On input-to-state stability for stochastic coupled control systems on networks

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Abstract

This paper investigates the input-to-state stability (ISS) properties for stochastic coupled control systems on networks (SCCSNs). With the help of Lyapunov method and graph theory, some inequality techniques and stochastic analysis skills, some sufficient criteria are obtained for $e^{\lambda t}$-weighted integral ISS in mean and almost sure exponential ISS of SCCSNs. These sufficient criteria are closely related to the topological property of the network anatomy. Finally, an example is given to illustrate the results.

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1. Introduction

Coupled control systems on networks have come to play an important role in many fields, such as artificial intelligence, neural networks, engineering, etc. [1–4]. Stability analysis for coupled control systems on networks is one of the most important problems in control theory and engineering [5–8]. However, coupled control systems on networks in many applications are often perturbed by environmental noise in the real world. Some tiny noise often lead to the failure of stability for coupled control systems on networks. Hence, stability analysis for stochastic coupled control systems on networks (SCCSNs) have attracted the attention of many researchers in the past few years and lots of results have been reported (see [9–13]).

On the other hand, control systems are often perturbed by controls and errors on observations. Thus, it is desirable for a control system not only to be stable, but also to display Sontag’s well-known ‘input-to-state stability’ (ISS) properties. The precise definition of ISS was introduced by Sontag in the later 1980s [14], and then he established some basic results [15,16]. Roughly speaking, so-called ISS means that the behavior of control systems should remain bounded when its inputs are bounded, and should tend to equilibrium when inputs tend to zero. In recent decades, ISS analysis of nonlinear control system has become one of the most active topic in nonlinear feedback analysis and design. Lots of important properties for ISS have been obtained by many researchers [17–22], and then these results have been applied in engineering, observer design, new small-gain theorems and control theory [23–26]. Thus, it is a necessary and meaningful task to discover the ISS properties for SCCSNs.

Among the methods that contributed to investigate stability of coupled control systems on networks, the Lyapunov method is the main one. Usually, linear matrix inequality and Lyapunov function are used to acquire stability criteria for coupled control systems on networks. However, how to construct an appropriate Lyapunov function for a specific coupled control systems on networks is still a difficult issue and this is the disadvantage of Lyapunov method. Recently, Li et al. investigated the global stability for a general coupled systems of differential equations on networks by graph theory [27,28]. By applying Kirchhoff’s matrix tree theorem in graph theory, a systematic approach was given to constructed Lyapunov function for coupled systems on networks. This technology has been successfully employed in the global stability for many mathematic models on networks,
such as, coupled oscillators model [29,30], multi-group model [31,32], and neural networks [33], etc. Moreover, this technology was also extended to many different systems, such as stochastic system [11,34,35], discrete-time system [36,37], and delay system [29,37,38]. But, to the best of the authors’ knowledge, few scholars use this technology to investigate ISS properties for SCCSNs.

Motivated by the above discussions, in this paper, we aim to investigate ISS properties for SCCSNs. Some sufficient criteria ensuring the $e^{t\alpha}$-weighted integral ISS in mean and almost sure exponential ISS for SCCSNs are established by applying Lyapunov method and graph theory, some inequality techniques, and stochastic analysis skills. Finally, an example is provided to illustrate our results.

The main contribution of this paper is as follows.

1. We apply graph theory and Lyapunov method to investigate ISS properties for SCCSNs.

2. By employing Kirchhoff’s matrix tree theorem in graph theory, some sufficient criteria are established which are closely related to the topological property of the network anatomy.

The rest of this paper is organized as follows. Some notations and preliminary results are given in next section. In Section 3, based on Lyapunov method and graph theory, the $e^{t\alpha}$-weighted integral ISS in mean and almost sure exponential ISS for SCCSNs are investigated. In Section 4, a numerical test is used to illustrate the results obtained in this paper. Finally, some conclusion remarks are drawn in Section 5.

2. Notations and preliminary results

Throughout this paper, $R$ denotes the set of all real numbers. Let $R_+ = [0, +\infty)$ and $Z_+ = \{1, 2, \ldots\}$. Set $R^n$ and $R^n \times m$ denote the $n$-dimensional real space and $n \times m$-dimensional real matrix space, respectively. The transpose of vectors and matrices are denoted by superscript “T”. For vector $x = (x_1, \ldots, x_n)^T \in R^n$. \|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ denotes the Euclidean norm. Let $L = \{1, 2, \ldots, l\}$. $M = \sum_{i=1}^{l} m_i (m_i \in Z^+)$, and $N = \sum_{i=1}^{l} n_i (n_i \in Z^+)$. Let $C^2_\Omega(R^n \times R_+; R)$ denote the family of all real-valued functions $V(x, t)$ defined on $R^n \times R_+$ such that they are continuously twice differentiable in $x$ and once in $t$. A function $\alpha : R_+ \rightarrow R_+$ is of class $\kappa$, if $\alpha$ is continuous, strictly increasing and $\alpha(0) = 0$. If $\alpha$ is also unbounded, then it is of class $\kappa_\infty$.

