Quasi-ISS/ISDS observers for interconnected systems and applications

Sergey Dashkovskiy\textsuperscript{a,}\textsuperscript{*}, Lars Naujok\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a} University of Applied Sciences Erfurt, Department of Civil Engineering, Germany
\textsuperscript{b} University of Bremen, Centre for Industrial Mathematics, P.O. Box 330440, 28334 Bremen, Germany

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\textbf{ABSTRACT}

This paper considers interconnected nonlinear dynamical systems and studies observers for such systems. For single systems the notion of quasi-input-to-state dynamical stability (quasi-ISDS) for reduced-order observers is introduced and observers are investigated using error Lyapunov functions. It combines the main advantage of ISDS over input-to-state stability (ISS), namely the memory fading effect, with reduced-order observers to obtain quantitative information about the state estimate error. Considering interconnections quasi-ISS/ISDS reduced-order observers for each subsystem are derived, where suitable error Lyapunov functions for the subsystems are used. Furthermore, a quasi-ISS/ISDS reduced-order observer for the whole system is designed under a small-gain condition, where the observers for the subsystems are used. As an application, we prove that quantized output feedback stabilization for each subsystem and the overall system is achievable, when the systems possess a quasi-ISS/ISDS reduced-order observer and a state feedback law that yields ISS/ISDS for each subsystem and therefor the overall system with respect to measurement errors. Using dynamic quantizers it is shown that under the mentioned conditions asymptotic stability can be achieved for each subsystem and for the whole system.

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1. Introduction

This paper is motivated by the work [1], where the notion of quasi-input-to-state stable (quasi-ISS) reduced order observers was introduced, by the works [2,3], where the stability property input-to-state dynamical stability (ISDS) was established and by the work [4], where the ISDS property for interconnected systems was investigated. The aim of this paper is to show the advantage of the ISDS-like observers over the ISS-like ones and to show that for large scale systems that can be decomposed into an interconnection of several subsystems the observers can be designed in a decentralized manner.

In practice, observers are used for systems, where the state or parts of the state cannot be measured due to uneconomic measurement costs or physical circumstances like high temperatures, where no measurement equipment is available, for example. We are interested in the topic under which conditions the designed observer guarantees that the state estimation error is stable. The used stability properties for observers are based on ISS, introduced in [5] and ISDS, and are called quasi-ISS and quasi-ISDS, respectively.

ISDS, which is equivalent to the ISS property, has some advantages over ISS. One of these advantages is the so-called memory fading effect. Fading memory estimates were first studied in [6] and further studied for example in [7]. It is known for ISS (ISDS) systems that the influence of the “older” signals on the current state is essentially smaller than the influence of the recent ones. However, the ISS estimation of trajectories does not take this into account. The advantage of the ISDS estimation is that it takes this dissipative property into account. In particular, if the input tends to zero, then the ISDS estimate will tend to zero, whereas the ISS estimate depends on the supremum norm of the input, moreover the fading rate is given by the ISDS.

This motivates the introduction of the quasi-ISDS property for observers, where the approaches of reduced-order observers and the ISDS property are combined and which have the advantage that the recent disturbance of the measurement is taken into account. We investigate under which conditions a quasi-ISDS reduced order observer can be derived for single nonlinear systems, where error Lyapunov functions (see [8,9]) are used. The design of observers in the context of this paper was investigated in [10,3,11,12,1], for example, and remarks on the equivalence of full order and reduced order observers can be found in [13].

Investigating interconnected systems it turns out that the ISS and ISDS properties can be studied in a decentralized way, provided that a small-gain condition is satisfied, see [14,15,4,16,17].

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\textsuperscript{*} Corresponding authors.

E-mail addresses: sergey.dashkovski@fh-erfurt.de (S. Dashkovskiy), larsnaujok@math.uni-bremen.de (L. Naujok).

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This allows to decompose a large system into an interconnection of several subsystems and to study their stability properties separately. Examples of such large-scale interconnections can be found in [15] in the ISS framework and in [4] in the ISDS framework. In this paper we show that if quasi-ISS/ISDS reduced-order observers for each subsystem of an interconnected system are given, then an observer for the whole system can be obtained under a small-gain condition.

Furthermore, the problem of stabilization of systems is investigated and we apply the presented approach to quantized output feedback stabilization for single and interconnected systems. The goal of stabilizing a system is an important problem in applications. Many approaches were performed during the last years and the design of stabilizing feedback laws is a popular research area, which is linked up with many applications. The stabilization using output feedback quantization was investigated in [18–21,9,22,11,1], for example. A quantizer is a device, which converts a real-valued signal into a piecewise constant signal. It may affect the process output or may also affect the control input.

Adapting the quantizer with a so-called zoom variable leads to dynamic quantizers, which have the advantage that asymptotic stability for single and interconnected systems can be achieved under certain conditions.

The paper is organized as follows: Section 2 contains some basic notions. The quasi-ISDS property is introduced in Section 3, where as the first main result of this paper a quasi-ISDS observer for single systems is presented. In Section 4 we consider interconnected systems and as the second result we derived quasi-ISDS/ISDS reduced-order observers for each subsystem. Furthermore, this section contains as the third main result the design of a quasi-ISDS/ISDS observer for the whole system using a small-gain condition. The quantized output feedback stabilization and the dynamic quantizers for interconnections can be found in Section 5. Finally, Section 6 concludes the paper and gives an outlook for future research activities.

2. Preliminaries

By $x'$ we denote the transposition of a vector $x \in \mathbb{R}^n$, $n \in \mathbb{N}$, furthermore $\mathbb{R}_+:=[0, \infty)$ and $\mathbb{R}_n^+$ denotes the positive orthant $\{x \in \mathbb{R}^n : x \geq 0\}$, where we use the standard partial order for $x, y \in \mathbb{R}^n$ given by $x \preceq y \iff x_i \leq y_i$, $i = 1, \ldots$, and $x \succeq y \iff \exists i: x_i < y_i$.

The relation $x \succ y$ for vectors is defined in the same way.

We denote the standard Euclidean norm in $\mathbb{R}^n$ by $\|\cdot\|$ and the supremum norm over an interval $[a, b]$, $a \leq b$ of a function $f$ by $\|f\|_{[a,b]}$.

We consider general nonlinear systems of the form

$$\dot{x} = f(x, u), \quad y = h(x),$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is an essentially bounded measurable control input, $y \in \mathbb{R}^p$ is the output, function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in $x$ uniformly in $u$ and function $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuously differentiable with locally Lipschitz derivative (called a $C^1$ function). In addition, it is assumed that $f(0,0)=0$ and $h(0)=0$ holds. Note that these considerations guarantee that a unique solution of system (1) exists.

A state observer for the system (1) is of the form

$$\dot{\hat{x}} = F(\hat{y}, \hat{\xi}, u), \quad \hat{x} = H(\hat{y}, \hat{\xi}, u),$$

where $\hat{\xi} \in \mathbb{R}^l$ is the observer state, $\hat{x} \in \mathbb{R}^k$ is the estimate of the system state $x$ and $\hat{y} \in \mathbb{R}^r$ is the measurement of $y$ that may be disturbed by $d \in \mathbb{R}^r: \hat{y} = y + d$, where $d$ is measurable and essentially bounded function. The function $F : \mathbb{R}^p \times \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}^l$ is locally Lipschitz in $\hat{y}$ and $\hat{\xi}$ uniformly in $u$ and function $H : \mathbb{R}^p \times \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}^l$ is continuously differentiable with locally Lipschitz derivative (called a $C^1$ function). In addition, it is assumed that $F(0,0,0)=0$ and $H(0,0,0)=0$ holds.

We denote the state estimation error by $\hat{x} = \hat{x} - x$.

For the next sections we need the following sets of comparison functions:

**Definition 2.1.** We define the following classes of functions:

- $\mathcal{K} := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ | \gamma \text{ is continuous, } \gamma(0) = 0 \text{ and strictly increasing}\}$
- $\mathcal{K}_\infty := \{\gamma \in \mathcal{K} | \gamma \text{ is unbounded}\}$
- $\mathcal{L} := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ | \gamma \text{ is continuous and strictly decreasing with } \lim_{t \to \infty} \gamma(t) = 0\}$
- $\mathcal{KL} := \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ | \beta \text{ is continuous, } \beta(t,0) = \beta(0,t) \in \mathcal{L}, \forall t, r \geq 0\}$
- $\mathcal{KLD} := \{\mu \in \mathcal{KL} | \mu(t,s) = \mu(\mu(r,t), s), \forall r, t, s \geq 0\}$

3. Quasi-ISDS observers for single systems

In this section, we introduce quasi-ISDS observers and give a motivating example for the introduction. Then, we show that the reduced-order observer designed in Theorem 1 in [1] is a quasi-ISDS observer provided that an error ISDS Lyapunov function exists.

