A Control Problem in Medical Image Processing

Analysis of a Generalized Conditional Gradient Method

Kristian Bredies, Dirk A. Lorenz, Peter Maaß

Zentrum für Technomathematik – Universität Bremen

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Outline

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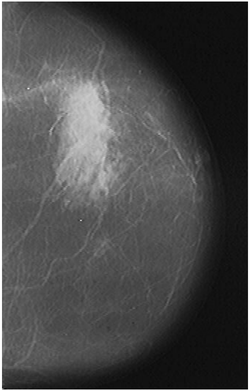
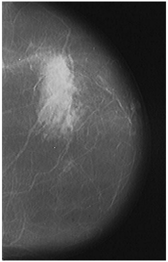
Mammography screening

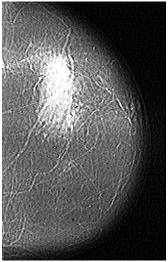


- Examination of X-ray scans for cancer diagnosis
- Mammographie Projekt Bremen
- DFG Priority Program 1114

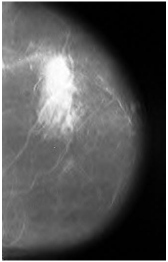
The examination: a sweep across scales



y_0 original



y_f fine scales



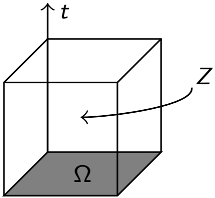
y_c coarse scale

Display: $y(t),\ t \in [0,1],\ y(0) = y_0$

$$y^*(t) = \begin{cases} y_f, & t \in [.2, .4] \\ y_c, & t \in [.6, .8] \end{cases}$$

Outline

Control problem



$$(*)\begin{cases} y_t - \Delta y &= u \\ \partial_\nu y &= 0 \text{ on } \partial\Omega,\ \Omega = [0,1]\times[0,1] \\ y(0,x) &= y_0(x) \end{cases}$$

$$\min_{u \in C_{\text{ad}}} \int |y(t,x) - y^*|^2 dt dx + \alpha \|u\|^2_{L^2(Z)}$$

subject to

- $u \in C_{\text{ad}}$, e. g. $-1 \leq u \leq 1$
- y solves $(*)$

Solution operator and optimal control problem

$$y_t - \Delta y = u,\ y(0) = y_0,\ \partial_\nu y = 0 \quad (*)$$

$$A: \begin{array}{ccc} L^2(Z) & \to & L^2(Z) \\ u & \mapsto & y \text{ solves } (*) \end{array}$$

compact operator (affine linear)

$$\min \|Au - y^*\|^2 + \alpha\|u\|^2$$
$$\text{subject to } u \in C_{\text{ad}}$$

Different tasks in different areas

$$J(u) = \|Au - y^*\|_X^2 + \alpha\|u\|_Y^p$$

Control problems:

- A (semi-)linear PDE
- L^2, H^s, L^∞
- Control: source terms

Inverse problems / image processing:

- A compact operator
- Choice of function spaces
- Choice of α
- $y_t - \operatorname{div}(p\nabla y) = u$
  diffusion parameters (edge adaptive)

Choice of function spaces

Image model:

$$\begin{array}{ccccccc} & \text{objects} & + & \text{texture} & + & \text{noise} \\ f = & u & + & v & + & w \end{array}$$

objects: $\|u\|_{BV} \sim$ length of edges
$B^1_{1,1} \subset BV \subset B^1_{1,\infty}$
(Y. Meyer, S. Osher, L. Vese)

texture: $v = \text{div}(\phi)$,
$\phi \in L^\infty$ vector field

noise: $w \in L^2, \; H^{-1}$

Close to image processing

$$J(u) = \|Au - y^*\|^2 + 2\alpha\|u\|^p_{B^s_{p,p}}, \; p \geq 1, \; s \geq 0$$

Daubechies, Defrise, De Mol, 2004

$\|u\|^p_{B^s_{p,p}} \asymp \sum w_{s,p,j}|\langle u|\psi_j\rangle|^p$, with $\{\psi_j\}$ wavelet base

Classical cases:

1. $p = 2$, A compact (regularization of inverse problems)
2. $p \neq 2$, $A = I$ (denoising in image processing)

Outline

1. Motivation: A control problem in medical imaging
   - Examination of mammograms
   - Formulation as a control problem
2. Algorithms from different fields
   - The method of surrogate functionals
   - The generalized conditional gradient method
3. Equivalences of methods
   - Equivalence: GCG $\iff$ surrogate functionals
4. Application, Summary
   - Application to mammography images
   - Summary

Inverse problems with non-quadratic penalties

Task: Minimize functionals of the form

$$J(u) = \|Au - y^*\|^2 + 2\alpha\|u\|^p_{B^s_{p,p}}, \; p \geq 1, \; s \geq 0.$$

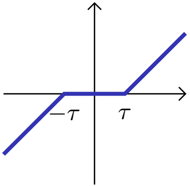
Problems:

1. A is compact (i. e. not stable invertible)
2. the penalty is not quadratic (i. e. the normal equation is not linear)

Deal with both: Surrogate functionals.

