Interdisciplinary Evaluation of Mathematical Methods for Signal and Image Processing

Zentrum für Technomathematik
Universität Bremen

DFG SPP 1114, Third annual meeting,
September 2004

Overview

- Organization SPP
- Test Examples
- Mathematical Analysis (D. Lorenz)

Organization SPP

- T. Köhler 01.08.2001 – 30.09.2003
- S. Fischer 01.10.2003 – 30.06.2004
- D. Lorenz, H. Thielemann 01.07.2004

Test Examples

- Basic Problems
- Complex Applications
- Research

`www.math.uni-bremen.de/zetem/DFG-Schwerpunkt`

Mathematical Analysis of Different Denoising Methods

Interpolation between soft and hard wavelet shrinkage

Dirk A. Lorenz
Zentrum für Technomathematik

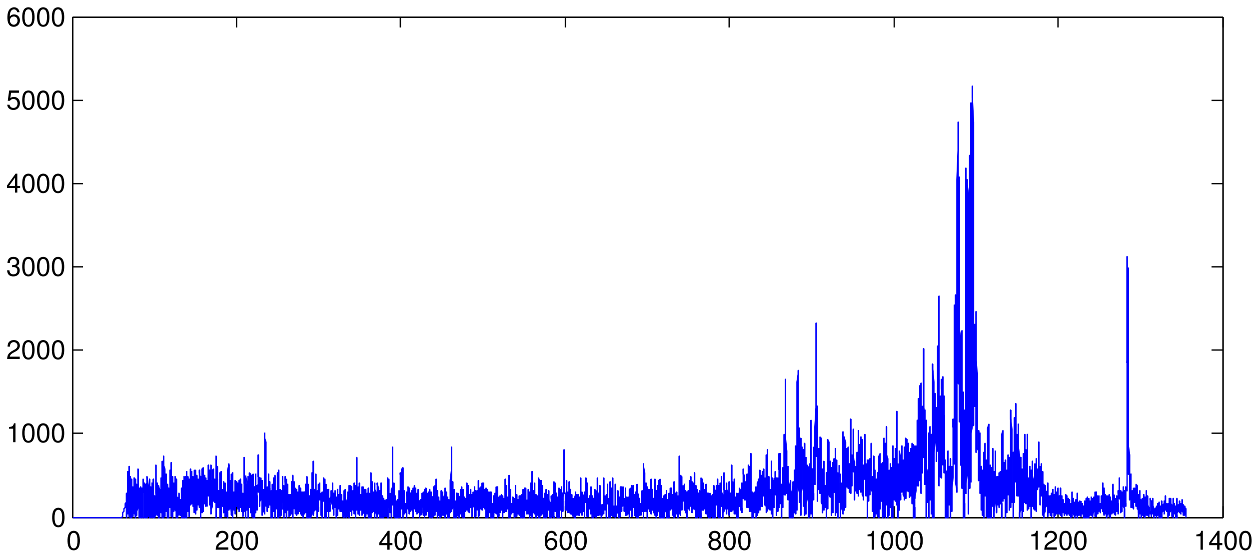
Overview

1. Nonlinear wavelet methods for denoising: soft and hard thresholding
2. Relations to variational methods for both of them
3. Interpolation between soft and hard thresholding

Nonlinear wavelet filtering

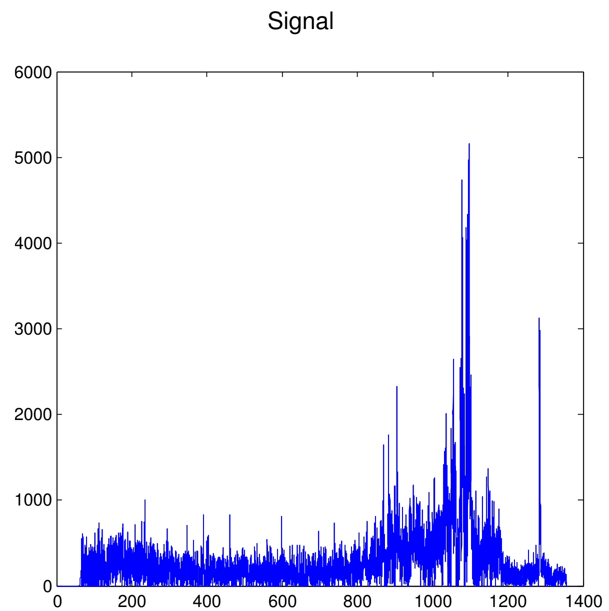
Problem: Data is noisy!