We recall the definition of quasi-ISS observers from [1]:

**Definition 3.1.** System (2) is called a quasi-ISS observer for the system (1), if there exists a function $\tilde{\beta} \in \mathcal{K}_\infty$ and for each $K > 0$, there exists a function $\tilde{y}_K^{\text{ISS}} \in \mathcal{K}_\infty$ s.t.

$$[\tilde{x}(t)] \leq \max\{\tilde{\beta}(\|\tilde{x}_0\|, t), \tilde{y}_K^{\text{ISS}}(\|d\|_{[0,t]}), \}$$

whenever $\|u\|_{[0,t]} \leq K$ and $\|x\|_{[0,t]} \leq K$.

Now we define quasi-ISDS observers.

**Definition 3.2.** System (2) is called a quasi-ISDS observer for the system (1), if there exist functions $\tilde{\mu} \in \mathcal{KLD}$, $\tilde{\eta} \in \mathcal{K}_\infty$ and for each $K > 0$ a function $\tilde{y}_K^{\text{ISDS}} \in \mathcal{K}_\infty$ such that

$$[\tilde{x}(t)] \leq \max\{\tilde{\mu}(\tilde{\eta}(\|\tilde{x}_0\|), t), \text{ ess sup } \tilde{\mu}(\tilde{y}_K^{\text{ISDS}}(\|d\|_{[0,t]}), t - \tau), \}$$

whenever $\|u\|_{[0,t]} \leq K$ and $\|x\|_{[0,t]} \leq K$.

Function $\tilde{\mu}$ called decay rate, describes the fading memory effect. Namely it shows how fast the influence of the “older” values of the disturbance $d$ on the state $\tilde{x}$ decays with time. This is an important advantage over the ISS property (3), where the estimation of $[\tilde{x}]$ depends on $d$ via $\|d\|_{[0,t]}$, which is nondecreasing in time. The notion of ISDS was introduced in [2]. In [4] the advantages of ISDS over ISS were discussed.

The motivation of the introduction of quasi-ISDS observers will be illustrated by the following example.

**Example 3.3.** Consider the system as in the Example 1 in [1]

$$\dot{x} = -x + x^2 u, \quad y = x,$$

where $\dot{x} = -\dot{x} + y^2 u$ is an observer. We consider the perturbed measurement $\tilde{y} = y + d$, with $d = e^{-\tau}\tilde{\mu}$. Then, the error dynamics becomes

$$\dot{x} = -\dot{x} + 2xud + ud^2.$$
This system is ISS and ISDS from $d$ to $\hat{x}$ when $u(t)$ and $x(t)$ are bounded. Let $u \equiv 1$ be constant, then the estimations of the error dynamics are displayed in Fig. 1 for $x_0 = \hat{x}_0 = 0.3$. The ISS estimate is chosen equal to 1, since $\hat{\beta}(\hat{x}_0, t) \leq \hat{\gamma}_0^k(\|d\|_{0, t})$ for a sufficient function $\hat{\beta}$, $\|d\|_{0, t} = 1$, $\hat{x}_0$ small enough and with $\hat{\gamma}_0^k = 1$. The ISDS estimation follows by choosing $\hat{\gamma}_0^k(d(t)) = \frac{(\alpha(t))^2}{1 + \alpha(t)}$ and $\hat{\mu}(r, t) = e^{-\epsilon(t)}r$, $r \geq 0$ and $\epsilon = 0.1$. Here, the quasi- ISS estimation takes the maximal value of $d$ into account, whereas the quasi- ISDS estimation possesses the so-called memory-fading effect. We see from Fig. 1 that the ISS property provides essentially better estimation of the state magnitude after a short transition time.

Remark 3.4. This example illustrates one of the advantages of the ISDS over ISS and motivates this paper, which aims to demonstrate the advantage of observers satisfying a quasi-ISDS property. This advantage becomes clear in case of fluctuating disturbances, which is typical, for example, in case of the quantization error due to transmission congestions. If the transmission network is overloaded this error is large, if we have a possibility to send signals more often, the error becomes smaller. The fading memory of the ISDS framework allows to keep track of these fluctuations.

In the following, we consider reduced-order observers. We assume that systems of the form (1) can be divided into one part, where the state can be measured and a second part, where the state cannot be measured. The practical meaning is the following: For systems it can be uneconomic to measure all of the states system, because the measurement equipment or the running costs for the measurement are very expensive, for example. Therefore, a part of the state is measured and the other part has to be estimated. Here, we use quasi-ISDS/ISDS reduced-order observers for the state estimation, where only the part of the state is estimated that is not measured.

We assume that there exists a global coordinate change $z = \phi(x)$ such that the system (1) is globally diffeomorphic to a system with linear form of the output

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} f_1(z_1, z_2, u) \\ f_2(z_1, z_2, u) \end{bmatrix} = f(z, u), \quad y = z_1,$$

where $z_1 \in \mathbb{R}^p$ and $z_2 \in \mathbb{R}^{N-p}$.

For the results in this paper we need the following assumption, where we use reduced-order error Lyapunov functions. Error Lyapunov functions were first introduced in [8] and in [9] the equivalence of the existence of an error Lyapunov function and the existence of an observer was shown.

**Assumption 3.5.** There exist a $C^1$ function $I : \mathbb{R}^p \to \mathbb{R}^{N-p}$, a $C^1$ function $V : \mathbb{R}^{N-p} \to \mathbb{R}_+$, and functions $\alpha_i \in \mathcal{K}_\infty$, $i = 1, \ldots, 4$ such that for all $e \in \mathbb{R}^{N-p}, z \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$

$$\alpha_1(|e|) \leq V(e) \leq \alpha_2(|e|),$$

$$\left| \frac{\partial V(e)}{\partial e} \right| \leq \alpha_4(V(e)),$$

$$\frac{\partial V(e)}{\partial e} \left( \hat{f}_2(z_1, e + z_2, u) - f_2(z_1, z_2, u) + \frac{\partial h(z_1)}{\partial z_1} f_1(z_1, e + z_2, u) - f_1(z_1, z_2, u) \right) \leq -\alpha_3(V(e)),$$

and there exists a function $\alpha \in \mathcal{K}_\infty$ such that $\alpha(s)\alpha_4(s) \leq \alpha(s)$, $s \in \mathbb{R}_+$.

**Remark 3.6.** Note that in [1] the properties of an error Lyapunov function are slightly different, namely on the right hand side of (8) and (9) we write $V(e)$ instead of $|e|$. However this change leads to an equivalent assumption due to (7).

Now, the following lemma can be stated, which was proved in [1]:

**Lemma 3.7.** Under Assumption 3.5, the system

$$\hat{\xi} = \hat{f}_2(\hat{y}, \hat{\xi} - l(\bar{y}), u) + \frac{\partial h(\hat{\xi})}{\partial z_1} f_1(\hat{y}, \hat{\xi} - l(\bar{y}), u),$$

$$\hat{z}_1 = \hat{y}, \quad \hat{z}_2 = \hat{\xi} - l(\bar{y})$$

becomes a quasi-ISDS reduced-order observer for the system (6), where $\bar{x} \in \mathbb{R}^{N-p}$ is the observer state and $\hat{z}_1, \hat{z}_2$ are the estimates of $z_1$ and $z_2$, respectively, and $\bar{y} = y + d = z_1 + d$, which is the measurement of $z_1$ disturbed by $d$.

Motivated by the advantages discussed above we will use quasi-ISDS observers and assume the following:

**Assumption 3.8.** Let $\varepsilon \in (0, 1)$ be given. There exist a $C^1$ function $I : \mathbb{R}^p \to \mathbb{R}^{N-p}$, a $C^1$ function $V : \mathbb{R}^{N-p} \to \mathbb{R}_+$, functions $\alpha, \tilde{\eta} \in \mathcal{K}_\infty$ and a $C^1$ function $\tilde{\mu} \in \mathcal{K}_{\mathcal{L}D}$ such that for all $e \in \mathbb{R}^{N-p}$, $z \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$

$$\frac{|e|}{1 + \varepsilon} \leq V(e) \leq \tilde{\eta}(|e|),$$

$$\left| \frac{\partial V(e)}{\partial e} \right| \leq \alpha(V(e)),$$

$$\frac{\partial V(e)}{\partial e} \left( \hat{f}_2(z_1, e + z_2, u) + \frac{\partial h(z_1)}{\partial z_1} f_1(z_1, e + z_2, u) + \frac{\partial h(z_1)}{\partial z_1} f_1(z_1, z_2, u) \right) - \left[ f_2(z_1, z_2, u) + \frac{\partial h(z_1)}{\partial z_1} f_1(z_1, z_2, u) \right] \leq - \left( 1 - \varepsilon \right) g(V(e))$$

where $g$ is given by $\frac{\partial}{\partial \tilde{\mu}} \tilde{\mu}(r, t) = - g(\tilde{\mu}(r, t), r, t > 0$). By properties of $\tilde{\mu}$ function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is non-negative and is assumed to be locally Lipschitz continuous. Further, we assume that there exists a function $\tilde{a} \in \mathcal{K}_\infty$ such that

$$\tilde{a}(s)\alpha(s) \leq \left(1 - \varepsilon\right) g(s), \quad s \in \mathbb{R}_+.$$
Theorem 3.9. Under Assumption 3.8 the system (10) becomes a quasi-ISDS reduced-order observer for the system (6).