Let $w(t)$ be a one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with $\Omega$ being a sample space, $\mathcal{F}$ being a $\sigma$-field, $\{\mathcal{F}_t\}_{t \geq 0}$ being a filtration satisfying the usual conditions and $\mathbb{P}$ being a probability measure. The mathematical expectation with respect to the given probability measure $\mathbb{P}$ is denoted by $\mathbb{E}(\cdot)$.

Then, we introduce some basic concepts on graph theory [39,40]. A directed graph or digraph $G = (H, E)$ contains a set $H = \{1, 2, \ldots, l\}$ of vertices and a set $E$ of arcs $(i, j)$ leading from initial vertex $i$ to terminal vertex $j$. A subgraph $H$ of $G$ is said to be spanning if $H$ and $G$ have the same vertex set. A digraph $G$ is weighted if each arc $(j, i)$ is assigned a positive weight $a_{ij}$. In our convention, $a_{ij} > 0$ and only if there exists an arc from vertex $j$ to vertex $i$ in $G$. The weight $W(H)$ of a subgraph $H$ is the product of the weights on all its arcs. A directed path $P$ in $G$ is a subgraph with distinct vertices $\{i_1, i_2, \ldots, i_m\}$ such that each of its sets of arcs is $\{(i_k, i_{k+1}) : k = 1, 2, \ldots, m - 1\}$. If $i_m = i_1$, we call $P$ a directed cycle. A connected subgraph $T$ is a tree if it contains no cycles, directed or undirected. A tree $T$ rooted at vertex $i$, called the root, if $i$ is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A subgraph $Q$ is unicyclic if it is a disjoint union of rooted trees whose roots form a directed cycle. Given a weighted digraph $G$ with $l$ vertices, define the weight matrix $A = (a_{ij})_{l \times l}$, whose entry $a_{ij}$ equals the weight of arc $(j, i)$ if it exists, and $0$ otherwise. Denote the directed graph with weight matrix $A$ as $(G, A)$. A digraph $G$ is strongly connected if for any pair of distinct vertices, there exists a directed path from one to the other. A weighted digraph $(G, A)$ is said to be balanced if $W(C) = W(-C)$ for all directed cycles $C$. Here, $-C$ denotes the reverse of $C$ and is constructed by reversing the direction of all arcs in $C$. For a unicyclic graph $Q$ with cycle $C_Q$, let $\hat{Q}$ be the unicyclic graph obtained by replacing $C_Q$ with $-C_Q$. Suppose that $(G, A)$ is balanced, then $W(Q) = W(\hat{Q})$. The Laplacian matrix of $(G, A)$ is defined as

$$L = \begin{pmatrix}
\sum_{k \neq 1} a_{1k} & -a_{12} & \cdots & -a_{1l} \\
-a_{21} & \sum_{k \neq 2} a_{2k} & \cdots & -a_{2l} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{l1} & -a_{l2} & \cdots & \sum_{k \neq l} a_{lk}
\end{pmatrix}.$$  

The following result is standard in graph theory, and customarily called Kirchhoff’s matrix tree theorem [39].

**Lemma 1.** Assume that $l \geq 2$. Let $c_i$ denote the cofactor of the $i$th diagonal element of $L$. Then

$$c_i = \sum_{T \in \mathcal{T}_i} W(T), \quad i \in \mathbb{L},$$

where $\mathcal{T}_i$ is the set of all spanning trees $T$ of $(G, A)$ that are rooted at vertex $i$, and $W(T)$ is the weight of $T$.