Tool for minimization: shrinkage

$$S_\tau(z) = \begin{cases} z - \tau & , \ z > \tau \\ 0 & , \ -\tau \leq z \leq \tau \\ z + \tau & , \ z < -\tau \end{cases}$$



soft shrinkage

Wavelet basis expansion:

$$u = \sum_j \langle u | \psi_j \rangle \psi_j$$

Shrinking the expansion:

$$\mathbf{S}_\tau(u) = \sum_j S_\tau(\langle u | \psi_j \rangle) \psi_j$$

The method of surrogate functionals

Theorem

The iteration

$$u_{n+1} = \mathbf{S}_{\alpha w,p}(u_n - A^*(Au_n - y^*))$$

converges strongly to a minimizer of

$$J(u) = \|Au - y^*\|^2 + 2\alpha \sum w_{s,p,j} |\langle u|\psi_j\rangle|^p.$$

Daubechies, Defries, De Mol, 2004.

▸ Very short sketch of proof ▸ Sketch of proof ▸ Continue

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Daubechies, Defries, De Mol, 2004.

▸ Very short sketch of proof ▸ Sketch of proof ▸ Continue

Proof: decouple A and $\|\cdot\|^p$, surrogate functional:

$$\bar{J}(u,a) = \|Au - y^*\|^2 + 2\alpha\|u\|^p_{B^s_{p,p}} + \left(\|u-a\|^2 - \|Au - Aa\|^2\right)$$

alternate minimization for u, a.

Outline

Classical: Conditional gradient method

$$\min_{u \in C_{\mathrm{ad}}} F(u) \text{ by}$$

Classical: Conditional gradient method

$\min_{u \in C_{\text{ad}}} F(u)$ by

1. directional derivative

$$\min_{v \in C_{\text{ad}}} \langle F'(u_n) | v \rangle$$

2. line search

$$\min_{s \in [0,1]} F(u_n + s(v_n - u_n))$$

3. update

$$u_{n+1} = u_n + s_n(v_n - u_n)$$

Dunn (1980), F convex, differentiable, e. g. $F(u) = \|Au - y^*\|^2$.

Preparing for generalization

$$I_{C_{\mathrm{ad}}}(u) = \begin{cases} 0 & u \in C_{\mathrm{ad}} \\ \infty & u \notin C_{\mathrm{ad}} \end{cases} \quad \rightarrow \quad \min F(u) + \Phi(u) \quad (\Phi(u) = I_{C_{\mathrm{ad}}}(u))$$

F: smooth, minimization hard, main ingredient

Φ: not differentiable, minimization easy, influence rather small

Classical: Conditional gradient method

$$\min_{u\in C_{\mathrm{ad}}} F(u) \text{ by}$$

1. directional derivative

$$\min_{v\in C_{\mathrm{ad}}} \langle F'(u_n)|v\rangle$$

2. line search

$$\min_{s\in[0,1]} F(u_n + s(v_n - u_n))$$

3. update

$$u_{n+1} = u_n + s_n(v_n - u_n)$$

Now: Generalized Conditional gradient method

$$\min_{u \in C_{\text{ad}}} F(u) \text{ by}$$

1. directional derivative

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2. line search

$$\min_{s \in [0,1]} F(u_n + s(v_n - u_n))$$

3. update

$$u_{n+1} = u_n + s_n(v_n - u_n)$$

Now: Generalized Conditional gradient method

$\min F(u) + \Phi(u)$ by

1. directional derivative

$$\min_{v \in C_{\mathrm{ad}}} \langle F'(u_n) | v \rangle$$

2. line search

$$\min_{s \in [0,1]} F(u_n + s(v_n - u_n))$$

3. update

$$u_{n+1} = u_n + s_n(v_n - u_n)$$

Now: Generalized Conditional gradient method

$\min F(u) + \Phi(u)$ by

1. directional derivative

$$\min_v \langle F'(u_n) | v \rangle + \Phi(v)$$

2. line search

$$\min_{s\in[0,1]} F(u_n + s(v_n - u_n))$$

3. update

$$u_{n+1} = u_n + s_n(v_n - u_n)$$

Now: Generalized Conditional gradient method

$\min F(u) + \Phi(u)$ by

1. directional derivative

$$\min_v \langle F'(u_n) | v \rangle + \Phi(v)$$

2. line search

$$\min_{s \in [0,1]} (F + \Phi)(u_n + s(v_n - u_n))$$

3. update

$$u_{n+1} = u_n + s_n(v_n - u_n)$$

Convergence of the generalized conditional gradient method

Theorem

Φ proper, convex, lsc.
F continuously differentiable, $F + \Phi$ coercive and

$$E_t = \{\Phi(u) \leq t\} \text{ compact for every } t.$$

Then: convergence to a stationary point of $F + \Phi$.