The proof goes along the lines of the proof of Lemma 3.7 in [1] with corresponding changes according to Definition 3.2 and Assumption 3.8:

Proof. We define \( \xi := z_l + l(z_l) \) and convert the system (6) into

\[
\dot{z}_1 = f_1(z_1, \xi - l(z_1), u), \quad y = z_1,
\]

\[
\dot{\xi} = f_2(z_1, \xi - l(z_1), u) + \frac{\partial l(z_1)}{\partial z_1} f_1(z_1, \xi - l(z_1), u)
=: F(z_1, \xi, u).
\]

With this \( F \) the dynamics of the observer state in (10) can be written as \( \dot{\hat{\xi}} = F(\bar{y}, \hat{\xi}, u) \), where \( \bar{y} = y + d \) and \( d \) is the measurement disturbance. Let \( e := \hat{\xi} - \xi \) and from Assumption 3.8 we obtain

\[
\dot{V}(e) = \frac{\partial V(e)}{\partial e} \left( \tilde{f}_2(z_1 + d, \hat{\xi} - l(z_1 + d), u) - \tilde{f}_2(z_1, \hat{\xi} - l(z_1), u) + \frac{\partial l(z_1 + d)}{\partial z_1} f_1(z_1 + d, \hat{\xi} - l(z_1 + d), u) - \frac{\partial l(z_1)}{\partial z_1} f_1(z_1, \hat{\xi} - l(z_1), u) \right)
\]

\[
\leq - \left( 1 - \varepsilon \right) g(V(e)) + \gamma f(y, \xi, u) - F(y, \xi, u).
\]

In [23] it was shown that there exist a continuous positive function \( \rho \) and \( \gamma \in \mathcal{K} \) s.t. \( |F(\bar{y}, \xi, u) - F(y, \xi, u)| \leq \rho(y, \xi, u) \gamma(|d|) \) that implies

\[
\dot{V}(e) \leq - \left( 1 - \varepsilon \right) g(V(e)) + \rho(y, \xi, u) \gamma(|d|),
\]

and for arbitrary \( \delta \in (0, 1) \) it follows

\[
\dot{V}(e) \leq - \left( 1 - \delta \right) \left( 1 - \varepsilon \right) g(V(e)) - \left( 1 - \varepsilon \right) \delta g(V(e)) + \alpha(y, \xi, u) \gamma(|d|) - \left( 1 - \varepsilon \right) \delta g(V(e))
\]

if \( 1 - \delta \left( 1 - \varepsilon \right) g(V(e)) > \alpha(y, \xi, u) \gamma(|d|) \). Note that by (14) we have

\[
(1 - \delta)(1 - \varepsilon) g(V(e)) \geq (1 - \delta) \alpha(V(e)) \gamma(|d|)
\]

and the previous inequality is guaranteed by \( 1 - \delta) \alpha(V(e)) > \rho(y, \xi, u) \gamma(|d|) \). This shows the implication

\[
V(e) \geq \tilde{\alpha}^{-1} \left( \frac{\rho(y, \xi, u) \gamma(|d|)}{1 - \delta} \right) \Rightarrow \dot{V}(e) \leq - \left( 1 - \varepsilon \right) \tilde{g}(V(e)),
\]

where \( \tilde{g}(r) := \delta g(r) \), \( \forall r > 0 \). By Theorem 3.5.8 in [3] and its proof this is equivalent to

\[
|e(t)| \leq \max \left\{ \tilde{\mu}(\tilde{\eta}(e(0))), \tau, \text{ess sup}_{\tau \in [0, t]} \tilde{\mu}(\tilde{\gamma}^{ISDS}(|d(\tau)|)), t - \tau \right\} \quad (15)
\]

under \( \|z\|_{[0, t]} \leq K, \|u\|_{[0, t]} \leq K \), where \( \tilde{\gamma}^{ISDS} \in \mathcal{K}_\infty \) is parametrized by \( K \). Now, we have

\[
\dot{z} = \left( \begin{array}{c} \tilde{z}_1 \\ \tilde{z}_2 \end{array} \right) := \left( \begin{array}{c} \tilde{z}_1 - z_1 \\ \tilde{z}_2 - z_2 \end{array} \right) = \left( \begin{array}{c} e - (l(y) - l(z_1)) \end{array} \right).
\]

By \( \theta_k \in \mathcal{K} \), parametrized by \( K \) such that \( |l(z_1 + d) - l(z_1)| \leq \theta_k(|d|), |z_1| \leq K \) it follows

\[
\|\tilde{z}\| \leq |e| + |d| + \theta_k(|d|) \quad \text{and} \quad |e| \leq \|\tilde{z}\| + \theta_k(|d|).
\]

Overall, combining (15) and (16) we have

\[
\|\tilde{z}\| \leq \max \left\{ \tilde{\mu}(\tilde{\eta}(e(0))), \tau, \text{ess sup}_{\tau \in [0, t]} \tilde{\mu}(\tilde{\gamma}^{ISDS}(|d(\tau)|)), t - \tau \right\}
+ |d(t)| + \theta_k(|d(t)|)
\leq \max \left\{ \tilde{\mu}(\tilde{\eta}(e(0))), \tau, \text{ess sup}_{\tau \in [0, t]} \tilde{\mu}(\tilde{\gamma}^{ISDS}(|d(\tau)|)), t - \tau \right\}
\]

+ \chi_k(|d(t)|),
\]

where \( \chi_k(s) := s + \theta_k(s), s \geq 0. \) Since \( \tilde{\mu} \) is a \( \mathcal{K} \mathcal{L} \mathcal{D} \)-function it follows

\[
\|\tilde{z}\| \leq \max \left\{ 2\tilde{\mu}(\tilde{\eta}(e(0))), \tau, \text{ess sup}_{\tau \in [0, t]} 2\tilde{\mu}(\tilde{\gamma}^{ISDS}(|d(\tau)|)), t - \tau \right\}
\]

\[
\text{ess sup}_{\tau \in [0, t]} 2\tilde{\mu}(\tilde{\gamma}^{ISDS}(|d(\tau)|)), t - \tau \right\}
\leq \max \left\{ 2\tilde{\mu}(\tilde{\eta}(e(0))), \tau, \text{ess sup}_{\tau \in [0, t]} 2\tilde{\mu}(\tilde{\gamma}^{ISDS}(|d(\tau)|)), t - \tau \right\}
\]

\[
\text{ess sup}_{\tau \in [0, t]} 2\tilde{\mu}(\tilde{\gamma}^{ISDS}(|d(\tau)|)), t - \tau \right\},
\]

where \( \tilde{\gamma}^{ISDS}(s) := \max(\tilde{\gamma}^{ISDS}(s), \chi_k(s)) \), and we used (16) and the inequality \( \alpha(a + b) \leq \max(\alpha(2a), \alpha(2b)) \) for \( a, b \geq 0. \) Furthermore, we have

\[
\|\tilde{z}\| \leq \max \left\{ 2\tilde{\mu}(\tilde{\eta}(e(0))), \tau, \text{ess sup}_{\tau \in [0, t]} 2\tilde{\mu}(\tilde{\gamma}^{ISDS}(|d(\tau)|)), t - \tau \right\}
\]

\[
\text{ess sup}_{\tau \in [0, t]} 2\tilde{\mu}(\tilde{\gamma}^{ISDS}(|d(\tau)|)), t - \tau \right\}
\leq \max \left\{ 2\tilde{\mu}(\tilde{\eta}(e(0))), \tau, \text{ess sup}_{\tau \in [0, t]} 2\tilde{\mu}(\tilde{\gamma}^{ISDS}(|d(\tau)|)), t - \tau \right\}
\]

\[
\text{ess sup}_{\tau \in [0, t]} 2\tilde{\mu}(\tilde{\gamma}^{ISDS}(|d(\tau)|)), t - \tau \right\},
\]

where \( \tilde{\gamma}^{ISDS}(s) := \max(\tilde{\gamma}^{ISDS}(s), \chi_k(s)) \). Finally, by definition of \( \tilde{\mu}(r, t) := 2\tilde{\mu}(r, t) \) and \( \tilde{\eta}(s) := \tilde{\eta}(2s) \) it follows

\[
\|\tilde{z}\| \leq \max \left\{ \tilde{\mu}(\tilde{\eta}(e(0))), \tau, \text{ess sup}_{\tau \in [0, t]} \tilde{\mu}(\tilde{\gamma}^{ISDS}(|d(\tau)|)), t - \tau \right\},
\]

which proves the assertion. \( \square \)

Remark 3.10. The decay rate and the gain of the definition of ISDS are the same as the ones using ISDS-Lyapunov functions, see [2]. Note that this is not the case for the definition of a quasi-ISDS observer and using error ISDS-Lyapunov functions. It remains as an open topic to investigate if it is possible to use a different error ISDS-Lyapunov function, from which the information about the decay rate and the gains of the error ISDS-Lyapunov function can be preserved for the quasi-ISDS estimation.
In the next section, we are going to extend the notion of quasi-ISS/ISDS observers for interconnected systems and provide tools to derive such kind of observers.

