g(y) \\ 1 \end{pmatrix}$$

for $z = (y, g(y)) \in W$, with $y \in W'$. Thus, applying (A.3) we can rewrite the right-hand side of (A.5) as

$$\begin{aligned} \int_{\partial\Omega} u(z)\cdot\nu(z)\,\mathrm{d}\sigma(z) &= \int_{W'} u(y,g(y))\cdot\frac{1}{\sqrt{|\nabla g(y)|^2+1}}\begin{pmatrix} -\nabla g(y) \\ 1 \end{pmatrix}\sqrt{1+|\nabla g(y)|^2}\,\mathrm{d}y \\ &= \int_{W'} u(y,g(y))\cdot\begin{pmatrix} -\nabla g(y) \\ 1 \end{pmatrix}\mathrm{d}y. \end{aligned}$$

In order to transform the left-hand side of (A.5) into the latter expression we extend u by $0 \in \mathbb{K}^n$ to a function on $\mathbb{R}^n$ and define $v\colon W' \times (-\infty, 0] \to \mathbb{K}^n$,

$$v(y,t) := u(y, t+g(y)).$$

From $u(y,t) = v(y, t-g(y))$ we then obtain

$$\operatorname{div} u(y,t) = \sum_{j=1}^{n-1} \partial_j v_j(y, t-g(y)) + \partial_n v(y,t-g(y))\cdot\begin{pmatrix} -\nabla g(y) \\ 1 \end{pmatrix}.$$

Therefore

$$\begin{aligned} \int_\Omega \operatorname{div} u(x)\,\mathrm{d}x &= \int_{W'}\int_{-\infty}^{g(y)} \operatorname{div} u(y,t)\,\mathrm{d}t\,\mathrm{d}y \\ &= \int_{W'}\int_{-\infty}^{0} \operatorname{div} u(y,t+g(y))\,\mathrm{d}t\,\mathrm{d}y \qquad (A.6) \\ &= \sum_{j=1}^{n-1}\int_{-\infty}^{0}\int_{W'} \partial_j v_j(y,t)\,\mathrm{d}y\,\mathrm{d}t + \int_{W'}\int_{-\infty}^{0} \partial_n v(y,t)\cdot\begin{pmatrix} -\nabla g(y) \\ 1 \end{pmatrix}\mathrm{d}t\,\mathrm{d}y. \end{aligned}$$

Since $\int_{W'} \partial_j \varphi(y)\,\mathrm{d}y = 0$ for all $\varphi \in C^1_{\mathrm{c}}(W')$ (see (4.1)), we conclude that the left-hand term in the last expression vanishes. In the right-hand term we apply the fundamental theorem of calculus; then we can continue (A.6) by

$$\int_\Omega \operatorname{div} u(x)\,\mathrm{d}x = \int_{W'} v(y,0)\cdot\begin{pmatrix} -\nabla g(y) \\ 1 \end{pmatrix}\mathrm{d}y = \int_{W'} u(y,g(y))\cdot\begin{pmatrix} -\nabla g(y) \\ 1 \end{pmatrix}\mathrm{d}y.$$

(ii) Now we treat the case of a general C^1-graph W. By definition, there exists an orthogonal matrix $B \in \mathbb{R}^{n\times n}$ such that $\widetilde{W} := B(W)$ is a standard C^1-graph with respect

to $\widetilde{\Omega} := B(\Omega)$. By step (i) there exists an open set $\widetilde{U} \subseteq \mathbb{R}^n$ with the following properties: $\widetilde{W} = \widetilde{U} \cap \partial\widetilde{\Omega}$, and

$$\int_{\widetilde{\Omega}} \operatorname{div} \tilde{u}(x)\,\mathrm{d}x = \int_{\partial\widetilde{\Omega}} \tilde{u}(z) \cdot \tilde{\nu}(z)\,\mathrm{d}\sigma(z) \tag{A.7}$$

for all $\tilde{u} \in C^1\big(\overline{\widetilde{\Omega}}; \mathbb{K}^n\big)$ with $\operatorname{spt} u \subseteq \widetilde{U}$, where $\tilde{\nu}$ denotes the outer unit normal for the set $\widetilde{\Omega}$.

Put $U := B^{-1}(\widetilde{U})$ and note that $W = U \cap \partial\Omega$. Let $u \in C^1(\overline{\Omega}; \mathbb{K}^n)$ be such that $\operatorname{spt} u \subseteq U$, and put $\tilde{u} := B \circ u \circ B^{-1}$. Then $\tilde{u} \in C^1\big(\overline{\widetilde{\Omega}}; \mathbb{K}^n\big)$ and $\operatorname{spt} u \subseteq \widetilde{U}$, so $\tilde{u}$ satisfies (A.7). From Exercise A.3 and the invariance of the Lebesgue measure under orthogonal transformations we obtain

$$\int_{\widetilde{\Omega}} \operatorname{div} \tilde{u}(x)\,\mathrm{d}x = \int_{\widetilde{\Omega}} \operatorname{div} u(B^{-1}x)\,\mathrm{d}x = \int_{\Omega} \operatorname{div} u(x)\,\mathrm{d}x.$$

On the other hand $\tilde{\nu}(Bz) = B\nu(z)$ for all $z \in W$, and the invariance of the measure σ under orthogonal transformations (see Remark A.1(d)) implies that

$$\begin{aligned}\int_{\partial\widetilde{\Omega}} \tilde{u}(z) \cdot \tilde{\nu}(z)\,\mathrm{d}\sigma(z) &= \int_{\partial\Omega} \tilde{u}(Bz) \cdot \tilde{\nu}(Bz)\,\mathrm{d}\sigma(z) = \int_{\partial\Omega} Bu(z) \cdot B\nu(z)\,\mathrm{d}\sigma(z)\\ &= \int_{\partial\Omega} u(z) \cdot \nu(z)\,\mathrm{d}\sigma(z).\end{aligned}$$

Plugging the above equalities into (A.7) we obtain (A.5). □

Proof of Theorem A.3. Let $y \in \partial\Omega$. Since Ω has C^1-boundary we can find a C^1-graph $W_y \subseteq \partial\Omega$ containing y. We choose an open set $U_y \subseteq \mathbb{R}^n$ corresponding to W_y with the properties described in Proposition A.4.

For the open covering $\{U_y;\ y \in \partial\Omega\}$ of the compact set $\partial\Omega$ there exists a finite subcovering $\{U_y;\ y \in F\}$. Supplementing this covering by the set Ω, we obtain a finite open covering of $\overline{\Omega}$ and can choose a subordinate partition of unity on $\overline{\Omega}$, i.e., functions $0 \leqslant \varphi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{spt}\varphi \subseteq \Omega$, $0 \leqslant \varphi_y \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{spt}\varphi_y \subseteq U_y$ for $y \in F$, and such that $\varphi + \sum_{y\in F} \varphi_y = 1$ on $\overline{\Omega}$; see Exercise 4.3(b).

Now let $u \in C^1(\overline{\Omega}; \mathbb{K}^n)$. Then $\int_\Omega \operatorname{div}(\varphi u)(x)\,\mathrm{d}x = 0$ by (4.1), and

$$\begin{aligned}\int_\Omega \operatorname{div} u(x)\,\mathrm{d}x &= \int_\Omega \operatorname{div}\Big(\big(\varphi + \sum_{y\in F} \varphi_y\big)u\Big)(x)\,\mathrm{d}x\\ &= \int_\Omega \operatorname{div}(\varphi u)(x)\,\mathrm{d}x + \sum_{y\in F} \int_\Omega \operatorname{div}(\varphi_y u)(x)\,\mathrm{d}x\\ &= \sum_{y\in F} \int_{\partial\Omega} \varphi_y(z) u(z) \cdot \nu(z)\,\mathrm{d}\sigma(z),\end{aligned}$$

where in the last step we have applied Proposition A.4. This shows (A.4) since $\sum_{y\in F} \varphi_y = 1$ on $\partial\Omega$, and the proof of the divergence theorem is complete. □

A.5 Remark. In Appendix E, Section E.2, it is shown that in fact the surface measure σ coincides with the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$. We avoided using this property because we wanted to stick to the more elementary definition of the surface measure by parametrisations. △

Notes

The divergence theorem plays an important role for applications of the theory of forms. In this book we rely on the theorem whenever traces and weak normal derivatives are involved; see Chapters 7, 8, 11 and 12. More generally, it is central for the theory of partial differential equations.

We apply the divergence theorem only in the elementary form of Theorem A.3, for bounded open sets with C^1-boundary. It was hard to find an accessible source for this version in the English literature. So, according to our principle to provide complete information for topics going beyond a certain level, we decided to include a complete proof.

We mention that in German textbooks on analysis one can find proofs; see [For83; §15], [Kab99; Theorem 20.3]. The main inspiration for our proof was the proof presented in [Tre75; Lemma 10.1]. A different access can be found in [ArUr23; Chapter 7]. For a more general version of the divergence theorem in the context of geometric measure theory we refer to [EvGa92; 5.8, Theorem 1].

Exercises

A.1 Let (X, τ) and (Y, ρ) be topological spaces, $\Phi\colon X \to Y$ a homeomorphism. Let $\mathcal{B}_\tau$ and $\mathcal{B}_\rho$ denote the σ-algebra of Borel subsets of X and Y, respectively.

(a) Show that $\mathcal{B}_\rho = \{\Phi(A);\, A \in \mathcal{B}_\tau\}$.

(b) Let $f\colon Y \to \mathbb{K}$. Show that f is Borel measurable on Y if and only if $f \circ \Phi$ is Borel measurable on X.

A.2 Let Ω be a set, $\mathcal{A}$ a σ-algebra on Ω. Let $\Omega_1, \dots, \Omega_m \in \mathcal{A}$ be such that $\bigcup_{j=1}^m \Omega_j = \Omega$. For each $j \in \{1, \dots, m\}$ let μ_j be a measure on $\mathcal{A}_j := \{A \in \mathcal{A};\, A \subseteq \Omega_j\}$ with $\mu_j(\Omega_j) < \infty$. Suppose that the measures μ_j are mutually consistent, i.e. for all $j, k \in \{1, \dots, m\}$, $A \in \mathcal{A}_j \cap \mathcal{A}_k$ one has $\mu_j(A) = \mu_k(A)$.

Show that there exists a unique measure μ on $\mathcal{A}$ such that $\mu(A) = \mu_j(A)$ for all $j = 1, \dots, m$, $A \in \mathcal{A}_j$.

Hint: Choose a family $(B_j)_{j=1,\dots,m}$ in $\mathcal{A}$ such that $B_j \in \mathcal{A}_j$ for all $j = 1, \dots, m$, $\bigcup_{j=1}^m B_j = \Omega$ and $B_j \cap B_k = \varnothing$ if $j \neq k$. Define $\mu(A) := \sum_{j=1}^m \mu_j(A \cap B_j)$.

A.3 Let $\Omega \subseteq \mathbb{R}^n$ be open, $u \in C^1(\Omega; \mathbb{K}^n)$, and let $B \in \mathbb{R}^{n\times n}$ be an invertible matrix, $\widetilde{\Omega} := B(\Omega)$. Put $\tilde{u} := B \circ u \circ B^{-1} \in C^1(\widetilde{\Omega}; \mathbb{K}^n)$, spelled out

$$\tilde{u}(y) := Bu(B^{-1}y) \qquad (y \in \widetilde{\Omega}).$$

Show that $\operatorname{div} \tilde{u} = (\operatorname{div} u) \circ B^{-1}$. (This fact is used in the proof of Proposition A.4 for orthogonal matrices B.)

Hints: 1. For $u \in C^1(\Omega; \mathbb{K}^n)$ the divergence can be written as $\operatorname{div} u(x) = \operatorname{tr} u'(x)$, where the latter expression denotes the trace of the derivative (Jacobi matrix) of $u'(x)$. 2. For matrices $A \in \mathbb{K}^{m\times n}$, $B \in \mathbb{K}^{n\times m}$ one has $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Appendix B

The Stone–Weierstrass theorem

The aim of this appendix is to prove the Stone–Weierstrass theorem, Theorem B.2. The proof is based on Stone's theorem, Theorem B.1, which is interesting and important in its own right. In a supplementary remark we mention that our treatment also provides a proof of the Weierstrass approximation theorem.

Throughout this appendix let K be a compact topological space. The norm $\|\cdot\|$ on $C(K)$ will be the supremum norm. For real-valued functions $f, g \in C(K)$ we define $f \vee g, f \wedge g \in C(K)$, $f \vee g(x) := \max\{f(x), g(x)\}$, $f \wedge g(x) := \min\{f(x), g(x)\}$ $(x \in K)$. (With these operations, the space $C(K;\mathbb{R})$ becomes a vector lattice.) If $F \subseteq C(K)$ is a finite subset, then $\bigvee F \in C(K)$ is defined by $\big(\bigvee F\big)(x) := \max\{f(x);\ f \in F\}$ $(x \in K)$, and similarly for '$\bigwedge$'.

A **sublattice** of $C(K;\mathbb{R})$ is a subset L with the property that $f \vee g, f \wedge g \in L$ for all $f, g \in L$. A **vector sublattice** of $C(K;\mathbb{R})$ is a subspace which is also a sublattice.

If L is a subset of $C(K)$, then we say that L **separates the points** of K if for all $x \neq y$ in K there exists a function $f \in L$ such that $f(x) \neq f(y)$.

B.1 Theorem (Stone). *Let L be a vector sublattice of $C(K;\mathbb{R})$ such that*

(i) *L separates the points of K,*

(ii) *L contains the constant function $\mathbf{1}$.*

Then L is dense in $C(K;\mathbb{R})$.

Proof. First we note that, given points $x \neq y$ in K and scalars $\alpha, \beta \in \mathbb{R}$, there exists $f \in L$ such that $f(x) = \alpha$, $f(y) = \beta$. Indeed, there exists $g \in L$ such that $g(x) \neq g(y)$, and then f can be obtained as a linear combination of g and $\mathbf{1}$.

Let $f \in C(K;\mathbb{R})$, and let $\varepsilon > 0$. We need to show that there exists $h \in L$ with $\|f - h\| \leqslant \varepsilon$. First, fix $x \in K$. For all $y \in K$ there exists $g_y \in L$ such that $g_y(x) = f(x)$, $g_y(y) = f(y)$. Then $U_y := [g_y > f - \varepsilon]$ is an open neighbourhood of y and from the open covering $\{U_y;\ y \in K\}$ of K we can choose a finite subcovering $\{U_y;\ y \in F\}$. Putting $h_x := \bigvee_{y\in F} g_y \in L$ we have found a function satisfying $h_x(x) = f(x)$, $h_x \geqslant f - \varepsilon$.

For all $x \in K$, the set $V_x := [h_x < f + \varepsilon]$ is an open neighbourhood of x. The open covering $\{V_x;\ x \in K\}$ of K contains a finite subcovering $\{V_x;\ x \in G\}$, and then the function $h := \bigwedge_{x\in G} h_x \in L$ satisfies $f - \varepsilon \leqslant h \leqslant f + \varepsilon$, i.e. $\|f - h\| \leqslant \varepsilon$. □

B.2 Theorem (Stone–Weierstrass). *Let L be a subalgebra of $C(K;\mathbb{K})$ such that*

(i) *L separates the points of K,*

(ii) *L contains the constant function $\mathbf{1}$.*

If $\mathbb{K} = \mathbb{C}$, we suppose additionally that
(iii) *for all $f \in L$ one has $\overline{f} \in L$.*
Then L is dense in $C(K;\mathbb{K})$.

Proof. Without loss of generality we can assume that L is closed. Indeed, it is easy to see that the closure of L is again a subalgebra of $C(K;\mathbb{K})$ satisfying the conditions (i), (ii) (and (iii)).

First we treat the case $\mathbb{K} = \mathbb{R}$. We show that L is a sublattice of $C(K;\mathbb{R})$; then Theorem B.1 implies the assertion. Let $f \in L$. By the Weierstrass approximation theorem there exists a sequence (p_n) of real polynomials such that $\sup_{|t|\leqslant\|f\|} \big||t| - p_n(t)\big| \to 0$ as $n \to \infty$. Then $\big\||f| - p_n \circ f\big\| \to 0$ as $n \to \infty$. Since $p_n \circ f \in L$ for all $n \in \mathbb{N}$, we conclude that $|f| \in L$. This has the consequence that for all $f, g \in L$ one has

$$f \vee g = \tfrac{1}{2}(f + g + |f - g|) \in L, \quad f \wedge g = \tfrac{1}{2}(f + g - |f - g|) \in L,$$

i.e. L is a sublattice of $C(K;\mathbb{R})$.

In the complex case one immediately deduces that $L_{\mathrm{r}} := L \cap C(K;\mathbb{R})$ is a subalgebra of $C(K;\mathbb{R})$ satisfying conditions (i) and (ii); hence the case treated above implies that L_{r} is dense in $C(K;\mathbb{R})$. Thus $L = L_{\mathrm{r}} + \mathrm{i}L_{\mathrm{r}}$ is dense in $C(K;\mathbb{C})$. □

B.3 Remarks. (a) In the proof of Theorem B.2 we quoted the Weierstrass approximation theorem, but in fact we did not need the full extent of the theorem since we only had to approximate the absolute value function on bounded intervals. Here we show how this can be done.

From basic analysis we recall that the power series

$$(1 + s)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} s^n$$

is uniformly absolutely convergent on $[-1, 1]$. Therefore the expression

$$|t| = \big(1 + (t^2 - 1)\big)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (t^2 - 1)^n$$

yields a series of polynomials converging to $|t|$, uniformly for $-1 \leqslant t \leqslant 1$. Clearly this implies that $|t|$ can be approximated uniformly on arbitrary bounded intervals.

(b) The Weierstrass approximation theorem can be obtained as a special case of Theorem B.2 if in the proof one uses the argument from part (a) to approximate the absolute value function. △

Notes

The foundations for the use of order and lattice properties in the approximation of continuous functions as in Theorems B.1 and B.2 go back to Stone [Sto37]; see in particular the discussion on the Weierstrass approximation theorem in [Sto37; p. 467]. A formulation of Theorem B.1 can be found in Kakutani [Kak41; Theorem 4], and [Sto48] contains further discussion of the topic.

Appendix C

Weyl's lemma

Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $u \in L_{1,\mathrm{loc}}(\Omega)$ satisfy $\Delta u = 0$ in the distributional sense. The issue of this appendix is to show that then u is a.e. equal to a harmonic function $\tilde{u}$, i.e. $\tilde{u} \in C^2(\Omega)$, $\Delta\tilde{u} = 0$.

C.1 Proposition. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $u \in C^2(\Omega)$, $\Delta u = 0$. Then u has the **mean value property**: if $x \in \Omega$, $r > 0$ are such that $B[x,r] \subseteq \Omega$, then*

$$u(x) = \frac{1}{r^{n-1}\sigma_{n-1}} \int_{\partial B(x,r)} u(y)\,\mathrm{d}\sigma(y) = \frac{1}{\sigma_{n-1}} \int_{S_{n-1}} u(x+r\xi)\,\mathrm{d}\sigma(\xi), \tag{C.1}$$

where σ_{n-1} is the $(n-1)$-dimensional volume of the unit sphere $S_{n-1} \subseteq \mathbb{R}^n$.

Proof. For $x \in \mathbb{R}^n$, $r > 0$ such that $B[x,r] \subseteq \Omega$ we compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r} \int_{S_{n-1}} u(x+r\xi)\,\mathrm{d}\sigma(\xi) &= \int_{S_{n-1}} \nabla u(x+r\xi)\cdot\xi\,\mathrm{d}\sigma(\xi) \\ &= \frac{1}{r^{n-1}} \int_{\partial B(x,r)} \nabla u(y)\cdot\nu(y)\,\mathrm{d}\sigma(y) \\ &= \frac{1}{r^{n-1}} \int_{B(x,r)} \Delta u(y)\,\mathrm{d}y = 0, \end{aligned} \tag{C.2}$$

where in the last step we have applied the divergence theorem, Theorem A.3. It follows that the expression on the right-hand side of (C.1) does not depend on r. This expression tends to $u(x)$ as $r \to 0$, which shows the assertion. □

C.2 Theorem (Weyl's lemma). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $u \in L_{1,\mathrm{loc}}(\Omega)$, $\Delta u = 0$ (in the distributional sense). Then there exists an infinitely differentiable harmonic representative $\tilde{u}$ of u.*

Proof. Below we show that for every $x \in \Omega$ there exist an open neighbourhood $U_x \subseteq \Omega$ and a representative $u_x \in C^\infty(U_x)$ of $u|_{U_x}$. Applying Exercise C.1 one then obtains an infinitely differentiable representative $\tilde{u}$ of u, and Remark 4.6 implies that $\tilde{u}$ is harmonic.

Let $x_0 \in \Omega$, and let $r_0 > 0$ be such that $B[x_0, 3r_0] \subseteq \Omega$. With $U_x := B(x_0, r_0)$ we show the property asserted above. For simplicity, and without loss of generality, we assume that $x_0 = 0$ and $r_0 = 1$.

Let $(\rho_k)_{k\in\mathbb{N}}$ be a delta sequence in $C_c^\infty(\mathbb{R}^n)$, and define $u_k\colon B(0,2)\to\mathbb{K}$ by

$$u_k(x) := \int_{B(0,3)} \rho_k(x-y)u(y)\,\mathrm{d}y \qquad (|x|<2,\ k\in\mathbb{N}).$$

Then $u_k\to u$ in $L_1(B(0,2))$, by Lemma 7.6(b). Moreover $u_k\in C^\infty(B(0,2))$,

$$\Delta u_k(x) = \int_{B(0,3)} \Delta\rho_k(x-y)u(y)\,\mathrm{d}y = \int_{B(0,3)} \rho_k(x-y)\Delta u(y)\,\mathrm{d}y = 0$$

for all $x\in B(0,2)$, where the middle equality holds because $\rho_k(x-\cdot)\in C_c^\infty(B(0,3))$. Hence Proposition C.1 implies that u_k satisfies the mean value property, for all $k\in\mathbb{N}$.

Let $\rho\in C_c^\infty(\mathbb{R}^n)_+$ with $\operatorname{spt}\rho\subseteq B(0,1)$, $\int\rho(x)\,\mathrm{d}x=1$, be a radial function, i.e. $\rho(x)=\rho(|x|e_1)$ for all $x\in B(0,1)$, where e_1 is the first unit vector in $\mathbb{R}^n$. Then the mean value property of u_k implies that, for $x\in B(0,1)$,

$$\begin{aligned}\rho * u_k(x) &= \int_{B(0,1)} \rho(y)u_k(x-y)\,\mathrm{d}y = \int_{r=0}^1 \rho(re_1)\int_{\partial B(0,r)} u_k(x-y)\,\mathrm{d}\sigma(y)\,\mathrm{d}r\\ &= \Big(\sigma_{n-1}\int_0^1 \rho(re_1)r^{n-1}\,\mathrm{d}r\Big)u_k(x) = \Big(\int_{B(0,1)}\rho(y)\,\mathrm{d}y\Big)u_k(x) = u_k(x).\end{aligned}$$

The convergence $u_k\to u$ in $L_1(B(0,2))$ implies that $(\rho * u_k)_k$ converges uniformly on $B(0,1)$ to $\rho * u$ (defined analogously to $\rho * u_k$ above); hence the C^∞-function $\rho * u$ is a representative of $u|_{B(0,1)}$. □

Notes

Theorem C.2 goes back to Hermann Weyl [Wey40; Lemma 2]. In modern language of partial differential operators one could state it by saying that the Laplace operator is a *hypoelliptic* differential operator. The mean value property of harmonic functions is a standard fact in potential theory and can be found in the standard literature on partial differential equations; see, for instance, [Eva10; Section 2.2.2]. We use the mean value property of harmonic functions and properties of convolutions to provide a simple direct proof of Weyl's lemma. We refer to [Rud91; Corollary of Theorem 8.12], where Weyl's lemma is obtained as a consequence of an important result concerning the regularity of distributional solutions of general elliptic equations.

We mention that Weyl's lemma also holds for the case when u is a distribution instead of a locally integrable function; with vitually the same proof as given for Theorem C.2.

Exercises

C.1 Let $\Omega\subseteq\mathbb{R}^n$ be an open set, and let $u\in L_{1,\mathrm{loc}}(\Omega)$. Suppose that for all $x\in\Omega$ there exist an open neighbourhood $U_x\subseteq\Omega$ and a function $u_x\in C(U_x)$ such that $u_x=u$ a.e. on U_x. Show that there exists a function $\tilde u\in C(\Omega)$ such that $\tilde u|_{U_x}=u_x$ for all $x\in\Omega$. Apply Exercise 4.1 to show that $\tilde u$ is a representative of u.

Conclude that $\tilde u$ is infinitely differentiable if every u_x is infinitely differentiable.

C.2 Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $u \in C(\Omega)$ be a function having the mean value property (C.1). Show that then u is an infinitely differentiable harmonic function. (Hint: For the property that $u \in C^\infty(\Omega)$ look at the proof of Theorem C.2; for 'harmonic' use (C.2) and argue by contradiction.)

C.3 Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set.