**Lemma 2.** [27]. Assume that $l \geq 2$. Let $c_i$ denote the cofactor of the $i$-th diagonal element of $L$. Then the following identity holds:

$$\sum_{i,j=1}^{l} c_{ij} F_{ij}(x_i, x_j) = \sum_{Q \in \mathcal{Q}} W(Q) \sum_{(i, r) \in \epsilon(C_Q)} F_{ir}(x_r, x_i).$$
3. Model description and main results

We consider SCCSNs based on a digraph $G$ with $(l \geq 2)$ vertices. The arc of digraph $G$ represents the interaction between two dynamic vertices of the networks. In the $i$-th vertex ($i \in L$), assign a nonlinear control system, whose dynamics is described by a $n_i$-dimensional stochastic differential equation (see Fig. 1):

$$dx_i(t) = f_i(x_i(t), u_i(t))dt + g_i(x_i(t), u_i(t))dw(t),$$

where $x_i \in R^{n_i}$, $u_i \in R^{n_i}$ are the system state and input, respectively, and $w(t)$ is a one-dimensional Brownian motion. Control or input $u_i : R_+ \rightarrow R^{n_i}$ are measurable locally essentially bounded functions. Here, $f_i, g_i : R^{n_i} \times R^{n_i} \rightarrow R^{n_i}$ are called the drift coefficient and diffusion coefficient. Let $P_{ij}, Q_{ij} : R^{n_i} \times R^{n_i} \rightarrow R^{n_i}$, $i, j \in L$, represent the influence of vertex $j$ on vertex $i$ for drift coefficient $f_i$ and diffusion coefficient $g_i$, respectively, and $P_{ij} = Q_{ij} = 0$ if there exists no arc from $j$ to $i$ in $G$. Thus, by replacing $f_i$ and $g_i$ by $f_i + \sum_{j=1}^{l} P_{ij}$ and $g_i + \sum_{j=1}^{l} Q_{ij}$, we get the following SCCSNs (see Fig. 2).

$$dx_i(t) = \left[f_i(x_i(t), u_i(t)) + \sum_{j=1}^{l} P_{ij}(x_j(t), x_j(t))\right]dt + \left[g_i(x_i(t), u_i(t)) + \sum_{j=1}^{l} Q_{ij}(x_i(t), x_j(t))\right]dw(t), \quad i \in L. \quad (2)$$

In this paper, we suppose that both $\sum_{j=1}^{l} P_{ij}$ and $\sum_{j=1}^{l} Q_{ij}$ satisfy certain conditions such that there exists a unique solution $x(t; x_0, u)$ to system (2) with initial data $x_0 = x(0) \in R^{n_i}$ and input $u = (u_1, \ldots, u_l)^T \in R^{n_i}$. For simplicity we write $x(t) = x(t; x_0, u)$ and suppose that $x = 0$ is a trivial solution of system (2). For any given $V_i(x_i, t) \in C^{2,1}(R^{n_i} \times R_+; R_+)$, define an operator $\mathcal{L}V_i$ from $R^{n_i} \times R_+$ to $R$ by

$$\mathcal{L}V_i(x_i, t) = \frac{\partial V_i(x_i, t)}{\partial t} + \frac{\partial V_i(x_i, t)}{\partial x_i} \left[f_i(x_i, u_i) + \sum_{j=1}^{l} P_{ij}(x_j, x_j)\right] + \frac{1}{2} \text{trace} \left[\left[g_i(x_i, u_i) + \sum_{j=1}^{l} Q_{ij}(x_i, x_j)\right]^T \frac{\partial^2 V_i(x_i, t)}{\partial (x_i)^2} \left[g_i(x_i, u_i) + \sum_{j=1}^{l} Q_{ij}(x_i, x_j)\right]\right].$$
where
\[
\frac{\partial V_i(x_i, t)}{\partial x_i} = \left(\frac{\partial V_i(x_i, t)}{\partial x_i^{(1)}}, \ldots, \frac{\partial V_i(x_i, t)}{\partial x_i^{(n)}}\right), \quad \frac{\partial^2 V_i(x_i, t)}{\partial (x_i^j)^2} = \left(\frac{\partial^2 V_i(x_i, t)}{\partial x_i^{(k)} \partial x_i^{(l)}}\right)_{n_i \times n_i}.
\]

The purpose of this paper is to develop \(e^{\lambda t}\)-weighted integral ISS in mean and almost sure exponential ISS for system (2). Now, let us introduce the two definitions as follows.

**Definition 1.** The system (2) is said to be almost sure exponential ISS if

\[
\limsup_{t \to \infty} \frac{1}{t} \ln(\|x(t)\|) < 0 \quad \text{a.s.}
\]

for all \(x_0 \in \mathbb{R}^n\) and \(u \in \mathbb{R}^m\).

This is well known that Lyapunov method plays a crucial role in the study of stability problem. However, it is very difficult to construct an appropriate Lyapunov function for system (2). In order to construct an appropriate Lyapunov function for system (2), we first give the concept of ISS-vertex-Lyapunov functions.