4. Quasi-ISS and quasi-ISDS observers for interconnected systems

The framework of ISS and ISDS properties is very useful in stability analysis of interconnections. Application examples of the small-gain conditions can be found for example in [15, 4]. As we will see here it can be used for a decentralized observer design for systems decomposed into an interconnection of several ones.

In this chapter we consider $n \in \mathbb{N}$ interconnected systems of the form

$$
\dot{x}_i = f_i(x_1, \ldots, x_n, u), \quad y_i = h_i(x_i),
$$

$i = 1, \ldots, n$, where $x_i \in \mathbb{R}^{n_i}$ is the state of the $i$th subsystem, $u_i \in \mathbb{R}^{m_i}$ are essentially bounded measurable control inputs, $y_i \in \mathbb{R}^{p_i}$ are the outputs, functions $f_i : \mathbb{R}^{n_i+p_i} \rightarrow \mathbb{R}^{n_i}$ are locally Lipschitz in $(x_1, \ldots, x_n)$ uniformly in $u_i$ and functions $h_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{p_i}$ are continuously differentiable with locally Lipschitz derivatives. In addition, it is assumed that $f_i(0, \ldots, 0) = 0$ and $h_i(0) = 0$.

The state observer of the $i$th subsystem is of the form

$$
\dot{\hat{x}}_i = \hat{F}_i(\hat{y}_i, \ldots, \hat{\hat{y}}_i, \hat{\hat{\hat{y}}}_i, \ldots, \hat{x}_n, u), \quad \hat{y}_i = \hat{h}_i(\hat{x}_i),
$$

$i = 1, \ldots, n$, where $\hat{x}_i \in \mathbb{R}^{n_i}$ is the observer state of the $i$th subsystem, $\hat{\hat{x}}_i$ is the estimate of the system state $x_i$, and $\hat{\hat{y}}_i \in \mathbb{R}^{p_i}$ is the measurement of $y_i$ that may be disturbed by $\hat{\hat{\hat{x}}}_i \in \mathbb{R}^{n_i}$: $\hat{y}_i = y_i + \hat{\hat{\hat{x}}}_i$, where $\hat{\hat{\hat{x}}}_i$ is measurable and essentially bounded. The function $F_i : \mathbb{R}^{n_i+p_i} \rightarrow \mathbb{R}^{n_i}, P = \sum P_i$, $L = \sum L_i$ is locally Lipschitz in $\hat{y}_i = (\hat{y}_1, \ldots, \hat{y}_n)$ and $\xi = (\hat{\hat{x}}_1, \ldots, \hat{\hat{\hat{x}}}_n)$ uniformly in $u_i$ and function $H_i : \mathbb{R}^{n_i} \times \mathbb{R}^{n_i+p_i} \rightarrow \mathbb{R}^{n_i}$ is continuously differentiable with locally Lipschitz derivative (called a $C^1_L$ function). In addition, it is assumed that $f_i(0, 0, 0) = 0$ and $H_i(0, 0, 0) = 0$ holds.

We denote the state estimation error of the $i$th subsystem by $\hat{x}_i - x_i$. The quasi-ISS/ISDS property for interconnected systems reads as follows:

1. The $i$th subsystem of (18) is called a quasi-ISS observer for the $i$th subsystem of (17), if there exists a function $\hat{\beta}_i \in \mathcal{L}$ and for each $K_i > 0$ there exist functions $(\gamma_{i L}^{K_i})^{\mathrm{ISS}}, (\gamma_{i IS}^{K_i})^{\mathrm{ISS}} \in \mathcal{K}_{\infty}$, $j = 1, \ldots, n$, $j \neq i$ such that

$$
|\hat{x}_i(t)| \leq \max \left\{ \hat{\beta}_i(|\hat{y}_1(t)|), \max_{j \neq i} \gamma_{i L}^{K_i}(|\hat{y}_j(t)|), \gamma_{i IS}^{K_i}(\|d_j(t)|_1), \gamma_{i IS}^{K_i}(\|d_j(t)|_{0,1}) \right\},
$$

whenever $\|u_i\|_{0,1} \leq K_i$ and $\|\hat{y}_j\|_{0,1} \leq K_i$, $j = 1, \ldots, n$.

2. The $i$th subsystem of (18) is called a quasi-ISDS observer for the $i$th subsystem of (17), if there exist functions $\mu_i \in \mathcal{L}_\infty$, $\tilde{n}_i \in \mathcal{K}_\infty$ and for each $K_i > 0$ there exist functions $(\gamma_{i L}^{K_i})^{\mathrm{IS}}, (\gamma_{i IS}^{K_i})^{\mathrm{IS}} \in \mathcal{K}_{\infty}$, $j = 1, \ldots, n$, $j \neq i$ such that

$$
|\hat{x}_i(t)| \leq \max \left\{ \mu_i(\tilde{n}_i(|\hat{y}_1(t)|)), \max_{j \neq i} \gamma_{i L}^{K_i}(|\hat{y}_j(t)|), \gamma_{i IS}^{K_i}(\|d_j(t)|_1), \gamma_{i IS}^{K_i}(\|d_j(t)|_{0,1}) \right\},
$$

whenever $\|u_i\|_{0,1} \leq K_i$ and $\|\hat{y}_j\|_{0,1} \leq K_i$, $j = 1, \ldots, n$.

We assume that there exists a global coordinate change $z_i = \phi_i(x_i)$ with $x = (x_1', \ldots, x_n')$, such that the $i$th subsystem of (17) is globally diffeomorphic to a system with linear output of the form

$$
\dot{\hat{z}}_i = \begin{bmatrix} \hat{z}_{1i} \\ \hat{z}_{2i} \end{bmatrix} = \begin{bmatrix} f_{1i}(z_{11}, \ldots, z_{1n}, z_{21}, \ldots, z_{2n}, u_i) \\ f_{2i}(z_{11}, \ldots, z_{1n}, z_{21}, \ldots, z_{2n}, u_i) \end{bmatrix} = f_i(z_1, \ldots, z_n, u_i),
$$

where $z_{1i} \in \mathbb{R}^{p_i}$ and $z_{2i} \in \mathbb{R}^{p_i}$, $i = 1, \ldots, n$.

For the design of a quasi-ISS/ISDS observer for each subsystem of (17) we assume:

**Assumption 4.1.** 1. For each $i = 1, \ldots, n$ there exist a $C^1_L$ function $l_i : \mathbb{R}^{p_i} \rightarrow \mathbb{R}^{p_i}$, a $C^1_L$ function $V_i : \mathbb{R}^{p_i} \rightarrow \mathbb{R}_+$, functions $\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, \alpha_{i4}, \gamma_i \in \mathcal{K}_\infty$, $j = 1, \ldots, n$, $j \neq i$ such that for all $e_i \in \mathbb{R}^{p_i}$, $z_{1i} \in \mathbb{R}^{p_i}$, $z_{2i} \in \mathbb{R}^{p_i}$ and $u_i \in \mathbb{R}^{m_i}$ it holds

$$
\alpha_{i1}(|e_i|) \leq V_i(e_i) \leq \alpha_{i2}(|e_i|),
$$

$$
V_i(e_i) \geq \max_{j \neq i} \gamma_{i}(V_j(e_j)),
$$

$$
\frac{\partial V_i}{\partial e_i}(e_i) \leq \alpha_{i3}(V_i(e_i)),
$$

and there exists $\alpha_i \in \mathcal{K}_\infty$ such that $\alpha_i(s) \alpha_{i4}(s) \leq \alpha_{i3}(s)$, $s \in \mathbb{R}_+$.

2. Let $\epsilon_i > 0$ be given. For each $i = 1, \ldots, n$ there exist a $C^1_L$ function $l_i : \mathbb{R}^{p_i} \rightarrow \mathbb{R}^{p_i}$, a $C^1_L$ function $V_i : \mathbb{R}^{p_i} \rightarrow \mathbb{R}_+$, functions $\alpha_i, \tilde{n}_i, \gamma_i \in \mathcal{K}_\infty$, $j = 1, \ldots, n$, $j \neq i$ and each $\tilde{\mu}_i \in \mathcal{L}_\infty$ such that for each $e_i \in \mathbb{R}^{p_i}$, $z_{1i} \in \mathbb{R}^{p_i}$, $z_{2i} \in \mathbb{R}^{p_i}$ and $u_i \in \mathbb{R}^{m_i}$

$$
\frac{|e_i|}{1 + e_i} \leq V_i(e_i) \leq \tilde{n}_i(|e_i|),
$$

$$
\frac{\partial V_i}{\partial e_i}(e_i) \leq \alpha_i(V_i(e_i)),
$$

and $\tilde{\mu}_i(s) \alpha_i(s) \leq \alpha_i(s)$, $s \in \mathbb{R}_+$.

