Proof of Theorem 4.15:

First, we prove the result for $\tau = 0$, i.e., in the case of testing a point hypothesis.

Due to Theorem 3.22, the MLE $\hat{\theta}_m$ admits an asymptotic expansion of the form

$$f_m \approx \frac{f(z_m^r, z_m)}{f(z_0, z_0)} = \frac{f(\hat{\theta}_m^r, \hat{\theta}_m)}{f(\theta_0, \theta_0)} + o_p(1),$$

where $z_m^r = 2m \log(1, I(z_m))$ and $z_m = 2m \log(1, I(\theta_0))$.

We define $\Delta_m(x) = \ln(\Theta_m)$ and $\hat{\Delta}_m(x) = \ln(\hat{\Theta}_m)$. 

The preceding theorem, in connection with $\Delta_0(x) = \ln(\Theta_0)$, $\lambda_m = \ln(\Theta_m)$, and Remark 12.2.1, in Lehmann and Romano (2005) [3],

yield that under $H_0$.

$\Delta_m(x) \rightarrow 0$ in probability for any fixed $c > 0$.

By the triangle inequality,

$2 \log \lambda_m \leq E [\log \lambda_m^2] - \frac{1}{2} \epsilon \log \lambda_m + \text{Enic}.$

as soon as $\| \hat{\Theta}_m - \Theta_0 \| = o_p(1)$.
But, using $I_\Psi$ we get

$$2 \mathbb{E} \left< \hat{\lambda}_m, z_m \right> = \frac{1}{2} \mathbb{E} \left< \hat{\lambda}_m, I(\theta_0) \hat{\lambda}_m \right>$$

$$= z_m^T I^{-1}(\theta_0) z_m + \sigma_{\theta_0}^m (\lambda).$$

$$= 2 \log (L_m, \hat{z}_m) \leq z_m^T I^{-1}(\theta_0) z_m + \tilde{\epsilon}_{\text{mic}}$$

if $|\hat{\lambda}_m| \leq c$, where $\tilde{\epsilon}_{\text{mic}} \rightarrow 0$ for any $c > 0$.

$$\Rightarrow P_{\theta_0} \left( 2 \log (L_m, \hat{z}_m) \geq x \right)$$

$$\leq P_{\theta_0} \left( z_m^T I^{-1}(\theta_0) z_m + \tilde{\epsilon}_{\text{mic}} \geq x, |\hat{\lambda}_m| \leq c \right)$$

$$+ P_{\theta_0} \left( |\hat{\lambda}_m| > c \right)$$

$$\leq P_{\theta_0} \left( z_m^T I^{-1}(\theta_0) z_m + \tilde{\epsilon}_{\text{mic}} \geq x \right) + P_{\theta_0} \left( |\hat{\lambda}_m| > c \right).$$

Now, notice under $\theta_0$ that $z_m^T I^{-1}(\theta_0) z_m \xrightarrow{D} \chi^2_2$ and $\hat{\lambda}_m \xrightarrow{D} \mathcal{N}(0, I^{-1}(\theta_0))$ as $m \rightarrow \infty$.

So, (34) tends to

$$P \left( \chi^2_2 \geq c \right) + P \left( |z_2| > c \right).$$

Let $c \rightarrow \infty$ to conclude

$$\lim_{m \rightarrow \infty} \sup_{\theta_0} P_{\theta_0} \left( 2 \log (L_m(\lambda)) \geq x \right)$$

$$\leq P \left( \chi^2_2 \geq x \right).$$
Similarly, deduce that
\[
\liminf_{n \to \infty} 2 \log \left( A_n (X) \right) \geq \mathbb{P}(X_0^2 \geq x),
\]
proving the case $r = 0$.

The proof for general $r$ relies on the same argumentation and the following additional considerations for normally distributed random vectors:

Suppose $X = (X_1, \ldots, X_R)^T$ is multivariate normal with unknown mean vector $\mu$, but known positive definite covariance matrix $\Sigma$. The likelihood function of $X$ is given by
\[
L(\mu, X) = \frac{1}{(2\pi)^{R/2} |\Sigma|^{1/2}} \exp \left[ -\frac{1}{2} (X - \mu)^T \Sigma^{-1} (X - \mu) \right].
\]

\[
= 2 \log(2 \log(\mathcal{A}_1 (X)))
\]
\[
\quad = - \inf_{\nu \in \mathbb{R}^R} (X - \nu)^T \Sigma^{-1} (X - \nu) + \inf_{\nu \in \mathcal{O}_0} (X - \nu)^T \Sigma^{-1} (X - \nu).
\]
Letting $z = z^{1/2} x$, we have, equivalently,

$$z \log \left( A_n(x) \right) = \inf_{y \in \mathbb{Q}} \| z - z^{-1/2} y \|^2.$$

As $z$ varies in $\mathbb{Q}^+$, $z^{-1/2} y$ varies in an $r$-dimensional subspace of $\mathbb{R}^q$, say $M$. If $P$ is the projection matrix onto $M$, then

$$z \log \left( A_n(x) \right) = \| (I_q - P) z \|^2.$$

Theorem 4.32.2 completes the argumentation.