**Definition 3.** Functions \(V_i(x_i, t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+^+)\) for \(i \in I\) are called ISS-vertex-Lyapunov functions for system (2) if the following holds:

**U1.** There exist functions \(\alpha_1^{(i)}, \alpha_2^{(i)} \in \kappa_\infty\), such that

\[
\alpha_1^{(i)}(\|x_i\|) \leq V_i(x_i, t) \leq \alpha_2^{(i)}(\|x_i\|).
\]

**U2.** For any \(j \in I\) and \(u_i \in \mathbb{R}^n\), there exist constants \(\alpha_j > 0, \lambda > 0, \Psi_i \in \kappa_\infty\) and functions \(F_j(x_i, x_j)\) such that

\[
\mathcal{L}V_i(x_i, t) \leq -\lambda V_i(x_i, t) + \sum_{j=1}^l a_{ij}F_j(x_i(t), x_j(t)) + \Psi_i(\|u_i(t)\|).
\]

**U3.** Along each directed cycle \(C\) of weighted digraph \((G, A)\), for all \(x_i \in \mathbb{R}^n, x_j \in \mathbb{R}^n\) there is

\[
\sum_{(i,j) \in E(C)} F_j(x_i, x_j) \leq 0. \tag{3}
\]

The main results of this paper are presented as follows.

**Theorem 1.** Suppose that system (2) admits ISS-vertex-Lyapunov functions \(V_i(i \in I)\), and digraph \((G, A)\) is strongly connected in which \(A = (a_{ij})_{I \times I}\), then system (2) is \(e^{\lambda t}\)-weighted integral ISS in mean for \(\lambda > 0\).

**Proof.** Clearly it holds if \(x_0 = 0\). So we only need to show it for \(x_0 \neq 0\). Let \(V(x, t) = \sum_{i=1}^n c_i V_i(x_i, t)\), where \(c_i\) denotes the cofactor of the \(i\)th diagonal element of \(L\). The property of strong connectedness of digraph \((G, A)\) implies that \(c_i > 0\) for any \(i \in I\). By the definition of \(\mathcal{L}V_i(x_i, t)\), we can easily get \(\mathcal{L}V_i(x_i, t) = \sum_{i=1}^n c_i \mathcal{L}V_i(x_i, t)\). Fix such \(x_0\), for each integer \(k \geq 1\), define a sequence of stopping time

\[
\tau_k = \inf\{t \geq 0 : \|x(t)\| \geq k\}.
\]

Obviously, \(\tau_k \to \infty\) almost surely as \(k \to \infty\). By Itô’s formula, one can show that

\[
d(e^{\lambda t}V(x, t)) = e^{\lambda t}[\lambda V(x, t) + \mathcal{L}V(x, t)]dt + e^{2\lambda t} \sum_{i=1}^n c_i \frac{\partial V_i(x_i, t)}{\partial x_i} \left( g_i(x_i, u_i) + \sum_{j=1}^l Q_{ij}(x_i, x_j) \right) dw(t).
\]

So we have

\[
e^{\lambda (t + \tau_k)}V(x(t + \tau_k), t + \tau_k) = V(x_0, 0) + \int_0^{t + \tau_k} e^{\lambda s} [\lambda V(x(s), s) + \mathcal{L}V(x(s), s)]ds
\]