(a) Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be harmonic on Ω. Show the **maximum principle**

$$\max_{x \in \overline{\Omega}} |u(x)| = \max_{x \in \partial\Omega} |u(x)|.$$

(Hint: Use the mean value property (C.1).)

(b) Show that

$$C_\Delta(\overline{\Omega}) := \big\{ u \in C(\overline{\Omega});\, u|_\Omega \text{ harmonic} \big\}$$

is a closed subspace of $C(\overline{\Omega})$, and conclude that $\big\{ u|_{\partial\Omega};\, u \in C_\Delta(\overline{\Omega}) \big\}$ is a closed subspace of $C(\partial\Omega)$. (Hint concerning the last statement: Part (a) shows that the trace mapping $T\colon C_\Delta(\overline{\Omega}) \to C(\partial\Omega),\ u \mapsto u|_{\partial\Omega}$ is a linear isometry.)

Comment. With the trace mapping T from above, the image $T(C_\Delta(\overline{\Omega}))$ is the set of 'admissible boundary values for the classical Dirichlet problem'. For $g \in T(C_\Delta(\overline{\Omega}))$ the Dirichlet problem $\Delta u = 0$, $u|_{\partial\Omega} = g$ has the unique solution $u = T^{-1}g$, and the mapping T^{-1} is an isometry, in particular continuous.

Appendix D

Hausdorff measure and an inequality due to Maz'ya

D.1 The Sobolev space $W_1^1(\Omega)$

We will need the Sobolev space $W_1^1(\Omega)$ and some of its properties. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. We define

$$W_1^1(\Omega) := \big\{u \in L_1(\Omega);\, \partial_j u \in L_1(\Omega)\ (j \in \{1,\dots,n\})\big\},$$

with norm

$$\|u\|_{1,1} := \|u\|_1 + \sum_{j=1}^n \|\partial_j u\|_1 .$$

Then the space $W_1^1(\Omega)$ is a separable Banach space. This is proved in the same way as the corresponding property for $H^1(\Omega)$; see Theorem 4.9 and its proof.

Analogously to H_{c}^1 and H_0^1 we define

$$W_{1,\mathrm{c}}^1(\Omega) := \big\{u \in W_1^1(\Omega);\, \operatorname{spt} u \text{ compact in } \Omega\big\},$$
$$W_{1,0}^1(\Omega) := \overline{W_{1,\mathrm{c}}^1(\Omega)}^{W_1^1(\Omega)}.$$

The property $W_{1,0}^1(\Omega) = \overline{C_{\mathrm{c}}^\infty(\Omega)}^{W_1^1(\Omega)}$ is proved in the same way as the corresponding property for H^1; see Theorem 4.14 and its proof.

We will need some lattice properties of W_1^1-functions. These are consequences of the properties treated in Section 9.3.

D.1 Lemma. *Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $u, v \in L_{1,\mathrm{loc}}(\Omega;\mathbb{R})$, $\nabla u, \nabla v \in L_{1,\mathrm{loc}}(\Omega;\mathbb{R}^n)$. Then $\nabla(u \wedge v) \in L_{1,\mathrm{loc}}(\Omega;\mathbb{R}^n)$, $|\nabla(u \wedge v)| \leqslant |\nabla u| + |\nabla v|$.*

Proof. Let $j \in \{1,\dots,n\}$. From Corollary 9.15 we recall $\partial_j u^+ = \mathbf{1}_{[u>0]}\partial_j u$. This implies

$$\partial_j u^- = \partial_j(u^+ - u) = (\mathbf{1}_{[u>0]} - 1)\partial_j u = -\mathbf{1}_{[u\leqslant 0]}\partial_j u;$$

hence

$$\partial_j |u| = \partial_j u^+ - \partial_j u^- = \big(\mathbf{1}_{[u>0]} - \mathbf{1}_{[u\leqslant 0]}\big)\partial_j u.$$

Applying this equality to $u \wedge v = \frac{1}{2}(u + v - |u - v|)$ we obtain

$$\partial_j(u \wedge v) = \frac{1}{2}\big(\partial_j u + \partial_j v - (\mathbf{1}_{[u>v]} - \mathbf{1}_{[u\leqslant v]})(\partial_j u - \partial_j v)\big) = \mathbf{1}_{[u\leqslant v]}\partial_j u + \mathbf{1}_{[u>v]}\partial_j v.$$

We conclude that

$$\nabla(u \wedge v) = \mathbf{1}_{[u\leqslant v]}\nabla u + \mathbf{1}_{[u>v]}\nabla v,$$

which implies the assertion. □

Lemma D.1 will be applied in the proof of Theorem D.9 in the following more specific situation.

D.2 Lemma. *Let $\Omega, \Omega_1 \subseteq \mathbb{R}^n$ be open sets, Ω_1 bounded, let $u \in C(\overline{\Omega}) \cap W_1^1(\Omega)$, $v \in C(\overline{\Omega_1}) \cap W_1^1(\Omega_1)$, $u, v \geqslant 0$, and assume that $u(x) < v(x)$ for all $x \in \partial\Omega_1 \cap \overline{\Omega}$. Put $v := \infty$ on $\mathbb{R}^n \setminus \overline{\Omega_1}$, and define $w := u \wedge v$ on $\overline{\Omega}$. Then $w \in C(\overline{\Omega}) \cap W_1^1(\Omega)$,*

$$\int_\Omega |\nabla w(x)|\,\mathrm{d}x \leqslant \int_\Omega |\nabla u(x)|\,\mathrm{d}x + \int_{\Omega_1} |\nabla v(x)|\,\mathrm{d}x. \tag{D.1}$$

Proof. The continuity of w follows from the continuity of w on $\overline{\Omega} \cap \overline{\Omega_1}$ and the continuity of $w = u$ on $\overline{\Omega} \setminus \Omega_1$. (Note that $\{\overline{\Omega} \cap \overline{\Omega_1}, \overline{\Omega} \setminus \Omega_1\}$ is a finite covering of $\overline{\Omega}$ by closed sets.)

Lemma D.1 shows that $w|_{\Omega\cap\Omega_1} \in W_1^1(\Omega \cap \Omega_1)$,

$$\int_{\Omega\cap\Omega_1} |\nabla w(x)|\,\mathrm{d}x \leqslant \int_{\Omega\cap\Omega_1} |\nabla u(x)|\,\mathrm{d}x + \int_{\Omega_1} |\nabla v(x)|\,\mathrm{d}x. \tag{D.2}$$

The hypotheses imply that there exists an open set $U \subseteq \Omega$ with $\partial\Omega_1 \cap \Omega \subseteq U$ such that $u(x) < v(x)$ for all $x \in U \cap \overline{\Omega_1}$. Then $w = u$ on $(\Omega \setminus \Omega_1) \cup U$, and therefore $w \in W_1^1(\Omega)$,

$$\int_\Omega |\nabla w(x)|\,\mathrm{d}x = \int_{\Omega\setminus\Omega_1} |\nabla u(x)|\,\mathrm{d}x + \int_{\Omega\cap\Omega_1} |\nabla w(x)|\,\mathrm{d}x. \tag{D.3}$$

Combining (D.2) and (D.3) one obtains (D.1). □

D.3 Proposition. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, $u \in W_1^1(\Omega) \cap C_0(\Omega)$. Then $u \in W_{1,0}^1(\Omega)$.*

Proof. It is sufficient to treat the case when u is real-valued. Splitting $u = u^+ - u^-$ we reduce the problem to the case $u \geqslant 0$.

For $s > 0$ one has $u - s \in W_1^1(\Omega)$, and Corollary 9.15 implies $\nabla(u-s)^+ = \mathbf{1}_{[u>s]}\nabla(u-s) = \mathbf{1}_{[u>s]}\nabla u$. Note that $u \leqslant s$ on a neighbourhood of $\partial\Omega$ because $u \in C_0(\Omega)$, and therefore $\operatorname{spt}(u-s)^+$ is compact, which shows that $(u-s)^+ \in W_{1,\mathrm{c}}^1(\Omega)$. For $s \to 0$ one has $(u-s)^+ \to u$, $\nabla(u-s)^+ \to \mathbf{1}_{[u>0]}\nabla u$ pointwise and dominated, hence in $L_1(\Omega)$, $L_1(\Omega;\mathbb{R}^n)$, respectively. Because of $\mathbf{1}_{[u>0]}\nabla u = \nabla u^+ = \nabla u$, this means that $(u-s)^+ \to u$ in $W_1^1(\Omega)$, hence $u \in W_{1,0}^1(\Omega)$. □

D.4 Remark. The same result as Proposition D.3 holds with W_1^1 replaced by H^1. This fact is remindful of Theorem 7.9, but note the differences concerning the quality of $\partial\Omega$ and the behaviour of u at the boundary. △

So far, we have treated lattice properties concerning differentiation only for real-valued functions. In order to include complex-valued functions in our treatment, we need the following formula concerning the differentiation of absolute values.

D.5 Proposition. *Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $u \in L_{1,\mathrm{loc}}(\Omega)$, $1 \leqslant j \leqslant n$, $\partial_j u \in L_{1,\mathrm{loc}}(\Omega)$. Then*

$$\partial_j |u| = \mathrm{Re}(\overline{\mathrm{sgn}\, u}\, \partial_j u) \in L_{1,\mathrm{loc}}(\Omega).$$

Proof. Given $\delta > 0$, the equality

$$\partial_j (|u|^2 + \delta)^{1/2} = \partial_j (\overline{u} u + \delta)^{1/2} = (|u|^2 + \delta)^{-1/2} \mathrm{Re}(\overline{u}\, \partial_j u)$$

is straightforward if u is continuously differentiable. Using the same arguments as in the proof of Proposition 9.13 one can easily show that this equality remains valid under the present hypotheses.

Letting $\delta \to 0$ and applying Lemma 4.10 we conclude that

$$\partial_j |u| = \mathrm{Re}\Big(\frac{\overline{u}}{|u|} \mathbf{1}_{[u \neq 0]}\, \partial_j u \Big) = \mathrm{Re}(\overline{\mathrm{sgn}\, u}\, \partial_j u). \qquad \square$$

The last issue in this section is Poincaré's inequality for the L_1-context. Here we say that a set $\Omega \subseteq \mathbb{R}^n$ is **contained in a slab** if there exist $\beta > 0$ such that after a suitable translation and rotation applied to Ω, one has $\Omega \subseteq (-\beta, \beta) \times \mathbb{R}^{n-1}$; in Section 5.4 we had used a more restricted version of this notion. We note that Theorem 5.12 is also valid with the present more general version, and that the proof given below is analogous to the proof of Theorem 5.12.

D.6 Theorem (Poincaré's inequality). *Assume that Ω is contained in a slab. Then*

$$\int_\Omega |u(x)|\, \mathrm{d}x \leqslant c_{\mathrm{P}} \int_\Omega |\nabla u(x)|\, \mathrm{d}x \qquad (u \in W^1_{1,0}(\Omega)),$$

where c_{P} is the width of the slab.

Proof. It was mentioned above that $W^1_{1,0}(\Omega)$ is the closure of $C^\infty_{\mathrm{c}}(\Omega)$ in $W^1_1(\Omega)$; therefore it suffices to prove the inequality for all $u \in C^\infty_{\mathrm{c}}(\Omega)$. Under translations and orthogonal transformations of the independent variable, the gradient of a function is transformed along with the function. (This is obvious for translations. Concerning orthogonal transformations for a function $u \in C^1(\Omega)$ and a matrix $B \in \mathbb{R}^{n \times n}$, the chain rule implies that $\nabla(u \circ B) = \big((u \circ B)'\big)^\top = \big((u' \circ B) B\big)^\top = B^\top (\nabla u \circ B)$.) As a consequence we may assume that $\Omega \subseteq (-\beta, \beta) \times \mathbb{R}^{n-1}$, where $2\beta > 0$ is the width of the slab. Let $h \in C^1[-\beta, \beta]$, $h(-\beta) = 0$. Then we estimate

$$\int_{-\beta}^{\beta} |h(x)|\, \mathrm{d}x = \int_{-\beta}^{\beta} \Big| \int_{-\beta}^{x} h'(y)\, \mathrm{d}y \Big|\, \mathrm{d}x \leqslant \int_{-\beta}^{\beta} \int_{-\beta}^{\beta} |h'(y)|\, \mathrm{d}y\, \mathrm{d}x = 2\beta \int_{-\beta}^{\beta} |h'(y)|\, \mathrm{d}y.$$

Let $u \in C^\infty_{\mathrm{c}}(\Omega)$. Applying the above estimate with $h(r) = u(r, x_2, \ldots, x_n)$ we obtain

$$\int_\Omega |u(x)|\, \mathrm{d}x \leqslant 2\beta \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{-\beta}^{\beta} |\partial_1 u(x_1, \ldots, x_n)|\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \leqslant 2\beta \int_\Omega |\nabla u(x)|\, \mathrm{d}x. \qquad \square$$

D.2 On the Hausdorff measure $\mathcal{H}_d$

In this section we present an auxiliary result concerning the approximation of the Hausdorff measure of dimension $d \in [0,\infty)$ on a metric space M. We start by introducing some notation and giving a short introduction to Hausdorff measures.

For a set $C \subseteq M$ we define $\operatorname{rd}(C) := \frac{1}{2}\operatorname{diam}(C)$, and for a countable collection $\mathcal{C}$ of subsets of M we define $\operatorname{rd}(\mathcal{C}) := \sup_{C\in\mathcal{C}} \operatorname{rd}(C)$ and

$$S_d(\mathcal{C}) := \omega_d \sum_{C\in\mathcal{C}} \operatorname{rd}(C)^d,$$

where

$$\omega_d := \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)},$$

which is the volume of the unit ball in $\mathbb{R}^d$ if $d \in \mathbb{N}_0$. (Even though the notation 'rd' should be remindful of 'radius', the reader should be aware that a set C will not necessarily be contained in a ball with radius $\operatorname{rd}(C)$.)

Let $B \subseteq M$. For $\delta > 0$ we put

$$\mathcal{H}_{d,\delta}(B) := \inf\big\{ S_d(\mathcal{C});\, \mathcal{C} \text{ countable covering of } B,\ \operatorname{rd}(\mathcal{C}) \leqslant \delta \big\}.$$

Then

$$\mathcal{H}^*_d(B) := \lim_{\delta\to 0} \mathcal{H}_{d,\delta}(B) = \sup_{\delta>0} \mathcal{H}_{d,\delta}(B)$$

is the **outer *d*-dimensional Hausdorff measure** of B; note that the limit exists and is equal to the supremum because $\mathcal{H}_{d,\delta}(B)$ is decreasing in δ. Carathéodory's construction of measurable sets yields a measure $\mathcal{H}_d$, the ***d*-dimensional Hausdorff measure**, and it turns out that Borel sets are measurable. For all of these properties (and more) we refer to [EvGa92; Chapter 2] and [Fed69; 2.10.2]. If $d \in \mathbb{N}$, and $E = \mathbb{R}^d \times \{0\} \subseteq \mathbb{R}^n$, then $\mathcal{H}_d$ is the Lebesgue measure on $\mathbb{R}^d \cong E$; see Appendix E.

Observe that, in the definition of $\mathcal{H}_{d,\delta}(B)$, one can also take the infimum over all countable *open* coverings of B and still obtain the same resulting value for $\mathcal{H}^*_d(B)$. Indeed, for $\varepsilon > 0$ there exists a countable covering $\mathcal{C}$ with $\operatorname{rd}(\mathcal{C}) \leqslant \delta/2$ and $S_d(\mathcal{C}) \leqslant \mathcal{H}^*_d(B) + \varepsilon/2$. Choose $(\varepsilon_C)_{C\in\mathcal{C}} \in (0,\infty)^{\mathcal{C}}$ such that $\sum_{C\in\mathcal{C}} \varepsilon_C \leqslant \varepsilon/2$. Then for all $C \in \mathcal{C}$ there exists an open set $U_C \supseteq C$ such that

$$\operatorname{rd}(U_C) \leqslant \delta \quad \text{and} \quad \omega_d \operatorname{rd}(U_C)^d \leqslant \omega_d \operatorname{rd}(C)^d + \varepsilon_C\,.$$

Then $\mathcal{U} := \{U_C;\, C \in \mathcal{C}\}$ is a countable open covering of B with $\operatorname{rd}(\mathcal{U}) \leqslant \delta$ and $S_d(\mathcal{U}) \leqslant \mathcal{H}^*_d(B) + \varepsilon$.

D.7 Proposition. *Let M be a metric space, $d \in [0,\infty)$, and assume that $\mathcal{H}_d(M) < \infty$. Let $\varepsilon > 0$.*

Then for all $\delta > 0$ there exists a countable partition $\mathcal{A}$ of M with $\operatorname{rd}(\mathcal{A}) \leqslant \delta$, consisting of Borel subsets of M and such that

$$\sum_{C\in\mathcal{A}} \big|\mathcal{H}_d(C) - \omega_d \operatorname{rd}(C)^d\big| \leqslant \varepsilon. \tag{D.4}$$

If M is compact, then the partition $\mathcal{A}$ can be chosen finite.

Proof. (i) The definition of $\mathcal{H}_d$ implies that there exists $\delta_\varepsilon > 0$ such that for all countable coverings $\mathcal{C}$ of M with $\operatorname{rd}(\mathcal{C}) \leqslant \delta_\varepsilon$ one has

$$\mathcal{H}_d(M) - \varepsilon \leqslant S_d(\mathcal{C}). \tag{D.5}$$

(ii) Next we show: for all $\delta > 0$ there exists a countable partition $\mathcal{A}$ of M with $\operatorname{rd}(\mathcal{A}) \leqslant \delta$, consisting of Borel subsets of M and such that

$$S_d(\mathcal{A}) \leqslant \mathcal{H}_d(M) + \varepsilon;$$

if M is compact, then the partition $\mathcal{A}$ can be chosen finite.

As pointed out above, there exists a countable open covering $\mathcal{U}$ of M with $\operatorname{rd}(\mathcal{U}) \leqslant \delta$ and $S_d(\mathcal{U}) \leqslant \mathcal{H}_d(M) + \varepsilon$. If M is compact, then there exists a finite subcovering of $\mathcal{U}$. A standard procedure to produce a pairwise disjoint covering yields the desired partition $\mathcal{A}$.

(iii) Let $0 < \delta \leqslant \delta_\varepsilon$ (from step (i)), and let $\mathcal{A}$ be a partition of M as in step (ii). Let $\mathcal{A}_1 \subseteq \mathcal{A}$; then $\mathcal{A}_1$ is a partition of $M_1 := \bigcup \mathcal{A}_1$. We show that

$$\mathcal{H}_d(M_1) - \varepsilon \leqslant S_d(\mathcal{A}_1) \leqslant \mathcal{H}_d(M_1) + 2\varepsilon. \tag{D.6}$$

Let $\mathcal{C}_2$ be a countable covering of $M_2 := M \setminus M_1$ with $\operatorname{rd}(\mathcal{C}_2) \leqslant \delta_\varepsilon$. Then the covering $\mathcal{A}_1 \cup \mathcal{C}_2$ of M satisfies $\operatorname{rd}(\mathcal{A}_1 \cup \mathcal{C}_2) \leqslant \delta_\varepsilon$, so from (D.5) we obtain

$$\mathcal{H}_d(M_1) + \mathcal{H}_d(M_2) - \varepsilon = \mathcal{H}_d(M) - \varepsilon \leqslant S_d(\mathcal{A}_1 \cup \mathcal{C}_2) \leqslant S_d(\mathcal{A}_1) + S_d(\mathcal{C}_2).$$

Since one can approximate $\mathcal{H}_d(M_2)$ arbitrarily well by $S_d(\mathcal{C}_2)$, choosing $\mathcal{C}_2$ suitably, this inequality implies

$$\mathcal{H}_d(M_1) - \varepsilon \leqslant S_d(\mathcal{A}_1),$$

the left-hand inequality of (D.6).

The application of the result obtained so far to the partition $\mathcal{A}_2 := \mathcal{A} \setminus \mathcal{A}_1$ of M_2 yields $\mathcal{H}_d(M_2) - \varepsilon \leqslant S_d(\mathcal{A}_2)$. Putting this inequality together with the inequality stated in step (ii) we obtain

$$S_d(\mathcal{A}_1) = S_d(\mathcal{A}) - S_d(\mathcal{A}_2) \leqslant \mathcal{H}_d(M) + \varepsilon - (\mathcal{H}_d(M_2) - \varepsilon) = \mathcal{H}_d(M_1) + 2\varepsilon,$$

the right-hand inequality of (D.6).

We now choose $\mathcal{A}_1 := \big\{C \in \mathcal{A};\, \omega_d \operatorname{rd}(C)^d \leqslant \mathcal{H}_d(C)\big\}$ and apply (D.6) with $\mathcal{A}_1$, M_1 and with $\mathcal{A}_2$, M_2 (as defined above):

$$\begin{aligned}
&\sum_{C\in\mathcal{A}} \big|\mathcal{H}_d(C) - \omega_d \operatorname{rd}(C)^d\big| \\
&\quad = \sum_{C\in\mathcal{A}_1} \big(\mathcal{H}_d(C) - \omega_d \operatorname{rd}(C)^d\big) + \sum_{C\in\mathcal{A}_2} \big(\omega_d \operatorname{rd}(C)^d - \mathcal{H}_d(C)\big) \\
&\quad = \big(\mathcal{H}_d(M_1) - S_d(\mathcal{A}_1)\big) + \big(S_d(\mathcal{A}_2) - \mathcal{H}_d(M_2)\big) \leqslant 3\varepsilon.
\end{aligned}$$

□

D.8 Remark. The crucial point of the inequality in Proposition D.7 is that not only is the sum $S_d(\mathcal{A})$ close to $\mathcal{H}_d(M)$, but the individual terms $\omega_d \operatorname{rd}(C)^d$ of the sum also approximate the corresponding terms $\mathcal{H}_d(C)$, with a small total error. △

D.3 Maz'ya's inequality

The following theorem contains the central result of this appendix. The inequality we show is not quite Maz'ya's inequality (12.12), but it is strong enough to allow the derivation of the inequality (12.11) in Section 12.6.

D.9 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with $\mathcal{H}_{n-1}(\partial\Omega) < \infty$. Then there exists a constant $c(n,\Omega)$ (depending only on n and the width of Ω) such that*

$$\|u\|_{L_1(\Omega)} \leqslant c(n,\Omega)\Big(\int_\Omega |\nabla u(x)|\,\mathrm{d}x + \int_{\partial\Omega} |u|\,\mathrm{d}\mathcal{H}_{n-1}\Big) \tag{D.7}$$

for all $u \in C(\overline{\Omega}) \cap W_1^1(\Omega)$.

Proof. Let $u \in C(\overline{\Omega}) \cap W_1^1(\Omega)$. It follows from Proposition D.5 that $|u| \in C(\overline{\Omega}) \cap W_1^1(\Omega)$ and that $\big|\nabla|u|\big| \leqslant |\nabla u|$. This shows that it is sufficient to treat the case $u \geqslant 0$.

Let $u \in C(\overline{\Omega}) \cap W_1^1(\Omega)$, $u \geqslant 0$, and let $\varepsilon > 0$. Then, by the (uniform) continuity of u, there exists $\delta \in (0,\varepsilon]$ such that $|u(x) - u(y)| < \varepsilon$ whenever $x, y \in \overline{\Omega}$, $|x-y| < \delta$. By Proposition D.7 there exists a finite partition $\mathcal{A}$ of $\partial\Omega$ with $\mathrm{rd}(\mathcal{A}) \leqslant \delta/4$, consisting of Borel sets and such that (D.4) holds. We choose a family $(x_C)_{C\in\mathcal{A}}$ with $x_C \in C$ for all $C \in \mathcal{A}$. (Recall that, by definition, the sets in a partition are supposed to be non-empty.)

Clearly $\{B_{\mathbb{R}^n}[x_C, \mathrm{diam}(C)]\,;\, C \in \mathcal{A}\}$ is a covering of $\partial\Omega$, where we use the notation $B[x,r]$ for the closed ball with centre x and radius r. For each $C \in \mathcal{A}$, $s \in (0, \delta/2)$ we define a function

$$\psi_{C,s}(x) := \begin{cases} (u(x_C) + \varepsilon)\,\frac{1}{s}\,\mathrm{dist}\big(x, B(x_C, \mathrm{diam}(C))\big) & \text{if } x \in B[x_C, \mathrm{diam}(C) + s], \\ \infty & \text{otherwise.} \end{cases}$$

Note that $\psi_{C,s} \in W_1^1(B(x_C, \mathrm{diam}(C) + s))$. Note also that $\mathrm{diam}(C) + s \leqslant \delta$, and hence $\psi_{C,s}(x) = u(x_C) + \varepsilon > u(x)$ for all $x \in \partial B(x_C, \mathrm{diam}(C) + s) \cap \overline{\Omega}$, $C \in \mathcal{A}$. These properties and Lemma D.2 – applied repeatedly – imply that

$$u_{\varepsilon,s} := u \wedge \inf_{C\in\mathcal{A}} \psi_{C,s} \quad \text{on } \overline{\Omega}$$

belongs to $C(\overline{\Omega}) \cap W_1^1(\Omega)$. Since $\delta \leqslant \varepsilon$, the function $u_{\varepsilon,s}$ coincides with u on $\Omega_\varepsilon := \{x \in \Omega\,;\, B(x,\varepsilon) \subseteq \Omega\}$, and as $u_{\varepsilon,s}$ vanishes on $\partial\Omega$, Proposition D.3 shows that $u_{\varepsilon,s} \in W_{1,0}^1(\Omega)$.