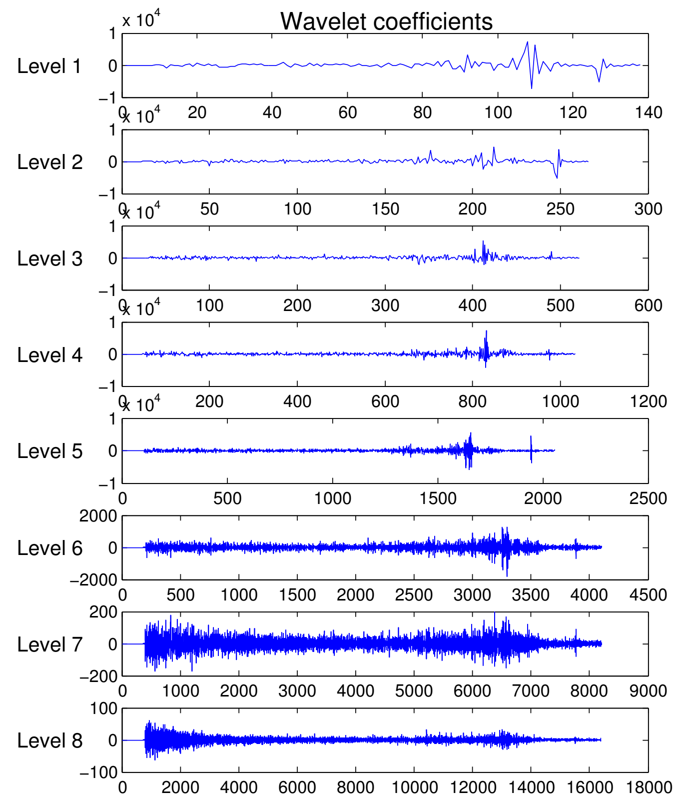
Example: Data from spectrum analysis of blood for cancer diagnosis by proteomics.



Goal: Separate and detect peaks, measure area under the peaks.
Denoising is an important step in preprocessing.