\[+ \int_0^{t + \tau_k} e^{\lambda s} \sum_{i=1}^n c_i \frac{\partial V_i(x_i(s), s)}{\partial x_i} \left( g_i(x_i(s), u_i(s)) + \sum_{j=1}^l Q_{ij}(x_i(s), x_j(s)) \right) dw(s).
\]
Then we get
\[
\mathbb{E}[e^{\lambda(t \wedge T_k)}V(x(t \wedge T_k), t \wedge T_k)] = V(x_0, 0) + \mathbb{E} \int_0^{T_k} e^{\lambda s}[\lambda V(x(s), s) + \mathcal{L}V(x(s), s)]ds
\]
\[
+ \mathbb{E} \int_0^{T_k} e^{\lambda s} \sum_{i=1}^l c_i \frac{\partial V_i(x_i(s), s)}{\partial x_i} \left( g_i(x_i(s), u_i(s)) + \sum_{j=1}^l Q_{ij}(x_i(s), x_j(s)) \right) dw(s)
\]
\[
= V(x_0, 0) + \mathbb{E} \int_0^{T_k} e^{\lambda s}[\lambda V(x(s), s) + \mathcal{L}V(x(s), s)]ds.
\]
Letting \( k \to \infty \) gives
\[
\mathbb{E}[e^{\lambda s}V(x(t), t)] = V(x_0, 0) + \sum_{i=1}^l c_i \mathbb{E} \int_0^t e^{\lambda s}[\lambda V_i(x_i(s), s) + \mathcal{L}V_i(x_i(s), s)]ds
\]
where \( \tilde{V}_1 = \max_{i \in \mathbb{K}} \{ \alpha_2^{(0)}(\cdot) \} \). On the other hand, we can easily find a convex function \( \tilde{\alpha}_1 \in \mathbb{K} \) such that \( \tilde{\alpha}_1(\cdot) \leq \alpha_1^{(0)}(\cdot) \). So using the property of convex function we obtain that
\[
V(x, t) \geq \sum_{i=1}^l c_i \tilde{\alpha}_1(\|x_i\|)
\]
\[
\geq \sum_{i=1}^l c_i \tilde{\alpha}_1(\|x_i\|)
\]
\[
= \sum_{i=1}^l c_i \sum_{j=1}^l \left( \frac{c_i \tilde{\alpha}_1(\|x_i\|)}{\sum_{k=1}^l c_k} \right)
\]
\[
\geq \sum_{i=1}^l c_i \tilde{\alpha}_1 \left( \frac{\min_{i \in \mathbb{K}} (\|x_i\|)}{\sum_{k=1}^l c_k} \right)
\]
Thus, by writing
\[
\tilde{\alpha}_1(\cdot) = \sum_{i=1}^l c_i \tilde{\alpha}_1 \left( \frac{\min_{i \in \mathbb{K}} (\|x_i\|)}{\sum_{k=1}^l c_k} \right), \quad \tilde{\alpha}_2(\cdot) = \sum_{i=1}^l c_i \tilde{\alpha}_2(\cdot)
\]
we get
\[
\tilde{\alpha}_1(\|x\|) \leq V(x, t) \leq \tilde{\alpha}_2(\|x\|).
\]
By (4), (5), and condition U2, it yields that
\[
e^{\lambda s} \mathbb{E}\tilde{\alpha}_1(\|x(t)\|) \leq \tilde{\alpha}_2(\|x_0\|) + \mathbb{E} \int_0^t e^{\lambda s} \left[ \sum_{i=1}^l c_i \Psi_i(\|u_i(s)\|) \right] + \sum_{i,j=1}^l c_{ij} F_{ij}(x_i(s), x_j(s)) \] \[ \] \[ ds. \]
In view of Lemma 2, condition U3, and the fact \( W(\Xi) \geq 0 \), we have
\[
e^{\lambda s} \mathbb{E}\tilde{\alpha}_1(\|x(t)\|) \leq \tilde{\alpha}_2(\|x_0\|) + \mathbb{E} \int_0^t e^{\lambda s} \sum_{i=1}^l c_i \Psi_i(\|u_i(s)\|) ds + \mathbb{E} \int_0^t e^{\lambda s} \left[ \sum_{\Xi \in \mathbb{X}} W(\Xi) \sum_{i,j \in \mathbb{K}} F_{ij}(x_i(s), x_j(s)) \right] ds
\]
\[
\quad \leq \tilde{\alpha}_2(\|x_0\|) + \int_0^t e^{\lambda s} \tilde{\Psi}(\|u(s)\|) ds,
\]
where \( \tilde{\Psi}(\cdot) = \sum_{i=1}^l c_i \max_{i \in \mathbb{K}} \{ \Psi_i(\cdot) \} \).
So, system (2) is $e^{\kappa t}$-weighted integral ISS in mean for $\lambda > 0$ with $\alpha_1(\cdot) := \alpha_1(\cdot)$, $\alpha_2(\cdot) := \alpha_2(\cdot)$, and $\Psi(\cdot) := \gamma(\cdot)$. The proof is complete. □

Now, we give the definition of $e^{\kappa t}$-weighted ISS in mean for SCSSNs as follows.

**Definition 4.** The system (2) is $e^{\kappa t}$-weighted ISS in mean for some $\lambda > 0$, if there exist $\alpha_1$, $\alpha_2$, $\gamma \in \kappa_\infty$ such that for any $u \in \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$, we have

$$e^{\kappa t}E[\alpha_1(||x(t)||)] \leq \alpha_2(||x_0||) + \sup_{s \in [0,t)} \{e^{\kappa s}\gamma(||u(s)||)\}.$$ 

**Remark 1.** Moreover, we can know that system (2) is $e^{\kappa t}$-weighted ISS in mean for $\lambda_0 \in (0, \lambda)$ from Theorem 1. The proof is very similar to that of Theorem 1, so we omit the details here.