The following computations prepare the application of Poincaré's inequality to $u_{\varepsilon,s}$. Lemma D.2 implies

$$\int_\Omega |\nabla u_{\varepsilon,s}(x)|\,\mathrm{d}x \leqslant \int_\Omega |\nabla u(x)|\,\mathrm{d}x + \sum_{C\in\mathcal{A}} \int_{B(x_C, \mathrm{diam}(C)+s)} |\nabla \psi_{C,s}(x)|\,\mathrm{d}x. \tag{D.8}$$

Observing that $\psi_{C,s} = 0$ on $B(x_C, \mathrm{diam}(C))$ and $|\nabla\psi_{C,s}| = (u(x_C) + \varepsilon)/s$ on the spherical shell $B(x_C, \mathrm{diam}(C) + s) \setminus B(x_C, \mathrm{diam}(C))$, we obtain

$$\int_{B(x_C, \mathrm{diam}(C)+s)} |\nabla \psi_{C,s}(x)|\,\mathrm{d}x = (u(x_C) + \varepsilon)\frac{1}{s}\omega_n\big((\mathrm{diam}(C) + s)^n - \mathrm{diam}(C)^n\big). \tag{D.9}$$

We note that, for $s \to 0$, the latter expression tends to

$$(u(x_C)+\varepsilon)\, n\omega_n \operatorname{diam}(C)^{n-1} = 2^{n-1}\frac{n\omega_n}{\omega_{n-1}}(u(x_C)+\varepsilon)\,\omega_{n-1}\operatorname{rd}(C)^{n-1}.$$

Recalling that $u|_{\Omega_\varepsilon} = u_{\varepsilon,s}|_{\Omega_\varepsilon}$, we conclude from Theorem D.6 that

$$\|u\|_{L_1(\Omega_\varepsilon)} \leqslant \|u_{\varepsilon,s}\|_{L_1(\Omega)} \leqslant c_{\mathrm{P}} \int_\Omega |\nabla u_{\varepsilon,s}(x)|\,\mathrm{d}x. \tag{D.10}$$

Inserting (D.8) and (D.9) into (D.10) and taking $s \to 0$ we obtain

$$\|u\|_{L_1(\Omega_\varepsilon)} \leqslant c_{\mathrm{P}}\Big(\int_\Omega |\nabla u(x)|\,\mathrm{d}x + 2^{n-1}\frac{n\omega_n}{\omega_{n-1}}\sum_{C\in\mathcal{A}}(u(x_C)+\varepsilon)\omega_{n-1}\operatorname{rd}(C)^{n-1}\Big). \tag{D.11}$$

Exploiting (D.4) we can estimate the sum on the right-hand side of (D.11) by

$$\begin{aligned}
&\sum_{C\in\mathcal{A}}(u(x_C)+\varepsilon)\big(\big|\omega_{n-1}\operatorname{rd}(C)^{n-1} - \mathcal{H}_{n-1}(C)\big| + \mathcal{H}_{n-1}(C)\big)\\
&\qquad\leqslant (\|u\|_\infty+\varepsilon)\varepsilon + \sum_{C\in\mathcal{A}}\int_C (u(x_C)+\varepsilon)\,\mathrm{d}\mathcal{H}_{n-1}\\
&\qquad\leqslant (\|u\|_\infty+\varepsilon)\varepsilon + \int_{\partial\Omega}(u+2\varepsilon)\,\mathrm{d}\mathcal{H}_{n-1}\,.
\end{aligned}$$

Because of this inequality, the estimate (D.11) implies

$$\begin{aligned}
\|u\|_{L_1(\Omega_\varepsilon)} \leqslant c_{\mathrm{P}}\Big(&\int_\Omega |\nabla u(x)|\,\mathrm{d}x\\
&+ 2^{n-1}\frac{n\omega_n}{\omega_{n-1}}\Big(\int_{\partial\Omega} u\,\mathrm{d}\mathcal{H}_{n-1} + \varepsilon(2\mathcal{H}_{n-1}(\partial\Omega) + \|u\|_\infty + \varepsilon)\Big)\Big).
\end{aligned}$$

Letting $\varepsilon \to 0$ we finally obtain

$$\|u\|_{L_1(\Omega)} \leqslant c_{\mathrm{P}}\Big(\int_\Omega |\nabla u(x)|\,\mathrm{d}x + 2^{n-1}\frac{n\omega_n}{\omega_{n-1}}\int_{\partial\Omega}|u|\,\mathrm{d}\mathcal{H}_{n-1}\Big). \tag{D.12}$$

□

From Theorem D.9 we now derive the inequality (12.11) presented in Section 12.6.

D.10 Corollary. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with $\mathcal{H}_{n-1}(\partial\Omega) < \infty$. Then*

$$\|u\|^2_{L_2(\Omega)} \leqslant 4c(n,\Omega)^2 \int_\Omega |\nabla u(x)|^2\,\mathrm{d}x + 2c(n,\Omega)\int_{\partial\Omega}|u|^2\,\mathrm{d}\mathcal{H}_{n-1}$$

for all $u \in C(\overline{\Omega}) \cap H^1(\Omega)$, with $c(n,\Omega)$ from Theorem D.9.

Proof. Let $u \in H^1(\Omega)$, $1 \leqslant j \leqslant n$. Then Proposition D.5 implies $\partial_j|u| = \operatorname{Re}(\overline{\operatorname{sgn} u}\,\partial_j u) \in L_2(\Omega)$. Using Corollary 9.15 we conclude that $\partial_j(|u| \wedge k) = \mathbf{1}_{[|u|<k]}\,\partial_j|u| \in L_2(\Omega)$ for all $k \in \mathbb{N}$, and $\partial_j(|u| \wedge k) \to \partial_j|u|$ in $L_2(\Omega)$ as $k \to \infty$. Then $(|u| \wedge k)^2 \to |u|^2$ in $L_1(\Omega)$, and

applying Corollary 9.13 we obtain $\partial_j(|u| \wedge k)^2 = 2(|u| \wedge k)\, \partial_j(|u| \wedge k) \to 2|u|\, \partial_j|u|$ in $L_1(\Omega)$ as $k \to \infty$. This shows that $|u|^2 \in W_1^1(\Omega)$, $\nabla|u|^2 = 2|u|\, \nabla|u|$.

Assume additionally that $u \in C(\overline{\Omega})$. We apply Theorem D.9 to $|u|$ and obtain

$$\|u\|_{L_2(\Omega)}^2 \leqslant c(n,\Omega)\Big(2\int_\Omega |u(x)|\big|\nabla|u|(x)\big|\,\mathrm{d}x + \int_{\partial\Omega} |u|^2\,\mathrm{d}\mathcal{H}_{n-1}\Big). \tag{D.13}$$

Applying the Cauchy–Schwarz inequality, Proposition D.5 and the Peter–Paul inequality we get

$$\int_\Omega |u(x)|\big|\nabla|u|(x)\big|\,\mathrm{d}x \leqslant \|u\|_2\Big(\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x\Big)^{1/2} \leqslant \gamma\|u\|_2^2 + \frac{1}{4\gamma}\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x$$

for all $\gamma > 0$. Inserting this inequality into (D.13), with $\gamma := \frac{1}{4c(n,\Omega)}$, and reshuffling terms we finally get

$$\|u\|_2^2 \leqslant 2c(n,\Omega)\Big(2c(n,\Omega)\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x + \int_{\partial\Omega} |u|^2\,\mathrm{d}\mathcal{H}_{n-1}\Big). \qquad \square$$

D.11 Remarks. (a) In the last step of the proof of Theorem D.9, in the estimate (D.10), we have applied the Poincaré inequality. There is another important inequality, estimating the L_q-norm, for $q = \frac{n}{n-1}$, in terms of the L_1-norm of the gradient, namely the **Sobolev inequality**

$$\|u\|_{L_q(\mathbb{R}^n)} \leqslant c(n)\int_{\mathbb{R}^n} |\nabla u(x)|\,\mathrm{d}x, \tag{D.14}$$

valid for all $u \in W_1^1(\mathbb{R}^n)$, with $c(n)$ only dependent on the dimension n. We refer to [Bre11; Theorem 9.9] or [Ada75; Remark 5.11] for this inequality. Applying (D.14) instead of the Poincaré inequality in the proof of Theorem D.9, one obtains as a result the 'better' inequality (12.12) (with a different value of $c(n)$). We put quotes around 'better' because clearly one can derive (D.7) from an inequality of the type (12.12), but then the constant in (D.7) will necessarily depend on the volume of Ω. In our derivation, however, that constant depends only on the dimension and on the width of Ω.

(b) Here we comment on the close relationship between (12.12), (D.14) and the **isoperimetric inequality**

$$\mathrm{vol}_n(A)^{(n-1)/n} \leqslant c(n)\mathcal{H}_{n-1}(\partial A) \tag{D.15}$$

for any measurable set $A \subseteq \mathbb{R}^n$ of finite n-dimensional volume, where

$$c(n) := \frac{1}{n\omega_n^{1/n}} = \frac{\Gamma(n/2+1)^{1/n}}{n\sqrt{\pi}}$$

is the **isoperimetric constant**. For a proof of (D.15) we refer to [Fed69; Theorem 3.2.43 and Remark 3.2.44] or [Maz11; Theorem 9.1.5 and Remark 9.2.2]. In fact, the isoperimetric constant $c(n)$ is also the optimal constant in (12.12) and (D.14).

We point out that (12.12) implies (D.15) for bounded open sets $A = \Omega$ (put $u = \mathbf{1}_\Omega$). Moreover, (12.12) implies (D.14) for functions $u \in W_{1,\mathrm{c}}^1(\mathbb{R}^n)$. On the other hand, assuming that one knows (D.15) for 'nice' sets, one can show (D.14) for C_c^∞-functions, using the

coarea formula of integration theory. Finally, one can extend (D.14) to L_1-functions 'of bounded variation', i.e. with distributional derivatives that are finite measures, and from this inequality one can derive (12.12). (This last step is rather non-trivial, and in a way contains the technical refinements one always encounters if one wants to show (D.15) or (12.12) for the general case.)

In view of the above comments one says that in some sense the inequalities (D.15), (12.12) and (D.14) are equivalent. △

Notes

Inequalities like (12.12), due to Maz'ya, or the general isoperimetric inequality are notorious for their challenging technical level. The inequalities become easier if one does not insist on the optimal constant. For the isoperimetric inequality, at one point one has to estimate the 'perimeter' of an open set Ω by $\mathcal{H}_{n-1}(\partial\Omega)$; see for instance [AFP00; Theorem 3.62]. The proof of that theorem inspired us to approximate integrals with respect to a Hausdorff measure by integrals of suitable 'simple functions' on partitions arising in connection with the definition of the Hausdorff measure (see Proposition D.7) and to express the integral over the resulting simple functions by integrals over the gradient of suitable auxiliary functions (see the proof of Theorem D.9).

We refer to [VoVo20b] for this treatment of Maz'ya's inequality as well as for further comments on the relation between Mazya's inequality, the isoperimetric inequality and the Sobolev inequality.

Appendix E

The n- and $(n-1)$-dimensional Hausdorff measures on $\mathbb{R}^n$

E.1 $\mathcal{H}_n$ on $\mathbb{R}^n$

In this section we show that the n-dimensional Hausdorff measure on $\mathbb{R}^n$ is the Lebesgue measure λ^n. For the definition of the Hausdorff measure and the notation used below we refer to Section D.3.

E.1 Theorem. *For all $n \in \mathbb{N}_0$ one has $\mathcal{H}_n = \lambda^n$.*

The assertion includes the statement that the measurable subsets coincide. This will be achieved by showing that the outer measures $\mathcal{H}_n^*$ and $\lambda^{n,*}$ (on arbitrary subsets of $\mathbb{R}^n$) are identical.

We recall that the outer n-dimensional Lebesgue measure is defined – for instance – by

$$\lambda^{n,*}(A) := \inf\Big\{ \sum_{R\in\mathcal{R}} \lambda^n(R)\,;\, \mathcal{R} \text{ countable collection of closed } n\text{-dimensional rectangles, } A \subseteq \bigcup \mathcal{R} \Big\},$$

for $A \subseteq \mathbb{R}^n$; see [Str94; Section 2.1]. The properties we will need below are that $\lambda^{n,*}$ is countably subadditive and that

$$\lambda^{n,*}(A) = \inf\{\lambda^n(U)\,;\, U \subseteq \mathbb{R}^n \text{ open, } A \subseteq U\} \qquad (A \subseteq \mathbb{R}^n);$$

see [Str94; Lemmas 2.1.2 and 2.1.4].

The most important tool for the aim formulated above is the **isodiametric inequality**

$$\lambda^{n,*}(A) \leqslant \omega_n \operatorname{rd}(A)^n, \tag{E.1}$$

valid for all bounded subsets $A \subseteq \mathbb{R}$, where $\operatorname{rd}(A) := \frac{1}{2}\operatorname{diam}(A)$. This inequality will be proved in Section E.3.

We will also need the following nice property concerning the Lebesgue measure.

E.2 Lemma. *Let $U \subseteq \mathbb{R}^n$ be an open set, and let $\delta > 0$. Then there exists a pairwise disjoint countable collection $\mathcal{B}$ of open balls $B \subseteq U$ with $\operatorname{rd}(\mathcal{B}) \leqslant \delta$ and $\lambda^n(U \setminus \bigcup \mathcal{B}) = 0$.*

Proof. We take it for granted from measure theory that there exists a pairwise disjoint countable collection $\mathcal{Q}$ of open cubes $Q \subseteq U$ of side lengths $\leqslant 2\delta$ and such that $\lambda^n(U \setminus \bigcup \mathcal{Q}) = 0$.

In view of this property it clearly suffices to show the assertion for the case when U is an open cube U_0 with side length $s \leqslant 2\delta$. Put $\mathcal{B}_0 := \{B[c, s/2]\}$, where c is the centre of the cube U_0; then $U_1 := U_0 \setminus \bigcup \mathcal{B}_0 = U_0 \setminus B[c, s/2]$ is an open set, with

$$q := \lambda^n(U_1)/\lambda^n(U_0) = 1 - 2^{-n}\omega_n < 1.$$

The open set U_1 can be covered, up to a null set, by a pairwise disjoint countable collection $\mathcal{U}_1$ of open cubes $Q \subseteq U_1$. Applying the previous procedure to each of these cubes one obtains a countable collection $\mathcal{B}_1$ of closed balls and a new open set $U_2 = U_1 \setminus \bigcup \mathcal{B}_1$ with $\lambda^n(U_2)/\lambda^n(U_1) = q$. Repeating the procedure we obtain a decreasing sequence (U_k) of open sets with

$$\lambda^n(U_k) \leqslant q\lambda^n(U_{k-1}) \leqslant \cdots \leqslant q^k\lambda^n(U_0) \to 0 \qquad (k \to \infty),$$

and this implies

$$\lambda^n\Big(U_0 \setminus \bigcup_{k\in\mathbb{N}_0} \bigcup \mathcal{B}_k\Big) = 0,$$

where $\mathcal{B}_k$ is the countable collection of balls corresponding to U_k. Replacing each of the (closed) balls in $\bigcup_{k\in\mathbb{N}} \mathcal{B}_k$ by its interior we obtain the desired countable collection $\mathcal{B}$ of open balls with the property $\lambda^n(U_0 \setminus \bigcup \mathcal{B}) = 0$. □

Proof of Theorem E.1. Let $A \subseteq \mathbb{R}^n$.

(i) First we show that $\mathcal{H}^*_n(A) \leqslant \lambda^{n,*}(A)$. There is nothing to show if $\lambda^{n,*}(A) = \infty$; so assume that $\lambda^{n,*}(A) < \infty$. Let $\delta > 0$, $\varepsilon > 0$. There exists an open set $U \supseteq A$ with $\lambda^n(U) \leqslant \lambda^{n,*}(A) + \varepsilon$. By Lemma E.2 there exists a pairwise disjoint countable collection $\mathcal{B}$ of open balls contained in U, with $\operatorname{rd}(\mathcal{B}) \leqslant \delta$ and

$$\sum_{B\in\mathcal{B}} \lambda^n(B) = \lambda^n(U) \leqslant \lambda^{n,*}(A) + \varepsilon.$$

As $U \setminus \bigcup \mathcal{B}$ is a Lebesgue-null set, there exists a countable covering $\mathcal{Q}$ of $U \setminus \bigcup \mathcal{B}$ with $\operatorname{rd}(\mathcal{Q}) \leqslant \delta$, consisting of cubes, and such that $\sum_{Q\in\mathcal{Q}} \lambda^n(Q) \leqslant \frac{1}{\omega_n}\big(\frac{2}{\sqrt{n}}\big)^n\varepsilon$. Then $\mathcal{B} \cup \mathcal{Q}$ is a countable covering of A, $\operatorname{rd}(\mathcal{B} \cup \mathcal{Q}) \leqslant \delta$, and

$$\begin{aligned} S_n(\mathcal{B} \cup \mathcal{Q}) &= \omega_n \sum_{B\in\mathcal{B}} \operatorname{rd}(B)^n + \omega_n \sum_{Q\in\mathcal{Q}} \operatorname{rd}(Q)^n \\ &= \lambda^n(U) + \omega_n\Big(\frac{\sqrt{n}}{2}\Big)^n \sum_{Q\in\mathcal{Q}} \lambda^n(Q) \leqslant \lambda^{n,*}(A) + 2\varepsilon, \end{aligned}$$

which shows that $\mathcal{H}_{n,\delta}(A) \leqslant \lambda^{n,*}(A) + 2\varepsilon$. Letting first $\varepsilon \to 0$ and then $\delta \to 0$ we obtain the assertion of the first step.

(ii) Now we show that $\lambda^{n,*}(A) \leqslant \mathcal{H}_n^*(A)$. Let $\mathcal{C}$ be a countable covering of A with $\mathrm{rd}(\mathcal{C}) \leqslant 1$. Then the isodiametric inequality (E.1) shows that

$$\lambda^{n,*}(A) \leqslant \sum_{C\in\mathcal{C}} \lambda^{n,*}(C) \leqslant \omega_n \sum_{C\in\mathcal{C}} \mathrm{rd}(C)^n,$$

by the countable subadditivity of $\lambda^{n,*}$. Taking the infimum on the right-hand side we obtain

$$\lambda^{n,*}(A) \leqslant \mathcal{H}_{n,1}(A) \leqslant \mathcal{H}_n^*(A). \qquad \square$$

E.2 $\mathcal{H}_{n-1}$ on $\mathbb{R}^n$

In this section we show that that the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$ is the surface measure σ described in Section 7.1.

E.3 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1-boundary. Then $\sigma(A) = \mathcal{H}_{n-1}(A)$ for all Borel sets $A \subseteq \partial\Omega$.*

The following lemma will be needed in the proof of Theorem E.3.

E.4 Lemma. *Let (M_1, ρ_1) and (M_2, ρ_2) be metric spaces, $\varphi\colon M_1 \to M_2$ Lipschitz continuous, with Lipschitz constant $L \geqslant 0$, and let $d \in [0, \infty)$. Then*

$$\mathcal{H}_d^*(\varphi(A)) \leqslant L^d \mathcal{H}_d^*(A) \tag{E.2}$$

for all $A \subseteq M_1$.

Proof. Let $\delta > 0$, and let $\mathcal{C}$ a covering of $A \subseteq M_1$ with $\mathrm{rd}(\mathcal{C}) \leqslant \delta$. Then

$$\varphi(\mathcal{C}) := \{\varphi(C)\,;\, C \in \mathcal{C}\}$$

is a covering of $\varphi(A)$, with $\mathrm{rd}(\varphi(\mathcal{C})) \leqslant L\delta$. We estimate

$$\mathcal{H}_{d,L\delta}(\varphi(A)) \leqslant S_d(\varphi(\mathcal{C})) = \omega_d \sum_{C\in\mathcal{C}} \mathrm{rd}(\varphi(C))^d \leqslant \omega_d L^d \sum_{C\in\mathcal{C}} \mathrm{rd}(C)^d = L^d S_d(\mathcal{C}).$$

Taking the infimum on the right-hand side we obtain $\mathcal{H}_{d,L\delta}(\varphi(A)) \leqslant L^d \mathcal{H}_{d,\delta}(A)$, which for $\delta \to 0$ yields (E.2). $\square$

Proof of Theorem E.3. (i) Let $z \in \partial\Omega$, and let $\nu(z)$ be the outer unit normal at z. In a first step we construct a special local parametrisation of $\partial\Omega$ around z. There exists an orthogonal matrix B such that $B\nu(z) = e_n$, the n-th unit vector. Applying B to $\partial\Omega$ one can see that there exists an open neighbourhood W of z in $\partial\Omega$ such that $B(W)$ is a standard C^1-graph, i.e., there exist an open set $W' \subseteq \mathbb{R}^{n-1}$ and a C^1-function $g\colon W' \to (a, b)$ such that $B(W) = \{(y, g(y))\,;\, y \in W'\}$. There exists $z' \in W'$ such that $Bz = (z', g(z'))$. Because of $B\nu(z) = e_n$ one has $\nabla g(z') = 0$; one could say that $B(W)$ is *flat at* Bz.

Next, let $\alpha > 1$. Reduce W' to a smaller convex open neighbourhood of z' by requiring $\sqrt{1 + |\nabla g(y')|^2} \leqslant \alpha$ for all $y' \in W'$, and reduce the set W accordingly. It is easy to see

that then the mapping $\varphi\colon W' \to B(W),\ y' \mapsto (y', g(y'))$ is Lipschitz continuous with Lipschitz constant α. Let $A \subseteq W$ be a Borel set, $A' := \varphi^{-1}(B(A))$. Then

$$\sigma(A) = \sigma(B(A)) = \int_{A'} \sqrt{1 + |\nabla g(y')|^2}\, \mathrm{d}y',$$

and by estimating the integrand below and above we obtain

$$\lambda^{n-1}(A') \leqslant \sigma(A) \leqslant \alpha\lambda^{n-1}(A'). \tag{E.3}$$

Let $\mathrm{pr}_{n-1}\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the canonical projection, $\mathrm{pr}_{n-1}(x_1, \dots, x_n) := (x_1, \dots, x_{n-1})$, and note that then $A' = \mathrm{pr}_{n-1}(B(A))$. From Lemma E.4 we conclude that

$$\mathcal{H}_{n-1}(A') = \mathcal{H}_{n-1}\big(\mathrm{pr}_{n-1}(B(A))\big) \leqslant \mathcal{H}_{n-1}(B(A)) = \mathcal{H}_{n-1}(\varphi(A')) \leqslant \alpha^{n-1}\mathcal{H}_{n-1}(A'),$$

which in view of Theorem E.1 and the obvious equality $\mathcal{H}_{n-1}(B(A)) = \mathcal{H}_{n-1}(A)$ can be rewritten as

$$\lambda^{n-1}(A') \leqslant \mathcal{H}_{n-1}(A) \leqslant \alpha^{n-1}\lambda^{n-1}(A'). \tag{E.4}$$

Combining the estimates (E.3) and (E.4) we obtain

$$\big|\sigma(A) - \mathcal{H}_{n-1}(A)\big| \leqslant (\alpha^n - 1)\lambda^{n-1}(A') \leqslant (\alpha^n - 1)\sigma(A).$$

(ii) In order to continue we rephrase the result of step (i) as follows. Let $\alpha > 1$. Then for every $z \in \partial\Omega$ there exists an open neighbourhood W_z in $\partial\Omega$ such that for every Borel set $A_z \subseteq W_z$ one has $\big|\sigma(A_z) - \mathcal{H}_{n-1}(A_z)\big| \leqslant (\alpha^n - 1)\sigma(A_z)$. From the open covering $\{W_z\,;\, z \in \partial\Omega\}$ of $\partial\Omega$ we choose a finite subcovering $\{W_z\,;\, z \in F\}$.

Let $A \subseteq \partial\Omega$ be a Borel set. Then there exists a partition $\{A_z\,;\, z \in F\}$ of A, with a Borel set $A_z \subseteq W_z$ for all $z \in F$, and we obtain

$$|\sigma(A) - \mathcal{H}_{n-1}(A)| \leqslant \sum_{z\in F}\big|\sigma(A_z) - \mathcal{H}_{n-1}(A_z)\big| \leqslant \sum_{z\in F}(\alpha^n - 1)\sigma(A_z) \leqslant (\alpha^n - 1)\sigma(\partial\Omega).$$

As the inequality between the outer terms of this chain of inequalities holds for all $\alpha > 1$, we conclude that $\sigma(A) = \mathcal{H}_{n-1}(A)$. □

E.5 Remarks. (a) The proof of Theorem E.3 can be modified to show that more generally one has equality of the d-dimensional Hausdorff measure and the "usual" d-dimensional surface measure on embedded d-dimensional manifolds, for $d = 1, \dots, n-1$.