The next theorem is a counterpart of Lemma 3.7 and Theorem 3.9 for the design of a quasi-ISS/ISDS reduced-order observer for each subsystem of an interconnected system.
Theorem 4.2. 1. Under Assumption 4.1 point 1, the system
\[
\dot{\hat{z}}_i = f_{2i}(\hat{y}_1, \ldots, \hat{y}_n, \hat{z}_1 - l_1(\hat{y}_1), \ldots, \hat{z}_n - l_n(\hat{y}_n), u_i)
+ \frac{\partial h_i}{\partial z_{1i}}(\hat{y}_1, \ldots, \hat{y}_n, \hat{z}_1)
- \frac{\partial h_i}{\partial z_{1i}}(\hat{y}_1, \ldots, \hat{z}_n - l_n(\hat{y}_n), u_i)
\]
(20)
\[
\dot{\hat{z}}_1 = \hat{z}_i, \quad \dot{\hat{z}}_i = \hat{z}_i - l_1(\hat{y}_1)
\]
becomes a quasi-ISS reduced-order observer for the ith subsystem of (19).

2. Under Assumption 4.1 point 2, the system (20) becomes a quasi-ISS reduced-order observer for the ith subsystem of (19).

Proof. The proof goes along the proof of Theorem 3.9 with changes according to the quasi-ISS/ISSPS property for interconnected systems and Assumption 4.1.

We define \(\hat{z}_i := z_{2i} + l_i(z_{1i})\). Then,
\[
\dot{\hat{z}}_{1i} = f_{1i}(z_{1i}, \ldots, z_{1i}, \hat{z}_1 - l_1(z_{1i}), \ldots, \hat{z}_n - l_n(z_{1n}), u_i),
\]
\[
\dot{\hat{z}}_i = f_{2i}(z_{1i}, \ldots, z_{ni}, \hat{z}_1 - l_1(z_{1i}), \ldots, \hat{z}_n - l_n(z_{1n}), u_i)
+ \frac{\partial h_i}{\partial z_{1i}}(z_{1i}, \ldots, z_{ni}, \hat{z}_1 - l_1(z_{1i}), \ldots, \hat{z}_n - l_n(z_{1n}), u_i)

\]
\[
\hat{z}_1 = z_{1i}, \quad i = 1, \ldots, n.
\]
The reduced-order observer (20) is written as \(\dot{\hat{z}}_i = F_i(\hat{y}_1, \ldots, \hat{y}_n, \hat{z}_1, \ldots, \hat{z}_n, u_i)\). Let \(e_i := \hat{z}_i - z_i\). We use the shorthand for \(j = 1, 2\)
\[
\dot{f}_{j1} = f_j(y_1, \ldots, y_n, e_1 - l_1(y_1) - d_1(y_1)(1 + d_1(y_2)), \ldots, e_n - l_n(y_n) - d_n(y_n), u_i),
\]
further, then we have whenever \(V_i(e_i) \geq \alpha_j \max_j V_j(V_j(e_j))\) holds, it follows
\[
\dot{V}_i(e_i) = \frac{\partial V_i}{\partial e_i}(e_i) \left( \dot{f}_{2i} + \frac{\partial h_i}{\partial z_{1i}}(z_{1i} + d_1)^2 \right)
- \frac{\partial h_i}{\partial z_{1i}}(z_{1i}),
\]
\[
\leq -\alpha_j V_i(e_i) + \frac{\partial V_i}{\partial e_i}(e_i) (F_i(\hat{y}_1, \ldots, \hat{y}_n, e_1, \ldots, e_n) - F_i(y_1, \ldots, y_n, e_1, \ldots, e_n))
\]
\[
\leq \rho_i(y_1, \ldots, y_n, e_1, \ldots, e_n) \max_j V_j(d_j)
\]
where \(\gamma_i \in \mathcal{K}\) and \(\rho_i\) is a continuous positive function such that
\[
|F_i(\hat{y}_1, \ldots, \hat{y}_n, e_1, \ldots, e_n, u_i) - F_i(y_1, \ldots, y_n, e_1, \ldots, e_n, u_i) |
\leq \rho_i(y_1, \ldots, y_n, e_1, \ldots, e_n, u_i) \max_j V_j(d_j)
\]
whose existence can be shown using the results in Appendix of [23]:
\[
[V_i(\hat{y}_1, \ldots, \hat{y}_n, e_1, \ldots, e_n, u_i) - F_i(y_1, \ldots, y_n, e_1, \ldots, e_n, u_i)]
\leq \rho_i(y_1, \ldots, y_n, e_1, \ldots, e_n, u_i) \gamma_i((d_1, \ldots, d_n)^T)
\]
\[
\leq \rho_i(y_1, \ldots, y_n, e_1, \ldots, e_n, u_i) \gamma_i(\max_j V_j(d_j))
\]
\[
= \rho_i(y_1, \ldots, y_n, e_1, \ldots, e_n, u_i) \gamma_i(\max_j V_j(d_j)).
\]
It follows that for an arbitrary \(\delta_i \in (0, 1)\), we have
\[
V_i(e_i) \geq \alpha_i^{-1}(1 - \delta_i)^{-1} \rho_i(y_1, \ldots, y_n, e_1, \ldots, e_n, u_i) \gamma_i(\max_j V_j(d_j))
\]
\[
\xi_n, u_i) \max_j V_j(d_j) \Rightarrow \dot{V}_i \leq -\delta_i \alpha_i \gamma_i(\max_j V_j(d_j))
\]
Under the conditions that \(|z_i| |u_i| \leq K_i\) and \(|(u_i)_{|0,i|} | \leq K_i, j = 1, \ldots, n\) it can be shown by standard arguments that there exist a function \(\hat{\beta}_i \in \mathcal{K}_L\), functions \(\hat{\gamma}_i^{K_i} \in \mathcal{K}_\infty\), \(j = 1, \ldots, n\), \(j \neq i\) such that for all \(t \in \mathbb{R}_+\) it holds
\[
|e_i(t)| \leq \max \left\{ \hat{\beta}_i(e_i(t)), \max_j \hat{\gamma}_j^{K_j}(\max_j V_j(d_j)) \right\}
\]
Recalling (20), we have that
\[
|z_i(t)| \leq |d_i| + |e_i| + \theta_i(|d_i|) \quad \text{and} \quad |e_i| \leq |z_i| + \theta_i(|d_i|),
\]
where \(\theta_i(|d_i|)\) is a class-\(\mathcal{K}\) function, parametrized by \(K_i\) such that \(|(z_i + d_i) - z_i| \leq \theta_i(|d_i|)\) when \(|z_i| \leq K_i\). Together with (21) we obtain
\[
|\dot{z}_i(t)| \leq \max \left\{ 3\hat{\beta}_i(e_i(t), t), 3\hat{\gamma}_i^{K_i}(\max_j V_j(d_j)) \right\}
\]
where \(\check{\gamma}_i^{K_i}(r) := \max_j \hat{\gamma}_j^{K_j}(r), \theta_i(r), t\). By \(\alpha(a + b) \leq \max(\alpha(2a), \alpha(2b))\) for any \(\alpha \in \mathcal{K}\) and any \(a, b \geq 0\) we have that
\[
|\dot{z}_i(t)| \leq \max \left\{ 3\hat{\beta}_i(2|z_i|, t), 3\hat{\gamma}_i^{K_i}(2\hat{\gamma}_i^{K_i}(\max_j V_j(d_j))), \right\}
\]
This proves that the system (20) is a quasi-ISS reduced order observer for the ith subsystem.

The proof for the quasi-ISDS reduced order observer for the ith subsystem follows the same steps as for a quasi-ISS reduced order observer. \(\square\)

Now, if we define \(M = \sum_i M_i \in \mathbb{R}^N\), \(z_1 := (z_1^T, \ldots, z_n^T)^T \in \mathbb{R}^N, z_2 := (z_2^T, \ldots, z_n^T)^T \in \mathbb{R}^{N-N}, u := (u_1^T, \ldots, u_n^T)^T \in \mathbb{R}^M, d = (d_1^T, \ldots, d_n^T)^T\) and \(f := (f_{11}^T, \ldots, f_{1n}^T)^T, f_{i1} := (f_{i1}^T, \ldots, f_{i1}^T)^T, f_{ij} := (f_{ij}^T, \ldots, f_{ij}^T)^T\), then the system (19) can be written as a system of the form (6).
We collect all gains \((\gamma_g^K) \) in a gain-matrix \( \Gamma_{\text{ISS}} \), which defines a map \( \tilde{\Gamma}_{\text{ISS}} : \mathbb{R}_{+}^{n} \rightarrow \mathbb{R}_{+}^{n} \) by

\[
\tilde{\Gamma}_{\text{ISS}}(s) := \left( \max_{j} (\gamma_{g}^{K_1})_{\text{ISS}}(s_j), \ldots, \max_{j} (\gamma_{g}^{K_n})_{\text{ISS}}(s_j) \right) \mathbf{T},
\]

\( \forall s \in \mathbb{R}_{+}^{n} \), with \((\gamma_{g}^{K})_{\text{ISS}} \equiv 0 \). We define \( \tilde{\Gamma}_{\text{ISS}} \) accordingly.

To prove the next theorem, which is the design of a quasi-ISS/ISDS reduced-order observer for the whole system we need the following: We say that \( \tilde{\Gamma} = \tilde{\Gamma}_{\text{ISS}} \) or \( \tilde{\Gamma} = \tilde{\Gamma}_{\text{ISS}} \) satisfies the small-gain condition if

\[
\tilde{\Gamma}(s) \leq s, \quad \forall s \in \mathbb{R}_{+}^{n} \setminus \{0\}.
\]

More details about this condition can be found in [14, 15], for example. For the proof of the following theorem we need the definition of an \( \Omega \)-path, see [15,24], for example.