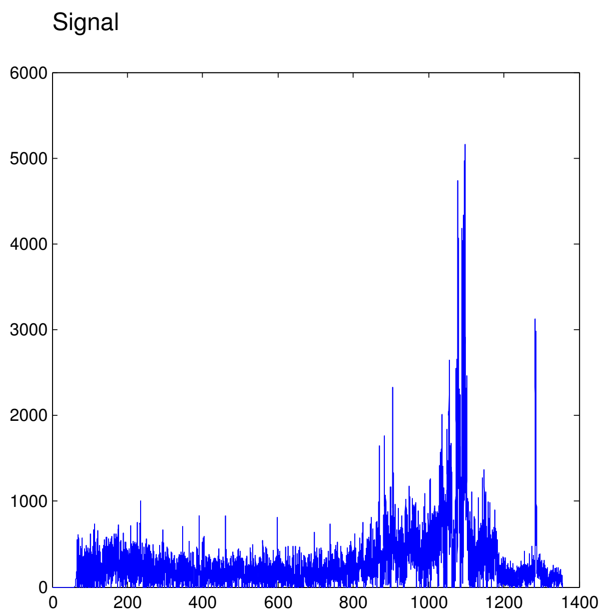
Noisy signal and wavelet transform

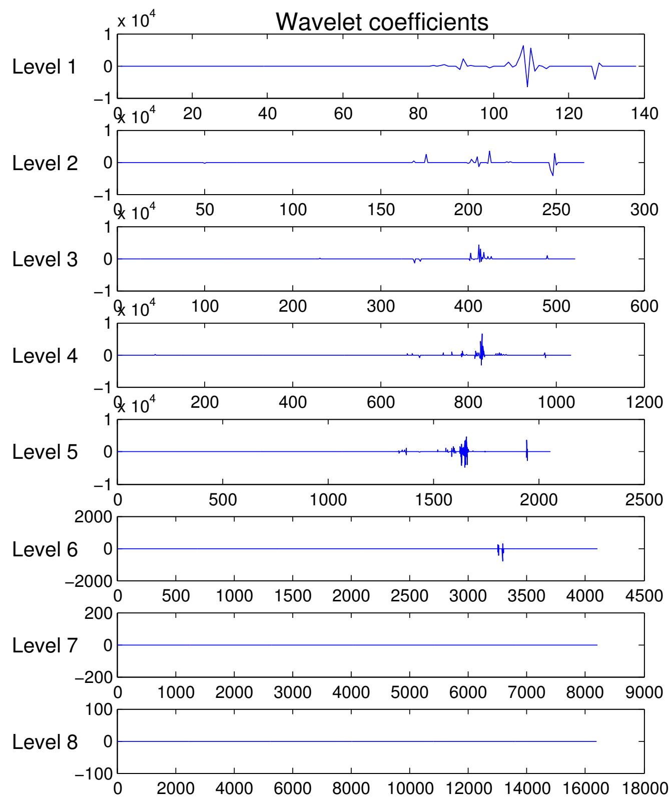


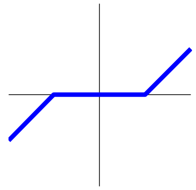


Idea: Reduce small wavelet coefficients.