**Remark 2.** In Theorem 1, we let digraph $(G, A)$ be strongly connected. Moreover, if we suppose digraph $(G, A)$ is balanced, then

$$\sum_{i,j=1}^n c_i a_{ij} F_{ij}(x_i, x_j) = \frac{1}{2} \sum_{Q \in \Phi} W(Q) \sum_{(i,j) \in E(Q)} (F_{ij}(x_i, x_j) + F_{ji}(x_j, x_i)).$$

Hence, (3) could be replaced by

$$\sum_{(i,j) \in E(Q)} (F_{ij}(x_i, x_j) + F_{ji}(x_j, x_i)) \leq 0. \quad (7)$$

So, we get the following corollary from Theorem 1.

**Corollary 1.** Suppose that digraph $(G, A)$ is balanced. Then the conclusion of Theorem 1 holds if (3) is replaced by (7).

**Remark 3.** We also note that if there exist functions $T_i$ and $T_j$ for every $i, j \in L$, such that

$$F_{ij}(x_i, x_j) \leq T_i(x_i) - T_j(x_j). \quad (8)$$

Then (3) clearly holds. So we can easily obtain Corollary 2 as follows.

**Corollary 2.** The conclusion of Theorem 1 holds if (3) is replaced by (8).

In the remainder of this section, we will discuss almost sure exponential ISS for system (2). This property is very important in control theory and neural networks [41–44].

**Theorem 2.** Let digraph $(G, A)$ be strongly connected. If system (2) admits ISS-vertex-Lyapunov functions $V_i(x_i, t), \chi_i \in \kappa_\infty, i \in L$ with

- $\alpha_1^i(||x_i||) = a_i ||x_i||^p$, where $p \geq 2$ and $a_i$ are positive constants.
- $V_2$. For any $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$, $\chi_i(||u_i||) \leq ||x_i||$.

Then system (2) is almost sure exponential ISS.

**Proof.** Because digraph $(G, A)$ is strongly connected, then $c_i > 0$ for any $i \in L$. Let $V(x, t) = \sum_{i=1}^l c_i V_i(x_i, t)$, where $c_i$ denotes the cofactor of the $i$-th diagonal element of $L$. By the definition of $L V_i(x_i, t)$, we can easily get $L V(x, t) = \sum_{i=1}^l c_i L V_i(x_i, t)$. By Itô’s formula we can derive that

$$d \ln V(x, t) = \left[ \frac{\sum_{i=1}^l c_i \partial V_i(\chi_i, t)}{V(x, t)} - \frac{1}{V^2(x, t)} \sum_{i=1}^l c_i \frac{\partial V_i(x_i, t)}{\partial x_i} \left( g_i(x_i, u_i) + \sum_{j=1}^l Q_{ij}(x_i, x_j) \right) \right] dt + \frac{1}{V(x, t)} \sum_{i=1}^l c_i \frac{\partial V_i(x_i, t)}{\partial x_i} \left( g_i(x_i, u_i) + \sum_{j=1}^l Q_{ij}(x_i, x_j) \right) dw(t).$$

This, together with conditions U2 and V2, implies

$$\ln V(x, t) \leq \ln V(x_0, 0) + \int_0^t \frac{1}{V(x(s), s)} \sum_{i=1}^l c_i \left[ -\lambda V_i(x_i(s), s) + c_i \partial V_i(\chi_i(s), s) + \sum_{j=1}^l Q_{ij}(x_i(s), x_j(s)) \right] ds + \Psi_i(||u_i(t)||) ds + \int_0^t \frac{1}{V(x(s), s)} \sum_{i=1}^l c_i \frac{\partial V_i(x_i(s), s)}{\partial x_i} \left( g_i(x_i(s), u_i(s)) + \sum_{j=1}^l Q_{ij}(x_i(s), x_j(s)) \right) dw(s) - N(t) \leq \ln V(x_0, 0) - \lambda t + \int_0^t \frac{1}{V(x(s), s)} \sum_{i=1}^l c_i a_{ij} F_{ij}(x_i(s), x_j(s)) ds.$$
This implies for all 0 holds for all 0 is a continuous martingale with initial value. This together with (9), implies Making use of Lemma 2, condition (9) and the fact W(Q) ≥ 0, we further obtain

\[ N(t) = \int_0^t \left( \sum_{i=1}^l c_i \frac{\partial V_i(x(s), t)}{\partial x_i} \right) \left( g_i(x(s), u(s)) + \sum_{j=1}^l Q_{ij}(x(s), x_j(s)) \right) \, ds. \]

Making use of Lemma 2, condition (U3) and the fact W(Q) ≥ 0, we further obtain

\[ \sum_{i=1}^l c_i a_{ij} F_{ij}(x(s), x_j(s)) = \sum_{Q \in \mathcal{Q}} W(Q) \sum_{(i,j) \in E \cap \mathcal{G}_Q} F_{ij}(x(s), x_j(s)) \leq 0. \]