**Definition 4.3.** A function \( \sigma = (\sigma_1, \ldots, \sigma_n)^{\mathbf{T}} : \mathbb{R}_{+}^{n} \rightarrow \mathbb{R}_{+}^{n} \), where \( \sigma_i \in \mathcal{X}_{\infty} \), is called an \( \Omega \)-path with respect to \( \tilde{\Gamma} \), if it possesses the following properties:

1. \( \sigma_i^{-1} \) is locally Lipschitz continuous on \((0, \infty)\);
2. For every compact set \( P \subset (0, \infty) \) there are finite constants \( 0 < K_1, K_2 \) such that for all points of differentiability of \( \sigma_i^{-1} \) we have
   \[
   0 < K_1 \leq (\sigma_i^{-1})'(r) \leq K_2, \quad \forall r \in P;
   \]
3. \( \tilde{\Gamma}(\sigma(r)) < \sigma(r), \quad \forall r > 0 \).

Note that if \( \tilde{\Gamma} \) satisfies the small-gain condition \((23)\), then there exists an \( \Omega \)-path \( \sigma = (\sigma_1, \ldots, \sigma_n)^{\mathbf{T}} \) with respect to \( \tilde{\Gamma} \)(see [15, Theorem 5.2 and [24]].

With these considerations a quasi-ISS/ISDS reduced-order observer is obtained in the following:

**Theorem 4.4.** Consider an interconnected system of the form \((19)\).
1. Assume that Assumption 4.1 point 1 and Theorem 4.2 point 1 hold true for each \( i = 1, \ldots, n \). If \( \tilde{\Gamma}_{\text{ISS}} \) satisfies the small-gain condition \((23)\), then the reduced-order error Lyapunov function \( V \) as in Assumption 3.5 is given by \( V = \max_{i} (\sigma_i^{-1}(V_i)) \) and the quasi-ISS reduced-order observer for the overall system is given by
   \[
   \dot{\hat{e}}_i = (\xi_i^{\mathbf{T}}, \ldots, \xi_n^{\mathbf{T}})^{\mathbf{T}},
   \]
   and
   \[
   \dot{\hat{z}}_i = f_j(\hat{y}, \dot{\hat{y}}) - l(\hat{y}), \quad u + \frac{\partial}{\partial z_i}(\hat{y})f_j(\hat{y}, \dot{\hat{y}}), \quad u
   \]
   \( \dot{\hat{z}}_i = \dot{\hat{y}} \), \( \dot{\hat{z}}_2 = \dot{\hat{y}} - l(\hat{y}) \).
2. Assume that Assumption 4.1 point 2 and Theorem 4.2 point 2 hold true for each \( i = 1, \ldots, n \). If \( \tilde{\Gamma}_{\text{ISS}} \) satisfies the small-gain condition \((23)\), then the reduced-order error Lyapunov function \( V \) as in Assumption 3.8 is given by \( V = \max_{i} (\sigma_i^{-1}(V_i)) \) and the quasi-SDS reduced-order observer for the overall system is given by \((26)\) and \((27)\).

**Proof.** Ad 1.: We define the error Lyapunov function candidate of the whole system by \( V(e) = \max_{i} (\sigma_i^{-1}(V_i(e_i))), e = (e_1, \ldots, e_n)^{\mathbf{T}} \). We have to verify the conditions from Assumption 3.5, from which with the help of Lemma 3.7 the observer defined by \((26)\) becomes a quasi-ISS reduced-order observer for the whole system of the form \((6)\).

Let \( I := \{ i \in \{1, \ldots, n\} | V(e) = \sigma_i(V(e_i)) \geq \max_{j \neq i} \sigma_j(V(e_j)) \} \). Fix an \( i \in I \). Let \( e_i := \xi_i - \xi_i \). From \((25)\) it follows

\[
V_i(e_i) = \sigma_i(V(e_i)) > \max_{j \neq i} (\gamma_{g}^{K_i})_{\text{ISS}}(\sigma_i(V(e_i))) = \max_{j \neq i} (\gamma_{g}^{K_i})_{\text{ISS}}(V_i(e_i)).
\]

We observe that there exist \( c_1 > 0, c_2 > 0 \) such that

\[
V(e) \geq \min_{i} (\alpha_i(\sigma_i(e_i))) =: \alpha_i(e), \quad \text{and}
\]

\[
V(e) \leq \max_{i} (\alpha_i(\sigma_i(e_i))) =: \sigma_i(e_i).
\]

Now, with \((24)\) it holds for each \( i \)

\[
\frac{\partial V(e)}{\partial e_i} = \left[ (\sigma_i^{-1}(V_i(e_i)))' \right] \frac{\partial V_i(e_i)}{\partial e_i} \leq \bar{K}_2 \alpha_4(V_i(e_i)) \leq \alpha_4(V(e)),
\]

where \( \alpha_4(r) := \min \bar{K}_2 \alpha_4(\sigma_i(r)), \bar{K}_2 \) is from \((24)\), i.e., \((8)\) is satisfied.

Furthermore, we have for each \( i \) that it holds

\[
\dot{V}(e) = (\sigma_i^{-1}(V_i(e_i)))' \frac{\partial V_i(e_i)}{\partial e_i} \leq -\alpha_3(V(e)) \leq -\alpha_3(\sigma_i^{-1}(V_i(e_i)) \leq -\alpha_3(\sigma_i^{-1}(V_i(e_i))).
\]

where \( \alpha_3(r) := \min \bar{K}_3 \alpha_3(\sigma_i(r)), \bar{K}_3 \) is from \((24)\), i.e., \((9)\) is satisfied.

Finally, we choose a function \( \alpha \in \mathcal{X}_{\infty} \), such that \( \alpha(\sigma_3(\sigma_i(e_i))) \leq \alpha_3(\sigma_i) \) and all the conditions from Assumption 3.5 are satisfied. By application of Lemma 3.7 our reduced-order observer of the whole system is quasi-ISS.

Ad 2.: We define

\[
\bar{V}(e) := \max_{i} \left( \sigma_i^{-1}(V_i(e_i)) \right),
\]

\[
\bar{\eta}(|e|) := \max_{i} \left( \sigma_i^{-1}(\eta_1(|e_i|)) \right), \quad \psi(|e|) := \min_{i} \sigma_i^{-1} \left( \frac{|e|}{\sqrt{1 + \varepsilon}} \right),
\]

where \( \bar{V} \) satisfies the conditions in Assumption 4.1 point 2. For \( i = 1, \ldots, n \). Let \( j \) be such that \(|e_i| = \bar{e}_j|e_j|\), \( \bar{e}_j \) is from \((24)\), i.e., \((9)\) is satisfied.

We have

\[
\max_{i} \sigma_i^{-1} \left( \frac{|e|}{1 + \varepsilon_i} \right) \geq \max_{i} \sigma_i^{-1} \left( \frac{|e|}{1 + \varepsilon_i} \right) \geq \max_{i} \sigma_i^{-1} \left( \frac{|e|}{\sqrt{1 + \varepsilon}} \right) \geq \min_{i} \sigma_i^{-1} \left( \frac{|e|}{\sqrt{1 + \varepsilon}} \right)
\]

where \( \varepsilon := \max, \varepsilon_i \) and we obtain

\[
\psi \left( \frac{|e|}{\sqrt{1 + \varepsilon}} \right) \leq \bar{V}(e) \leq \bar{\eta}(|e|).
\]

Then, \( \frac{\bar{V}(e)}{\sqrt{1 + \varepsilon}} \leq \alpha(\bar{V}(e)) \) holds with \( \alpha(r) := \max, \bar{K}_2 \sigma_3(r), r > 0 \). Furthermore, we have

\[
\dot{\bar{V}}(e) \leq -(1 - \varepsilon)\tilde{g}(\bar{V}(e))
\]
with $\varepsilon = \max \{ \varepsilon_i \}$ and $\hat{g}(r) := \min \tilde{K}_i |g_i(\sigma_i r), \ r > 0$, is positive definite and locally Lipschitz. From (30) we get
\[
\frac{|e|}{1 + \varepsilon} \leq \psi^{-1}(\psi(e)) \leq \psi^{-1}(\eta(|e|))
\]
and we define $V(e) := \psi^{-1}(\psi(e))$ as the reduced-order error Lyapunov function candidate with $\eta(|e|) := \psi^{-1}(\eta(|e|))$. By the previous calculations for $V(e)$ it holds
\[
V(\dot{e}) \leq -(1 - \varepsilon)\psi(V(e)) .
\]
where $g(r) := (\psi^{-1}'(\psi(r)) \hat{g}(\psi(r))$ is locally Lipschitz continuous. Altogether, $V(\dot{e})$ satisfies all conditions in Assumption 3.8. Hence, $V(e)$ is the reduced-order error Lyapunov function of the whole system and by application of Theorem 3.9 a quasi-ISDS reduced-order observer of the whole system can be obtained. □

**Remark 4.5.** Note that for large $n$ the function $\psi$ in (28) becomes "small" and hence the rates and gains of the quasi-ISDS property defined by $\psi^{-1}$ become "large", which is not desired in applications. To avoid this kind of conservativeness one can use the maximum norm $\| \cdot \|_\infty$, instead of the Euclidean one in the definitions above and in Theorem 4.4. In this case, the division by $\sqrt{n}$ in (29) can be avoided and we get (28) with $\psi(|e|_\infty) = \min \sigma_i^{-1}(|e|_\infty)$.