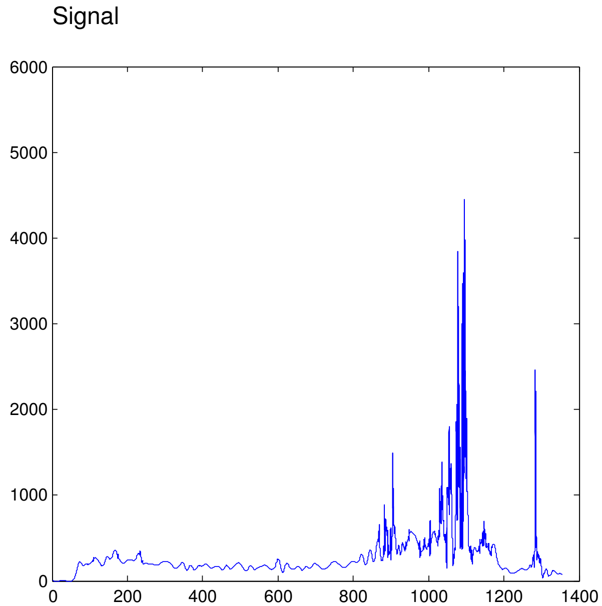
Soft shrinkage of the coefficients

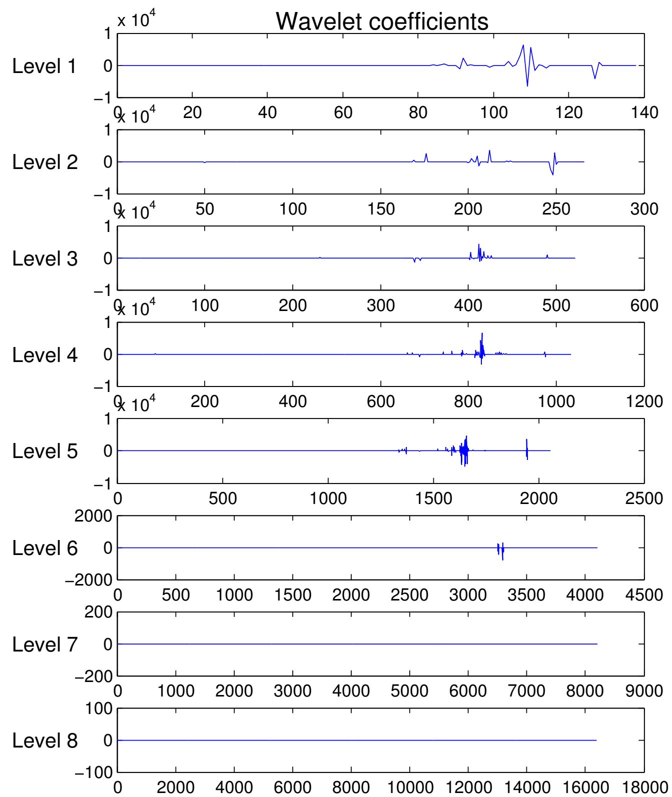


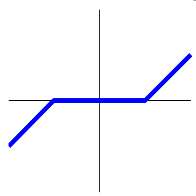


The soft shrinkage function:  $S_\lambda(x) = (|x| - \lambda)_+ \operatorname{sign} x$