K. Bredies, D. L., P. Maaß, 2005

Remark: F need not to be convex.

▸ Sketch of proof

Outline

The surrogate method as a gradient method

Corollary

The minimization of

$$\|Au - y^*\|^2 + 2\alpha \sum w_{s,p,j} |\langle u|\psi_j\rangle|^p$$

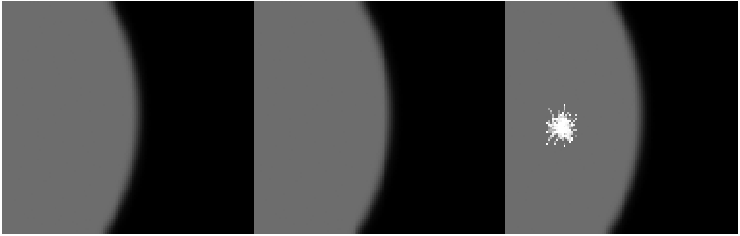
by means of the surrogate method is the same as the application of the generalized conditional gradient method for

$$F(u) = \|Au - y^*\|^2 - \lambda\|u\|^2,$$
$$\Phi(u) = \lambda\|u\|^2 + 2\alpha \sum w_{p,j} |\langle u|\psi_j\rangle|^p$$

for $\lambda = 1$ without line search.

Outline

Application: artificial mammogram



y_0 $y(t)$ y_f

Outline

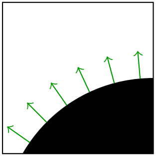
Summary

1. Presentation of mammography images can be improved by optimal control of PDEs.
2. Non standard penalty terms arise from image processing.
3. The generalized conditional gradient method converges even for non-convex functionals.
4. The surrogate method from inverse problems is equivalent to a generalized conditional gradient method.

Part I

Appendix

Different choice of control



- Edge adaptive:

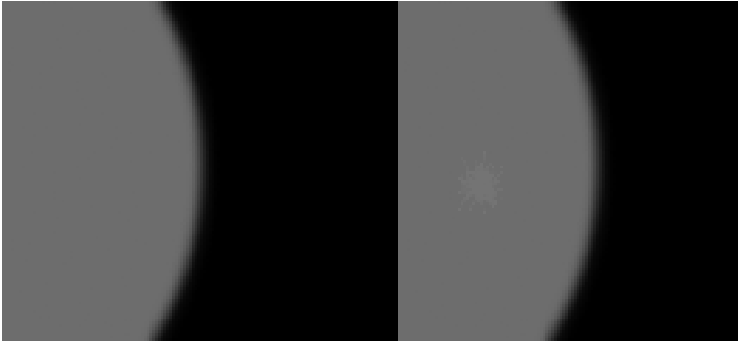
$$y_t - \operatorname{div}((I - p \otimes p)\nabla y) = u$$

- Two control terms: source u and diffusion tensor p

Tasks:

- $(p, u) \in L^2(Z, \mathbb{R}^2) \times L^2(Z, \mathbb{R}) \;\leftrightarrow\; y \in ?$
- solution operator $A : L^2(Z, \mathbb{R}^2) \times L^2(Z, \mathbb{R}) \;\rightarrow ?$

Optimal control vs. linear interpolation



Convergence of the surrogate method

Sketch of proof.

1. Define the surrogate functional

$$\bar{J}(u,a) = \|Au - y^*\|^2 + 2\alpha \|u\|_{B^s_{p,p}}^p + \left(\|u-a\|^2 - \|Au - Aa\|^2\right).$$

2. Iteration $u_{n+1} = \operatorname{argmin}_u \bar{J}(u, u_n) \to$ shrinkage
3. Opial's fixed point theorem $\to$ weak conv. to a fixed point (non-expansive, asymptotically regular, fixed point exists)
4. More work, use special structure $\to$ strong convergence

□

▸ Back to theorem

Convergence of the generalized conditional gradient method

Sketch of proof.

1. Necessary first order condition:

$$\forall v : \langle F'(u)|u\rangle + \Phi(u) = \min_v \langle F'(u)|v\rangle + \Phi(v)$$

2. Condition not fulfilled: $(F+\Phi)(u^{n+1}) < (F+\Phi)(u^n)$
3. $\Psi(u) := \langle F'(u)|u\rangle + \Phi(u) - \big(\min_v \langle F'(u)|v\rangle + \Phi(v)\big)$ is lsc
4. $\Psi(u^n) \to 0$

□

Back to theorem