(b) Much more generally, it is shown in the literature that the image of an injective Lipschitz continuous map $\varphi\colon U \to \mathbb{R}^n$, where $U \subseteq \mathbb{R}^d$ (with $d \in \{1, \dots, n-1\}$) is a bounded open set, has d-dimensional Hausdorff measure given by $\mathcal{H}_d(\varphi(U)) = \int_U \sqrt{\det(\varphi'(y)^\top \varphi'(y))}\, \mathrm{d}y$. This fact and the formula carry the name of "area formula" in geometric measure theory. (It deserves to be mentioned that, by Rademacher's theorem, the derivative occurring in the formula exists almost everywhere.) We refer to [EvGa92; 3.3, Theorem 1], [AFP00; Theorem 2.71], for the full formulation and the proof of the area formula. △

E.3 The isodiametric inequality

The issue of this section is to give a proof of the isodiametric inequality (E.1).

E.6 Theorem. *Let $A \subseteq \mathbb{R}^n$ be a bounded set. Then*

$$\lambda^{n,*}(A) \leqslant \omega_n \operatorname{rd}(A)^n. \tag{E.5}$$

E.7 Remarks. (a) Note that a set $A \subseteq \mathbb{R}^n$ need not be contained in a ball of radius $\operatorname{rd}(A)$; an example in $\mathbb{R}^2$ is the isolateral triangle.

(b) If A is symmetric with respect to the origin, i.e. $A = -A$, then $2|x| = |x - (-x)| \leqslant \operatorname{diam}(A)$ for all $x \in A$, hence $A \subseteq B[0, \operatorname{rd}(A)]$, and (E.5) holds. △

The idea of the proof of Theorem E.6 is the reduction to the case of Remark E.7(b), by successive symmetrisation of A. The one-dimensional symmetrisation of a bounded set $B \subseteq \mathbb{R}$ is the open interval

$$S(B) := \big(-\tfrac{1}{2}\lambda^{1,*}(B), \tfrac{1}{2}\lambda^{1,*}(B)\big)$$

(which is empty if B is a null set). Now let $A \subseteq \mathbb{R}^n$ be bounded, and let $(e_1, \dots, e_n)$ denote the standard orthonormal basis of $\mathbb{R}^n$. We define the **Steiner symmetrisation** $S_n(A)$ of A with respect to the coordinate hyperplane $e_n^\perp$ by

$$S_n(A) := \bigcup_{\check{x} \in \mathbb{R}^{n-1}} \{\check{x}\} \times S(A_{\check{x}}),$$

where, for $\check{x} \in \mathbb{R}^{n-1}$, we use the notation

$$A_{\check{x}} := \{x_n \in \mathbb{R};\, (\check{x}, x_n) \in A\}.$$

(Usually, the Steiner symmetrisation is defined with closed intervals; we found it more convenient to work with open intervals.) The Steiner symmetrisation consists in subdividing $\mathbb{R}^n$ into one-dimensional fibers and then applying the one-dimensional symmetrisation in each fiber. Clearly $S_n(A)$ is symmetric with respect to $e_n^\perp$.

The symmetrisations $S_k(A)$ of A with respect to the coordinate hyperplanes $e_k^\perp$, for $k = 1, \dots, n-1$, are defined analogously.

E.8 Lemma. *Let $A \subseteq \mathbb{R}^n$ be a bounded set.*

(a) *Then* $\operatorname{diam}(S_n(A)) \leqslant \operatorname{diam}(A)$.

(b) *If $k \in \{1, \dots, n-1\}$, and A is symmetric with respect to $e_k^\perp$, then $S_n(A)$ is symmetric with respect to $e_k^\perp$ as well.*

(c) *If A is a Borel set, then $S_n(A)$ is a Borel set, and $\lambda^n(S_n(A)) = \lambda^n(A)$.*

Proof. (a) Let $x = (\check{x}, x_n)$, $y = (\check{y}, y_n) \in S_n(A)$, and put $r_1 := \inf A_{\check{x}}$, $r_2 := \sup A_{\check{x}}$ and $s_1 := \inf A_{\check{y}}$, $s_2 := \sup A_{\check{y}}$. Then

$$\begin{aligned} |x_n - y_n| &\leqslant |x_n| + |y_n| < \tfrac{1}{2}\lambda^{1,*}(A_{\check{x}}) + \tfrac{1}{2}\lambda^{1,*}(A_{\check{y}}) \\ &\leqslant \tfrac{1}{2}(r_2 - r_1) + \tfrac{1}{2}(s_2 - s_1) = \tfrac{1}{2}(r_2 - s_1) + \tfrac{1}{2}(s_2 - r_1) \\ &\leqslant \max\{r_2 - s_1, s_2 - r_1\}. \end{aligned}$$

Therefore

$$\begin{aligned}|x-y|^2 &= |\check{x}-\check{y}|^2 + (x_n-y_n)^2 \\ &< |\check{x}-\check{y}|^2 + \max\bigl\{(r_2-s_1)^2,(s_1-r_2)^2\bigr\} \\ &= \max\bigl\{|(\check{x},r_2)-(\check{y},s_1)|^2, |(\check{y},s_2)-(\check{x},r_1)|^2\bigr\} \\ &\leqslant \operatorname{diam}(A)^2,\end{aligned}$$

where the last inequality holds because the points $(\check{x},r_1),(\check{x},r_2),(\check{y},s_1),(\check{y},s_2)$ belong to the closure of A. This shows the assertion.

(b) The symmetry of A with respect to $e_k^\perp$ implies that $A_{R_k\check{x}} = A_{\check{x}}$ for all $\check{x}\in\mathbb{R}^{n-1}$, where R_k denotes the reflection of $\mathbb{R}^{n-1}$ in the $(n-2)$-dimensional coordinate hyperplane $e_k^\perp$. Then it follows from the definition of $S_n(A)$ that $S_n(A)$ is symmetric, too.

(c) In this part of the proof, 'measurable' will always mean 'Borel measurable'. From Fubini's theorem we know that the function

$$f\colon \mathbb{R}^{n-1}\to\mathbb{R},\quad \check{x}\mapsto \int_{\mathbb{R}} \mathbf{1}_A(\check{x},x_n)\,\mathrm{d}x_n = \lambda^1(A_{\check{x}})$$

is measurable. It follows that the function $g\colon \mathbb{R}^n\to\mathbb{R}$, $g(\check{x},x_n) := \frac{1}{2}f(\check{x}) - |x_n|$ ($\check{x}\in\mathbb{R}^{n-1}$, $x_n\in\mathbb{R}$), is measurable, and therefore the set $S_n(A) = [g>0]$ is measurable. Then Fubini's theorem implies

$$\lambda^n(A) = \int_{\mathbb{R}^n} \mathbf{1}_A(x)\,\mathrm{d}x = \int_{\check{x}\in\mathbb{R}^{n-1}} \lambda^1(A_{\check{x}})\,\mathrm{d}\check{x} = \lambda^n(S_n(A)). \qquad \square$$

Proof of Theorem E.6. Without loss of generality we suppose that A is closed (and hence measurable). Put $A_0 := S_1(\dots S_{n-1}(S_n(A))\dots)$; then Lemma E.8(c) implies $\lambda^n(A) = \lambda^n(A_0)$. Moreover, by Lemma E.8(b), A_0 is symmetric whith respect to all the coordinate hyperplanes $e_1^\perp,\dots,e_n^\perp$, hence with respect to the origin, and therefore $\lambda^n(A_0)\leqslant \omega_n \operatorname{rd}(A_0)^n$ by Remark E.7(b). Finally, Lemma E.8(a) shows that $\operatorname{rd}(A_0)\leqslant\operatorname{rd}(A)$, and this concludes the proof. $\square$

Notes

The contents of this appendix are standard subjects of geometric measure theory and can be found, for instance, in [Fed69], [EvGa92], [AFP00]. In Section E.2 we treat the equality of the surface measure and the $(n-1)$-dimensional Hausdorff measure in less generality than in the indicated references, thereby allowing an easier access. For comments on the general case we refer to Remark E.5(b). Our proof of the isodiametric inequality in Section E.3 is a reformulation of the proof given in [EvGa92; Section 2.2].

Exercises

E.1 Let (M_1,ρ_1), (M_2,ρ_2) be metric spaces, $d\in[0,\infty)$, $\varphi\colon M_1\to M_2$ Hölder continuous with exponent $\alpha\in(0,\infty)$, Hölder constant $H\geqslant 0$, i.e.

$$\rho_2(\varphi(x),\varphi(y))\leqslant H\rho_1(x,y)^\alpha \qquad (x,y\in M_1).$$

Show that $\mathcal{H}^*_{d/\alpha}(\varphi(A)) \leqslant H^{d/\alpha}\mathcal{H}^*_d(A)$, for all $A \subseteq M_1$. (See Lemma E.4 for a special case of this assertion.)

Appendix F

The spectral theorem for self-adjoint operators

In this appendix we prove the spectral theorem, which we have already stated in Chapter 13.

F.1 Theorem. *Let H be a Hilbert space, and let A be a self-adjoint operator in H. Then there exist a measure space $(\Omega, \mathcal{A}, \mu)$, a measurable function $\alpha\colon \Omega \to \mathbb{R}$ and a unitary operator $J\colon H \to L_2(\Omega, \mathcal{A}, \mu)$ such that*

$$A = J^{-1} M_\alpha J.$$

Clearly, the spectral theorem is only of interest in non-trivial Hilbert spaces. For this reason we exclude the Hilbert space $H = \{0\}$ in the present chapter.

F.1 The spectral theorem for bounded self-adjoint operators

Throughout this section, H will be a (real or complex) Hilbert space, and $A \in \mathcal{L}(H)$ will be a self-adjoint operator. It will be important to remember that $\sigma(A) \subseteq \mathbb{R}$; in the complex case this holds since the operators $\pm \mathrm{i}A$ are m-accretive, by Remark 3.18. (See also Exercise 6.1.) We start with a preparatory lemma of a general nature.

F.2 Lemma. *Assume that $\sigma(A) \subseteq [0, \infty)$. Then A is accretive.*

Proof. Let $x \in H$, $f(\lambda) := \big((\lambda + A)^{-1} x \,\big|\, x\big)$ for $\lambda > 0$. Then $f(\lambda) \to 0$ as $\lambda \to \infty$, and $f'(\lambda) = -\big((\lambda + A)^{-2} x \,\big|\, x\big) = -\big\|(\lambda + A)^{-1} x\big\|^2 \leqslant 0$ for all $\lambda > 0$ (note that $(\lambda + A)^{-1}$ is self-adjoint). This implies that $\big((\lambda + A)^{-1} x \,\big|\, x\big) \geqslant 0$ for all $\lambda > 0$. Therefore, for all $x \in H$, $\lambda > 0$ we obtain

$$\big((\lambda + A)x \,\big|\, x\big) = \big((\lambda + A)^{-1}(\lambda + A)x \,\big|\, (\lambda + A)x\big) \geqslant 0.$$

Letting $\lambda \to 0$ we obtain $(Ax \,|\, x) \geqslant 0$. □

The proof of the spectral theorem relies on some basic properties of the functional calculus for self-adjoint operators. The function spaces needed below will consist of real- or complex-valued functions, according to whether H is a real or complex Hilbert space.

We start by associating with each polynomial $p = a_0 + a_1 t + \cdots + a_n t^n$ the operator $p(A) := a_0 I + a_1 A + \cdots + a_n A^n$. Let $\mathcal{P}$ denote the set of polynomials. Then it is straightforward that the mapping $\mathcal{P} \ni p \mapsto p(A) \in \mathcal{L}(H)$ is an algebra homomorphism. The following theorem states further properties and extends this homomorphism to the continuous functions on the spectrum of A.

F.3 Theorem. *There exists an algebra homomorphism* $\Phi\colon C(\sigma(A)) \to \mathcal{L}(H)$ *satisfying*

$\Phi(p) = p(A)$ *for all polynomials, in particular* $\Phi(1) = I$, $\Phi(\mathrm{id}) = A$,

$\Phi(\bar{f}) = \Phi(f)^*$ *for all* $f \in C(\sigma(A))$,

$\Phi(f) \geqslant 0$ *for all* $f \in C(\sigma(A))$, $f \geqslant 0$,

$\|\Phi(f)\| \leqslant \|f\|_\infty$ *for all* $f \in C(\sigma(A))$.

Proof. If p is a polynomial, then we define $\Phi(p) := p(A)$. Then obviously $\Phi(\bar{p}) = \Phi(p)^*$ for all $p \in \mathcal{P}$.

Let $p \in \mathcal{P}$ be a real polynomial, i.e. all the coefficients of p are real, and assume that $p(t) \neq 0$ for all $t \in \sigma(A)$. Then one can decompose p as

$$p(t) = c \prod_{j \in J} (t - \alpha_j) \prod_{k \in K} \big((t - \beta_k)^2 + \gamma_k^2\big),$$

with finite sets J, K and $c \neq 0$, $\alpha_j \in \mathbb{R} \setminus \sigma(A)$ $(j \in J)$, $\beta_k, \gamma_k \in \mathbb{R}$, $\gamma_k \neq 0$ $(k \in K)$. The latter, quadratic factors correspond to non-real zeros of p – occurring in pairs of complex conjugate zeros. It follows that $p(A)$ is invertible in $\mathcal{L}(H)$, because all the factors $A - \alpha_j$ $(j \in J)$ and $(A - \beta_k)^2 + \gamma_k^2$ $(k \in K)$ are invertible. (Concerning the latter factors, note that $(A - \beta_k)^2$ is m-accretive.)

Now let $p \in \mathcal{P}$ be a real polynomial with $p(t) \geqslant 0$ for all $t \in \sigma(A)$; then $p(A)$ is self-adjoint, hence $\sigma(p(A)) \subseteq \mathbb{R}$. If $\lambda < 0$, then one has $\lambda - p(t) < 0$ for all $t \in \sigma(A)$, and $\lambda - p(A)$ is invertible in $\mathcal{L}(H)$ by the reasoning given above, i.e. $\lambda \in \rho(p(A))$. This shows that $\sigma(p(A)) \subseteq [0, \infty)$, and Lemma F.2 implies that $p(A) \geqslant 0$.

The last property implies that, if $p, q \in \mathcal{P}$ are real polynomials satisfying $p|_{\sigma(A)} \leqslant q|_{\sigma(A)}$, then $p(A) \leqslant q(A)$. Therefore, if $p \in \mathcal{P}$, then

$$\begin{aligned} \|p(A)x\|^2 = (\bar{p}(A)p(A)x \,|\, x) &= \big(|p|^2(A)x \,\big|\, x\big) \\ &\leqslant \big((\||p|_{\sigma(A)}\|_\infty^2 1)(A)x \,\big|\, x\big) = \||p|_{\sigma(A)}\|_\infty^2 \|x\|^2 \end{aligned}$$

for all $x \in H$, and this shows that $\|p(A)\| \leqslant \|p|_{\sigma(A)}\|_\infty$.

We have shown the asserted properties for Φ, defined on $\mathcal{P}$. The set $\mathcal{P}$ is dense in $C(\sigma(A))$ by the Stone–Weierstrass theorem, Theorem B.2 (or by the Weierstrass approximation theorem, see Exercise F.1). Since Φ is contractive, we can thus extend Φ by continuity to $C(\sigma(A))$, and the asserted properties carry over. □

We will use the notation $f(A) := \Phi(f)$ for all $f \in C(\sigma(A))$. The following theorem is the spectral theorem for the special case in which there exists a **cyclic vector** (for A), i.e. an element $x \in H$ satisfying (F.1) below.

F.4 Theorem. *Assume that there exists $x \in H$ such that*

$$H = \overline{\operatorname{lin}\{A^n x;\, n \in \mathbb{N}_0\}}. \tag{F.1}$$

Then there exist a Borel measure μ on $\sigma(A)$ and a unitary operator $J\colon H \to L_2\big(\sigma(A), \mu\big)$ such that $f(A) = J^{-1} M_f J$ for all $f \in C(\sigma(A))$.

Proof. We start with the crucial observation that

$$C(\sigma(A)) \ni f \mapsto (f(A)x \,|\, x) \in \mathbb{K}$$

defines a positive linear functional on $C(\sigma(A))$. By the Riesz–Markov theorem – see Appendix G – there exists a unique Borel measure μ on $\sigma(A)$ such that

$$(f(A)x \,|\, x) = \int_{\sigma(A)} f \,\mathrm{d}\mu \qquad \big(f \in C(\sigma(A))\big). \tag{F.2}$$

This implies that for all $f, g \in C(\sigma(A))$ one has

$$\big(f(A)x \,\big|\, g(A)x\big) = \big((f\overline{g})(A)x \,\big|\, x\big) = \int_{\sigma(A)} f\overline{g} \,\mathrm{d}\mu.$$

This equality shows that the mapping

$$\big(C(\sigma(A)), \|\cdot\|_{L_2(\sigma(A),\mu)}\big) \ni f \mapsto f(A)x \in H_0 := \big\{f(A)x;\, f \in C(\sigma(A))\big\}$$

is isometric, hence bijective, with (well-defined) isometric inverse

$$J_0\colon H_0 \to L_2\big(\sigma(A), \mu\big), \qquad J_0(f(A)x) := f \quad \big(f \in C(\sigma(A))\big).$$

(In the above reasoning we consider $C(\sigma(A))$ as a subspace of $L_2(\sigma(A), \mu)$, with a.e. equal functions identified. See also Exercise F.2.) As H_0 is dense in H, by (F.1), and $C(\sigma(A))$ is dense in $L_2\big(\sigma(A), \mu\big)$, by Exercise G.2, the continuous extension J of J_0 is a unitary operator $J\colon H \to L_2\big(\sigma(A), \mu\big)$.

Let $f \in C(\sigma(A))$. If $g, h \in C(\sigma(A))$, $y := g(A)x$, $z := h(A)x$ $(\in H_0)$, then

$$(f(A)y \,|\, z) = \big(f(A)g(A)x \,\big|\, h(A)x\big) = \int_{\sigma(A)} fg\overline{h} \,\mathrm{d}\mu = (fJy \,|\, Jz)_{L_2(\sigma(A),\mu)}\,.$$

This extends to

$$(f(A)y \,|\, z) = (fJy \,|\, Jz)_{L_2(\sigma(A),\mu)}$$

for all $y, z \in H$, which shows that $Jf(A) = M_f J$. □

Now, for arbitrary $x \in H \setminus \{0\}$ we define

$$H_x := \overline{\operatorname{lin}\{A^n x;\, n \in \mathbb{N}_0\}} = \overline{\{p(A)x;\, p \in \mathcal{P}\}} = \overline{\{f(A)x;\, f \in C(\sigma(A))\}}.$$

Obviously H_x is invariant under A, and therefore under $f(A)$ for all $f \in C(\sigma(A))$. The subspace H_x is the **cyclic subspace** generated by x, and clearly Theorem F.4 can be

applied to the restriction A_x of A to H_x. We denote by $\check{\mu}_x := \mu$ the (unique) measure satisfying (F.2) with A replaced by A_x.

We extend $\check{\mu}_x$ to a Borel measure μ_x on $\sigma(A)$ ($\supseteq \sigma(A_x)$), putting $\mu_x(B) := \check{\mu}_x(B \cap \sigma(A_x))$ for all Borel subsets of $\sigma(A)$. Then the mapping $f \mapsto f|_{\sigma(A_x)}$ from $L_2(\sigma(A), \mu_x)$ to $L_2(\sigma(A_x), \check{\mu}_x)$ is an isometric isomorphism. Thus one still has a unitary operator $J_x \colon H_x \to L_2(\sigma(A), \mu_x)$ such that $f(A_x) = J_x^{-1} M_f J_x$ for all $f \in C(\sigma(A))$. The measure μ_x is the **spectral measure** associated with x; note that

$$\mu_x(\sigma(A)) = \check{\mu}_x(\sigma(A_x)) = \int_{\sigma(A_x)} 1 \,\mathrm{d}\check{\mu}_x = \|x\|^2 \tag{F.3}$$

by (F.2). (In the situation of Theorem F.4, μ is the spectral measure associated with x.)

If $y \in H$ is orthogonal to H_x, then

$$\big(f(A)y \,\big|\, g(A)x\big) = \big(y \,\big|\, (\bar{f}g)(A)x\big) = 0$$

for all $f, g \in C(\sigma(A))$, and this implies that $H_y \perp H_x$.

F.5 Proposition. *There exists a set $X \subseteq H \setminus \{0\}$ such that $H_x \perp H_y$ for all $x, y \in X$ with $x \neq y$ and such that $H = \bigoplus_{x \in X} H_x$.*

Proof. The set

$$\mathcal{X} := \big\{ X \subseteq H \setminus \{0\} ;\, H_x \perp H_y \text{ for all } x, y \in X \text{ with } x \neq y \big\}$$

is ordered by inclusion, and any chain $\mathcal{C}$ in $\mathcal{X}$ is dominated, by $\bigcup_{X \in \mathcal{C}} X$. Zorn's lemma implies that the $\mathcal{X}$ has a maximal element X. For this set X one has $H = \bigoplus_{x \in X} H_x$, because otherwise there would exist $0 \neq y \in \big(\bigoplus_{x \in X} H_x\big)^\perp$, and then $X \cup \{y\} \in \mathcal{X}$, in contradiction to the maximality of X. □

F.6 Remark. It is not hard to see that the set X in Proposition F.5 is countable if and only if H is separable. If H is separable one does not need Zorn's lemma for the proof, because one can construct the set X, starting with a countable dense subset of H. △

Proof of Theorem F.1 for the case of bounded self-adjoint operators.
Let $X \subseteq H \setminus \{0\}$ be a set with the properties stated in Proposition F.5. For each $x \in X$ let μ_x be the spectral measure associated with x, and let A_x be the restriction of A to H_x. Then there exists a unitary operator $J_x \colon H_x \to L_2(\sigma(A), \mu_x)$ such that $A_x = J_x^{-1} M_{\mathrm{id}} J_x$; see Theorem F.4 and the comments after its proof. We compose the family $(J_x)_{x \in X}$ to a unitary operator

$$J := \bigoplus_{x \in X} J_x \colon \bigoplus_{x \in X} H_x \,(= H) \to \bigoplus_{x \in X} L_2\big(\sigma(A), \mu_x\big)$$

in the canonical way. (Recall that for any $y = (y_x)_{x \in X} \in \bigoplus_{x \in X} H_x$ the set $\{x \in X ;\, y_x \neq 0\}$ is countable.) Composing $A = \bigoplus_{x \in X} A_x$ we then obtain

$$A = \bigoplus_{x \in X} \big(J_x^{-1} M_{\mathrm{id},x} J_x\big),$$

where $M_{\mathrm{id},x}$ denotes the operator of multiplication by id on $L_2\big(\sigma(A),\mu_x\big)$.

In order to complete the proof we define $\Omega := \sigma(A)\times X = \bigcup_{x\in X}\sigma(A)_x$, where in the last equality we have denoted the set $\sigma(A)\times\{x\}$ by $\sigma(A)_x$. Given $x\in X$, we use the bijection

$$\sigma(A)\to\sigma(A)_x,\quad \lambda\mapsto(\lambda,x)$$

to transfer the measure μ_x from $\sigma(A)$ to $\sigma(A)_x$; slightly abusing notation, we retain the name μ_x for the image measure. We define

$$\begin{aligned}\mathcal{A} &:= \big\{B\subseteq\Omega;\ B\cap\sigma(A)_x \text{ measurable } (x\in X)\big\},\\ \mu(B) &:= \sum_{x\in X}\mu_x(B\cap\sigma(A)_x)\qquad (B\in\mathcal{A}).\end{aligned}$$

It is easy to see that then $(\Omega,\mathcal{A},\mu)$ is a measure space and that

$$L_2(\Omega,\mathcal{A},\mu)=\bigoplus_{x\in X}L_2\big(\sigma(A)_x,\mu_x\big),$$

with $f\in L_2(\Omega,\mu)$ corresponding to $\big(f|_{\sigma(A)_x}\big)_{x\in X}\in\bigoplus_{x\in X}L_2\big(\sigma(A)_x,\mu_x\big)$. Finally, it is also easy to see that the functions $\mathrm{id}\colon\sigma(A)_x\to\mathbb{R}$, for $x\in X$, merge to a measurable function $\alpha\colon\Omega\to\mathbb{R}$, $\alpha(t,x):=t$ for $(t,x)\in\Omega=\sigma(A)\times X$, and that then

$$A=J^{-1}M_\alpha J. \qquad \square$$

F.7 Remarks. (a) If H is separable, then the set X can be chosen such that μ is finite. Indeed, as $\mu(\Omega)=\sum_{x\in X}\mu(\sigma(A)_x)=\sum_{x\in X}\mu_x(\sigma(A))=\sum_{x\in X}\|x\|^2$ (recall (F.3)), one only has to choose X such that $\sum_{x\in X}\|x\|^2<\infty$, and this can be arranged if X is countable.