5. **Applications**

In this section, we investigate the stabilization of single and interconnected systems subject to quantization. At first, we consider single systems and combine the quantized output feedback stabilization with the ISDS property as in Chapter V in [1]. Then, we consider large-scale systems and give a counterpart to Proposition 1 in [1] for such kind of systems. Furthermore, we investigate dynamic quantizers, where the quantizers can be adapted by a zooming variable. This leads to asymptotic stability of the overall closed-loop system.

The question how to stabilize a system plays an important role in the analysis of control systems. The answer is a challenging one and the design of stabilizing feedback laws is a popular research area, which is linked up with many applications. In this section we are using quantized output feedback stabilization according to the results in [18,19,11]. A quantizer is a device, which converts a real-valued signal into a piecewise constant signal. It may affect the process output or may also affect the control input.

First, we consider one single system of the form (6). By an output quantizer we mean a piecewise constant function $q : \mathbb{R}^p \rightarrow \Omega$, where $\Omega$ is a finite subset of $\mathbb{R}^p$. The quantization error is denoted by
\[
d := q(y) - y , \tag{31}
\]
and the quantizer’s range $M > 0$ by $|y| \leq M$, which implies the error bound $|d| \leq \Delta, \Delta > 0$. This condition is referred to saturation in the literature.

Now, suppose that Assumption 3.8 holds and a quasi-ISDS observer has been designed as in Theorem 3.9. With $d$ as in (31) the observer acts on the quantized output measurements $\hat{y} = q(y)$. Furthermore, suppose that a controller and the observer is in the form $u = k(z)$. We can now define a quantized output feedback law by
\[
u = k(\hat{z}) = k(z + \hat{z}) ,
\]
where $\hat{z}$ is the state estimate generated by the observer and $\hat{z} := \hat{z} - z$ is the state estimation error. We impose on the feedback law:

**Assumption 5.1.** The system
\[
\dot{\hat{z}} = f(z, k(\hat{z})) = f(z, k(z + \hat{z}))
\]
is ISDS, i.e.,
\[
|z(t)| \leq \max \left\{ \hat{\mu}_i(\hat{\eta}(|z_i(t)|), t), \sup_{t \in [0, t]} \hat{\mu}_i(\hat{\eta}(|\hat{z}_i(t)|), t), \sup_{t \in [0, t]} \hat{\mu}_i(\hat{\eta}(|\hat{z}_i(t)|), t - t) \right\} \tag{32}
\]
for some $\hat{\mu}_i \in \mathcal{K}_L$, $\hat{\eta}$ and $\hat{\psi}_i^{\text{ISDS}} \in \mathcal{K}_\infty$.

For a detailed discussion for the case of an ISS controller we refer to [11]. The overall closed-loop system obtained by combining the plant, the observer and the control law can be written as
\[
\dot{\hat{z}} = f(z, k(z + \hat{z}))
\]
and we define $V(\dot{e}) := \psi^{-1}(\psi(e))$ as the reduced-order error Lyapunov function candidate with $\eta(|e|) := \psi^{-1}(\eta(|e|))$. By the previous calculations for $V(e)$ it holds
\[
V(\dot{e}) \leq -(1 - \varepsilon)\psi(V(e)) .
\]
where $g(r) := (\psi^{-1}'(\psi(r)) \hat{g}(\psi(r))$ is locally Lipschitz continuous. Altogether, $V(\dot{e})$ satisfies all conditions in Assumption 3.8. Hence, $V(e)$ is the reduced-order error Lyapunov function of the whole system and by application of Theorem 3.9 a quasi-ISDS reduced-order observer of the whole system can be obtained. □

**Remark 4.5.** Note that for large $n$ the function $\psi$ in (28) becomes "small" and hence the rates and gains of the quasi-ISDS property defined by $\psi^{-1}$ become "large", which is not desired in applications. To avoid this kind of conservativeness one can use the maximum norm $\| \cdot \|_\infty$, instead of the Euclidean one in the definitions above and in Theorem 4.4. In this case, the division by $\sqrt{n}$ in (29) can be avoided and we get (28) with $\psi(|e|_\infty) = \min \sigma_i^{-1}(|e|_\infty)$.
It can be verified, if Assumption 5.2 point 1 holds for the ith subsystem of the overall closed-loop system obtained by combining the plant, the observer and the control law that it holds
\[
\left| \frac{z_i(t)}{\xi_i(t)} \right| \leq \max \left\{ \beta_i \left( \frac{z_i^0}{\xi_i^0}, t \right), \max_{\beta_i} (y^K_i)_{\text{ISS}} \times (\|d_i\|_{[0,1.1]}, (y^K_i)_{\text{ISS}}(\|d_i\|_{[0,1.1]})) \right\}
\]
for \( \beta_i \in \mathcal{K}_\mathcal{L} \), \((y^K_i)_{\text{ISS}}, (y^K_i)_{\text{ISS}} \in \mathcal{K}_\infty, \|z_i\|_{[0,1.1]} \leq K_i, j = 1, \ldots, n \) and \( \|u_i\|_{[0,1.1]} = \|k_i(\hat{z}_i)\|_{[0,1.1]} \leq K_i \).

If Assumption 5.2 point 2 holds for the ith subsystem of the overall closed-loop system obtained by combining the plant, the observer and the control law, then it holds
\[
\left| \frac{z_i(t)}{\xi_i(t)} \right| \leq \max \left\{ \mu_i \left( \eta_i \left( \frac{z_i^0}{\xi_i^0} \right), t \right), \right. \\
\left. \max_{\nu_i} \nu_i(d_i(t), \nu_i(d_i(t)) \right\}
\]
for \( \mu_i \in \mathcal{K}_\mathcal{L} \), \( \eta_i \), \((y^K_i)_{\text{ISS}}, (y^K_i)_{\text{ISS}} \in \mathcal{K}_\infty, \|z_i\|_{[0,1.1]} \leq K_i, j = 1, \ldots, n \) and \( \|u_i\|_{[0,1.1]} = \|k_i(\hat{z}_i)\|_{[0,1.1]} \leq K_i \), where
\[
\nu_i(d_i(t), \nu_i(d_i(t)) := \text{ess sup} \mu_i((y^K_i)_{\text{ISS}}(d_i(t))), (d_i(t)), \nu_i(t), \nu_i(d_i(t)) = \text{ess sup} \mu_i((y^K_i)_{\text{ISS}}(d_i(t))), (d_i(t)), \nu_i(t).
\]

Furthermore, we can show that if the small-gain condition (23) is satisfied for \( R = ((y^K_i)_{\text{ISS}}_{n \times n} \) or \( R = ((y^K_i)_{\text{ISS}}_{n \times n} \), respectively, which defines a map as in (22), then for the overall system it holds
\[
1. \left| \frac{z(t)}{\xi(t)} \right| \leq \max \left\{ \beta_i \left( \frac{z_i^0}{\xi_i^0}, t \right), (y^K_i)_{\text{ISS}}(\|d_i\|_{[0,1.1]})) \right\}.
\]
\[
2. \left| \frac{z(t)}{\xi(t)} \right| \leq \max \left\{ \mu_i \left( \eta_i \left( \frac{z_i^0}{\xi_i^0} \right), t \right), \right. \\
\left. \max_{\nu_i} \nu_i(d_i(t), \nu_i(d_i(t)) \right\}
\]
Let \( \kappa_i^0 \in \mathcal{K}_\infty \) with the property \( |l_i(z_i)| \leq \kappa_i^0(|z_i|), \forall z_i, \kappa_i^0 \in \mathcal{K}_\infty \) such that \( |k_i(z_i)| \leq \kappa_i^0(|z_i|), \forall z_i \) and define \( K_i := \max \left\{ M_i, \kappa_i^0(2M_i + \Delta_i + \kappa_i^0(M_i + \Delta_i)) \right\} \). With \( z = (z_1^0, \ldots, z_n^0)^T \) in [1] for inter-connected systems.

**Proposition 5.3.** Let \( y^K_i \equiv (y^K_i)_{\text{ISS}} \equiv (y^K_i)_{\text{ISS}} \) and \( y^K_i \equiv (y^K_i)_{\text{ISS}} \equiv (y^K_i)_{\text{ISS}} \).