This together with (9), implies

\[ \ln V(x, t) \leq \ln V(x_0, 0) - \frac{\lambda t}{2} + M(t) - N(t), \quad (10) \]

where

\[ M(t) = \int_0^t \left( \sum_{i=1}^l c_i \frac{\partial V_i(x(s), s)}{\partial x_i} \right) \left( g_i(x(s), u(s)) + \sum_{j=1}^l Q_{ij}(x(s), x_j(s)) \right) \, ds. \]

is a continuous martingale with initial value M(0) = 0. Assign \( \varepsilon \in (0, 2) \) arbitrarily and let \( n \in \mathbb{Z}^+ \). By the exponential martingale inequality, we have

\[ \mathbb{P} \left( \sup_{0 \leq t \leq n} \left[ M(t) - \frac{\varepsilon}{2} N(t) \right] > \frac{2}{\varepsilon} \log n \right) \leq \frac{1}{n^2}. \]

Applying the Borel–Cantelli lemma we see that for almost all \( \omega \in \Xi \), there is an integer \( n_0 = n_0(\omega) \) such that if \( n \geq n_0 \),

\[ M(t) \leq \frac{\varepsilon}{2} N(t) + \frac{2}{\varepsilon} \log n \]

holds for all \( 0 \leq t \leq n \). Substituting this into (10), we obtain that

\[ \ln V(x, t) \leq \ln V(x_0, 0) - \frac{\lambda t}{2} + \frac{2}{\varepsilon} \ln n - \left( 1 - \frac{\varepsilon}{2} \right) N(t) \leq \ln V(x_0, 0) - \frac{\lambda t}{2} + \frac{2}{\varepsilon} \ln n \]

for all \( 0 \leq t \leq n, n \geq n_0 \) almost surely. Consequently, for almost all \( \omega \in \Xi \), if \( n - 1 \leq t \leq n \) and \( n \geq n_0 \),

\[ \frac{1}{t} \ln V(x, t) \leq -\frac{\lambda}{2} + \frac{1}{n - 1} \left[ \ln V(x_0, 0) + \frac{2}{\varepsilon} \ln n \right]. \]

This implies

\[ \limsup_{t \to \infty} \frac{1}{t} \ln V(x, t) \leq -\frac{\lambda}{2} \quad \text{a.s.} \quad (11) \]

Finally, using condition (V1) we get

\[ V(x, t) = \sum_{i=1}^l c_i V_i(x_i, t) \geq \sum_{i=1}^l c_i a_i \|x_i\|^p \]
4. Numerical test

The proof is complete. □

Combining (11) and (12), we obtain that
\[
\limsup_{t \to \infty} \frac{1}{t} \ln \|x(t)\| \leq -\frac{\lambda}{2p} < 0 \text{ a.s.}
\]
The proof is complete.

To close this section, let us make three important remarks.

Remark 4. Coupled control systems on networks are inherently difficult to understand as structural complexity. We focus on the relationship between the network anatomy and the global dynamics of the network. In other words, how does the network anatomy influence the global dynamics of the network? Theorems 1 and 2 desire the strong connectedness of the network which is a condition on topological property of the network anatomy. Recently, some scholars [44] have explored global dynamics of coupled systems on networks when the network is not strongly connected. It is worth discussing ISS properties for SCCSNs when the network is not strongly connected in the future.

Remark 5. From the proof of Theorems 1 and 2, we construct the Lyapunov function \( V(x, t) \) for SCCSNs as
\[
V(x, t) = \sum_{i=1}^{3} c_i V_i(x_i, t).
\]
It is closely associated with the weight of digraph \((G, A)\) and the ISS-vertex-Lyapunov functions \(V_i(x_i, t)\). Hence, discovering the ISS-vertex-Lyapunov functions \(V(x_i, t)\) is a strategic point in the study of ISS properties for SCCSNs. This condition is not difficult to be satisfied in practice. In fact, coupled control systems on networks are very complex. To make some improvement, many large-scale control systems have lessened certain complications. The simple and nearly identical nonlinear control systems are coupled together in regular ways. These simplifications could avoid some problems of structural complexity and the system’s potentially formidable dynamics of some coupled control systems could be studied intensively. In different application fields, ISS-vertex-Lyapunov \(V(x_i, t)\) has not a uniform form of constructing. Thus, we are interested in the SCCSNs which could be derived by coupling some classical control systems whose ISS-vertex-Lyapunov functions \(V_i(x_i, t)\), \(i \in \mathbb{L}\), are known or constructed without difficulty.