(b) The measure space $(\Omega,\mathcal{A},\mu)$ obtained in the proof of Theorem F.1 for the case of bounded operators is easily seen to be **semi-finite**, i.e., every set $B\in\mathcal{A}$ with $\mu(B)=\infty$ contains a set $C\in\mathcal{A}$ with $0<\mu(C)<\infty$. Indeed, if $B\in\mathcal{A}$ – in the proof given above – is such that $\mu(B)=\infty$, then there exists $x\in X$ with $0<\mu_x(B\cap\sigma(A)_x)$ $(<\infty)$. This property persists in the case of unbounded self-adjoint operators; see the first paragraph of the proof for this case in the next section. This means that the property of semi-finiteness of the measure space could have been part of the statement of Theorem F.1.

(c) We refer to Exercise F.6 for the proof that any measure μ can be modified to a semi-finite measure without changing the space $L_2(\mu)$. $\triangle$

F.2 The spectral theorem for self-adjoint operators; the general case

In this section we use the spectral theorem for bounded self-adjoint operators to give a proof for the general case. As before, let H be a (real or complex) Hilbert space.

We begin with some preliminary material. The main tool is the following result of von Neumann.

F.8 Theorem. *Let G be a Hilbert space, A a closed operator from H to G,* $\mathrm{dom}(A)$ *dense. Then A^*A is an accretive self-adjoint operator in H.*

Proof. Let $z \in H$. From $A^\perp = -(A^*)^{-1}$ in $H \oplus G$ (see Section 6.1) we conclude that $(z,0) \in H \oplus G$ can be written as $(z,0) = (x,Ax) + (-A^*y, y)$, with suitable $x \in \mathrm{dom}(A)$, $y \in \mathrm{dom}(A^*)$. This implies $z = x - A^*y = x + A^*Ax$. So we have shown that $\mathrm{ran}(I + A^*A) = H$. Clearly

$$(A^*Ax \,|\, x) = (Ax \,|\, Ax) = (x \,|\, A^*Ax) \geqslant 0 \qquad \big(x \in \mathrm{dom}(A^*A) \subseteq \mathrm{dom}(A)\big). \tag{F.4}$$

Therefore Proposition 6.9 implies that $I + A^*A$ is self-adjoint, and (F.4) also implies that A^*A is accretive. □

F.9 Remark. Note that the assertion of Theorem F.8 also contains the information that $\mathrm{dom}(A^*A)$ is dense in H – a property that is not a priori obvious. △

Given an accretive self-adjoint operator C, we will use the notation $(I + C)^{-1/2}$ as a shorthand for the (accretive) square root of the bounded accretive self-adjoint operator $(I + C)^{-1}$. This square root exists by Theorem F.3.

F.10 Proposition. *Let A be a self-adjoint operator in H. Then A^2 is an accretive self-adjoint operator, $B := A(I + A^2)^{-1/2}$ belongs to $\mathcal{L}(H)$ and is a self-adjoint contraction, and -1 and 1 are not eigenvalues of B.*

Proof. From $A^* = A$ and Theorem F.8 we conclude that A^2 is an accretive self-adjoint operator; hence $(I + A^2)^{-1}$ is an accretive self-adjoint operator.

Let $x \in \mathrm{dom}(A)$, $y := (I + A^2)^{-1}x$. Then $y \in \mathrm{dom}(A^3)$,

$$A(I + A^2)^{-1}x = Ay = (I + A^2)^{-1}A(I + A^2)y = (I + A^2)^{-1}Ax,$$

and therefore $A\,p\big((I+A^2)^{-1}\big)x = p\big((I+A^2)^{-1}\big)Ax$ for all polynomials p. Since $(I+A^2)^{-1/2}$ is approximated by polynomials in $(I + A^2)^{-1}$ and since A is closed, it follows that $(I + A^2)^{-1/2}x \in \mathrm{dom}(A)$ and

$$A(I + A^2)^{-1/2}x = (I + A^2)^{-1/2}Ax. \tag{F.5}$$

This implies that

$$\begin{aligned}\big\|A(I + A^2)^{-1/2}x\big\|^2 &= \big((I + A^2)^{-1}Ax \,\big|\, Ax\big) = \big(A^2(I + A^2)^{-1}x \,\big|\, x\big) \\ &\leqslant \big((I + A^2)(I + A^2)^{-1}x \,\big|\, x\big) = \|x\|^2.\end{aligned}$$

As $\mathrm{dom}(A)$ is dense in H and A is closed, we conclude that $\mathrm{ran}\big((I + A^2)^{-1/2}\big) \subseteq \mathrm{dom}(A)$ and $B \in \mathcal{L}(H)$, $\|B\| \leqslant 1$.

For all $x, y \in \mathrm{dom}(A)$ one has

$$\big(A(I + A^2)^{-1/2}x \,\big|\, y\big) = \big((I + A^2)^{-1/2}Ax \,\big|\, y\big) = \big(x \,\big|\, A(I + A^2)^{-1/2}y\big),$$

which, again by the denseness of $\mathrm{dom}(A)$, shows that B is symmetric, hence self-adjoint.

Since (F.5) holds for all $x \in \mathrm{dom}(A)$, we obtain

$$I - B^2 = I - A^2(I + A^2)^{-1} = (I + A^2)^{-1}. \tag{F.6}$$

Hence, if $x \in H$ satisfies $Bx = \pm x$, then $0 = x - B^2x = (I + A^2)^{-1}x$, and this implies $x = 0$. □

Proof of Theorem F.1, general case.
We put $B := A(I + A^2)^{-1/2}$ – see Proposition F.10 – and recall that B is a contractive self-adjoint operator. We note that (F.6) implies that

$$B = A(I + A^2)^{-1/2} = A(I - B^2)^{1/2}.$$

We know from Theorem F.1 for bounded self-adjoint operators and its proof – see Section F.1, in particular Remark F.7(b) – that B is unitarily equivalent to a multiplication operator M_β on $L_2(\mu)$ for some semi-finite measure space $(\Omega, \mathcal{A}, \mu)$, with measurable $\beta\colon \Omega \to (-1, 1)$. (For the property that β takes its values in $(-1, 1)$ we apply properties (i) and (iii) of Remark 13.21, taking into account the contractivity of B and the injectivity of $\pm I - B$.)

In order to simplify notation we now assume – without loss of generality – that $H = L_2(\mu)$ and $B = M_\beta$. Then we have $M_\beta = B = A(I - B^2)^{1/2} = AM_{(1-\beta^2)^{1/2}}$, and composing on the right with $M_{(1-\beta^2)^{-1/2}}$ we obtain

$$M_\beta M_{(1-\beta^2)^{-1/2}} = A|_{\mathrm{ran}(I-B^2)^{1/2}} \subseteq A.$$

One easily sees that $M_\beta M_{(1-\beta^2)^{-1/2}} = M_\alpha$, where $\alpha := \beta(1 - \beta^2)^{-1/2}$; indeed, this is an immediate consequence of $\mathrm{dom}(M_\alpha) = \mathrm{dom}(M_{(1-\beta^2)^{-1/2}})$, which in turn follows from the equality $\alpha^2 + 1 = 1/(1 - \beta^2)$.

The conclusion is that $M_\alpha \subseteq A$, and since both M_α and A are self-adjoint one obtains $M_\alpha = A$. □

F.11 Remark. The idea of the proof of Theorem F.1 for the general case is to find an injective function $f\colon \sigma(A) \to \mathbb{R}$ such that $f(A)$ becomes a bounded self-adjoint operator (to which the result for the bounded case can be applied). The sophisticated choice $f(t) = t(1 + t^2)^{-1/2}$ above is needed when $\sigma(A) = \mathbb{R}$. If there exists a point $\lambda \in \mathbb{R} \setminus \sigma(A)$, then one can use $f(t) := (\lambda - t)^{-1}$ and argue as above, thereby reducing the proof to the bounded self-adjoint operator $(\lambda - A)^{-1}$. △

Notes

The version of the spectral theorem we have presented in Theorem 13.20 is well-known and well-established. Nevertheless we have been at a loss finding a complete and simple proof in the literature, without the restriction to separable or complex Hilbert spaces. This is why we decided to include a complete proof.

The proof we have given draws on various sources. The proof of the functional calculus for bounded operators, Theorem F.3, is inspired by Riesz–Sz.-Nagy [RiNa68; Kap. VII, Nr. 106], with a shortcut made possible by our Lemma F.2. The procedures to find the spectral measure for elements in the Hilbert space (Theorem F.4), the decomposition of the Hilbert space into cyclic subspaces (Proposition F.5) and the construction of the measure space for the representation of the operator as a multiplication operator in Section F.1 are rather standard and have no special sources. The proof of Theorem F.8 is taken from Kato [Kat80; V, Theorem 3.24]. To our knowledge, the method for deriving the spectral

theorem for unbounded operators from the case of bounded operators using the operator $A(I + A^2)^{-1/2}$, as presented in Section F.2, appeared for the first time in Schmüdgen's book [Sch12].

Exercises

F.1 Let $K \subseteq \mathbb{R}$ be a compact set, and let $a, b \in \mathbb{R}$, $a < b$, $K \subseteq [a, b]$. Show that for any $f \in C(K)$ there exists a function $\tilde{f} \in C[a, b]$ such that $\tilde{f}|_K = f$, $\|\tilde{f}\|_\infty = \|f\|_\infty$. Conclude that every function $f \in C(K)$ can be approximated by polynomials, uniformly on K. (Hint: There exists a pairwise disjoint family $(U_j)_{j\in N}$, with N countable, of open intervals such that $K = \mathbb{R} \setminus \bigcup_{j\in N} U_j$.)

We note that the above statement is a special case of the Tietze extension theorem; see for instance [Haa14; Theorem 15.15].

F.2 Let H be a Hilbert space, $A \in \mathcal{L}(H)$ self-adjoint.

(a) Show that the mapping $\Phi\colon C(\sigma(A)) \to \mathcal{L}(H),\ f \mapsto f(A)$ from Theorem F.3 is injective.

Hint: Assume that there exists $f \in C(\sigma(A))$, $f \neq 0$ such that $f(A) = 0$. Choose $\lambda \in \sigma(A)$ such that $f(\lambda) \neq 0$. Show that $(\lambda - A)^2 = ((\lambda - \mathrm{id})^2 + |f|^2)(A)$ is invertible in $\mathcal{L}(H)$, and conclude that $\lambda \in \rho(A)$.

(b) Let $x \in H$ be a cyclic vector for A. Show that the mapping $C(\sigma(A)) \ni f \mapsto f(A)x \in H$ is injective.

F.3 Prove the statements in Remark F.6.

F.4 Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $H := L_2(\mu)$. Let $\alpha\colon \Omega \to \mathbb{R}$ be measurable, $A := M_\alpha$, and define $\mathrm{dom}^\infty(A) := \bigcap_{n\in\mathbb{N}} \mathrm{dom}(A^n)$.

For $a \leqslant b$ put $\lambda := \frac{a+b}{2}$, $r := \frac{b-a}{2}$, and define

$$H_{[a,b]} := \big\{ f \in L_2(\mu);\ f = \mathbf{1}_{[|\alpha-\lambda|\leqslant r]} f \big\}.$$

Show that

$$H_{[a,b]} = \big\{ f \in \mathrm{dom}^\infty(A);\ \|(A-\lambda)^n f\| \leqslant r^n \|f\|\ (n \in \mathbb{N}) \big\}. \tag{F.7}$$

Convince yourself that this shows that the spaces $H_{[a,b]}$ in (13.13) do not depend on the representation of A as a multiplication operator in Theorem 13.20; note that in the description (F.7) the function α does not appear.

F.5 Let A be a self-adjoint operator in a Hilbert space H. Let $(\Omega_j, \mathcal{A}_j, \mu_j)$, α_j, J_j be as in Theorem 13.20, for $j = 1, 2$. Show that then

$$J_1^{-1} M_{f\circ\alpha_1} J_1 = J_2^{-1} M_{f\circ\alpha_2} J_2 \tag{F.8}$$

for all Borel measurable functions $f\colon \mathbb{R} \to \mathbb{R}$, by implementing the following steps.

(i) It is sufficient to show the assertion for all bounded Borel measurable functions.

(ii) Let F denote the set of all bounded Borel measurable functions $f\colon \mathbb{R} \to \mathbb{R}$ for which (F.8) holds. Show that F is an algebra (under pointwise addition and multiplication

of functions), that F contains the indicator functions of bounded closed intervals (use the independence of the spaces $H_{[a,b]}$, Exercise F.4) and the function $\mathbf{1}$, and that for all bounded, pointwise convergent sequences in F the limit belongs to F (a consequence of the dominated convergence theorem).

(iii) Use (ii) to show that the set $\mathcal{B} := \{B \subseteq \mathbb{R};\, \mathbf{1}_B \in F\}$ is the σ-algebra of Borel subsets of $\mathbb{R}$. Conclude that F is the set of all bounded Borel measurable functions (by using the fact that every bounded Borel measurable function can be obtained as the pointwise limit of a bounded sequence in $S(\mathcal{B}) = \operatorname{lin}\{\mathbf{1}_B;\, B \in \mathcal{B}\}$).

Note. Stripping hypotheses to a minimum, we include the following two observations. Let Ω be a set, S an algebra of functions $f\colon \Omega \to \mathbb{R}$. Then $\mathcal{B} := \{B \subseteq \Omega;\, \mathbf{1}_B \in S\}$ is a ring of subsets of Ω (see the hint to Exercise 10.5(a)). If additionally $\mathbf{1} \in S$, then $\Omega \in \mathcal{B}$, and $\mathcal{B}$ is an algebra of subsets (see step (iii) above).

F.6 Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. We define

$$\mathcal{A}_{\text{fin}} := \{A \in \mathcal{A};\, \mu(A) < \infty\},$$

and for $A \in \mathcal{A}$ we put

$$\mu_{\text{sf}}(A) := \sup\{\mu(B);\, B \in \mathcal{A}_{\text{fin}},\ B \subseteq A\}.$$

Show the following properties.

(a) μ_{sf} is a semi-finite measure, and $\mu_{\text{sf}}(A) = \mu(A)$ for all $A \in \mathcal{A}_{\text{fin}}$. Every μ-null set is a μ_{sf}-null set.

(b) Every μ-integrable function $f\colon \Omega \to [0, \infty)$ is μ_{sf}-integrable. If $f\colon \Omega \to [0, \infty)$ is μ_{sf}-integrable, then there exists a set $A \in \mathcal{A}$ with $\mu_{\text{sf}}(\Omega \setminus A) = 0$ such that $\mathbf{1}_A f$ is μ-integrable.

(c) If $f \in L_2(\mu)$, then every μ-representative of f belongs to $L_2(\mu_{\text{sf}})$, and the mapping $L_2(\mu) \to L_2(\mu_{\text{sf}})$, thereby defined, is an isometric isomorphism. (Hint: Given $f \in L_2(\mu_{\text{sf}})$, use part (b) to find a μ-square-integrable representative of f.)

Appendix G

The Riesz–Markov representation theorem

The aim of this appendix is to prove the Riesz–Markov theorem in the special context of compact metric spaces. By $C(K)$ we denote the space of real-valued continuous functions on a compact space K.

G.1 Theorem. *Let K be a compact metric space, and let $\eta\colon C(K) \to \mathbb{R}$ be a positive linear functional. Then there exists a unique Borel measure μ on K such that*

$$\eta(f) = \int f \,\mathrm{d}\mu$$

for all $f \in C(K)$.

Throughout this appendix we will assume that K is a compact topological space; only at the very end it will be assumed additionally that K is a metric space. Also, η will be assumed to be a positive linear functional on $C(K)$; see Section G.1 for the definition.

G.2 Remark. It is of interest to mention the 'complex version' of the Riesz–Markov theorem. In that case one has a positive linear functional $\eta\colon C(K;\mathbb{C}) \to \mathbb{C}$, which means that its restriction η_0 to $C(K;\mathbb{R})$ is positive in the sense defined in Section G.1.

Applying Theorem G.1 to η_0 one obtains a measure μ as above, and then one has

$$\eta(f) = \eta_0(\operatorname{Re} f) + \mathrm{i}\,\eta_0(\operatorname{Im} f) = \int \operatorname{Re} f \,\mathrm{d}\mu + \mathrm{i} \int \operatorname{Im} f \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu,$$

for all $f \in C(K;\mathbb{C})$. △

G.1 Elementary properties of positive linear functionals on $C(K)$

Besides the algebra structure of $C(K) = C(K;\mathbb{R})$ – we will only treat the case of real scalars – we will also use the lattice operations $f \vee g$ and $f \wedge g$ for $f, g \in C(K)$ (and in fact for any functions $f, g\colon K \to \mathbb{R}$),

$$f \vee g(x) := \max\{f(x), g(x)\}, \quad f \wedge g(x) := \min\{f(x), g(x)\} \qquad (x \in K).$$

By $C(K)_+ := \{f \in C(K);\ f \geqslant 0\}$ we will denote the positive cone of $C(K)$.

A linear functional $\eta\colon C(K) \to \mathbb{R}$ is called **positive** if $\eta(f) \geqslant 0$ for all $f \in C(K)_+$. It is important that positive functionals automatically have the following 'continuity property'.

G.3 Proposition. *The functional η is σ-continuous, i.e., whenever (f_n) is a decreasing sequence in $C(K)_+$ with $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in K$, then $\lim_{n\to\infty} \eta(f_n) = 0$.*

The essential step in the proof is Dini's lemma, as follows.

G.4 Lemma. *Let (f_n) be a decreasing sequence in $C(K)_+$ with $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in K$. Then $\lim_{n\to\infty} \|f_n\|_\infty = 0$.*

Proof. Let $\varepsilon > 0$. For all $x \in K$ there exists $n_x \in \mathbb{N}$ such that $f_{n_x}(x) < \varepsilon$, and therefore $U_x := [f_{n_x} < \varepsilon]$ is an open neighbourhood of x. From the open covering $(U_x)_{x\in K}$ of K we can choose a finite subcovering $(U_x)_{x\in F}$. Put $n_\varepsilon := \max_{x\in F} n_x$. Because of $f_{n_\varepsilon} \leqslant f_{n_x}$ for all $x \in F$ we obtain $f_{n_\varepsilon} \leqslant \varepsilon$, $\|f_{n_\varepsilon}\|_\infty \leqslant \varepsilon$. □

Proof of Proposition G.3. From Lemma G.4 and the positivity of η we obtain $0 \leqslant \eta(f_n) \leqslant \|f_n\|_\infty \eta(\mathbf{1}_K) \to 0$ as $n \to \infty$. □

G.2 The upper integral and $\mathcal{L}_1(K, \eta)$

Recall that η is supposed to be a positive linear functional on $C(K)$. Given a function $f \in [0,\infty)^K$, we call a sequence (φ_n) a **hull sequence for** f if (φ_n) is a monotone increasing sequence in $C(K)_+$ and $\lim_{n\to\infty} \varphi_n(x) \geqslant f(x)$ for all $x \in K$. We define the upper integral of f by

$$\eta^*(f) := \inf\Big\{ \lim_{n\to\infty} \eta(\varphi_n);\ (\varphi_n) \text{ a hull sequence for } f \Big\} \in [0,\infty].$$

We mention that in these definitions we allow the limit of an increasing sequence in $[0,\infty)$ to be ∞.

G.5 Proposition. (a) *For all $f \in [0,\infty)^K$, $a \geqslant 0$ one has*

$$\eta^*(af) = a\,\eta^*(f).$$

(b) *For all $f, g \in [0,\infty)^K$ one has*

$$\eta^*(f+g) \leqslant \eta^*(f) + \eta^*(g).$$

(c) *If $f, g \in [0,\infty)^K$, $f \leqslant g$, then*

$$\eta^*(f) \leqslant \eta^*(g).$$

Proof. (a) and (c) are obvious. (For the proof of (a), note first that the case $a = 0$ is trivial, and then consider the case $a > 0$.)

(b) If (φ_n) and (ψ_n) are hull sequences for f and g, respectively, then $(\varphi_n + \psi_n)$ is a hull sequence for $f + g$. This implies that

$$\eta^*(f+g) \leqslant \lim_{n\to\infty} \eta(\varphi_n + \psi_n) = \lim_{n\to\infty} \eta(\varphi_n) + \lim_{n\to\infty} \eta(\psi_n).$$

Using the definition of $\eta^*(f)$ and $\eta^*(g)$ one obtains the asserted inequality. □

Next we show that the subadditivity of η^* stated in Proposition G.5(b) can be reinforced to the following version of countable subadditivity.

G.6 Proposition. *Let (f_n) be a sequence in $[0,\infty)^K$ with $\sum_{n=1}^\infty f_n \in [0,\infty)^K$ (where the sum should be interpreted as taken pointwise). Then*

$$\eta^*\Big(\sum_{n=1}^\infty f_n\Big) \leqslant \sum_{n=1}^\infty \eta^*(f_n).$$

Proof. The inequality is trivial if the right-hand side is ∞. Assume that the right-hand side is less than ∞, and let $\varepsilon > 0$. Then for all $n \in \mathbb{N}$ there exists a hull sequence $(\varphi_{nk})_k$ for f_n such that $\sup_k \eta(\varphi_{nk}) \leqslant \eta^*(f_n) + 2^{-n}\varepsilon$. We put

$$\psi_k := \sum_{n=1}^k \varphi_{nk} \qquad (k \in \mathbb{N}).$$

Then (ψ_k) is a hull sequence for $\sum_{n=1}^j f_n$ for each $j \in \mathbb{N}$, hence for $\sum_{n=1}^\infty f_n$, and

$$\eta(\psi_k) = \sum_{n=1}^k \eta(\varphi_{nk}) \leqslant \sum_{n=1}^k \big(\eta^*(f_n) + 2^{-n}\varepsilon\big) \leqslant \sum_{n=1}^\infty \eta^*(f_n) + \varepsilon,$$

for all $k \in \mathbb{N}$. Hence

$$\eta^*\Big(\sum_{n=1}^\infty f_n\Big) \leqslant \sup_k \eta(\psi_k) \leqslant \sum_{n=1}^\infty \eta^*(f_n) + \varepsilon. \qquad \square$$

G.7 Proposition. *For all $\varphi \in C(K)_+$ one has $\eta^*(\varphi) = \eta(\varphi)$.*

Proof. The constant sequence $(\varphi)_{n\in\mathbb{N}}$ is a hull sequence for φ, and this implies the inequality $\eta^*(\varphi) \leqslant \eta(\varphi)$.

If (φ_n) is a hull sequence for φ, then $(\varphi_n \wedge \varphi)$ is a hull sequence for φ as well, and $(\varphi - \varphi_n \wedge \varphi)_{n\in\mathbb{N}}$ is decreasing, with $\varphi - \varphi_n \wedge \varphi \to 0$ pointwise as $n \to \infty$; hence Proposition G.3 implies

$$\eta(\varphi) = \lim_{n\to\infty} \eta(\varphi_n \wedge \varphi) \leqslant \lim_{n\to\infty} \eta(\varphi_n).$$

This shows that $\eta(\varphi) \leqslant \eta^*(\varphi)$. $\square$

Now we define $q\colon \mathbb{R}^K \to [0,\infty]$ by

$$q(f) := \eta^*(|f|) \qquad (f \in \mathbb{R}^K).$$

Then Proposition G.5 implies that $\mathcal{L} := \big\{f \in \mathbb{R}^K;\, q(f) < \infty\big\}$ is a subspace of $\mathbb{R}^K$ and that the restriction of q to $\mathcal{L}$ is a semi-norm. Proposition G.7 shows that η (defined on $C(K)$) is continuous with respect to q. We define the space of **integrable functions** $\mathcal{L}_1(K,\eta)$ as the closure of $C(K)$ in the semi-normed space $(\mathcal{L}, q)$; then η has a (unique) continuous extension to $\mathcal{L}_1(K,\eta)$, which we will still denote by η.

In the following theorem we collect properties that will be needed in order to define the measure μ on a suitable σ-algebra. In particular, the version of the monotone convergence theorem stated in part (c) will be responsible for the σ-additivity of the measure.

G.8 Theorem. (a) *The functional η is positive on $\mathcal{L}_1(K,\eta)$.*

(b) *If $f, g \in \mathcal{L}_1(K,\eta)$, then $|f|, f \vee g, f \wedge g \in \mathcal{L}_1(K,\eta)$.*

(c) *Let (f_n) be a monotone increasing sequence in $\mathcal{L}_1(K,\eta)_+$ such that $f := \sup_{n\in\mathbb{N}} f_n \in [0,\infty)^K$ and $\sup_{n\in\mathbb{N}} \eta(f_n) < \infty$. Then $f \in \mathcal{L}_1(K,\eta)$ and $\eta(f) = \sup_{n\in\mathbb{N}} \eta(f_n)$.*

Proof. (a) Let $f \in \mathcal{L}_1(K,\eta)_+$, and let (φ_n) be a sequence in $C(K)$ such that $q(f-\varphi_n) \to 0$ as $n \to \infty$. Then $|f - \varphi_n^+| \leqslant |f - \varphi_n|$ for all $n \in \mathbb{N}$; hence $q(f - \varphi_n^+) \to 0$ as $n \to \infty$, which implies $\eta(f) = \lim_{n\to\infty} \eta(\varphi_n^+) \geqslant 0$.