1. Assume, \( \max \{|y^K_i|, |y^K_i|, |y^K_i| \} \leq M_i \) and
\[
\left| \frac{z_i^0}{\xi_i^0} \right| < E_i^0
\]
where \( E_i^0 > 0 \) is such that \( \beta_i(0, 0) = \mu_i(M_i, 0) = M_i \). Then, the corresponding solution of the ith subsystem of the overall closed-loop system satisfies
\[
\limsup_{t \to \infty} \left| \frac{z_i(t)}{\xi_i(t)} \right| \leq \max \left\{ |y^K_i|, |y^K_i| \right\}.
\]
2. Assume that point 1 holds for all \( i = 1, \ldots, n \). Define \( M := \max M_i, \Delta := \max \Delta_i, K := \max K_i \), and suppose that \( \Gamma = (\gamma^K_i)_{n \times n} \) satisfies the small-gain condition (23). Then, the corresponding solution of the overall closed-loop system satisfies
\[
\limsup_{t \to \infty} \left| \frac{z(t)}{\xi(t)} \right| \leq \gamma^K(\Delta).
\]

**Proof.** Ad 1.: Note that it holds \( |y_i(t)| = |z_i(t)| \leq M_i, |q_i(z_i(t))| \leq |z_i(t)| \leq M_i \). As long as it holds
\[
\left| \frac{z(t)}{\xi(t)} \right| \leq M_i,
\]
we have \( |z_i(t)| \leq M_i \leq K_i \) and
\[
\left| \frac{z(t)}{\xi(t)} \right| = |\kappa_i(z_i(t))| \leq \kappa_i^0(|z_i(t)|) \leq \kappa_i^0(M_i + \Delta_i + K_i(M_i + \Delta_i)) \leq K_i.
\]

Define
\[
T := \sup \left\{ t \geq 0 : \left| \frac{z(t)}{\xi(t)} \right| < M_i \right\} \leq \infty.
\]
This is well-defined, since (37) and \( \beta_i(0, 0) \geq E_i^0 = \mu_i(M_i, 0) \) hold. It follows that (33) or (34), respectively, is true for \( t \in [0, T] \) and we obtain
\[
\left| \frac{z(t)}{\xi(t)} \right| < M_i, \quad \forall t \in [0, T]
\]
using the requirements of this proposition. Now, assume that \( T \) is finite. Therefore, there must exist a \( T^* \) such that
\[
\left| \frac{z(t)}{\xi(t)} \right| = M_i.
\]
But from (33) or (34), respectively, and from the above calculations it holds
\[
\left| \frac{z(t)}{\xi(t)} \right| < M_i,
\]
which contradicts the assumption that \( T \) is finite. It follows that \( T \) is infinite and from the fact that \( z_i \) and \( \hat{z}_i \) are continuous the estimation (40) holds for all \( t \geq 0 \).

Now, since \( \beta_i \in \mathcal{K}_\mathcal{L} \) for every \( \epsilon > 0 \) there exists \( T(\epsilon) \) such that
\[
\beta_i \left( \left| \frac{z(t)}{\xi(t)} \right|, t \right) \leq \epsilon, \quad \forall t \geq T(\epsilon)
\]
and therefore
\[
\left| \frac{z(t)}{\xi(t)} \right| \leq \max \left\{ \max_{i,j} \gamma^K_j(\Delta), \gamma^K_j(\Delta) \right\}, \forall t \geq T(\epsilon),
\]
which proves point 1.

Ad 2.: This follows by the same steps as for the proof of point 1 using (35) or (36), respectively, under the small-gain condition. □

Now, we are going to improve the mentioned results in order to get smaller bounds [38] and [39] using a dynamic quantizer, see [19,11].

Here, we obtain asymptotic convergence in (38) and (39) and we use the zooming-in strategy. We consider single systems of the form (6) and the dynamic quantizer
\[
q_i(y) = \lambda q_i \left( \frac{y}{\lambda} \right),
\]
where \( \lambda > 0 \) is the zoom variable. The range of this quantizer is \( M_i \) and \( \Delta \lambda \) is the quantization error. If we increase \( \lambda \), it is referred to the zooming-out strategy and corresponds to a larger range and quantization error and by decreasing \( \lambda \), we obtain a smaller
range and smaller quantization error, referred to the zooming-in strategy. The parameter $\lambda$ can be updated continuously, but the update of $\lambda$ for discrete instants of time has some advantages, see [19]. Using a discrete time the dynamics of the system change suddenly and is referred to hybrid feedback stabilization, which was investigated in [19].

Considering interconnected systems of the form (17) or (19), respectively, the dynamic quantizer of the $i$th subsystem is defined by

$$q_i^*(y_i) := \lambda_i q_i \left( \frac{y_i}{\lambda_i} \right),$$

$\lambda_i > 0$ with range $M_{z_i}$ and quantization error $\Delta_i \lambda_i$. Note that to get contraction of the bound (38) it must hold that

$$\max_{j \neq i} \max_{\Delta_i \lambda_i} (\gamma_{ij} (\Delta_i), \gamma_{ji} (\Delta_i)) \leq E_0$$

and

$$\beta_i \max_{j \neq i} \max_{\Delta_i \lambda_i} (\gamma_{ij} (\Delta_i), \gamma_{ji} (\Delta_i)), 0) = 0 < M_i.$$

Using this, we can find a $\lambda_i < 1$ such that

$$\beta_i \max_{j \neq i} \max_{\Delta_i \lambda_i} (\gamma_{ij} (\Delta_i), \gamma_{ji} (\Delta_i)), 0) = 0 < M_i.$$

For $E_i^0 > 0$ such that $\beta_i (E_i^0, 0) = \mu_i (M_i, 0) = M_i$, there is a time $t$ for which

$$\left( z(t)^{T} \bar{E}(t) \right) < E_i^0.$$

Define $K_i^z := \max \left\{ M_{z_i, k}^\nu (2M_i \lambda_i + \Delta_i \lambda_i + \kappa_i^j (M_{z_i} + \Delta_i \lambda_i) \right\}$ and applying the same analysis as in Proposition 5.3, we obtain a smaller bound

$$\max_{j \neq i} \max_{\Delta_i \lambda_i} (\gamma_{ij} (\Delta_i), \gamma_{ji} (\Delta_i))$$

(41)

Then, we can choose a smaller value of $\lambda_i$ and repeat the procedure. Theoretically, we can decrease $\lambda_i$ to 0 and obtain asymptotic contractiveness of the $i$th subsystem. Practically, the choice of the size of $\lambda_i$ depends on limitations, which determines the size of the bound (41), see [11]. The described procedure can be applied to the overall closed-loop system and we also obtain a smaller bound in (39) provided that $\Gamma = (\gamma_{ij}^{K_i})_{n \times n}$ Satisfies the small-gain condition. If we can decrease $\lambda_i$ to 0, for all $i = 1, \ldots, n$, we obtain asymptotic contractiveness of the overall closed-loop system. We summarize the observations in the following corollary:

**Corollary 5.4.** 1. Under a dynamic quantizer of the form $q_i^*(y_i) := \lambda_i q_i \left( \frac{y_i}{\lambda_i} \right), \lambda_i > 0$, it holds for the corresponding solution of the $i$th subsystem of the overall closed-loop system:

$$\lambda_i \rightarrow 0, \quad \Rightarrow \lim_{t \rightarrow \infty} \sup \left( z(t)^{T} \bar{E}(t) \right) \rightarrow 0.$$

2. Assume that point 2 holds for all $i = 1, \ldots, n$. Define $\lambda := \max \lambda_i$. If $\Gamma = (\gamma_{ij}^{K_i})_{n \times n}$ satisfies the small-gain condition, then for the corresponding solution of the overall closed-loop system it holds:

$$\lambda \rightarrow 0, \quad \Rightarrow \lim_{t \rightarrow \infty} \left( z(t)^{T} \bar{E}(t) \right) \rightarrow 0.$$

**Proof.** Ad point 1: Applying Proposition 5.3, point 1 by using the dynamic quantizer $q_i^*(y_i) = \lambda_i q_i \left( \frac{y_i}{\lambda_i} \right), \lambda_i > 0$, we obtain (41). If $\lambda_i \rightarrow 0$, the assertion follows.

Ad point 2: This follows, using Proposition 5.3, point 2. □

6. Conclusions

We have introduced the quasi-ISDS property for observers, which main advantage over ISS is the memory fading effect. This was demonstrated in an example. We have shown how to derive quasi-ISS/ISDS reduced-order observers for subsystems of interconnected systems. They were used to design a quasi-ISS/ISDS reduced-order observer for the overall system under a small-gain condition.

As an application we have shown that quantized output feedback stabilization for a subsystem is achievable, under the assumptions that the subsystem possesses a quasi-ISS/ISDS reduced-order observer and a state feedback controller providing ISS/ISDS with respect to measurement errors. If this holds for all subsystems of the large-scale system and the small-gain condition is satisfied, then quantized output feedback stabilization is also achievable for the overall system. The obtained bounds can be improved by using dynamic quantizers. We have shown that asymptotic convergence can be achieved for each subsystem and for the overall system provided that a small-gain condition is satisfied.

Future works are for example the investigation of the design of nonlinear output feedback control or nonlinear observers to satisfy the small-gain condition and the application of the results in this paper to the design of dynamic quantized interconnected control systems. Also the investigation of the observers used in this paper for systems with time-delays is of interest for future research as far as time-delays occur in many real-world applications.

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**References**