Remark 6. Recently, coupled control systems on networks have attracted the attention of many scholars. In [45, 46], Jin et al. investigated a class of complex networks with non-ideal coupling networks. With the help of adaptive control techniques, asymptotic synchronization problem for a class of nonlinearly coupled complex networks against network deterioration has been solved. How to use the results in this paper to efficiently solve stability problem for some non-ideal coupling networked systems will be the topic of future research.

4. Numerical test

We demonstrate the usefulness and applicability of the developed theory by means of a numerical simulation example. Given a digraph \( G \) with 3 vertices, we get the following SCCSNs:
\[
dx_i(t) = \left[ f_i(x_i, u_i) + \sum_{j=1}^{3} P_{ij}(x_i, x_j) \right] dt + \left[ g_i(x_i, u_i) + \sum_{j=1}^{3} Q_{ij}(x_i, x_j) \right] dw(t), \quad i = 1, 2, 3,
\]
where \( x_i \in \mathbb{R}^1, u_i \in \mathbb{R}^1 \) are the system state and input, respectively, and \( w(t) \) is a scalar Brownian motion. For convenience, we choose
\[
f_i(x_i, u_i) = -2x_i + u_i, \quad g_i(x_i, u_i) = x_i + u_i, \quad P_{ij}(x_i, x_j) = x_j - x_i, \quad Q_{ij} = 0.
\]
Let $V_i(x_i, t) = x_i^2$. Then we compute $\mathcal{L} V_i(x_i, t)$ as follows.

$$
\mathcal{L} V_i(x_i, t) = 2x_i \left[ f_i(x_i, u_i) + \sum_{j=1}^{3} P_{ij}(x_i, x_j) \right] + \left[ g_i(x_i, u_i) + \sum_{j=1}^{3} Q_{ij}(x_i, x_j) \right]^2
$$

$$
= 2x_i \left[ -2x_i + u_i + \sum_{j=1}^{3} (x_j - x_i) \right] + (u_i + x_i)^2
$$

$$
\leq -4x_i^2 + 3(u_i^2 + u_i^2) + \sum_{j=1}^{3} 2x_i(x_j - x_i)
$$

$$
\leq -x_i^2 + 3u_i^2 + \sum_{j=1}^{3} (x_j^2 - x_i^2)
$$

$$
= -\lambda V_i(x_i, t) + \sum_{j=1}^{3} a_{ij} F_{ij}(x_i, x_j) + \Psi_i(||u_i||).
$$

where $\lambda = 1$, $\Psi_i(||u_i||) = 3u_i^2$, $a_{ij} = 1$ and $F_{ij}(x_i, x_j) = x_j^2 - x_i^2$. We could check that all conditions in Theorem 1 are satisfied. Thus system (13) is $e^t$-weighted integral ISS in 2-th mean. Now we denote $x_1(0) = 2$, $x_2(0) = 3$, $x_3(0) = 4$ as the initial conditions associated with system (13). We can get $\mathbb{E} \|x(t)\|^2 \leq 180.25e^{-t} + 60.75$ when input $u_i = 1.5$ (see Fig. 3), and $\mathbb{E} \|x(t)\|^2 \leq 241e^{-t}$.
when input $u_i = 0$ (see Fig. 4). From Fig. 3, we can see that system state $x$ remain bounded, when input $u$ remain bounded. From Fig. 4, we can see that system state $x$ converge to zero when input $u = 0$. Moreover, let $p = 2, \alpha_i^0(\|x_i\|) = 0.5x_i^2$, and $\chi_i(\|z\|) = 0.3\|z\|$. We choose $x_1(0) = 0.13, x_2(0) = 0.15, x_3(0) = 0.17$ as the initial value of system (13). It is easy to see that conditions $V_1$ and $V_2$ hold. Hence, system (13) is almost sure exponential ISS (see Figs. 5–7).
5. Conclusion

In this paper, we investigate the ISS properties for SCCSNs. Combining graph theory and Lyapunov method we have obtained some sufficient criteria for $e^{\lambda t}$-weighted integral ISS in mean and almost sure exponential ISS of SCCSNs. These sufficient criteria are closely related to the topological property of the network anatomy. A numerical example has been given to illustrate the effectiveness of our main results. Moreover, our approach in this paper can be used to investigate ISS properties for coupled control systems on networks perturbed by some other types of environmental noise, such as SCCSNs with Markovian switching and SCCSNs with Lévy noise, etc. How to use our approach to solve these problems will be the topic of future research.

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