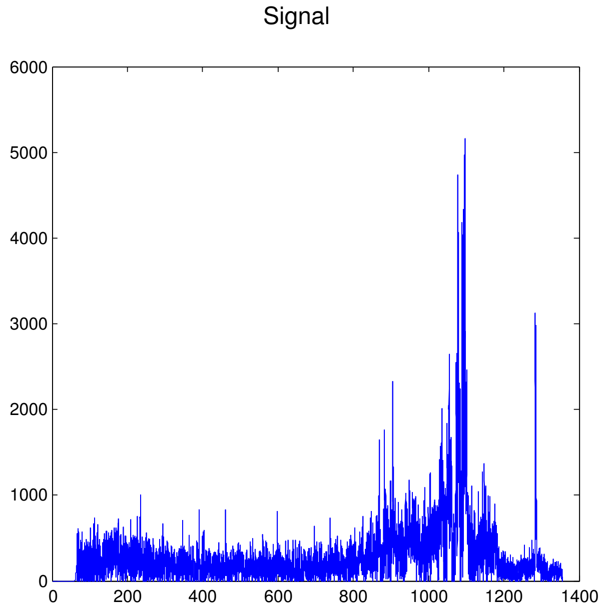
Denoising by soft shrinkage

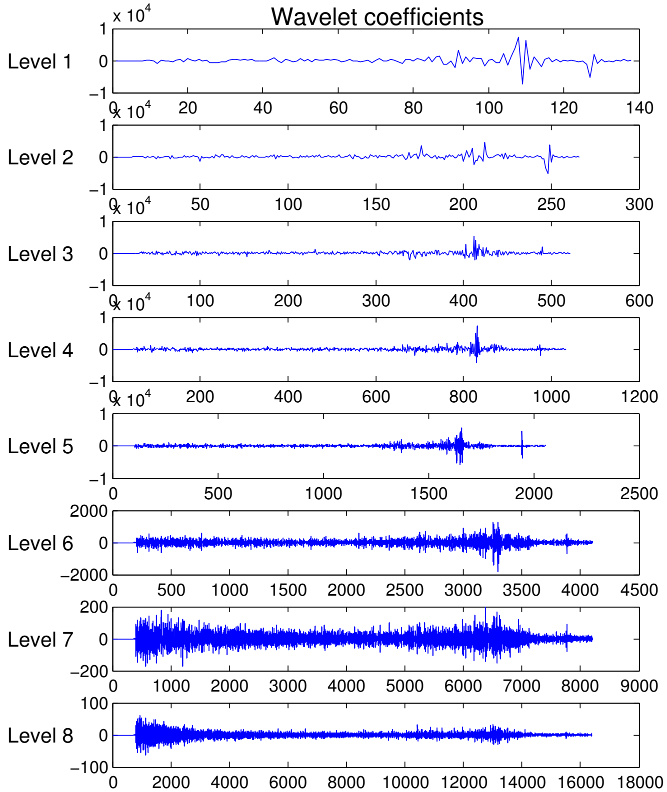




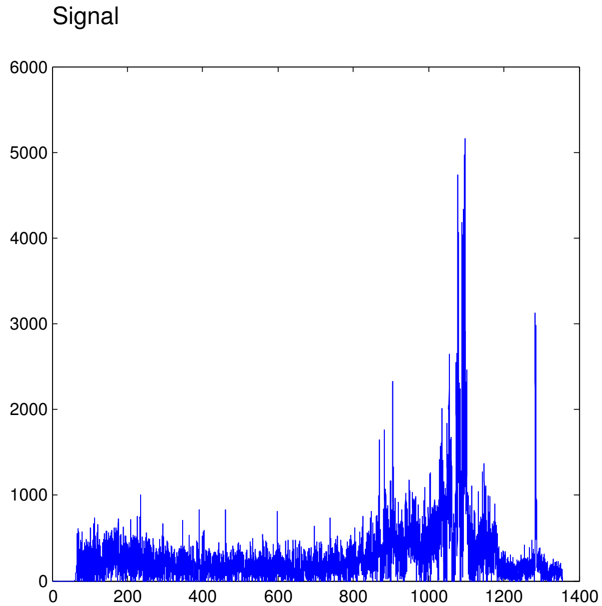
The soft shrinkage function:  $S_\lambda(x) = (|x| - \lambda)_+ \operatorname{sign} x$