(b) There exists a sequence (φ_n) in $C(K)$ such that $\eta^*(|f - \varphi_n|) \to 0$ $(n \to \infty)$. This implies that $\eta^*\big(\big||f| - |\varphi_n|\big|\big) \leqslant \eta^*(|f - \varphi_n|) \to 0$ $(n \to \infty)$; hence $|f| \in \mathcal{L}_1(K,\eta)$. The equalities $f \vee g = \frac{1}{2}(f + g + |f - g|)$ and $f \wedge g = \frac{1}{2}(f + g - |f - g|)$ show the remaining assertions.

(c) We supplement the sequence (f_n) by $f_0 := 0$. The relation

$$\sum_{j=1}^{n} \eta^*(f_n - f_{n-1}) = \sum_{j=1}^{n} \eta(f_n - f_{n-1}) = \eta(f_n) \leqslant \sup_{n\in\mathbb{N}} \eta(f_n) < \infty$$

implies that the series $\sum_{j=1}^{\infty} \eta^*(f_n - f_{n-1})$ is convergent. Hence by the countable subadditivity of η^*, see Proposition G.6, we obtain

$$\eta^*(f - f_n) = \eta^*\Big(\sum_{j=n+1}^{\infty} (f_j - f_{j-1})\Big) \leqslant \sum_{j=n+1}^{\infty} \eta^*(f_j - f_{j-1}) \to 0 \qquad (n \to \infty).$$

This shows that $f \in \mathcal{L}_1(K,\eta)$ and $\eta(f_n) \to \eta(f)$ $(n \to \infty)$, as asserted. □

G.3 The measure μ and the proof of the Riesz–Markov theorem

We define the collection $\mathcal{A}$ of **η-measurable sets**,

$$\mathcal{A} := \{A \subseteq K\,;\, \mathbf{1}_A \in \mathcal{L}_1(K,\eta)\},$$

and for $A \in \mathcal{A}$ we define $\mu(A) := \eta(\mathbf{1}_A)$.

G.9 Proposition. *The collection $\mathcal{A}$ is a σ-algebra, and μ is a measure on $\mathcal{A}$.*

Proof. From $0, \mathbf{1}_K \in \mathcal{L}_1(K,\eta)$ we obtain $\varnothing, K \in \mathcal{A}$; also $\mu(\varnothing) = \eta(0) = 0$. If $A, B \in \mathcal{A}$, then $\mathbf{1}_{A\cup B} = \mathbf{1}_A \vee \mathbf{1}_B \in \mathcal{L}_1(K,\eta)$, and hence $A \cup B \in \mathcal{A}$; if additionally $A \cap B = \varnothing$, then $\mu(A \cup B) = \eta(\mathbf{1}_A + \mathbf{1}_B) = \eta(\mathbf{1}_A) + \eta(\mathbf{1}_B) = \mu(A) + \mu(B)$.

If $A \in \mathcal{A}$, then $\mathbf{1}_{K\setminus A} = \mathbf{1}_K - \mathbf{1}_A \in \mathcal{L}_1(K,\eta)$, hence $K \setminus A \in \mathcal{A}$.

Finally, if (A_n) is an increasing sequence in $\mathcal{A}$, $A := \bigcup_n A_n$, then $\sup_n \mathbf{1}_{A_n} = \mathbf{1}_A$, and $\sup_n \eta(\mathbf{1}_{A_n}) \leqslant \eta(\mathbf{1}_K) < \infty$. Therefore, Theorem G.8(c) implies that $\mathbf{1}_A \in \mathcal{L}_1(K,\eta)$ and $\mu(A) = \eta(\mathbf{1}_A) = \lim_{n\to\infty} \eta(\mathbf{1}_{A_n}) = \lim_{n\to\infty} \mu(A_n)$.

These properties show that $\mathcal{A}$ is a σ-algebra and that μ is a measure on $\mathcal{A}$. □

We now come to the proof of the Riesz–Markov theorem.

Proof of Theorem G.1. For the existence part let $\mathcal{A}$ and μ be as defined above. Let $\mathcal{B}_0$ be the σ-algebra generated by the collection

$$\big\{[f>0];\, f\in C(K)\big\};$$

then $\mathcal{B}_0$ is the σ-algebra of **Baire sets**. We show that $\mathcal{B}_0\subseteq\mathcal{A}$; for this it is sufficient to show that $[f>0]\in\mathcal{A}$ for all $f\in C(K)$. Let $f\in C(K)$, and put $f_n:=\big(nf\wedge 1\big)\vee 0$. Then (f_n) is an increasing sequence in $C(K)_+$, $\mathbf{1}_{[f>0]}=\sup_n f_n$, and Theorem G.8(c) implies that $\mathbf{1}_{[f>0]}\in\mathcal{L}_1(K,\eta)$, i.e. $[f>0]\in\mathcal{A}$. Let μ_0 denote the restriction of μ to $\mathcal{B}_0$.

For $A\in\mathcal{B}_0$ one has $\eta(\mathbf{1}_A)=\mu_0(A)=\int\mathbf{1}_A\,\mathrm{d}\mu_0$. By linearity of η and the integral, the equality $\eta(f)=\int f\,\mathrm{d}\mu_0$ extends to all functions $f\in\mathcal{S}(\mathcal{B}_0):=\mathrm{lin}\{\mathbf{1}_A;\ A\in\mathcal{B}_0\}$ (the simple functions over $\mathcal{B}_0$). Given $f\in C(K)$, there exists a sequence (f_n) in $\mathcal{S}(\mathcal{B}_0)$ with $\|f-f_n\|_\infty\to 0$ as $n\to\infty$; see Exercise G.1. Using Theorem G.8(a) we conclude that $|\eta(f)-\eta(f_n)|\leqslant\|f-f_n\|_\infty\eta(\mathbf{1}_K)\to 0$ as $n\to\infty$, and also $|\int f\,\mathrm{d}\mu_0-\int f_n\,\mathrm{d}\mu_0|\leqslant\|f-f_n\|_\infty\mu_0(K)\to 0$ as $n\to\infty$. This shows $\eta(f)=\int f\,\mathrm{d}\mu_0$.

In order to complete the proof of existence it remains to show that $\mathcal{B}_0=\mathcal{B}$, where $\mathcal{B}$ is the σ-algebra of Borel sets, and for this property it is sufficient to show that each open subset U of K is of the form $U=[f>0]$, for some $f\in C(K)$. At this point we use the metric d on the space K. In fact, putting

$$f(x):=\mathrm{dist}(x,K\setminus U)\qquad(x\in K),$$

the distance of x to $K\setminus U$, we have a function $f\in C(K)$ satisfying $U=[f>0]$.

For the uniqueness part we assume that μ and ν are two Borel measures representing η, i.e. $\eta(f)=\int f\,\mathrm{d}\mu=\int f\,\mathrm{d}\nu$ for all $f\in C(K)$. Then $\mu+\nu$ is a Borel measure. Let $B\subseteq K$ be a Borel set. Then $\mathbf{1}_B\in L_1(\mu+\nu)$, and the denseness of $C(K)$ in $L_1(\mu+\nu)$ – see Exercise G.2 – implies that there exists a sequence (f_n) in $C(K)$ converging to $\mathbf{1}_B$ in $L_1(\mu+\nu)$. Now $\|f\|_{L_1(\mu)}\leqslant\|f\|_{L_1(\mu+\nu)}$ and $\|f\|_{L_1(\nu)}\leqslant\|f\|_{L_1(\mu+\nu)}$ for all $f\in L_1(\mu+\nu)$. It follows that the sequence (f_n) converges to $\mathbf{1}_B$ in $L_1(\mu)$ and $L_1(\nu)$; hence

$$\mu(B)=\int\mathbf{1}_B\,\mathrm{d}\mu=\lim_{n\to\infty}f_n\,\mathrm{d}\mu=\lim_{n\to\infty}f_n\,\mathrm{d}\nu=\int\mathbf{1}_B\,\mathrm{d}\nu=\nu(B).$$

This shows that μ and ν coincide on the σ-algebra $\mathcal{B}$ of Borel subsets of K. □

We add a comment on the uniqueness part of the above proof, for the case that K is a non-metric compact space. In view of the "Additional remark" at the end of Exercise G.2, the proof of uniqueness given above yields the uniqueness of μ on the σ-algebra $\mathcal{B}_0$ of Baire subsets of K.

Notes

The Riesz–Markov representation theorem, often simply called the 'Riesz representation theorem', is a classical result going back to F. Riesz (1909), A. Markov (1938) and

A. Kakutani (1941) in versions of increasing generality. We have been at a loss finding a concise proof just containing the bare facts. This is why we decided to include a proof for the mininal version we need in the proof of the spectral theorem; see Appendix F. Our proof is a reduced version of the Daniell integral, first extending the given positive linear functional η to a large class of functions, but developing only part of integration theory along this line.

We point out that this proof – applied to the case of (locally) compact spaces – only yields a Baire measure, which for the case of compact metric spaces is automatically a Borel measure. A modification of the extension of the functional η, admitting arbitrary directed sets as 'increasing hull nets' instead of increasing sequences in the space of continuous functions (with compact support, in the case of locally compact spaces), yields a Borel measure with the desired properties for the case of non-metric spaces as well. This is the procedure presented in Bourbaki [Bou65; Chap. IV, §1]. A discussion comparing the methods and the respective results can be found in Floret [Flo81; Anhang]. A discussion concerning the set of different Borel measures leading to the desired representation for continuous functions can be found in Bauer [Bau90; §28 and §29].

Exercises

G.1 Let K be a compact topological space, and let $\mathcal{B}_0$ be the σ-algebra on K generated by the collection $\{[f > 0]\,;\, f \in C(K)\}$.

Let $f \in C(K)$, $\varepsilon > 0$. Show that there exists a function $g \in \mathcal{S}(\mathcal{B}_0)$ $(= \operatorname{lin}\{\mathbf{1}_A\,;\, A \in \mathcal{B}_0\})$ such that $\|f - g\|_\infty \leqslant \varepsilon$. (Hint: Subdivide the interval $[\min f, \max f]$ into intervals of length $\leqslant \varepsilon$ and show that the preimages of these intervals under f belong to $\mathcal{B}_0$.)

G.2 Let K be a compact metric space, and let μ be a Borel measure on K. (Note that then $\mu(K) < \infty$.) Let $1 \leqslant p < \infty$. The issue of this exercise is to show that $C(K)$ is dense in $L_p(\mu)$. Without loss of generality we only treat the case $\mathbb{K} = \mathbb{R}$.

We say that $X \subseteq L_p(\mu)$ is a vector sublattice if X is a subspace, and $f \vee g, f \wedge g \in X$ for all $f, g \in X$, or equivalently, $|f| \in X$ for all $f \in X$. See also Appendix B.

(a) Let X be a vector sublattice of $L_p(\mu)$. Show that the closure of X in $L_p(\mu)$ is again a vector sublattice. (Hint: Show that $L_p(\mu) \ni f \mapsto |f| \in L_p(\mu)$ is Lipschitz continuous.)

(b) Let X be a closed vector sublattice of $L_p(\mu)$ containing the function 1. Show that $\mathcal{A} := \{A\,;\, \mathbf{1}_A \in X\}$ is a σ-algebra.

(c) For $f \in C(K)$, $n \in \mathbb{N}$ put $f_n := (nf^+) \wedge 1$, and show that $f_n \to \mathbf{1}_{[f>0]}$ $(n \to \infty)$ in $L_p(\mu)$.

(d) Show that $C(K)$ is dense in $L_p(\mu)$. (Hint: Let X be the closure of $C(K)$ in $L_p(\mu)$. Show that $\mathcal{A}$ from (b) contains the σ-algebra $\mathcal{B}$ of Borel subsets of K. Conclude that $X = L_p(\mu)$.)

Additional remark: These properties also hold if K is a non-metric compact space and μ is a finite Baire measure on K, i.e. a measure on the σ-algebra $\mathcal{B}_0$ of Baire sets satisfying $\mu(K) < \infty$. Then in part (d) one has to replace the σ-algebra $\mathcal{B}$ by $\mathcal{B}_0$.

Appendix H

Singular integrals and the Bogovskiĭ operator

The issue of this appendix is the proof of Theorem 16.10. The most important tool and key result is Theorem H.5, concerning convolution operators with singular kernels.

H.1 The Fourier transformation

For $f \in L_1(\mathbb{R}^n)$ we define the **Fourier transform** of f,

$$Ff(\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\xi\cdot x} f(x)\,\mathrm{d}x \qquad (\xi \in \mathbb{R}^n).$$

Then $\hat{f}\colon \mathbb{R}^n \to \mathbb{C}$ is continuous (by the dominated convergence theorem) and bounded, $\|\hat{f}\|_\infty \leqslant \frac{1}{(2\pi)^{n/2}}\|f\|_1$, and the **Fourier transformation** $F\colon L_1(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$ is a bounded linear operator. More strongly, the range of F is a subset of $C_0(\mathbb{R}^n)$.

We will need the following important facts concerning the Fourier transformation; see [Duo01; Chap. 1] or [Yos68; Chap. 6], for instance.

H.1 Remarks. (a) For $f \in L_1 \cap L_2(\mathbb{R}^n)$ one has $\hat{f} \in L_2(\mathbb{R}^n)$, $\|\hat{f}\|_2 = \|f\|_2$, and F extends from $L_1 \cap L_2(\mathbb{R}^n)$ to a unitary operator $F_2\colon L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$.

(b) If $f, g \in L_1(\mathbb{R}^n)$, then $F(f * g) = (2\pi)^{n/2}\hat{f}\hat{g}$, and the same same formula holds if $f \in L_1(\mathbb{R}^n)$, $g \in L_2(\mathbb{R}^n)$. (For the convolution $f * g$ recall Proposition 6.23.)

(c) For all $f \in C_c^\infty(\mathbb{R}^n)$, $\alpha \in \mathbb{N}_0^n$ one has

$$\widehat{\partial^\alpha f}(\xi) = (-\mathrm{i}\xi)^\alpha \hat{f}(\xi) \qquad (\xi \in \mathbb{R}^n).$$

(This property makes the Fourier transformation a fundamental tool in the theory of partial differential equations.)

(d) If $f \in L_2(\mathbb{R}^n)$, $j \in \{1, ..., n\}$, and the function $\xi \mapsto \xi_j \hat{f}(\xi)$ belongs to $L_2(\mathbb{R}^n)$, then f has a distributional derivative $\partial_j f$ belonging to $L_2(\mathbb{R}^n)$, and $\|\partial_j f\|_2 = \|\xi \mapsto \xi_j \hat{f}(\xi)\|_2$.

Indeed, for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ one has

$$\begin{aligned}(f \,|\, \partial_j\varphi) &= \big(\hat{f} \,\big|\, \widehat{\partial_j\varphi}\big) = \big(\hat{f} \,\big|\, \xi \mapsto (-\mathrm{i}\xi_j\hat{\varphi}(\xi)\big) = \big(\xi \mapsto \mathrm{i}\xi_j\hat{f}(\xi) \,\big|\, \hat{\varphi}\big) \\ &= \big(F_2^{-1}(\xi \mapsto \mathrm{i}\xi_j\hat{f}(\xi)) \,\big|\, \varphi\big).\end{aligned}$$

This shows that $-F_2^{-1}(\xi \mapsto \mathrm{i}\xi_j\hat{f}(\xi)) \in L_2(\mathbb{R}^n)$ is the distributional derivative $\partial_j f$. $\triangle$

H.2 Singular integrals of convolution type

In this section we discuss convolution operators mapping $L_2(\Omega)$ to $H^1(\Omega)$, for suitable open sets $\Omega \subseteq \mathbb{R}^n$. We will be working with the following version of the convolution of functions.

H.2 Remark. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and put $R := \operatorname{diam}(\Omega)$. Considering $L_2(\Omega)$ in the canonical way as a subspace of $L_2(\mathbb{R}^n)$, we define the convolution $k * f$ of $k \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$ and $f \in L_2(\Omega)$ by

$$k * f(x) := \int_\Omega k(x-y)f(y)\,\mathrm{d}y = \int (\mathbf{1}_{B(0,R)}k)(x-y)f(y)\,\mathrm{d}y \qquad (x \in \Omega).$$

In other words, $k * f = (\mathbf{1}_{B(0,R)}k) * f|_\Omega$. Proposition 6.23 implies that $k * f$ is defined a.e. on Ω and that $\|k * f\|_{L_2(\Omega)} \leqslant \|k\|_{L_1(B(0,R))}\|f\|_{L_2(\Omega)}$. △

A function $k\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{K}$ is **positively homogeneous of degree** $\alpha \in \mathbb{R}$ if $k(\lambda x) = \lambda^\alpha k(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$, $\lambda > 0$. Equivalently, one could formulate that, with $k_0 := k|_{S_{n-1}}\colon S_{n-1} \to \mathbb{K}$, one has $k(x) = |x|^\alpha k_0\big(\frac{x}{|x|}\big)$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

Our first result, Theorem H.4 below, is concerned with a non-singular case that can be treated by using classical convolution inequalities. For the proof we will need the following observation.

H.3 Remark. Let $k \in C^1(\mathbb{R}^n \setminus \{0\})$ be positively homogeneous of degree $\alpha > 1-n$. Then for all $R > 0$ the function k belongs to $W_1^1(B(0,R))$. The proof of this statement is delegated to Exercise H.3. For the definition of the Sobolev space $W_1^1(\Omega)$ we refer to Section D.1. △

H.4 Theorem. *Let $k \in C^1(\mathbb{R}^n \setminus \{0\})$ be positively homogeneous of degree $\alpha > 1-n$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Then the convolution $f \mapsto k * f$ defines a bounded linear operator $T\colon L_2(\Omega) \to H^1(\Omega)$, and there exists a constant $c_{\alpha,R} > 0$, only depending on α and $R := \operatorname{diam}(\Omega)$ (and the dimension n), such that*

$$\|T\| \leqslant c_{\alpha,R} \max\big\{\big\|k|_{S_{n-1}}\big\|_\infty, \big\|(\nabla k)|_{S_{n-1}}\big\|_\infty\big\}. \tag{H.1}$$

Proof. Recall Remark H.2 for the definition of the convolution $k * f$ and the estimate $\|k * f\|_{L_2(\Omega)} \leqslant \|k\|_{L_1(B(0,R))}\|f\|_{L_2(\Omega)}$, for $f \in L_2(\Omega)$.

Now let $f \in C_c^\infty(\Omega)$. The gradient $\nabla(k * f)$ on Ω is given by

$$\nabla(k * f)(x) = k * \nabla f(x) = \int_\Omega k(x-y)\nabla f(y)\,\mathrm{d}y = \int (\mathbf{1}_{B(0,R)}\nabla k)(x-y)f(y)\,\mathrm{d}y,$$

where in the last step we have used Remark H.3. From

$$\left|\int \mathbf{1}_{B(0,R)}\nabla k(x-y)f(y)\,\mathrm{d}y\right| \leqslant \int \big(\mathbf{1}_{B(0,R)}|\nabla k|\big)(x-y)|f(y)|\,\mathrm{d}y$$

it follows that

$$\|\nabla k * f\|_{L_2(\Omega;\mathbb{K}^n)} \leqslant \big\|\big(\mathbf{1}_{B(0,R)}|\nabla k|\big) * |f|\big\|_{L_2(\Omega)} \leqslant \big\||\nabla k|\big\|_{L_1(0,R)}\|f\|_{L_2(\Omega)}.$$

Now, $\|k\|_{L_1(B(0,R))}$ can be estimated by

$$\begin{aligned}\|k\|_{L_1(B(0,R))} &\leqslant \int_{B(0,R)} |x|^\alpha \,\mathrm{d}x\, \|k|_{S_{n-1}}\|_\infty = \sigma_{n-1} \int_0^R r^{n+\alpha-1}\,\mathrm{d}r\, \|k|_{S_{n-1}}\|_\infty \\ &= \sigma_{n-1}\frac{1}{n+\alpha} R^{n+\alpha} \|k|_{S_{n-1}}\|_\infty\,,\end{aligned}$$

where σ_{n-1} denotes the $(n-1)$-dimensional volume of S_{n-1}. From Exercise H.2 we know that ∇k is positively homogeneous of degree $\alpha-1$; hence as above one obtains

$$\big\||\nabla k|\big\|_{L_1(0,R)} \leqslant \sigma_{n-1}\frac{1}{n+\alpha-1} R^{n+\alpha-1} \big\|(\nabla k)|_{S_{n-1}}\big\|_\infty\,.$$

This shows that

$$\|k * f\|_{H^1(\Omega)} \leqslant c_{\alpha,R} \max\big\{\big\|k|_{S_{n-1}}\big\|_\infty, \big\|(\nabla k)|_{S_{n-1}}\big\|_\infty\big\} \|f\|_{L_2(\Omega)}\,,$$

with

$$c_{\alpha,R} := \sigma_{n-1}\Big(\frac{1}{n+\alpha} R^{n+\alpha} + \frac{1}{n+\alpha-1} R^{n+\alpha-1}\Big).$$

As $C_c^\infty(\Omega)$ is dense in $L_2(\Omega)$ we obtain the asserted properties; see Exercise H.4. □

The following main result of the present section is the version of Theorem H.4 for $\alpha = 1-n$. The touchy issue in this result is that the derivatives of the function k have a non-integrable singularity at the point 0. This means that the convolution with $\partial_j k$ is not easily interpreted as a mapping on L_2.

H.5 Theorem. *Let $k \in C^1(\mathbb{R}^n \setminus \{0\})$ be positively homogeneous of degree $1-n$, and let $R > 0$.*

Then $k_R := k\mathbf{1}_{B(0,R)} \in L_1(\mathbb{R}^n)$, $k_R * f \in H^1(\mathbb{R}^n)$ for all $f \in L_2(\mathbb{R}^n)$, and there exists $M_R \geqslant 0$ such that

$$\|k_R * f\|_{H^1(\mathbb{R}^n)} \leqslant M_R \|f\|_2 \qquad (f \in L_2(\mathbb{R}^n)). \tag{H.2}$$

More explicitly, there exist constants b and b', only depending on n, such that the estimate holds with

$$M_R := (\sigma_{n-1}R + b)\|k|_{S_{n-1}}\|_\infty + b'\|(\nabla k)|_{S_{n-1}}\|_\infty\,.$$

If $\Omega \subseteq \mathbb{R}^n$ is a bounded open set, $R := \operatorname{diam}(\Omega)$, then

$$\|k * f\|_{H^1(\Omega)} \leqslant M_R \|f\|_{L_2(\Omega)} \qquad (f \in L_2(\Omega)). \tag{H.3}$$

In the proof we will need the inequality

$$\left|\int_a^b \frac{\mathrm{e}^{-\mathrm{i}r}}{r}\,\mathrm{d}r\right| \leqslant \ln\frac{1}{a} + 2 \qquad (0 < a < 1,\ b > a). \tag{H.4}$$

Indeed, if $b \leqslant 1$ then $\big|\int_a^b \frac{\mathrm{e}^{-\mathrm{i}r}}{r}\,\mathrm{d}r\big| \leqslant \int_a^1 \frac{1}{r}\,\mathrm{d}r = \ln\frac{1}{a}$. If $b > 1$, then we split the integral, estimate $\big|\int_a^1 \frac{\mathrm{e}^{-\mathrm{i}r}}{r}\,\mathrm{d}r\big| \leqslant \ln\frac{1}{a}$ and use integration by parts to obtain

$$\left|\int_1^b \frac{\mathrm{e}^{-\mathrm{i}r}}{r}\,\mathrm{d}r\right| \leqslant \left|\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}r}}{r}\right|_1^b - \int_1^b \frac{-\mathrm{i}\mathrm{e}^{-\mathrm{i}r}}{r^2}\,\mathrm{d}r\right| \leqslant \frac{1}{b} + 1 + 1 - \frac{1}{b} = 2.$$

Proof of Theorem H.5. (i) By the hypotheses there exists $c \geqslant 0$ such that $|k(x)| \leqslant c|x|^{1-n}$ for all $x \in \mathbb{R}^n \setminus \{0\}$. This implies that $k_R \in L_1(\mathbb{R}^n)$, $\|k_R\|_1 \leqslant \sigma_{n-1} R \|k|_{S_{n-1}}\|_\infty$ (see the proof of Theorem H.4), and from Proposition 6.23 it follows that $k_R * f \in L_2(\mathbb{R}^n)$, $\|k_R * f\|_2 \leqslant \sigma_{n-1} R \|k|_{S_{n-1}}\|_\infty \|f\|_2$ for all $f \in L_2(\mathbb{R}^n)$.

(ii) In order to estimate the derivatives of $k_R * f$, we show that there exist constants b, $b' > 0$, depending only on n, such that

$$(2\pi)^{n/2} |\xi \widehat{k_R}(\xi)| \leqslant b \|k|_{S_{n-1}}\|_\infty + b' \|(\nabla k)|_{S_{n-1}}\|_\infty \qquad (\xi \in \mathbb{R}^n). \tag{H.5}$$

Observe that the estimate (H.5) is invariant under orthogonal transformations of $\mathbb{R}^n$. (Note that $\hat{f}(A\cdot) = \widehat{f(A\cdot)}$ for all $f \in L_1(\mathbb{R}^n)$ and all orthogonal matrices.) Thus, it suffices to prove (H.5) for $\xi = te_1$, with $t > 0$ and the first unit vector e_1.