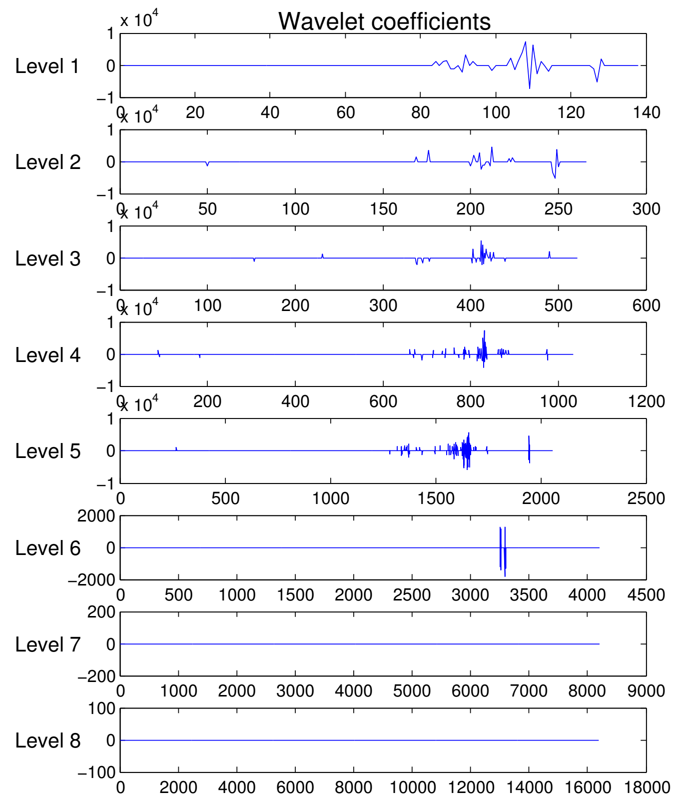
Noisy signal and wavelet transform



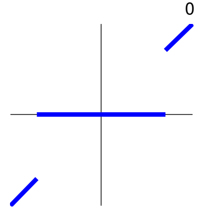


Hard shrinkage of the coefficients



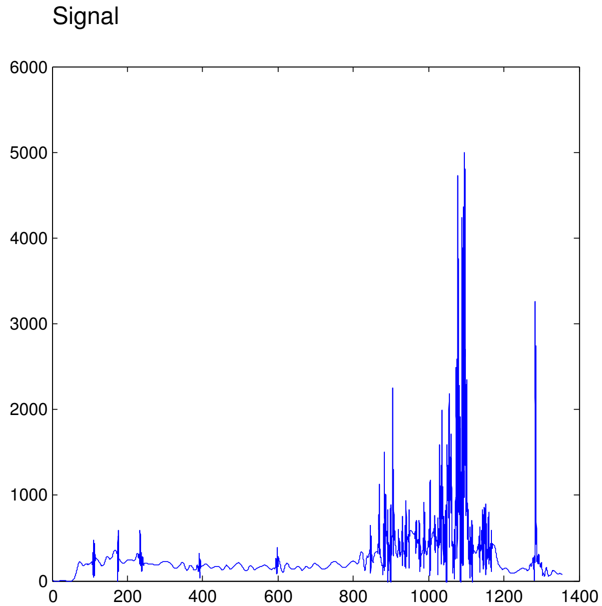


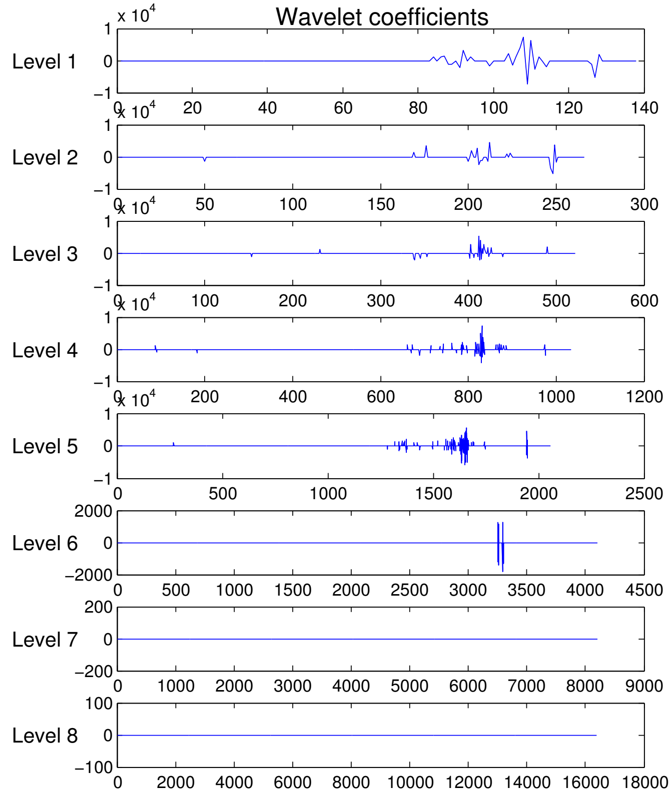
The hard shrinkage function:

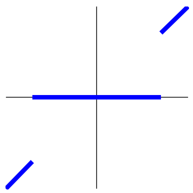


$$S_\lambda(x) = x(1 - \chi_{[-\lambda,\lambda]}(x))$$

Denoising by hard shrinkage





The hard shrinkage function:  $S_\lambda(x) = x(1 - \chi_{[-\lambda,\lambda]}(x))$

Motivation for soft shrinkage

It is well known, the shrinkage methods perform very well. (Asymptotic optimality, shown by Johnstone and Donoho.)

But why does they work so well? Is there a mathematical motivation for shrinkage methods?

The soft shrinkage is the solution of a variational problem.

Theorem: [Chambolle et al. 1998]:
The soft shrinkage of a function f, defined by

$$u = \sum_{j,k} S_\lambda(\langle f \,|\, \psi_{j,k}\rangle)\psi_{j,k}$$

is the minimizer of the functional

$$\|u - f\|_2^2 + 2\lambda |u|_{B^{d/2}_{1,1}}.$$

Shrinkage of single coeffcients: soft shrinkage

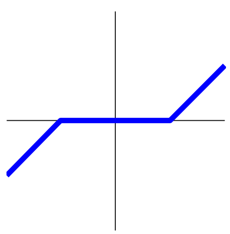
Besov spaces can be characterized through the decay of wavelet coefficients.

The variational problem $\bar{u} = \operatorname{argmin}_u \|u - f\|_{L^2}^2 + 2\lambda |u|_{B_{p,p}^s}^p$
for $s = d/2$, $p = 1$ gives a minimization problem for each wavelet coefficient:

$$\bar{u}_{j,k} = \underset{u_{j,k}}{\operatorname{argmin}} (u_{j,k} - f_{j,k})^2 + 2\lambda |u_{j,k}|, \qquad u_{j,k} = \langle u | \, \psi_{j,k} \rangle \, .$$

The shrinkage function appears as

$$S_\lambda : x \mapsto \underset{w}{\operatorname{argmin}} (x - w)^2 + 2\lambda \phi(w) \, :$$