By the positive $(1-n)$-homogeneity of k we obtain

$$(2\pi)^{n/2} t \widehat{k_R}(te_1) = \int_{B(0,R)} t \mathrm{e}^{-\mathrm{i}te_1 \cdot x} k(x) \, \mathrm{d}x = \int_{B(0,tR)} \mathrm{e}^{-\mathrm{i}x_1} k(x) \, \mathrm{d}x = \lim_{\varepsilon \to 0} h(\varepsilon, tR), \tag{H.6}$$

where

$$h(\varepsilon, s) := \int_{\Omega_{\varepsilon,s}} \mathrm{e}^{-\mathrm{i}x_1} k(x) \, \mathrm{d}x, \quad \Omega_{\varepsilon,s} := B(0,s) \setminus B[0,\varepsilon] \qquad (0 < \varepsilon < s).$$

Let $\alpha \in \{0,1\}$ and $0 < \varepsilon < s$. Applying Gauss' theorem, Theorem 7.3, we compute

$$\begin{aligned} -\mathrm{i}h(\varepsilon, s) &= \int_{\Omega_{\varepsilon,s}} \partial_1 (\mathrm{e}^{-\mathrm{i}x_1} - \alpha) k(x) \, \mathrm{d}x \\ &= \int_{\partial\Omega_{\varepsilon,s}} \nu_1(x) (\mathrm{e}^{-\mathrm{i}x_1} - \alpha) k(x) \, \mathrm{d}\sigma(x) - \int_{\Omega_{\varepsilon,s}} (\mathrm{e}^{-\mathrm{i}x_1} - \alpha) \partial_1 k(x) \, \mathrm{d}x. \end{aligned} \tag{H.7}$$

Since $\nu(x) = \pm x/|x|$ and ρ is positively $(1-n)$-homogeneous, the first integral on the right-hand side equals $\int_{S_{n-1}} y_1 \big((\mathrm{e}^{-\mathrm{i}sy_1} - \alpha) - (\mathrm{e}^{-\mathrm{i}\varepsilon y_1} - \alpha) \big) k(y) \, \mathrm{d}\sigma(y)$. Using generalised polar coordinates in $\mathbb{R}^n$ we see that the second integral equals $\int_{S_{n-1}} \int_\varepsilon^s (\mathrm{e}^{-\mathrm{i}ry_1} - \alpha) \partial_1 k(ry) r^{n-1} \, \mathrm{d}r \, \mathrm{d}\sigma(y)$. Observing that $\partial_1 k$ is positively homogeneous of degree $-n$ (see Exercise H.2(a)), we conclude that

$$-\mathrm{i}h(\varepsilon, s) = \int_{S_{n-1}} y_1 (\mathrm{e}^{-\mathrm{i}sy_1} - \mathrm{e}^{-\mathrm{i}\varepsilon y_1}) k(y) \, \mathrm{d}\sigma(y) - \int_{S_{n-1}} \int_\varepsilon^s \frac{\mathrm{e}^{-\mathrm{i}ry_1} - \alpha}{r} \, \mathrm{d}r \, \partial_1 k(y) \, \mathrm{d}\sigma(y),$$

and therefore

$$|h(\varepsilon, s)| \leqslant \int_{S_{n-1}} 2|y_1| \, \mathrm{d}\sigma(y) \, \|k|_{S_{n-1}}\|_\infty + \int_{S_{n-1}} \left| \int_\varepsilon^s \frac{\mathrm{e}^{-\mathrm{i}ry_1} - \alpha}{r} \, \mathrm{d}r \right| \mathrm{d}\sigma(y) \, \|(\nabla k)|_{S_{n-1}}\|_\infty .$$

In the case $s \leqslant 1$ we choose $\alpha = 1$ and use the estimate $\left| \frac{\mathrm{e}^{-\mathrm{i}ry_1} - 1}{r} \right| \leqslant |y_1|$ to obtain

$$|h(\varepsilon, s)| \leqslant \int_{S_{n-1}} 2|y_1| \, \mathrm{d}\sigma(y) \, \|k|_{S_{n-1}}\|_\infty + \int_{S_{n-1}} |y_1| \, \mathrm{d}\sigma(y) \, \|(\nabla k)|_{S_{n-1}}\|_\infty . \tag{H.8}$$

In the case $0 < \varepsilon < 1 < s$ we use the decomposition $h(\varepsilon, s) = h(\varepsilon, 1) + h(1, s)$. The first term $h(\varepsilon, 1)$ is estimated as above, whereas for the second term $h(1, s)$ we choose $\alpha = 0$ and compute

$$\left| \int_1^s \frac{\mathrm{e}^{-\mathrm{i}ry_1}}{r} \, \mathrm{d}r \right| = \left| \int_1^s \frac{\mathrm{e}^{-\mathrm{i}r|y_1|}}{r} \, \mathrm{d}r \right| = \left| \int_{|y_1|}^{s|y_1|} \frac{\mathrm{e}^{-\mathrm{i}r}}{r} \, \mathrm{d}r \right| \leqslant \ln \frac{1}{|y_1|} + 2,$$

where in the last estimate we have used (H.4). Therefore

$$|h(1, s)| \leqslant \int_{S_{n-1}} 2|y_1| \, \mathrm{d}\sigma(y) \, \|k|_{S_{n-1}}\|_\infty + \int_{S_{n-1}} \left(\ln \frac{1}{|y_1|} + 2 \right) \mathrm{d}\sigma(y) \, \|(\nabla k)|_{S_{n-1}}\|_\infty \, .$$

(We refer to Exercise H.6 for the finiteness $\int_{S_{n-1}} \ln \frac{1}{|y_1|} \, \mathrm{d}\sigma(y) < \infty$.) Combining this estimate with (H.8) for $s = 1$ or using (H.8) by itself we conclude that

$$|h(\varepsilon, s)| \leqslant b \|k|_{S_{n-1}}\|_\infty + b' \|(\nabla k)|_{S_{n-1}}\|_\infty$$

for $\varepsilon < \min\{s, 1\}$, with $b := \int_{S_{n-1}} 4|y_1| \, \mathrm{d}\sigma(y)$ and $b' := \int_{S_{n-1}} \big(|y_1| + \ln \frac{1}{|y_1|} + 2 \big) \, \mathrm{d}\sigma(y)$. Now applying (H.6) we obtain (H.5).

(iii) We now show the estimate on the derivatives of $k_R * f$. Using Remarks H.1(d), (b), (a) and inequality (H.5) we obtain

$$\begin{aligned} \|\nabla(k_R * f)\|_2 &= \big\| \xi \mapsto \xi \widehat{k_R * f}(\xi) \big\|_2 = \big\| \xi \mapsto \xi (2\pi)^{n/2} \widehat{k_R}(\xi) \hat{f}(\xi) \big\|_2 \\ &\leqslant \big\| \xi \mapsto (2\pi)^{n/2} \xi \widehat{k_R}(\xi) \big\|_\infty \big\| \hat{f} \big\|_2 \\ &\leqslant \big(b \|k|_{S_{n-1}}\|_\infty + b' \|(\nabla k)|_{S_{n-1}}\|_\infty \big) \|f\|_2 \end{aligned}$$

for all $f \in L_2(\mathbb{R}^n)$.

Combining this estimate with the estimate for $\|k_R * f\|_2$ presented in step (i) of the proof we obtain (H.2).

(iv) For the last statement of the theorem we recall from Remark H.2 that $k * f = k_R * f$ on Ω, so (H.3) is immediate from (H.2). □

H.6 Remark. In Exercise H.2(b) the reader is asked to show that in the situation of Theorem H.5 one has $\int_{S_{n-1}} \partial_j k(y) \, \mathrm{d}\sigma(y) = 0$ for all $j \in \{1, \dots, n\}$. Thus it turns out that in formula (H.7) of the proof of Theorem H.5 there is no need to introduce the parameter α. Showing (H.7) without α is sufficient, because in the last term of the formulas one can always add it, and this is what is needed in a later part of the proof. △

H.3 Proof of Theorem 16.10

For easy reference we recall the definition of the Bogovskiĭ operator,

$$\begin{aligned} Bf(x) &:= \int f(y)(x - y) \int_1^\infty \rho(y + r(x - y)) r^{n-1} \, \mathrm{d}r \, \mathrm{d}y \\ &= \int z \int_0^\infty \rho(x + tz)(t + 1)^{n-1} \, \mathrm{d}t \, f(x - z) \, \mathrm{d}z, \end{aligned}$$

for $x \in \mathbb{R}^n$, $f \in C_c^\infty(\mathbb{R}^n)$. We also recall from the hypotheses of Theorem 16.10 that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set, star-shaped with respect to a ball $B(x^0, r_0) \subseteq \Omega$, and that $\rho = \rho_0 * \tilde{\rho}$, where $\operatorname{spt} \rho_0 \subseteq B(0, r_0/2)$, $\operatorname{spt} \tilde{\rho} \subseteq B(x^0, r_0/2)$, $\int \rho_0(x)\,\mathrm{d}x = \int \tilde{\rho}(x)\,\mathrm{d}x = 1$.

Proof of Theorem 16.10. Recall that the operator B maps $C_c^\infty(\Omega)$ to $C_c^\infty(\Omega; \mathbb{K}^n)$. In order to prove the theorem it is sufficient to find a constant $M \geqslant 0$ such that $\|Bf\|_{H^1(\Omega;\mathbb{K}^n)} \leqslant M\|f\|_{L_2(\Omega)}$ for all $f \in C_c^\infty(\Omega)$. We now fix $f \in C_c^\infty(\Omega)$ and proceed in three steps to show this inequality.

(i) Using the definition of ρ we rewrite Bf, for $x \in \mathbb{R}^n$, as

$$\begin{aligned} Bf(x) &= \int_z z \int_0^\infty \int_y \rho_0(y)\tilde{\rho}(x + tz - y)\,\mathrm{d}y\,(t+1)^{n-1}\,\mathrm{d}t\, f(x-z)\,\mathrm{d}z \\ &= \int_w \rho_0(x-w) \int_z z \int_0^\infty \tilde{\rho}(w+tz)(t+1)^{n-1}\,\mathrm{d}t\, f(x-z)\,\mathrm{d}z\,\mathrm{d}w \\ &= \int_w \rho_0(x-w)\, k_w * f(x)\,\mathrm{d}w, \end{aligned}$$

where ρ_0, $\tilde{\rho}$ and f are considered as functions on $\mathbb{R}^n$ with value zero outside their supports, and where

$$k_w(z) := z \int_0^\infty \tilde{\rho}(w+tz)(t+1)^{n-1}\,\mathrm{d}t;$$

recall Remark H.2 for the definition of $k_w * f$. In fact, we are only interested in the values of Bf on Ω. Observing that $\rho_0(x-w) = 0$ for all $x \in \Omega$, $w \notin \Omega_0 := \Omega + B(0, r_0/2)$ we obtain

$$Bf(x) = \int_{\Omega_0} \rho_0(x-w)\, k_w * f(x)\,\mathrm{d}w \qquad (x \in \Omega). \tag{H.9}$$

(ii) In this step we show that the partial derivatives of Bf can be obtained by differentiation under the integral in (H.9), and we explain how to exploit this in the proof.

Put $R := \operatorname{diam}(\Omega)$. Defining

$$k_{w,R} := \mathbf{1}_{B(0,R)} k_w \qquad (w \in \mathbb{R}^n),$$

we obtain

$$k_w * f(x) = \int_\Omega k_w(x-y) f(y)\,\mathrm{d}y = k_{w,R} * f(x) \qquad (w \in \mathbb{R}^n,\ x \in \Omega). \tag{H.10}$$

As in step (ii) of the proof of Theorem 16.7 we rewrite k_w as

$$k_w(z) = \frac{z}{|z|^n} \int_0^\infty \tilde{\rho}\Big(w + s\frac{z}{|z|}\Big)(s + |z|)^{n-1}\,\mathrm{d}s \tag{H.11}$$

to see that there exists $c \geqslant 0$ such that

$$|k_{w,R}(z)| \leqslant \frac{c}{|z|^{n-1}} \qquad (z \in \mathbb{R}^n,\ w \in \Omega_0);$$

hence $k_{w,R} \in L_1(\mathbb{R}^n; \mathbb{K}^n)$. It is easy to see that, for $|z| \neq 0$, the function $\Omega_0 \ni w \mapsto k_{w,R}(z)$ is continuous. (Observe that the domain of the s-integration in (H.11) is contained in the

bounded interval $[0,R]$ because $\operatorname{spt}\tilde\rho(w+\cdot)=\operatorname{spt}\tilde\rho-w\subseteq B(x^0-w,r_0/2)\subseteq B(0,R)$.) This implies that the mapping $\Omega_0\ni w\mapsto k_{w,R}\in L_1(\mathbb{R}^n;\mathbb{K}^n)$ is bounded and continuous, by the dominated convergence theorem.

From Lemma 4.1 we know that $k_{w,R}*f$ is continuously differentiable and $\partial_j(k_{w,R}*f)=k_{w,R}*\partial_j f$. Proposition 6.23 implies that the linear operator ${}^*f\colon L_1(\mathbb{R}^n)\to C_{\mathrm{b}}(\mathbb{R}^n)$, $h\mapsto h*f$ is bounded, and $\|{}^*f\|_{\mathcal{L}(L_1,C_{\mathrm{b}})}\leqslant\|f\|_\infty$. Combining these facts we infer that the function $\Omega_0\ni w\mapsto k_{w,R}*f\in C^1_{\mathrm{b}}(\mathbb{R}^n;\mathbb{K}^n)$ is bounded and continuous. Taking into account that the function $\Omega_0\ni w\mapsto\rho_0(\cdot-w)\in C^1_{\mathrm{b}}(\mathbb{R}^n)$ is bounded and continuous, we conclude that the function

$$\Omega_0\ni w\mapsto\rho_0(\cdot-w)(k_{w,R}*f)\in C^1_{\mathrm{b}}(\mathbb{R}^n;\mathbb{K}^n)$$

is bounded and continuous as well; see Exercise H.7(b).

In order to obtain the estimate we are aiming for, we define the function $F\colon\Omega_0\to H^1(\Omega;\mathbb{K}^n)$,

$$F(w):=\rho_0(\cdot-w)\,k_w*f|_\Omega\qquad(w\in\Omega_0).$$

Then (H.10) implies that $F(w)=\rho_0(\cdot-w)\,k_{w,R}*f|_\Omega$, and in view of the properties mentioned above we can apply Exercise H.8 to conclude that

$$\begin{aligned}\|Bf\|_{H^1(\Omega;\mathbb{K}^n)}&=\Big\|x\mapsto\int_{\Omega_0}F(w)(x)\,\mathrm{d}w\Big\|_{H^1(\Omega;\mathbb{K}^n)}\leqslant\operatorname{vol}_n(\Omega_0)\sup_{w\in\Omega_0}\|F(w)\|_{H^1(\Omega;\mathbb{K}^n)}\\&=\operatorname{vol}_n(\Omega_0)\sup_{w\in\Omega_0}\|\rho_0(\cdot-w)\,k_w*f\|_{H^1(\Omega;\mathbb{K}^n)}\,.\end{aligned}$$

By Exercise H.7(a) we can estimate

$$\|\rho_0(\cdot-w)\,k_w*f\|_{H^1(\Omega;\mathbb{K}^n)}\leqslant\sqrt{2}\Big(\|\rho_0\|_\infty^2+\sum_{j=1}^n\|\partial_j\rho_0\|_\infty^2\Big)^{1/2}\|k_w*f\|_{H^1(\Omega;\mathbb{K}^n)}\,.$$

As a result we obtain

$$\|Bf\|_{H^1(\Omega;\mathbb{K}^n)}\leqslant C_R\sup_{w\in\Omega_0}\|k_w*f\|_{H^1(\Omega;\mathbb{K}^n)}\,,\tag{H.12}$$

where $C_R:=\operatorname{vol}_n(\Omega_0)\sqrt{2}\big(\|\rho_0\|_\infty^2+\sum_{j=1}^n\|\partial_j\rho_0\|_\infty^2\big)^{1/2}$.

Establishing a suitable estimate for the supremum in (H.12) will be the issue of the following step (iii) of the proof.

(iii) In order to establish the connection to Section H.2 on singular integrals we write $k_w=\sum_{m=0}^{n-1}k_w^m$, where

$$k_w^m(z):=\binom{n-1}{m}z\int_0^\infty\tilde\rho(w+tz)t^m\,\mathrm{d}t.$$

It is easy to see that k_w^m is positively homogeneous of degree m, for all $m\in\{0,\dots,n-1\}$.

We will use the elementary inequality

$$\|k_w^m*f\|_{H^1(\Omega;\mathbb{K}^n)}\leqslant\sqrt{n}\sup_{l=1,\dots,n}\|(k_w^m)_l*f\|_{H^1(\Omega)}\,,\tag{H.13}$$

where $(k_w^m)_l$ are the components of the vector-valued kernels k_w^m. In order to apply Theorems H.4 and H.5 to the right-hand side of (H.13), we need estimates for the components $(k_w^m)_l$ on S_{n-1}.

Let $l \in \{1, \dots, n\}$. For $m \in \{0, \dots, n-1\}$, $w \in \Omega_0$ we have

$$\|(k_w^m)_l|_{S_{n-1}}\|_\infty \leqslant c_m := \tbinom{n-1}{m} \|\tilde{\rho}\|_\infty \tfrac{1}{m+1} R^{m+1}$$

and

$$\begin{aligned}\|\nabla(k_w^m)_l|_{S_{n-1}}\|_\infty &= \tbinom{n-1}{m} \sup_{z \in S_{n-1}} \left| \int_0^\infty \tilde{\rho}(w+tz) t^m \,\mathrm{d}t \, e_l + z_l \int_0^\infty \nabla\tilde{\rho}(w+tz) t^{m+1} \,\mathrm{d}t \right| \\ &\leqslant c_m' := \tbinom{n-1}{m} \big(\|\tilde{\rho}\|_\infty \tfrac{1}{m+1} R^{m+1} + \|\nabla\tilde{\rho}\|_\infty \tfrac{1}{m+2} R^{m+2} \big).\end{aligned}$$

For $m = 0, \dots, n-2$ we now can apply Theorem H.4 to obtain

$$\|(k_w^m)_l * f\|_{H^1(\Omega;\mathbb{K}^n)} \leqslant c_{m,R} \max\{c_m, c_m'\} \|f\|_{L_2(\Omega)} \qquad (w \in \Omega_0).$$

For $m = n-1$ we apply (H.3) of Theorem H.5 and obtain

$$\big\|(k_w^{n-1})_l * f\big\|_{H^1(\Omega;\mathbb{K}^n)} \leqslant \big((\sigma_{n-1} + b) c_{n-1} + b' c_{n-1}'\big) \|f\|_{L_2(\Omega)} \qquad (w \in \Omega_0).$$

Summing up over $m = 0, ..., n-1$, using (H.13), and applying (H.12) we conclude that

$$\|Bf\|_{H^1(\Omega;\mathbb{K}^n)} \leqslant C_R \sqrt{n} \left(\sum_{m=0}^{n-2} c_{m,R} \max\{c_m, c_m'\} + (\sigma_{n-1} + b) c_{n-1} + b' c_{n-1}' \right) \|f\|_{L_2(\Omega)},$$

and this completes the proof of the theorem. □

H.4 Comments

The 'traditional' way to show the continuity of the Bogovskiĭ operator – see Theorem 16.10 – is to invoke the theory developed in [CaZy56]; this goes back already to Bogovskiĭ's original paper [Bog79]. In our attempt to provide complete information on the topics treated in the present book we realised that we could avoid part of the treatment of singular integrals by considering the special case of the Bogovskiĭ operator where the function ρ in (16.6) is of the form $\rho = \rho_0 * \tilde{\rho}$ described in Theorem 16.10. In order explain the difference between the traditional way and our proof we briefly sketch part of the theory of convolutions with singular kernels.

Let $k\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{K}$ be measurable and positively homogeneous of degree $-n$, and assume that $k|_{S_{n-1}} \in L_1(S_{n-1})$. Then

$$K\varphi(x) := k * \varphi(x) := \lim_{\varepsilon \to 0+} \int_{|y|>\varepsilon} k(y) \varphi(x-y) \,\mathrm{d}y$$

can exist for all $x \in \mathbb{R}^n$, $\varphi \in C_c^\infty(\mathbb{R}^n)$ only if

$$\int_{S_{n-1}} k(\xi) \,\mathrm{d}\sigma(\xi) = 0; \tag{H.14}$$

see [Duo01; Proposition 4.1]. If k is an odd function (which implies that (H.14) holds), then K acts as a bounded operator on $L_p(\mathbb{R}^n)$, for all $p \in (1,\infty)$; see [Duo01; Corollary 4.8]. If k satisfies (H.14), the odd part of $k|_{S_{n-1}}$ belongs to $L_1(S_{n-1})$ and the even part of $k|_{S_{n-1}}$ belongs to $L_q(S_{n-1})$ for some $q > 1$, then K is bounded on $L_p(\mathbb{R}^n)$ for all $p \in (1,\infty)$; see [Duo01; Theorem 4.12]. (We note that the proof of our Theorem H.5 is closely related to methods of the proof of the results mentioned above. Our restriction to the case $p = 2$ facilitates the use of the Fourier transformation.)

For the application to the Bogovskiĭ operator one needs the more general concept of 'extended convolution operators'. In this case one has a measurable function $k\colon \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{K}$, $k(x,\cdot)$ positively homogeneous of degree $-n$ for all $x \in \mathbb{R}^n$, and one wants to define

$$K\varphi(x) := \lim_{\varepsilon\to 0+} \int_{|y|>\varepsilon} k(x,y)\varphi(x-y)\,\mathrm{d}y$$

for $x \in \mathbb{R}^n$, $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$. Assume additionally that

$$\sup_{x\in\mathbb{R}^n} |k(x,\cdot)|_{S_{n-1}}| \in L_1(S_{n-1})$$

and that $k(x,\cdot)$ is an odd function for all $x \in \mathbb{R}^n$. Then K as defined above is bounded on $L_p(\mathbb{R}^n)$ for all $p \in (1,\infty)$; see [Duo01; Theorem 4.16]. Alternatively, assume that there exists $q > 1$ such that

$$\sup_{x\in\mathbb{R}^n} \|k(x,\cdot)\|_q < \infty$$

and that $k(x,\cdot)$ satisfies (H.14) for all $x \in \mathbb{R}^n$. Then K is bounded on $L_p(\mathbb{R}^n)$ for all $p \in [q/(q-1),\infty)$; see [CaZy56; Theorem 2].

In the proof of the continuity of the Bogovskiĭ operator (in the case that ρ is not necessarily of the form $\rho = \rho_0 * \tilde{\rho}$) one uses a decomposition as in step (iii) of our proof in Section H.3. Then the critical terms are of the form

$$\begin{aligned} k(x,y) &= \frac{\partial}{\partial y_j}\left(y_l \int_0^\infty \rho(x+ty)t^{n-1}\,\mathrm{d}t\right) \\ &= \delta_{jl}\int_0^\infty \rho(x+ty)t^{n-1}\,\mathrm{d}t + y_l\int_0^\infty \partial_j\rho(x+ty)t^n\,\mathrm{d}t \qquad (j,l\in\{1,\dots,n\}), \end{aligned}$$

responsible for the $L_2(\Omega)$-boundedness of the operator mapping a function f to the ∂_j-derivative of the l-th component of Bf. In this kernel, the function $k(x,\cdot)$ is positively homogeneous of degree $-n$, and $k(x,\cdot)$ satisfies (H.14) for all $x \in \mathbb{R}^n$ (because of Exercise H.2(b), but $k(x,\cdot)$ is not odd; so one has to apply Theorem 2 of [CaZy56]. In our more special case where ρ in the Bogovskiĭ operator is of the form $\rho = \rho_0 * \tilde{\rho}$ we avoid the application of [CaZy56; Theorem 2] and only have to use the version of [Duo01; Theorem 4.12] presented in Theorem H.5.

Notes

The information regarding the Fourier transformation is standard; we only include it for easy reference. The Bogovskiĭ operator appeared first in [Bog79], in connection with the

treatment of the Stokes operator, and Bogovskiĭ refers to [CaZy56] for the property that B maps $L^0_p(\Omega)$ to $W^1_{p,0}(\Omega;\mathbb{K})$. A more detailed description of this application of [CaZy56] can be found in [Gal11; Section III.3]. For more comments on singular integrals and the difference between the treatments of the general and our more special Bogovskiĭ operator we refer to Section H.4.

Exercises

H.1 Let $f \in L_2(\mathbb{R}^n)$, $j \in \{1,\dots,n\}$, $\partial_j f \in L_2(\mathbb{R}^n)$. Show that $[\xi \mapsto \xi_j \hat{f}(\xi)] \in L_2(\mathbb{R}^n)$.

Note. Combined with Remark H.1(d) this shows that $\partial_j f \in L_2(\mathbb{R}^n)$ if and only if $[\xi \mapsto \xi_j \hat{f}(\xi)] \in L_2(\mathbb{R}^n)$.

H.2 Let $\alpha \in \mathbb{R}$, and let the function $k \in C^1(\mathbb{R}^n \setminus \{0\})$ be positively homogeneous of degree α. Let $j \in \{1,\dots,n\}$.

(a) Show that $\partial_j k$ is positively homogeneous of degree $\alpha - 1$.

(b) If $\alpha = 1-n$, show that $\int_{S_{n-1}} \partial_j k(y)\,\mathrm{d}\sigma(y) = 0$. Hint: Choose $\zeta \in C^\infty_{\mathrm{c}}(0,\infty)_+$, and show that $\int_0^\infty \zeta(r) r^{-1}\,\mathrm{d}r \int_{S_{n-1}} \partial_j k(y)\,\mathrm{d}\sigma(y) = \int_0^\infty \zeta(r) \int_{S_{n-1}} \partial_j k(ry) r^{n-1}\,\mathrm{d}\sigma(y)\,\mathrm{d}r = \dots = -\int_0^\infty \zeta'(r)\,\mathrm{d}r \int_{S_{n-1}} y_j k(y)\,\mathrm{d}\sigma(y) = 0$.

H.3 Give a proof of Remark H.3. Hint: Choose $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$, $\psi = 1$ in a neighboourhood of 0. Show that $k_m := (1 - \psi(m\cdot))k \in W^1_1(B(0,R))$ $(m \in \mathbb{N})$, $k_m \to k$ in $W^1_1(B(0,R))$ $(m \to \infty)$. (Observe Exercise H.2.)

H.4 Let X, Y, Z be Banach spaces, $Y \hookrightarrow Z$, $T \in \mathcal{L}(X,Z)$, and suppose that there exist a dense subspace $D \subseteq X$ and a constant $c \geqslant 0$ such that $\|Tx\|_Y \leqslant c\|x\|_X$ for all $x \in D$. Show that $\mathrm{ran}(T) \subseteq Y$, $T \in \mathcal{L}(X,Y)$, $\|T\|_{\mathcal{L}(X,Y)} \leqslant c$.

H.5 Let $k \in W^1_1(\mathbb{R}^n)$. Show that $k * f \in H^1(\mathbb{R}^n)$ for all $f \in L_2(\mathbb{R}^n)$, and $Tf := k * f$ $(f \in L_2(\mathbb{R}^n))$ defines a bounded operator $T\colon L_2(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$, with $\|T\| \leqslant \big(\|k\|_1^2 + \sum_{j=1}^n \|\partial_j k\|_1^2\big)^{1/2} \|f\|_2$. (Hint: Use the denseness of $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ in $W^1_1(\mathbb{R}^n)$ to show that $\partial_j(k * f) = (\partial_j k) * f$.)

Note. This property is similar to Theorem H.4.

H.6 (a) Let $g\colon (0,1] \to [0,\infty)$ be a decreasing function, $\int_0^1 g(r)\,\mathrm{d}r < \infty$. Show that $\int_{S_{n-1}} g(|y_1|)\,\mathrm{d}\sigma(y) < \infty$.

Hint: Estimate $\int_{S_{n-1}} g(|y_1|)\,\mathrm{d}\sigma(y) \leqslant n \int_0^1 r^{n-1} \int_{S_{n-1}} g(|ry_1|)\,\mathrm{d}\sigma(y)\,\mathrm{d}r$, use generalised polar coordinates, and estimate the resulting integral over $B(0,1)$ by the integral over $[-1,1]^n$.

(b) Show that $\int_{S_{n-1}} \ln \frac{1}{|y_1|}\,\mathrm{d}\sigma(y) < \infty$.

H.7 Let $m, n \in \mathbb{N}$, and let $\Omega \subseteq \mathbb{R}^n$ be an open set.

(a) Show that $C^1_{\mathrm{b}}(\Omega) \times H^1(\Omega;\mathbb{K}^m) \ni (\varphi, g) \mapsto \varphi g \in H^1(\Omega;\mathbb{K}^m)$ defines a continuous bilinear mapping,

$$\|\varphi g\|_{H^1} \leqslant \sqrt{2}\Big(\|\varphi\|_\infty^2 + \sum_{j=1}^n \|\partial_j \varphi\|_\infty^2\Big)^{1/2} \|g\|_{H^1}\,.$$

(b) Show that $C^1_{\mathrm{b}}(\Omega) \times C^1_{\mathrm{b}}(\Omega;\mathbb{K}^m) \ni (\varphi, g) \mapsto \varphi g \in C^1_{\mathrm{b}}(\Omega;\mathbb{K}^m)$ defines a continuous bilinear mapping.

H.8 Let $\Omega \subseteq \mathbb{R}^n$, $\Omega' \subseteq \mathbb{R}^m$ be bounded open sets. Let $F\colon \Omega' \to C^1_{\mathrm{b}}(\Omega)$ be bounded and continuous.

(a) Show that $f(x) := \int_{\Omega'} F(w)(x)\,\mathrm{d}w$ defines a function $f \in C^1_{\mathrm{b}}(\Omega)$, with $\partial_j f(x) = \int_{\Omega'} \partial_j(F(w))(x)\,\mathrm{d}w$ for all $x \in \Omega$, $j = 1, \dots, n$.

(b) Show that

$$\|f\|_{H^1(\Omega)} \leqslant \int_{\Omega'} \|F(w)\|_{H^1(\Omega)}\,\mathrm{d}w.$$

Hint: Exploit the equality $\|f\|_{H^1} = \sup\bigl\{|(f\,|\,g)_{H^1}|\,;\, g \in H^1,\ \|g\|_{H^1} \leqslant 1\bigr\}$.

Note. If we had the Bochner integral for the general context at our disposal, the inequality would be elementary.

(c) Convince yourself that the corresponding property holds if $C^1_{\mathrm{b}}(\Omega)$ is replaced by $C^1_{\mathrm{b}}(\Omega;\mathbb{K}^n)$.

Appendix I

The fixed point theorems of Brouwer and Schauder

The fixed point theorems proved in this appendix are important for the existence of solutions for nonlinear problems. In contrast to the Banach fixed point theorem, which also provides uniqueness of solutions, the fixed point theorems treated here only provide existence. The Schauder fixed point theorem is used in Chapter 19. The Brouwer fixed point theorem, interesting and important in its own right, is needed for the proof of Schauder's theorem.

I.1 Brouwer's fixed point theorem

Throughout this section, we fix $n \in \mathbb{N}$, and we will denote the closed unit ball $B_{\mathbb{R}^n}[0,1]$ in $\mathbb{R}^n$ by B. As a consequence, $\mathring{B}$ will be the open unit ball, the interior of B. Slightly abusing the usual terminology, we will call a function continuously differentiable on B if it belongs to $C^1(\overline{B(0,1)})$.

I.1 Theorem (Brouwer). *Let f be a continuous mapping from B to itself. Then f has a fixed point, i.e., there exists $x \in B$ such that $f(x) = x$.*

In a first step we will show that it is sufficient to show the theorem under the stronger hypothesis that f is continuously differentiable on B. As the whole procedure of the proof is based on contradiction, we will formulate the following proposition 'negatively', i.e., by starting with an assumption which – in the end – we want to disprove.

I.2 Proposition. *Assume that f is a continuous self-map of B without a fixed point. Then there exists a continuously differentiable mapping $\tilde{f}\colon B \to \mathring{B}$ without a fixed point.*

Proof. The compactness of B implies that $\varepsilon := \inf\{|f(x) - x|\,;\, x \in B\} > 0$. Clearly one can find $r \in (0,1)$ such that $|rf(x) - f(x)| \leqslant \varepsilon/3$ for all $x \in B$. The Stone–Weierstrass theorem (Theorem B.2) implies that rf can be approximated by a mapping $\tilde{f}$ which is a polynomial in each component, $\bigl|\tilde{f}(x) - rf(x)\bigr| \leqslant \min\{\varepsilon/3, (1-r)/2\}$. (The set of polynomials on B is an algebra that separates points and contains the function 1; hence, the polynomials are dense in $C(B)$.) As a consequence, $\bigl|\tilde{f}(x)\bigr| \leqslant 1 - (1-r)/2$ and $\bigl|\tilde{f}(x) - x\bigr| \geqslant \varepsilon/3$ for all $x \in B$, and $\tilde{f}$ is as asserted. □

We will see below that the existence of a mapping $\tilde{f}$ as in Proposition I.2 would entail the existence of a continuously differentiable retraction of g of B to ∂B, i.e. a continuously differentiable mapping g from B to $\mathbb{R}^n$ with $g(B) \subseteq \partial B$ and $g|_{\partial B} = \mathrm{id}_{\partial B}$. The following result shows that such a retraction does not exist. In fact, this result is the key point in the proof of Brouwer's fixed point theorem and contains the main technical difficulty.

I.3 Proposition. *There is no continuously differentiable retraction of B to ∂B.*

Proof. The proof proceeds via contradiction. We assume that there exists a continuously differentiable retraction g of B to ∂B. We define a homotopy $g\colon [0,1] \times B \to B$ between id_B and g by

$$g_t := (1-t)\,\mathrm{id}_B + tg \qquad (t \in [0,1]).$$

Then $g_0 = \mathrm{id}_B$, $g_1 = g$ and $g_t|_{\partial B} = \mathrm{id}_{\partial B}$ $(t \in [0,1])$.

We will show the properties

(i) $t \to p(t) := \int_B^\circ \det g_t'(x)\,\mathrm{d}x$ is a polynomial,

(ii) $p(0) = \omega_n > 0$, and there exists $\varepsilon \in (0,1]$ such that $p|_{[0,\varepsilon]}$ is constant,

(iii) $p(1) = 0$.

Obviously these properties cannot hold simultaneously, which means that the contradiction will be established.

(i) The entries of the matrix $g_t'(x)$ are given by $(1-t)\delta_{jk} + t\partial_k g_j(x)$ $(j,k = 1\ldots,n)$; hence $\det g_t'(x)$ is a polynomial in t,

$$a_n(x)t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x),$$

with continuous coefficients $a_j\colon B \to \mathbb{R}$ $(j = 0,\ldots,n)$. Integration over $\mathring{B}$ yields the assertion.

(ii) Put $c := \|g'\|_\infty$ $(= \sup_{|x|\leqslant 1} |g'(x)|$, where $|g'(x)|$ denotes the Euclidean matrix norm). Then, for $t \in [0,1]$, $x, y \in B$ one obtains

$$|g_t(x) - g_t(y)| \geqslant (1-t)|x-y| - tc|x-y| = (1-t(c+1))|x-y|;$$

hence g_t is injective for $0 \leqslant t < 1/(c+1)$. For all $x \in \mathring{B}$ one has $\det g_0'(x) = 1$, hence the uniform continuity of $(t,x) \mapsto \det g_t'(x)$ on $[0,1] \times B$ implies that there exists $\varepsilon \in \big(0, 1/(c+1)\big)$ such that $\det g_t'(x) > 0$ for all $t \in [0,\varepsilon]$, $x \in \mathring{B}$. This shows that, for $t \in [0,\varepsilon]$, the mapping $g_t|_B^\circ$ is a diffeomorphism from $\mathring{B}$ to the open subset $g_t(\mathring{B})$ of $\mathring{B}$. From

$$\partial\big(g_t(\mathring{B})\big) = \overline{g_t\big(\mathring{B}\big)} \setminus g_t\big(\mathring{B}\big) \subseteq g_t(B) \setminus g_t\big(\mathring{B}\big) = g_t\big(\partial\mathring{B}\big) = \partial\mathring{B}$$

we conclude that in fact $g_t\big(\mathring{B}\big) = \mathring{B}$.

We now apply the transformation formula: for all $t \in [0,\varepsilon]$ we have

$$p(t) = \int_{\mathring{B}} \det g_t'(x)\,\mathrm{d}x = \int_{g_t(\mathring{B})} 1\,\mathrm{d}x = \lambda^n\big(g_t(\mathring{B})\big) = \lambda^n(\mathring{B}).$$

(iii) $g(\mathring{B}) \subseteq \partial B$ and the theorem of local invertibility imply that $\det g'(x) = 0$ for all $x \in \mathring{B}$; hence $p(1) = 0$. □

In order to construct a retraction as in Proposition I.3 from a (hypothetical) self-map of B without a fixed point, we need the following lemma.

I.4 Lemma. *For $y \in \mathring{B}$, $z \in \mathbb{R}^n \setminus \{0\}$, the half-ray $y + (0,\infty)z$ meets the unit sphere $S_{n-1} = \partial B$ in precisely one point $\varphi(y,z) \in S_{n-1}$. The thereby defined function $\varphi\colon \mathring{B} \times (\mathbb{R}^n \setminus \{0\}) \to \partial B$ is continuously differentiable.*

Proof. For $y \in \mathring{B}$, $z \in \mathbb{R}^n \setminus \{0\}$, the function $s \mapsto |y+sz|^2 = |y|^2 + 2(y \,|\, z)s + |z|^2 s^2$ is quadratic, equals $|y|^2 < 1$ for $s = 0$, and tends to ∞ as $s \to \infty$; hence there is a unique $s > 0$ such that $|y+sz|^2 = 1$. This solution is given by

$$s(y,z) = |z|^{-2}\Big(-(y \,|\, z) + \sqrt{(y \,|\, z)^2 + (1-|y|^2)|z|^2}\Big),$$

which depends continuously differentiably on (y,z). This shows that $\varphi(y,z) = y + s(y,z)z$ satisfies the assertion. □

Proof of Theorem I.1. Assume that there exists a continuous self-map f of B without a fixed point. Proposition I.2 implies that, without loss of generality, we may assume that f is a continuously differentiable mapping from B to $\mathring{B}$. Define $g\colon B \to \partial B$ by $g(x) := \varphi(f(x), x - f(x))$, with φ from Lemma I.4. Then g is a continuously differentiable retraction of B to ∂B. This is a contradiction to Proposition I.3.

Hence, an f as assumed above cannot exist. □

I.5 Corollary. (a) *Let $K \subseteq \mathbb{R}^n$ be homeomorphic to B. Then every continuous self-map of K has a fixed point.*

(b) *Let $K \neq \varnothing$ be a compact convex subset of $\mathbb{R}^n$. Then every continuous self-map of K has a fixed point.*

Proof. (a) Let $h\colon B \to K$ be a homeomorphism, and let f be a continuous self-map of K. Then $h^{-1} \circ f \circ h$ is a continuous self-map of B, and a fixed point x of $h^{-1} \circ f \circ h$ is mapped to a fixed point $h(x)$ of f.

(b) Provided with the Euclidean scalar product, $\mathbb{R}^n$ is a Hilbert space. Let $P\colon \mathbb{R}^n \to K$ be the minimising projection onto K. Let $R > 0$ be such that $K \subseteq B[0,R]$, and let f be a continuous self-map of K. Then $f \circ P$ is a continuous self-map of $B[0,R]$, and a fixed point $x \in B[0,R]$ of $f \circ P$ will automatically belong to $f(K) \subseteq K$, hence be a fixed point of f. □

I.6 Remark. Having Theorem I.1 at our disposal we can reinforce Propoposition I.3 to the statement that there is no continuous retraction of B to ∂B. Indeed, if g were a continuous retraction of B to ∂B, then $-g$ would be a continuous self-map of B without a fixed point. △

I.2 Schauder's fixed point theorem

I.7 Theorem (Schauder). *Let X be a Banach space, let $K \neq \varnothing$ be a compact convex subset of X, and let f be a continuous self-map of K. Then f has a fixed point.*

Proof. We may assume that the scalar field is $\mathbb{R}$. Assume, for a contradiction, that f has no fixed point. Then

$$\varepsilon := \inf\{\|f(x) - x\|\,;\, x \in K\} > 0. \tag{I.1}$$

For convenience, and without loss of generality, we assume that $\varepsilon = 1$. (This can be achieved by replacing the norm $\|\cdot\|$ on X by $\frac{1}{\varepsilon}\|\cdot\|$.) Put

$$\alpha(x) := (1 - \|x\|)^+ \qquad (x \in X).$$

There exist $x_1, \dots, x_n \in K$ such that $\{B(x_j, 1)\,;\, j = 1, \dots, n\}$ is a covering of K. For $x \in K$, $j \in \{1, \dots, n\}$ put $\alpha_j(x) := \alpha(x - x_j)$,

$$\beta_j(x) := \frac{\alpha_j(x)}{\alpha_1(x) + \cdots + \alpha_n(x)} \ (\geqslant 0).$$

Note that the denominator in the expression defining β_j is strictly positive, and observe that $\sum_{j=1}^n \beta_j(x) = 1$ $(x \in K)$.

Let C be the convex hull of $\{x_1, \dots, x_n\}$ and observe that $C \subseteq K$ is a compact convex subset of the finite-dimensional space $\operatorname{lin}\{x_1, \dots, x_n\}$. Then

$$g\colon K \to C, \quad x \mapsto \sum_{j=1}^n \beta_j(x) x_j$$

is continuous. Note that $\beta_j(x) = 0$ if $\|x - x_j\| \geqslant 1$, hence

$$\|x - g(x)\| = \left\| \sum_{j=1}^n \beta_j(x)(x - x_j) \right\| < 1 \qquad (x \in K). \tag{I.2}$$

Brouwer's fixed point theorem, in the form of Corollary I.5(b), implies that the continuous self-map $g \circ f|_C$ of C has a fixed point $x_f \in C$. Using (I.2) we obtain

$$\|f(x_f) - x_f\| = \|f(x_f) - g(f(x_f))\| < 1,$$

a contradiction to (I.1). □

Instead of supposing compactness of the set on which the mapping f is defined, one can also suppose that the mapping has relatively compact range, as follows.

I.8 Theorem (Schauder). *Let X be a Banach space, $C \subseteq X$ a closed convex subset, and let f be a continuous self-map of C, with the property that $f(C)$ is relatively compact in X. Then f has a fixed point.*

Proof. The closed convex hull of a relatively compact set in a Banach space is compact; see Exercise I.1. Thus, the closed convex hull $K := \overline{\operatorname{co}(f(C))}$ is compact, and it is also a subset of C. Hence, the restriction of f to K is a continuous self-map of K, and Theorem I.7 implies that f has a fixed point. □

Notes

The Brouwer fixed point theorem originally appeared in [Bro11; Satz 4]. A combinatorial proof, on the basis of "Sperner's lemma", [Spe28], was given in [KKM29]. Meanwhile multitudinous proofs can be found in the literature. Most of them belong to the area of homology theory and use the degree of mappings in $\mathbb{R}^n$. Our proof, which only relies on 'elementary analysis', is along the lines of the proof in [DuSc58; Section V.12]; see also [Heu98; XXVII, 228]. Another interesting proof is given in [Mil78], where also other sources are indicated.

The Schauder fixed point theorem appeared in [Sch30]. Our proof follows [Rud91; Theorem 5.28], where in fact Tikhonov's fixed point theorem [Tyc35] is proved, the generalisation of Schauder's fixed point theorem to locally convex topological vector spaces.

Exercises

I.1 Let X be a Banach space, $C \subseteq X$ a compact subset. Show that the closed convex hull of C is compact. (Hint: First show the claim for finite sets C. Then use the precompactness of C to show that $\mathrm{co}(C)$ is precompact as well.)

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Index of notation

$\mathbb{N}$	the set of natural numbers $\{1, 2, \dots\}$
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{R}$	the field of real numbers
$\mathbb{C}$	the field of complex numbers
$\mathbb{K}$	$\mathbb{R}$ or $\mathbb{C}$
$A^\top$	transpose of a matrix $A \in \mathbb{K}^{m\times n}$
$\operatorname{lin} B$	linear span of a subset B of a vector space
$B(x, r)$	open ball with centre x, radius r, page 16
$B[x, r]$	closed ball with centre x, radius r, page 16
Σ_θ	open sector of (semi-)angle $\theta \in (0, \pi/2]$, page 31
$\Sigma_{\theta,0}$	$\Sigma_\theta \cup \{0\}$, page 31
Σ_0	$(0, \infty)$, page 36
V'	dual space of a Hilbert space V, page 55
V^*	antidual space of a Hilbert space V, page 55
$L_2(\mu)_+$	cone of functions $0 \leqslant u \in L_2(\mu)$, page 112
$\operatorname{diam}(C)$	diameter of a set C in a metric space
$\operatorname{rd}(C)$	$\frac{1}{2}\operatorname{diam}(C)$ for a set C in a metric space, should be remindful of 'radius', page 282
$\operatorname{rd}(\mathcal{C})$	$\sup_{C\in\mathcal{C}} \operatorname{rd}(C)$, for a countable collection $\mathcal{C}$ of subsets of a metric space, page 282
$\mathcal{L}(X, Y)$	space of bounded linear operators from X to Y, with Banach spaces X, Y, page 3
s-lim	limit in the strong operator topology, page 3
$\operatorname{dom}(A)$	domain of A, page 5

$\operatorname{ran}(A)$	range of A, page 5
$\ker(A)$	kernel of A, page 5
$\operatorname{num}(A)$	numerical range of A, page 36
e^{tA}	the C_0-semigroup generated by a generator A, page 10
A^*	adjoint of the operator A, page 67
A^*	adjoint of the linear relation A, page 180
$A \sim (a, j)$	operator A associated with (a, j), page 149
tr	trace operator, assigning boundary values to functions, page 87
Δ_D	Dirichlet Laplacian, page 48
Δ_N	Neumann Laplacian, page 91
Δ_β	Robin Laplacian, page 92
Δ_β	Robin Laplacian for rough domains, page 156
$C(G)$	space of continuous functions with values in $\mathbb{K}$, for a locally compact set $G \subseteq \mathbb{R}^n$, page 13
$C_{\mathrm{b}}(G)$	space of bounded continuous functions, page 13
$C_{\mathrm{c}}(G)$	space of continuous functions with compact support, page 13
$C_0(G)$	closure of $C_{\mathrm{c}}(G)$ in $C_{\mathrm{b}}(G)$, page 13
$C^k(\Omega)$	space of k times continuously differentiable functions on an open set $\Omega \subseteq \mathbb{R}^n$, for $k \in \mathbb{N}_0$
$C^\infty(\Omega)$	space of infinitely differentiable functions
$C^k(\overline{\Omega})$	space of functions $u \in C(\overline{\Omega}) \cap C^k(\Omega)$, with derivatives up to order k continuously extendable to $\overline{\Omega}$, for bounded open $\Omega \subseteq \mathbb{R}^n$, page 83
$C^\infty(\overline{\Omega})$	$\bigcap_{k\in\mathbb{N}} C^k(\overline{\Omega})$, for bounded open $\Omega \subseteq \mathbb{R}^n$, page 85
$L_{1,\mathrm{loc}}(\Omega)$	space of locally integrable functions, page 39
$H^1(\Omega)$	Sobolev space of L_2-functions with distributional first order derivatives in L_2, page 44
$H^1_{\mathrm{c}}(\Omega)$	functions in $H^1(\Omega)$ with compact support, page 45
$H^1_0(\Omega)$	closure of $H^1_{\mathrm{c}}(\Omega)$ in $H^1(\Omega)$, page 45

$H^{-1}(\Omega)$	antidual of $H^1_0(\Omega)$ in the Gelfand triple $\big(H^1_0(\Omega), L_2(\Omega), H^{-1}(\Omega)\big)$, page 207
$W^1_1(\Omega)$	Sobolev space of L_1-functions, with distributional first order derivatives in L_1, page 279
$W^1_{1,\mathrm{c}}(\Omega),\ W^1_{1,0}(\Omega)$	analogous to $H^1_{\mathrm{c}}(\Omega),\ H^1_0(\Omega)$, page 279
$H^1(a,b;H)$	vector-valued Sobolev space, page 224
$\mathbf{1}_A$	indicator function of a subset A of a set Ω, $\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in \Omega \setminus A. \end{cases}$
$[f \leqslant g]$	$\{x \in \Omega;\, f(x) \leqslant g(x)\}$, for a set Ω and functions $f, g\colon \Omega \to \mathbb{R}$. Similarly for $[f < g]$, $[f = g]$ etc.
$f \vee g$, $f \wedge g$	$f \vee g(x) := \max\{f(x), g(x)\}$, $f \wedge g(x) := \min\{f(x), g(x)\}$ $(x \in \Omega)$, for a set Ω and functions $f, g\colon \Omega \to \mathbb{R}$.
f^+, f^-	positive and negative part of a real-valued function f, $f^+ := f \vee 0$, $f^- := (-f)^+$.
sgn	signum function, page 113
$\rho * u$	convolution of ρ and f, page 39
spt f	support of a continuous function, page 13
spt f	support of an $L_{1,\mathrm{loc}}$-function, page 45
$\lvert\alpha\rvert$	$\sum_{j=1}^n \alpha_j$, order of the multiindex $\alpha \in \mathbb{N}_0^n$
$\partial_j u$	partial derivative with respect to the j'th variable, for a differentiable function u of n variables
$\partial^\alpha u$	$\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} u$, for a multiindex $\alpha \in \mathbb{N}_0^n$
$\partial^\alpha f$	distributional derivative of $f \in L_{1,\mathrm{loc}}(\Omega)$, page 42
∇u	gradient of a function $u \in C^1(\Omega)$ or $u \in H^1(\Omega)$, $\nabla u := (u')^\top$, page 61
$\partial_\nu u$	normal derivative, page 84
$\partial_\nu u$	weak normal derivative, page 90
$\partial_\nu u$	weak normal derivative, page 155

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