$\phi(x) = |x|$:  $\rightsquigarrow$ soft shrinkage

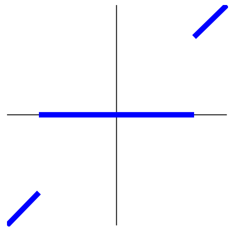
Shrinkage of single coefficients: Hard shrinkage

The variational problem

$$\bar{u} = \underset{u}{\operatorname{argmin}} \, \|u - f\|_{L^2}^2 + 2\lambda \# \{u_{j,k} \neq 0\}$$

is solved by

$$\bar{u}_{j,k} = S_{\sqrt{2\lambda}}(f_{j,k})$$

Shrinkage function S_λ:  $\rightsquigarrow$ hard shrinkage

The above functional can be reformulated. For every coefficient we have:

$$\underset{u_{j,k}}{\operatorname{argmin}} (u_{j,k} - f_{j,k})^2 + 2\lambda\phi(u_{j,k}) \text{ where } \phi(x) = x^0 = \begin{cases} 1 & , x \neq 0 \\ 0 & , x = 0. \end{cases}$$

Interpolation between soft and hard shrinkage

The soft and hard shrinkage are “extremal” cases of the minimizer of the functional

$$\operatorname*{argmin}_u \sum_{j,k} (u_{j,k} - f_{j,k})^2 + 2\lambda\phi(u_{j,k})$$

with

$$\phi(x) = |x|^p$$

$p = 1$: Soft shrinkage
$p = 0$: Hard shrinkage

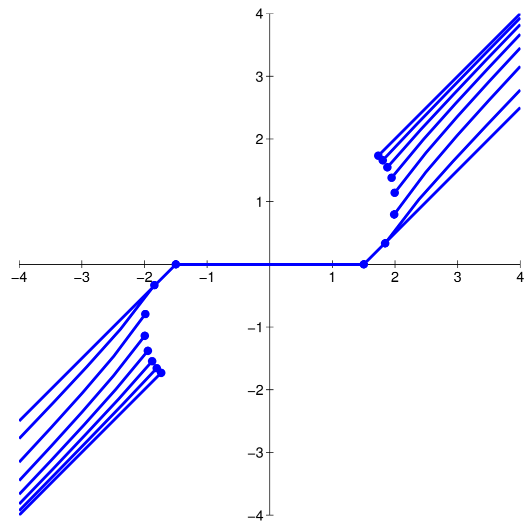
What happens for $0 < p < 1$?

Interpolation between soft and hard shrinkage

Minimization of the functional

$$\operatorname*{argmin}_u \sum_{j,k} (u_{j,k} - f_{j,k})^2 + 2\lambda |u_{j,k}|^p$$

yields in shrinkage of the coefficients with different shrinkage functions which look like



$p = 0, 0.15, 0.3, 0.45, 0.6, 0.75, 0.9, 1,\ \lambda = 1.5.$

Interpolation between soft and hard shrinkage

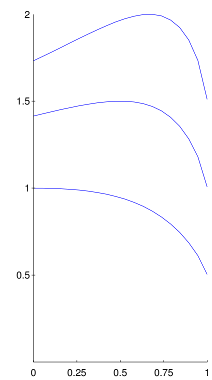
Explicit formula for the shrinkage functions:

For $0 < p < 1$ the shrinkage function is given by

$$S_\lambda(x) = 0 \text{ for } |x| \leq \lambda_{\text{eff}} = \frac{2-p}{2-2p}\Big(2\lambda(1-p)\Big)^{\frac{1}{2-p}}$$

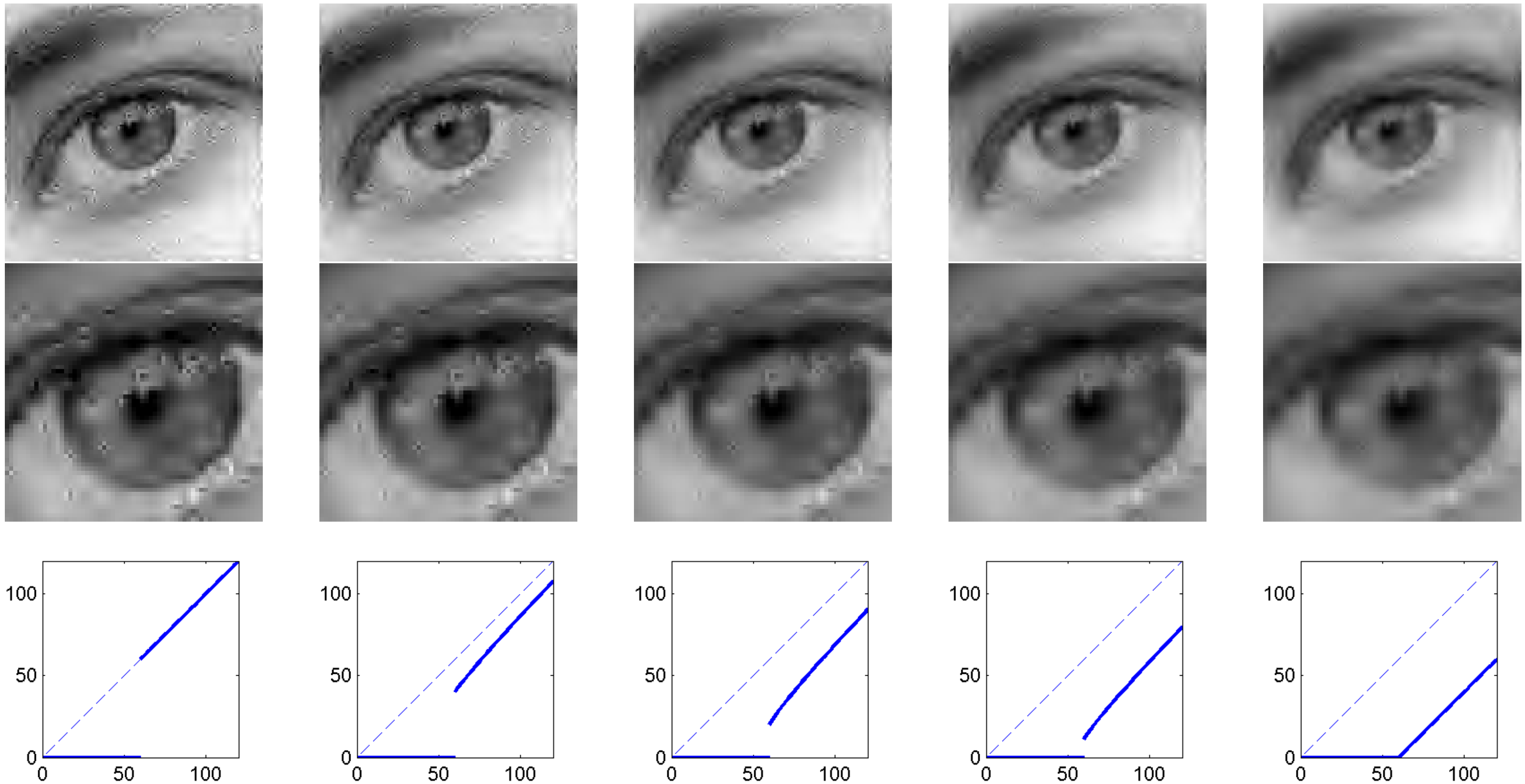
$$S_\lambda(x) = \begin{cases} \text{the value of largest absolute value of the inverse} \\ \text{mapping of } x \mapsto x + \lambda p |x|^{p-1} \operatorname{sign}(x) \text{ for } |x| \geq \lambda_{\text{eff}}. \end{cases}$$

The effective threshold λ_{eff} for $\lambda = 0.5, 1, 1.5$



Illustration

Illustration



$p = 0$ $p = 0.1$ $p = 0.2$ $p = 0.5$ $p = 1$

Summary

- Soft wavelet shrinkage is the solution of a variational problem with a Besov constraint $B^{d/2}_{1,1}$ or ℓ^1 constraint.
- Hard wavelet shrinkage corresponds to an "ℓ^0" constraint.
- $0 < p < 1$ results in an interpolation between hard and soft shrinkage in a "natural" way.